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# Summary: Lecture 5

Summary for the chapters $X$ and $X$. [5]

## Reduction

**Examples of NP-problems:**

- Travelling Salesman Problem
- SATISFIABLE
- REACHBILITY (in P)
- CIRCUIT VALUE (in P)

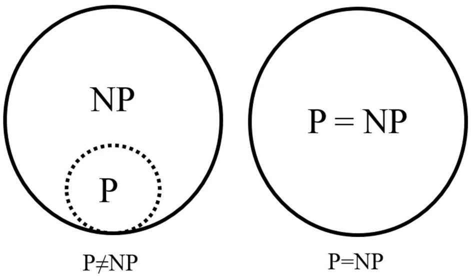


Figure 1: P and NP sets [3]

- reduction: a problem is at least as hard as another
- problem $A$ is at least as hard as problem $B$ if $B$ reduces to $A$
- $B$ reduces to $A$ if there is a transformation $R$
  - $R$ produces for every input $x$ of $B$ an equivalent input $R(x)$ of $A$
  - the answer of input $x$ on $B$ and input $R(x)$ on $A$ have to be the same
- to solve $B$ on input $x$, $A$ can be solved instead with input $R(x)$

**Reduction**

Problem $A$ is at least as hard as problem $B$ if $B$ reduces to $A$.

**Transformation function:**

- tranformation function $R$ should not be too hard to compute
  $\rightarrow$ $R$ should be limited
- efficient reduction $R$: $\log n$ space bounded

  **Transformation function**

  A language $L_1$ is reducible to $L_2$ if there is a function $R$ computable by a deterministic Turing Machine in space $O(\log n)$ and $x \in L_1 \Leftrightarrow R(x) \in L_2$.

  $R$ is called a reduction from $L_1$ to $L_2$.

- A Turing Machine $M$ that computes a reduction $R$ halts for all inputs $x$ after a polynomial number of steps.
  - there are $O(n \cdot c^{\log n})$ possible configurtions for $M$ on an input of length $n$
  - deterministic: no configuration can be repeated
  - computation of length at most $O(n^k)$

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## Reduction HAMILTONIAN PATH to SATISFIABLE

**Problem: HAMILTON PATH**

The Hamiltonian Path problem asks whether there is a route in a directed graph $G$ from a start node to an ending node, visiting each node exactly once. (Is there a path in $G$ that visits each node one?) [1]

**Problem: SAT**

The SAT (satisfiability) problem is the problem of determining if there exists an interpretation that satisfies a given Boolean formula. [6]

- HAMILTON PATH can be reduced to SAT
  $\rightarrow$ demonstrates HAMILTON PATH is not significantly harder that SAT
- construct a boolean expression $R(G)$ that is satisfiable only if $G$ has a Hamilton path
  $\rightarrow$ write a logical formular that only becomes true when HP is true
- instance: Graph $G = (V, E)$ with $n$ nodes $(1, 2, ..., n)$
- $R(G)$ has $n^2$ boolean variables $x_{i,j}$ then
- node $j$ is the $i$th node in the HAMILTON PATH
- $R(G)$ is in conjuctive normal form (CNF: $(a \vee b) \wedge (\neg a \vee c)$)
- conjuncted clauses of $R(x)$:
  - each node $j$ must appear in the path $x_{1,j} \vee x_{2,j} \vee ... \vee x_{n,j}$ – for every node $j$
  - no node $j$ appears twice in the path: $\neg x_{i,j} \vee \neg x_{k,j}$ for all $i, j, k$ with $i \neq k$
  - every position $i$ on the path must be occupied – $x_{i,1} \vee x_{i,2} \vee ... \vee x_{i,n}$ for each $i$
  - no two nodes $j$ and $k$ occupy the same position in the path – $\neg x_{i,j} \vee \neg x_{i,k}$ for all $i, j, k$ with $j \neq k$
  - nonadjacent nodes $i$ and $j$ cannot be adjacent in the path – $\neg x_{k,i} \vee \neg x_{k+1,j}$ for all $(i, j) \notin E$ and $k = 1, 2, ..., n - 1$

  [4]

**Proof idea:**

- to show:
  - for any graph $G$, $R(G)$ has a satisfying truth assignment only if and only if $G$ has a Hamilton path
  - $R$ can be computed in space $\log n$
- 

TODO
proof
Questions:

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**Boolean Circuit**

A Boolean circuit is a mathematical tree model for logic formulas.
Boolean circuits are defined in terms of the logic gates they contain. For example, a circuit might contain binary AND and OR gates and unary NOT gates, or be entirely described by binary NAND gates.

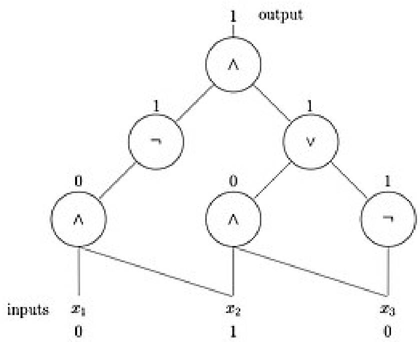


Figure 2: Boolean circuit example [2]

## Reduction REACHABILITY PATH to CIRCUIT VALUE

**Problem: GRAPH REACHABILITY**

Given a graph $G$ and two nodes $n_1, n_2 \in V$, is there path from $n_1$ to $n_2$?
A graph $G = (V, E)$ is a finite set $V$ of nodes and a set $E$ of edges as node pairs.

REACHIBILITY can be nondeterministically solved in space $\log n$.

**Problem: CIRCUIT VALUE**

The CIRCUIT VALUE Problem is the problem of computing the output of a given Boolean circuit on a given input.
In terms of time complexity, it can be solved in linear time (topological sort).
The problem is closely related to the SAT (Boolean Satisfiability) problem which is complete for NP and its complement, which is complete for co-NP.

TODO
Questions:

## Reduction CIRCUIT SAT to SAT

TODO
Questions:

## Further examples

TODO
Questions:

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## Closedness under Composition

TODO
Questions:

## References

[1] J. Baumgardner, K. Acker, O. Adefuye, and et al. "Solving a Hamiltonian Path Problem with a bacterial computer". In: *J Biol Eng* 3.11 (2009). DOI: `https://doi.org/10.1186/1754-1611-3-11`.

[2] *Image source: Boolean Circuit.* `https://upload.wikimedia.org/wikipedia/en/thumb/d/df/Three_input_Boolean_circuit.jpg/300px-Three_input_Boolean_circuit.jpg`.

[3] *Image source: P-NP sets.* `https://www.techno-science.net/actualite/np-conjecture-000-000-partie-denouee-N21607.html`.

[4] Prof. Yuh-Dauh Lyuu. *Lecture slides.* `https://www.csie.ntu.edu.tw/~lyuu/complexity/2011/20111018.pdf`. 2011.

[5] Christos H. Papadimitriou. *Computational Complexity.* Addison-Wesley Publishing Company, 1994.

[6] Prof. Dr. Thomas Schwentick. *Lecture slides in Grundbegriffe der theoretischen Informatik.* `https://www.cs.tu-dortmund.de/nps/de/Studium/Ordnungen_Handbuecher_Beschluesse/Modulhandbuecher/Archiv/Bachelor_LA_GyGe_Inf_Modellv/_Module/INF-BfP-GTI/index.html`.