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Summary: Lecture 10

Summary for the chapters *11.1* up to page 245 and *11.2* (page 258 optional). [5, 1]

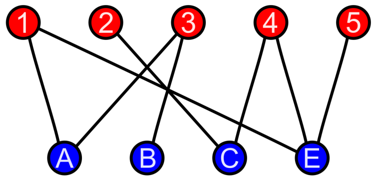
Randomized algorithms

Algorithms based on randomization.
(The algorithm employs a degree of randomness as part of its logic or procedure.)

Bipartite matching

Bipartite Graph

A graph $G = (U, V, E)$ is called bipartite if the vertices can be divided into two disjoint and independent sets U and V. (There are no edges between two elements of U or two elements of V).



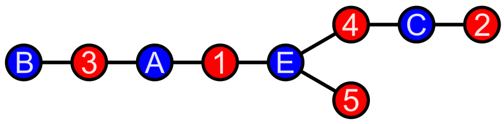


Figure 1: Examples of bipartife graphs with U and V marked in red and blue [2]

Problem: BipartiteMatching

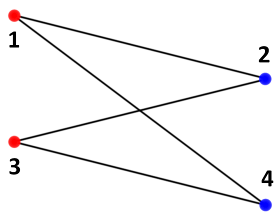
Given: Bipartite graph $G = (U, V, E)$.

Is there a perfect matching $M \subseteq E$ such that for any two edges (u, v) and (u', v') in M $u \neq u'$ and $v \neq v'$.

I other words: A matching in a Bipartite Graph is a set of the edges chosen in such a way that no two edges share an endpoint. The matching M is called perfect if for every node in V there is some edge in M. [3, 5, 4]

- construct bipartite graph with n nodes as $n \times n$ matrix A
- the element $A_{i,j}$ is a variable $x_{i,j}$ if $(i, j) \in E$
- the element $A_{i,j}$ is 0 if $(i, j) \notin E$

Example:



$$A = \begin{pmatrix} 0 & x_{1,2} & 0 & x_{1,4} \\ x_{2,1} & 0 & x_{2,3} & 0 \\ 0 & x_{3,2} & 0 & x_{3,4} \\ x_{4,1} & 0 & x_{4,3} & 0 \end{pmatrix}$$

Questions:
Is EVERY node cotained in the subset M when it is a perfect matching? Does a perfect matching then only exist with an even number of vertices and $|U| = |V|$?

Determinant calculation

Leibniz-formula:

$$A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \qquad |A| = a{\cdot}e{\cdot}i+b{\cdot}f{\cdot}g+c{\cdot}d{\cdot}h-g{\cdot}e{\cdot}c-h{\cdot}f{\cdot}a-i{\cdot}d{\cdot}b$$

$$B = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \qquad |B| = 1{\cdot}5{\cdot}9+2{\cdot}6{\cdot}7+3{\cdot}4{\cdot}8-7{\cdot}5{\cdot}3-8{\cdot}6{\cdot}1-9{\cdot}4{\cdot}2 = 0$$

$$|A| = \sum_{\pi} \sigma(\pi) \prod_{i=1}^{n} A_{i,\pi(i)}$$

- $\sigma(\pi)$ decides if $+$ or $-$
- leads to $n!$ summands
- Example: $n = 3$
  $3! = 6$ summands
  6 permutations for π

$+1$: $(\ 1\ 2\ 3\ )$ $(\ 2\ 3\ 1\ )$ $(\ 3\ 1\ 2\ )$

-1: $(\ 3\ 2\ 1\ )$ $(\ 2\ 1\ 3\ )$ $(\ 1\ 3\ 2\ )$

$\rightarrow |A| = A_{1,1} \cdot A_{2,2} \cdot A_{3,3} + A_{2,1} \cdot A_{3,2} \cdot A_{1,3} + ...$

Gaussian elimination:

- Gauß algorithm for solving LSE (linear systems of equations)
- allowed operations:
  - addition of rwos
  - subtraction of rows
  - multiply row with integer x
  - divide row by integer x
  - switch to rows
- wanted: upper triangular form (all entries below the diagonal 0)
- determinant is the product of the diagonal entries
- Example:

$$\begin{pmatrix} 1 & 3 & 2 & 5 \\ 1 & 7 & -2 & 4 \\ -1 & -3 & -2 & 2 \\ 0 & 1 & 6 & 2 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 3 & 2 & 5 \\ 0 & 4 & -4 & -1 \\ 0 & 0 & 0 & 7 \\ 0 & 1 & 6 & 2 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 3 & 2 & 5 \\ 0 & 4 & -4 & -1 \\ 0 & 0 & 0 & 7 \\ 0 & 0 & 7 & 2\frac{1}{4} \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 3 & 2 & 5 \\ 0 & 4 & -4 & -1 \\ 0 & 0 & 7 & 2\frac{1}{4} \\ 0 & 0 & 0 & 7 \end{pmatrix}$$

Figure 2: Examples gaussian elimination [5]

$|A| = 1 \cdot 4 \cdot 7 \cdot 7 = 196$

Symbolic Determinants

Symbolic matrix:

- matrix with variables instead of numerical entries
- Example:

$$\begin{pmatrix} x & w & z \\ z & x & w \\ y & z & 0 \end{pmatrix} \Rightarrow \begin{pmatrix} x & w & z \\ 0 & \frac{x^2-zw}{x} & \frac{wx-z^2}{x} \\ 0 & \frac{zx-wy}{x} & -\frac{zy}{x} \end{pmatrix} \Rightarrow$$
$$\begin{pmatrix} x & w & z \\ 0 & \frac{x^2-zw}{x} & \frac{wx-z^2}{x} \\ 0 & 0 & -\frac{yz(xz-xw)+(zx-wy)(wx-z^2)}{x(x^2-zw)} \end{pmatrix}$$

Figure 3: Examples gaussian elimination with symbolic matrix [5]

- subdeterminants have exponentially many terms
- wanted result: know if determinant is 0

Monte Carlo algorithm

Check if determinant of symbolic matrix is 0:

- use arbitrary integers for the variables
  → numerical matrix
- calculate determinant of numerical matrix:
  - if not 0:
    determinant of symbolic matrix is not 0
  - if 0:
    determinant of symbolic matrix is probably 0
    - numbers could be chosen such that the numrical determinant is 0 even though the symbolic one is not 0

Monte Carlo algorithm

Randomized algorithm for deciding if a graph G has a perfect matching with calculating the determinant of the corresponding matrix A to G.

- choose m random integers $i_1, ..., i_m$ between 0 and $2m$
- compute the determinant $|A|(i_1, ..., i_m)$ with the Gaussian elimnation
- if $|A|(i_1, ..., i_m) \neq 0$ reply *G has a perfect matching*
- if $|A|(i_1, ..., i_m) = 0$ reply *G has probably no perfect matching*

- if perfect matching found: decision is reliable and final
- if perfect matching not found: possibility of false negative

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Monte Carlo algorithm

(Algorithm above) decides whether a symbolic matrix is **not** indentically to zero.

Reducing chance of false negatives:

- perform many independent experiments
- chose each time random integers (independently)
- repeat k times the evaluation of the determinant of the symbolic matrix
  - answer always zero:
    chance that G hs no perfect matching is higher $(1 - (\frac{1}{2})^k)$
  - answer different from zero once:
    perfect matching exists

Monte Carlo algorithm:

- Monte Carlo algorithm has no false positives
- probability of false negatives is bounded
- time needed always polynomieal

Randomized complexity classes

Monte Carlo Turing Machine

A polynomial Monte Carlo Turing Machine M that decides a language L is a nondeterministic Turing Machine with exactly two choices in each step and the following conditions:

- if $x \in L$ then at least half of the computations on x halt with *yes*
- if $x \notin L$ then all computations halt with *no*

- randomized algorithms can be modeled with an ordinary non-deterministic Turing Machine with different interpretation of the meaning of accepting the input
- no false positive answers+
- probability of false negatives is at most $\frac{1}{2}$

RP

Complexity class of all languages that are decided with polynomial Monte Carlo Turing Machines is denoted as RP.

- RP lies between P and NP (P $\subseteq$ RP $\subseteq$ NP)

ZPP

LasVegas algorithm

The LasVegas algorithm as a Monte Carlo algorithm and its complement (one has no false positives and one has no false negatives). It runs k independent experiments on both and the right answer will come up. (Either a positive answer from the one with no false positives or a negative answer from the one with no false negatives.)

- RP: no false positive answers but false negative answers possible
- coRP: no false negative answers but false positive answers possible
- RP $\cap$ coRP seems interesting:
  - problem in this class has two Monte Carlo algorithms:
    * one has no false positives
    * one has no false negatives
  - run enough experiments on both: right answer will come up
    (either a positive answer from the one with no false positives or a negative answer from the one with no false negatives)
  - correct answer will be known for sure
  - execute both algorithms independent k times: probability that the correct answer is not obtained is $\frac{1}{2^k}$

  $\rightarrow$ called LasVegas algorithm

ZPP

RP $\cap$ coRP

Complexity class of all languages that are decided with LasVegas algorithms is denoted as RP.

PP

Problem: MAJSAT

Given: Boolean expression φ

Is it true that the majority of the 2^n truth assignments to its variables satisfy it?

- MajSat is probably not in NP
- MajSat is probably not in RP
- MajSat is in PP

PP

The class PP contains all languages L, such that there is a nondeterminsic polynomial Turing Machine M such that for all inputs x:

$x \in L$ only if more than half of the computations end up accepting (*yes*).

M decides L by majority.

- PP is a syntactic class (not a semantic class)
- NP $\subseteq$ PP

Syntactic class:

- has a complete language L
- P and NP

Semantic class:

- no complete problems
- there is no easy way to tell whether a machine always halts with a certified output

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BPP

Answers (*yes* and *no*) are correct with probability $\frac{3}{4}$.

- unclear whether BPP $\subseteq$ NP
- BPP is closed under complement: coBPP $=$ BPP
- BPP is a semantic class

Relations between randomized complexity classes

$$\textsf{RP} \subseteq \textsf{NP} \subseteq \textsf{PP}$$

Figure 4: Relations between classes from the lecture slides [1]

- unclear whether BPP $\subseteq$ NP

$$\begin{tikzcd} & \textsf{RP} \arrow[dr, phantom, "\subseteq"] & \\ \textsf{P} \subseteq \textsf{ZPP} \arrow[ur, phantom, "\subseteq"] \arrow[dr, phantom, "\subseteq"] & & \textsf{BPP} = \textsf{coBPP} \subseteq \textsf{PP} \\ & \textsf{coRP} \arrow[ur, phantom, "\subseteq"] & \end{tikzcd}$$

Figure 5: Relations between classes from the lecture slides [1]

- RP $\subseteq$ BPP:
  - every language in RP has a BPP algorithm:
    * run algorithm twice to assure that the probability of false negatives is less than $\frac{1}{4}$
    * probability of false positives is 0 $(0 \leq \frac{1}{4})$
- BPP $\subseteq$ PP:
  - majority of PP: $> \frac{1}{2}$
  - majority of BPP: $> \frac{3}{4}$

$\rightarrow$ RP $\subseteq$ BPP $\subseteq$ PP

References

[1] Martin Berglund. *Lecture notes in Computational Complexity.*

[2] *Bipartite graph image source.* `https://en.wikipedia.org/wiki/Bipartite_graph`.

[3] GeeksforGeeks. *Maximum Bipartite Matching.* `https://www.geeksforgeeks.org/maximum-bipartite-matching/`, last opened: 09.12.22.

[4] Swastik Kopparty. *Bipartite Graphs and Matchings.* `https://sites.math.rutgers.edu/~sk1233/courses/graphtheory-F11/matching.pdf`, last opened 09.12.22. 2011.

[5] Christos H. Papadimitriou. *Computational Complexity.* Addison-Wesley Publishing Company, 1994.