PROPERTIES OF POINT ESTIMATORS AND METHODS OF ESTIMATION

Maximum likelihood estimation 2

Not all MLE problems are straightforward

- The “typical steps to finding an MLE” often work.
- But sometimes they are not sufficient.
- We will also consider how to find an MLE when there are two or more parameters to estimate.

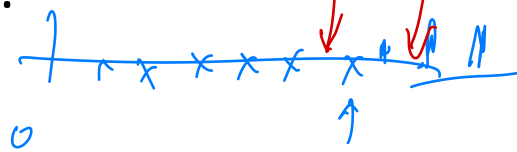
An example that is not quite so straightforward

$f_Y(y) = \begin{cases} \frac{1}{\theta} & 0 < y < \theta \\ 0 & \text{o.w.} \end{cases}$

- Example 6: $Y_1, Y_2, \dots, Y_n \sim iid\ U(0, \theta)$. Find the MLE of θ.

$\theta > 0$

NOT POSSIBLE FOR θ



$L(\theta) = \prod_{i=1}^{n} \frac{1}{\theta}$

$y_i \in [0, \theta]$, $i = 1, \dots, n$

If any $y_i < 0$ or $\boxed{\text{any } y_i > \theta}$ then $L(\theta) = 0$

$= \frac{1}{\theta^n}$

$\ell(\theta) = \log L(\theta) = -n \log(\theta)$

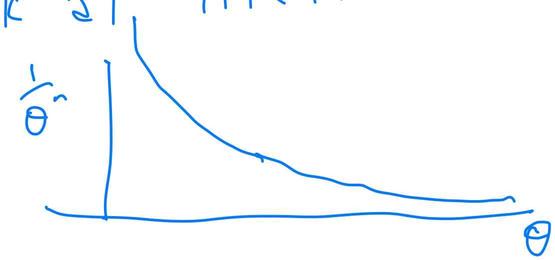
$\frac{d}{d\theta} \ell(\theta) = -\frac{n}{\theta} \overset{set}{=} 0$ solve for θ.

not true for any $\theta > 0$

$L(\theta)$

$\boxed{\hat{\theta} = Y_{(n)}}$

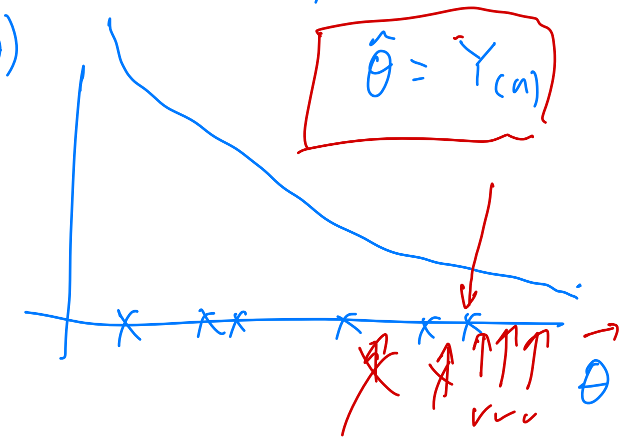
Look at likelihood function



decreasing functions

how to maximize?

make θ as small as possible!



MLE is $\hat{\theta} = Y_{(n)}$

Before, we found that $\underbrace{\frac{n+1}{n} Y_{(n)}}$ is unbiased for θ

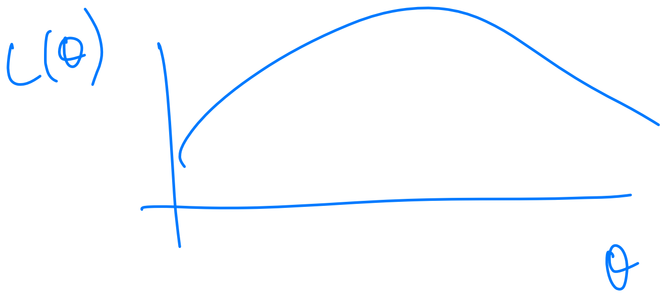
$$E\left[\frac{n+1}{n} Y_{(n)}\right] = \theta$$

$$\frac{n+1}{n} E\left[Y_{(n)}\right] = \theta$$

$$E\left[Y_{(n)}\right] = E\left[\hat{\theta}\right] = \frac{n}{n+1}\theta$$

So MLE is biased!!

$$\frac{n}{n+1}\theta \longrightarrow \theta \quad \text{as } n \to \infty$$



MLE when there are multiple parameters

- With two unknown parameters, the likelihood function is a surface (function of two variables).
- To maximize the likelihood in this case, we have to take the partial derivative of the likelihood w.r.t. each parameter, set each to zero, solve equations simultaneously.

where is $L(\theta_1, \theta_2)$ flat?

- Example 7: $Y_1, Y_2, \dots, Y_n \sim iid\ N(\mu, \sigma^2)$. Find MLEs of μ and σ^2. ν

Let $\nu = \sigma^2$

pdf: $\frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(y-\mu)^2}{2\sigma^2}}$

$\frac{1}{\sqrt{2\pi}\sqrt{\nu}} e^{-\frac{(y-\mu)^2}{2\nu}}$

$$L(\mu, \nu) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{\nu}} e^{-\frac{(y_i-\mu)^2}{2\nu}} = (2\pi)^{-\frac{n}{2}} \nu^{-\frac{n}{2}} \exp\left\{-\frac{\sum (y_i-\mu)^2}{2\nu}\right\}$$

$$\ell(\mu, \nu) = -\frac{n}{2}\log(2\pi) - \frac{n}{2}\log \nu - \frac{\sum (y_i-\mu)^2}{2\nu}$$

$$\frac{\partial}{\partial \mu} \ell(\mu,\nu) = \cancel{-}\frac{1}{\cancel{2}\nu} \cancel{2} \sum (y_i-\mu)^1 \cancel{(-1)} = \frac{1}{\nu}\sum(y_i-\mu) \overset{set}{=} 0$$

$$\frac{\partial}{\partial \nu} \ell(\mu,\nu) = -\frac{n}{2\nu} - \frac{\sum(y_i-\mu)^2}{2\nu^2}(-1) = -\frac{n}{2\nu} + \frac{\sum(y_i-\mu)^2}{2\nu^2} \overset{set}{=} 0$$

set $=0$ solve for μ, ν

① $\frac{1}{\nu}\sum(y_i-\mu)=0$

$\sum(y_i-\mu)=0$

$\sum y_i - n\mu = 0$

$n\mu = \sum y_i$

$\mu = \frac{1}{n}\sum y_i$

$\mu = \bar{y}$!!

② $-\frac{n}{2\nu} + \frac{\sum(y_i-\mu)^2}{2\nu^2} = 0$ $\times 2\nu$

$-n + \frac{\sum(y_i-\mu)^2}{\nu} = 0$

$n = \frac{\sum(y_i-\mu)^2}{\nu}$

$\nu = \frac{1}{n}\sum(y_i-\mu)^2$

replace unknown μ with its MLE $\bar{Y}$

$\nu = \frac{1}{n}\sum(y_i-\bar{y})^2$

biased for σ^2!

Is $(\bar{Y}, \frac{1}{n}\sum(Y_i-\bar{Y})^2)$ a maximum? ☆

tricky - calculate the "Jacobian"

MLE's ARE: $\hat{\mu} = \bar{Y}$, $\hat{\sigma}^2 = \hat{\nu} = \boxed{\frac{1}{n}\sum(Y_i-\bar{Y})^2}$