2.165. Robotics Fall 25
Problem Set # 1.

Problem 1.

Given : $l_1 = 4.$ $l_2 = 3$
Joints Angles : $\theta_1.$ θ_2
Endpoint Velocity : $V = [v_x . v_y]^T$

(a)

$$J(\theta) = \begin{bmatrix} -l_1 \sin\theta - l_2 \sin(\theta_1+\theta_2) & -l_2 \sin(\theta_1+\theta_2) \\ l_1 \cos\theta + l_2 \cos(\theta_1+\theta_2) & l_2 \cos(\theta_1+\theta_2) \end{bmatrix}$$

$$\dot{q} = J^{-1} P$$

$$\dot{\theta}_1 = \frac{v_x \cos(\theta_1+\theta_2) + v_y \sin(\theta_1+\theta_2)}{\sin\theta_2}$$

$$\dot{\theta}_2 = -\frac{v_x[\cos\theta_1 + \cos(\theta_1+\theta_2)] + v_y[\sin\theta_1 + \sin(\theta_1+\theta_2)]}{\sin\theta_2}$$

(b) $\det(J) = l_1 l_2 S_2 = 0$

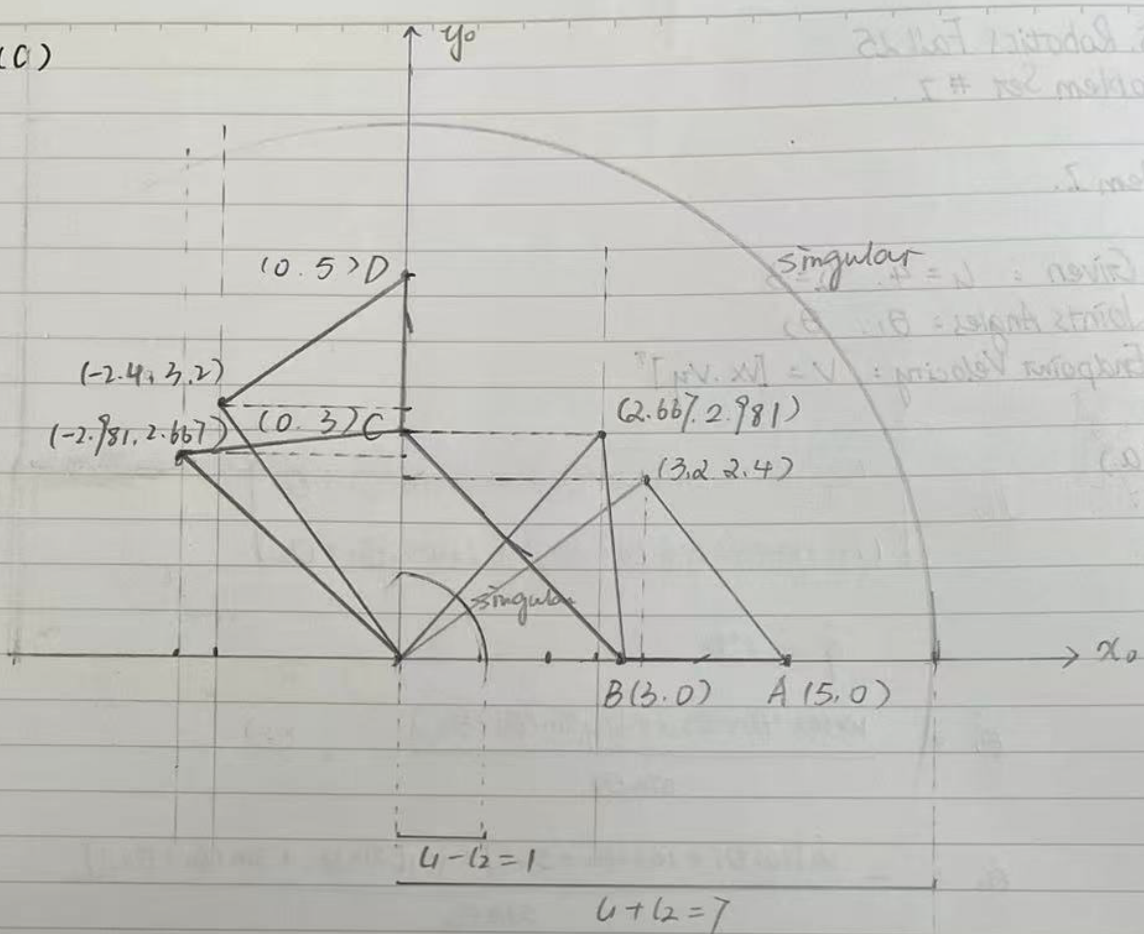
$$\sin\theta_2 = 0$$

$$\theta_2 = 0 \text{ or } \pi$$

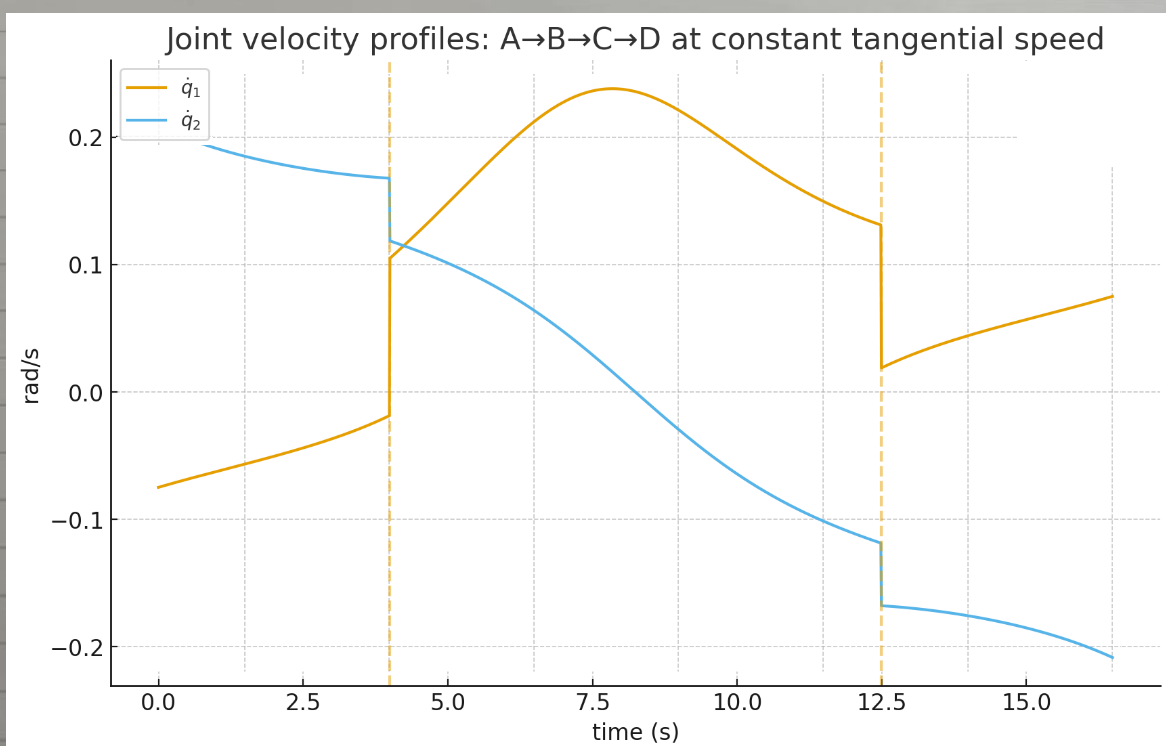
$$\dot{P} = \dot{\theta}_1 \begin{bmatrix} -l_1 \sin\theta_1 \\ l_1 \cos\theta_1 \end{bmatrix} + \dot{\theta}_2 \begin{bmatrix} -l_2 \sin(\theta_1+\theta_2) \\ l_2 \cos(\theta_1+\theta_2) \end{bmatrix}$$

Therefore, singular configurations occur for $\theta_2 = 0$ or $\theta_2 = \pi$
When the arm is fully extended or fully contracted
The corresponds to the endpoint positions shown the boundary of the reachable space and singular.

(c)



y_0
x_0
(0.5) D
(-2.4, 3.2)
(-2.981, 2.667)
(0.3) C
(2.667, 2.981)
(3.2 2.4)
singular
singular
B(3.0)
A (5.0)
$l_1 - l_2 = 1$
$l_1 + l_2 = 7$



Joint velocity profiles: A→B→C→D at constant tangential speed
$\dot{q}_1$
$\dot{q}_2$
rad/s
time (s)

Problem 2.

Given: $l_1 = l_2 = 3m$

Joint stiffness: $k_1 = 1 \times 10^5 Nm/rad$

$k_2 = 2 \times 10^5 Nm/rad$

Configuration: $\theta_1 = 30°$, $\theta_2 = 60°$

(a) $C_x = J K_q^{-1} J^T$

$$K_q = \begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix}$$

$$J = \begin{bmatrix} -l_1 \sin\theta_1 - l_2 \sin(\theta_1+\theta_2) & -l_2 \sin(\theta_1+\theta_2) \\ l_1 \cos\theta_1 + l_2 \cos(\theta_1+\theta_2) & l_2 \cos(\theta_1+\theta_2) \end{bmatrix}$$

$\sin(30°) = 0.5$

$\cos(30°) \approx 0.87$

$\sin(90°) = 1$

$\cos(90°) = 0$

$J_{11} = -l_1 \sin\theta_1 - l_2 \sin(\theta_1+\theta_2) = -1.5 - 3 = -4.5$

$J_{12} = -l_2 \sin(\theta_1+\theta_2) = -3$

$J_{21} = l_1 \cos\theta_1 + l_2 \cos(\theta_1+\theta_2) = 2.598$

$J_{22} = l_2 \cos(\theta_1+\theta_2) = 0$

$$J = \begin{bmatrix} -4.5 & -3 \\ 2.598 & 0 \end{bmatrix}$$

$$K_q^{-1} = \begin{bmatrix} 1/k_1 & 0 \\ 0 & 1/k_2 \end{bmatrix} = \begin{bmatrix} 1e-5 & 0 \\ 0 & 5e-6 \end{bmatrix} \text{ rad/Nm}$$

$C_x = J K^{-1}_q J^T$

$$C_x = J K_q^{-1} J^T$$

$$= \begin{bmatrix} -4.5 & -3 \\ 2.598 & 0 \end{bmatrix} \begin{bmatrix} 1e-5 & 0 \\ 0 & 5e-6 \end{bmatrix} \begin{bmatrix} -4.5 & 2.598 \\ -3 & 0 \end{bmatrix}$$

$$C_{11} = (-4.5e-5)(-4.5) + (-1.5e-5)(-3) = 2.475e-4 \text{ m/N}$$
$$C_{12} = (-4.5e-5)(2.598) + (-1.5e-5)(0) = -1.1691e-4 \text{ m/N}$$
$$C_{21} = (2.598e-5)(-4.5) + 0 \cdot (-3) = -1.1691e-4 \text{ m/N}$$
$$C_{22} = (2.598e-5)(2.598) + 0 \cdot 0 = 6.75e-5 \text{ m/N}$$

$$C_x \approx \begin{bmatrix} 2.475 \times 10^{-4} & -1.169 \times 10^{-4} \\ -1.169 \times 10^{-4} & 6.75 \times 10^{-5} \end{bmatrix} \text{ m/N}$$

(b) $\det(C_x - \lambda I) = 0$

$$\begin{vmatrix} 2.475e-4-\lambda & -1.169e-4 \\ -1.169e-4 & 6.75e-5-\lambda \end{vmatrix} = 0$$

$$(2.475e-4-\lambda)(6.75e-5-\lambda) - (-1.169e-4)^2 = 0$$

$$\lambda = \frac{3.15e-4 \pm \sqrt{(3.15e-4)^2 - 4(3.04e-9)}}{2}$$

$$\lambda_1 \approx 3.0495e-4 \Rightarrow \lambda_{max}$$
$$\lambda_2 \approx 9.9e-6 \Rightarrow \lambda_{min}$$

Eigenvectors (direction of max/min compliance)

$$(C_x - \lambda I)v = 0$$

when $\lambda_1 = 3.05e-4$

$$-5.75e-5x - 1.169e-4y = 0 \Rightarrow y/x = 0.492$$

$$V_1 = V_{max} = \frac{1}{\sqrt{1+0.492^2}}[1, 0.492] = [0.8972, 0.4416]$$

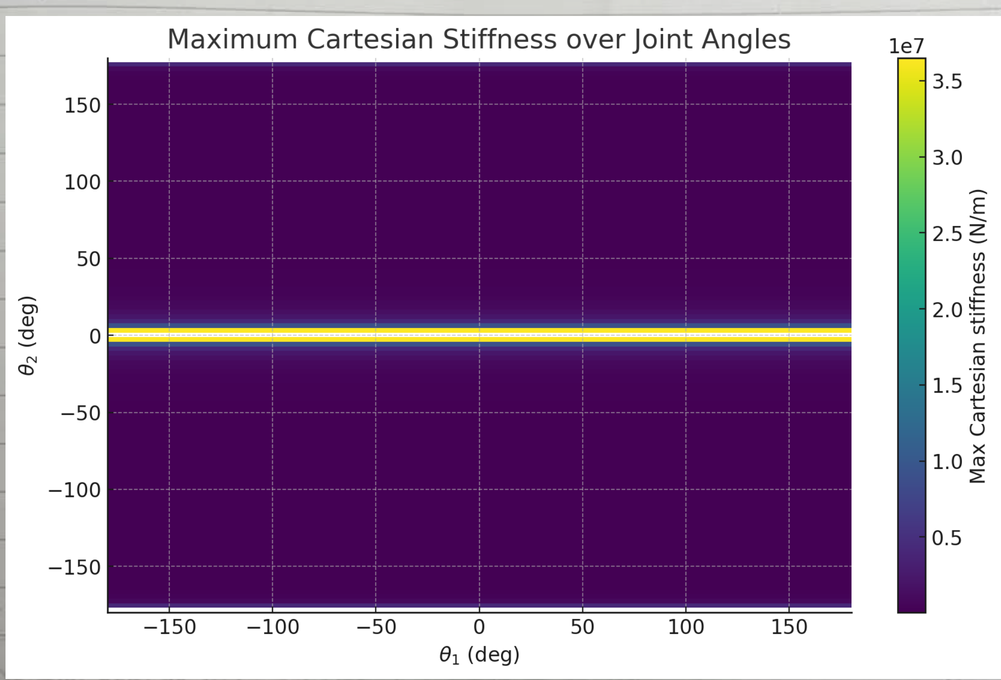
when $\lambda_2 = 9.9e-6$

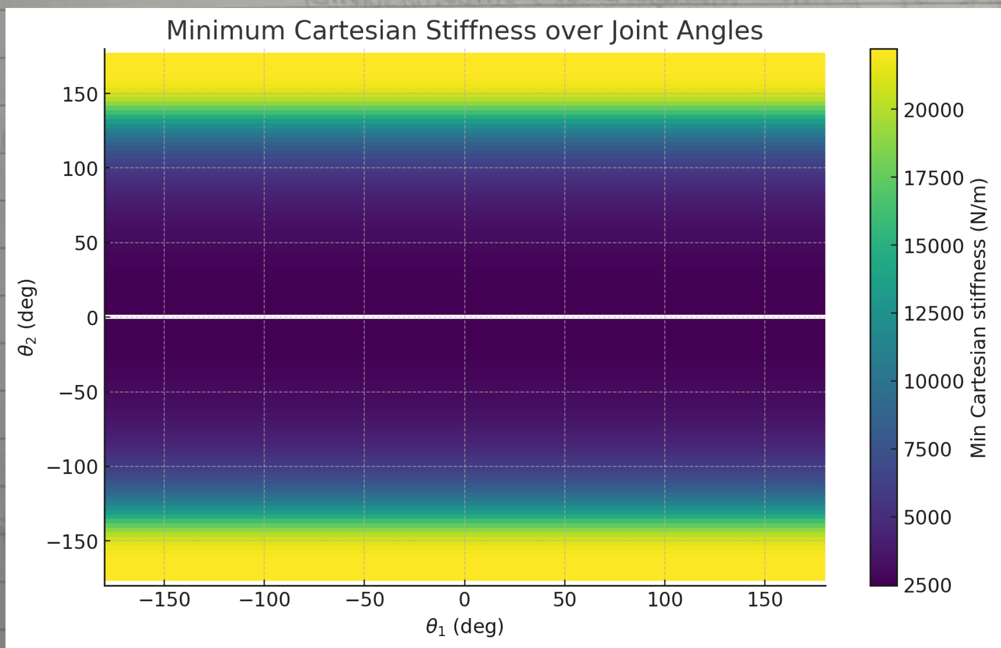
$$2.376e-4x - 1.169e-4y = 0 \Rightarrow y/x = 2.03$$

$$V_2 = V_{min} = \frac{1}{\sqrt{1+2.03^2}}[1, 2.03] = [0.4416, 0.8972]$$

· The direction of max compliance is [0.8972, 0.4416]
The direction of min compliance is [0.4416, 0.8972]

(C)

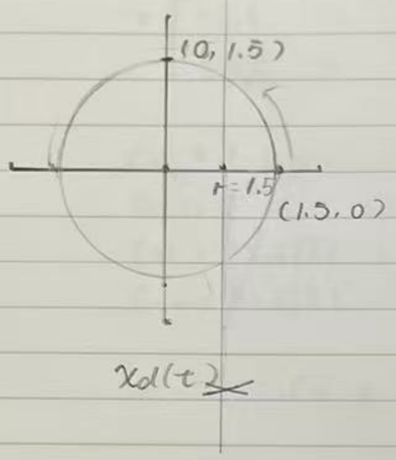




Problem 3.

Given: $l_1 = l_2 = 3m$ $\quad r = 1.5$

~~$\theta_1 = 30°$, $\theta_2 = 60°$~~ $\quad$ Inverse Kinematics



Part I. Explain Why define.

$$\dot{q}_r = \dot{q}_e - \lambda(q - q_e)$$

The invers kinematics problem requires finding the joint angles $q(t) = [\theta_1, \theta_2]^T$ corresponding to a desired end-effector trajectory $x_d(t)$. For a non-linear two-degree-of-freedom manipulator, directly solving

$$x_d(t) = f(q(t))$$

In the composite variable method, a desired joint configuration ~~q_e~~ $q_e(t)$ is introduced and updated according to the differiential equation:

$$\dot{q}_e = \dot{q}_r + \lambda(q - q_e)$$

standard linear form:

$$\dot{q}_e + \lambda q_e = \dot{q}_r + \lambda q$$

which becomes linear differential equation with respect to q_e.
Therefore, the equation can be explicitly solved at each time step to obtain $q_e(t)$

Part 2 Plot of q_e.