

## Summary: Lecture 8

Summary for the chapters 9.3 and 9.4. [2, 1]

### Undirected graph

An undirected graph  $G$  is a pair of sets  $(V, E)$  where  $V$  is the finite set of nodes and  $E$  is a set of unordered pairs in  $V$  that are symmetric:

$$\forall i, j \in V, i \neq j : (i, j) \in E \Rightarrow (j, i) \in E$$

### IndependentSet

#### IndependentSet

Input: An undirected Graph  $G = (V, E)$  and a number  $k$ .

Question: Is there a set  $I \subseteq V$  of  $k = |I|$  nodes with no edges in between? (INDEPENDENTSET)

#### 3SAT

Like the SAT problem, 3SAT is determining the satisfiability of a formula in CNF where each clause is limited to at most three literals.

INDEPENDENTSET is NP-complete.

#### Proof idea:

- triangle construction: any independent set can contain at most one node of the triangle

$$(x_1 \vee x_2 \vee x_3) \wedge (\neg x_1 \vee \neg x_2 \vee \neg x_3) \wedge (\neg x_1 \vee x_2 \vee x_3)$$

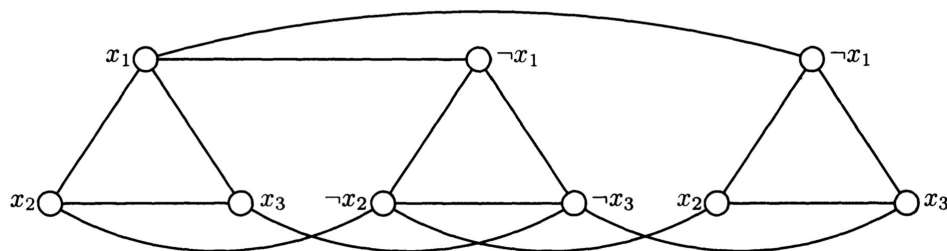


Figure 1: Graph with triangles [2]

- consider only graphs whose nodes can be partitioned in  $m$  disjoint triangles  
→ independent set can contain at most  $m$  nodes (one from each triangle)
- reduction from 3SAT to INDEPENDENTSET
- construct graph of formula  $\phi$ :
  - each literal as a node
  - clauses as triangles
  - edges between nodes in different triangles if they correspond to the same literal (negated)
  - $K = m$  ( $m$  clauses)

- given: instance  $\phi$  of 3SAT with  $m$  clauses  $C_1, \dots, C_m$
- each clause  $C_i = (\alpha_{i1} \vee \alpha_{i2} \vee \alpha_{i3})$  (with  $\alpha$  as boolean variables or negation of those)
- reduction  $R$  constructs a graph:  $R(\phi) = (G, K)$  where  $K = m$  and  $G = (V, E)$
- nodes  $V = \{v_{ij} : i = 1, \dots, m; j = 1, 2, 3\}$   
nodes for each of the  $m$  clauses ( $i$ ) for each of the 3 literals ( $j$ )
- edges  $E = \{[v_{ij}, v_{ik}] : i = 1, \dots, m; j \neq k\} \cup \{[v_{ij}, v_{lk}] : i \neq l, \alpha_{ij} = \neg \alpha_{lk}\}$   
edges between the nodes in one clause (triangle edges)  
edges between nodes with the same corresponding literal, but negated
- there is an independent set  $I$  of  $K$  nodes in  $G$  only if  $\phi$  is satisfiable
- $I$  must contain a node from each triangle
- negated literals are connected:  $I$  cannot contain a literal and its negation
- $I$  is a truth assignment of  $\phi$ :
  - true literals: nodes in  $I$
  - one true literal per clause

## HamiltonPath is NP-complete

### HamiltonPath

A HAMILTONPATH is a path in a graph that visits each node exactly once.

HAMILTONPATH is NP-complete.

### Proof idea:

- reduction from 3SAT to HAMILTONPATH
- given: formula  $\phi$  in CNF with  $n$  variables  $x_1, \dots, x_n$  and  $m$  clauses  $C_1, \dots, C_m$  with each 3 variables
- construct a graph  $R(\phi)$  that has a hamilton path only if  $\phi$  is satisfiable:
- boolean variables:
  - choice between true and false
  - all occurrences of  $x$  must have the same value (and  $\neg x$  the opposite)
  - use *choice* gadget (like flip flop)
- XOR:

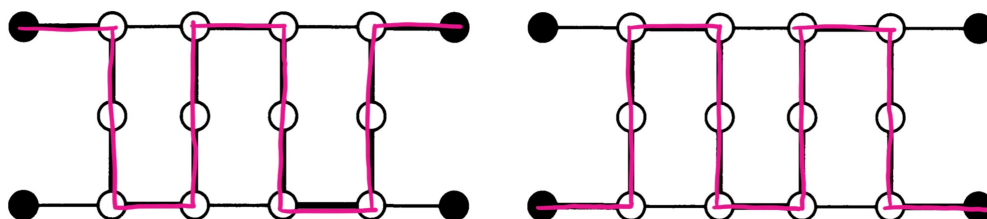


Figure 2: XOR subgraph from the book [2] with the relevant edges marked additionally

- use *consistency* gadget
- because of hamilton path: there are only two ways to traverse through this sub graph (as shown above)
- leads to exclusive or (XOR)

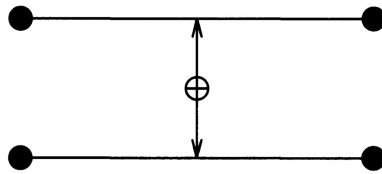


Figure 3: XOR connecting two independent edges (*consistency* gadget) [2]

- clauses:
  - triangles for clause construction
  - one side for each literal
  - if literal is false: hamilton path traverses triangle side
  - at least one literal need to be true: else all three edges of triangle will be traversed and this is not a hamilton path
- put everything together as graph  $G$ :
  - $G$  has  $n$  copies of the *choice* gadget as a chain (one for each variable)

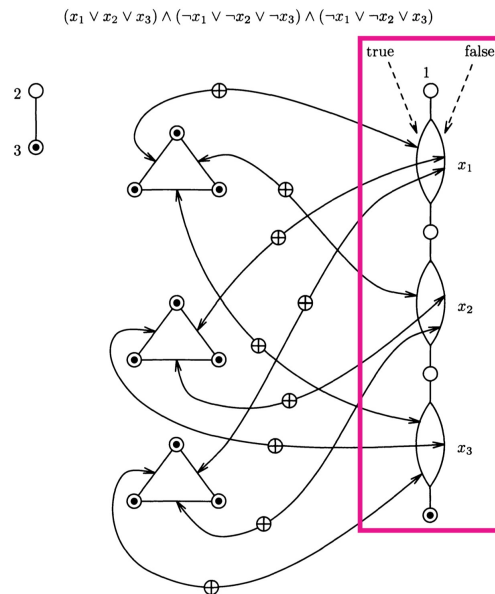


Figure 4: *Choice* gadgets marked in graph from the book [2]

- $G$  has  $m$  triangles (one for each clause) with edges for each clause in the triangle

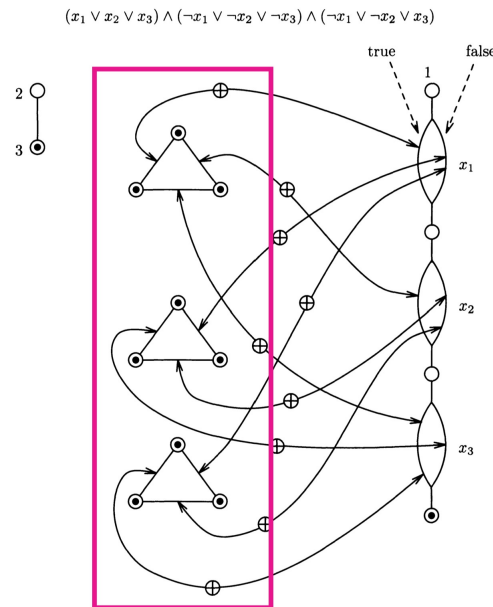


Figure 5: Clauses marked in graph from the book [2]

- finally all  $3m$  nodes of the triangles, the last node of the chain of *choice* gadgets and a new node 3 are connected with all possible edges
- a single node 2 is connects to the node 3
- graph has a hamilton path only if  $\phi$  has a satisfying truth assignment

TODO

Questions:

## TSP(D)

### TSP(D)

TSP(D) is a decision version of TSP.

Input: A  $n \times n$  distance matrix and a bound  $B \in \mathbb{N}$

Question: Is there a round tour of length  $\leq B$  that visits all *cities*?

TSP(D) is NP-complete.

**Proof idea:**

- budget of nodes is  $B = |V| + 1$

TODO

Questions:

## Knapsack

### Knapsack

KNAPSACK is NP-complete.

- filled in in one dimensional array onthe board

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TODO  
Questions:

## References

- [1] Martin Berglund. *Lecture notes in Computational Complexity*.
- [2] Christos H. Papadimitriou. *Computational Complexity*. Addison-Wesley Publishing Company, 1994.