INTRODUCTION TO PROBABILITY:

Conditional probability and the Law of Total Probability

Conditional probability

- Experiment: randomly select a 50-year old
- Event A="the person contracts lung cancer in the next 20 years"
- Event B="the person is a smoker"
- We can think of calculating $P(A)$ and $P(B)$.
- Suppose we know that event B has occurred (we selected a smoker).
- What is the probability that *this person* contracts lung cancer?
- Is this the same as $P(A)$?

Conditional probability definition

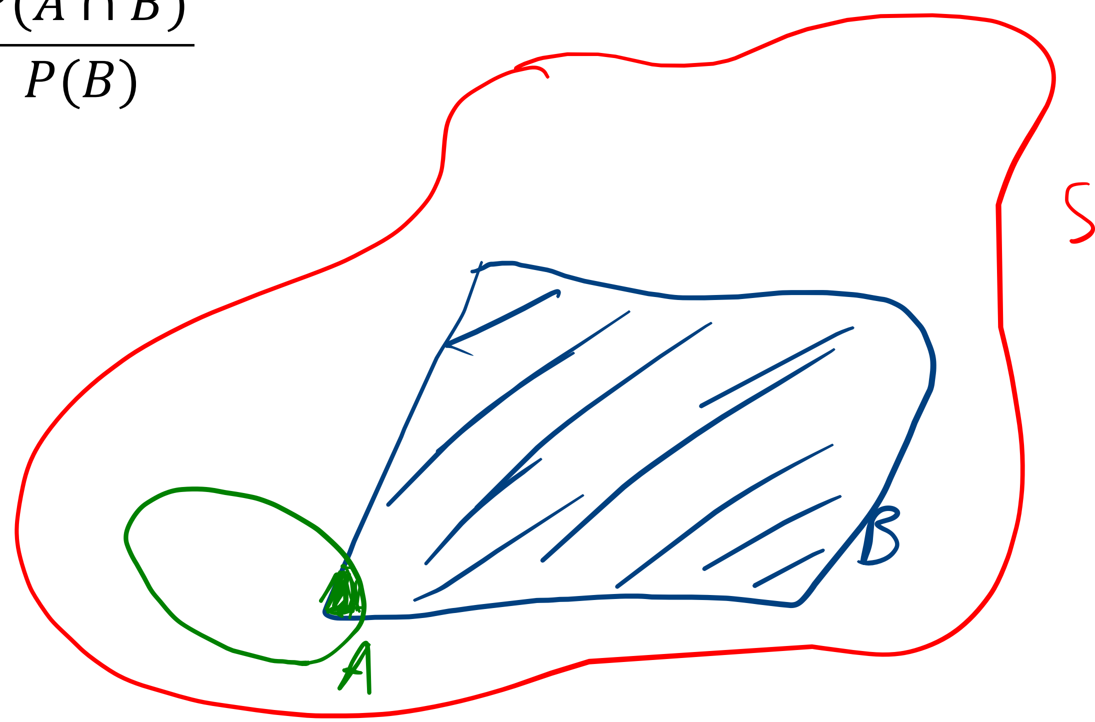
- Probability of event A given that event B has occurred is denoted: $P(A|B)$

intersection "and"

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

new S (B)

P(A|B)



Example 1

$S = \{1, 2, 3, 4, 5, 6\}$ $\frac{1}{6}$

- Experiment: Rolling a fair die one time
- $A = \{5\}$ $P(A) = \frac{1}{6}$
- $B =$ "odd number" $= \{1,3,5\}$ $P(B) = \frac{3}{6} = \frac{1}{2}$
- What is $P(A)$? What is $P(B)$?
- Given that B occurs, what is the probability of A? (What is $P(A|B)$?)

$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{\frac{1}{6}}{\frac{3}{6}} = \frac{1}{3}$ $P(A \cap B) = P(\{5\}) = \frac{1}{6}$

"new S" $(B) = \{1, 3, 5\}$

$P(A|B) = \frac{1}{3}$

Independence

def

- INDEPENDENCE: Two events A and B are independent if any of the following conditions hold
  - $P(A|B) = P(A)$
  - $P(B|A) = P(B)$
  - $P(A \cap B) = P(A)P(B)$

(Otherwise the events are dependent.)

Example 1 revisited

- Experiment: Rolling a fair die one time
- $A = \{5\}$
- B ="odd number"$= \{1,3,5\}$
- $C = \{4,5\}$
- Are A and C independent?

$P(C) = \frac{2}{6} = \frac{1}{3}$

$P(C|A) = \frac{P(C \cap A)}{P(A)} = \frac{P(\{5\})}{\frac{1}{6}} = \frac{\frac{1}{6}}{\frac{1}{6}} = 1$

$P(C) \neq P(C|A)$

- Are B and C independent?

$P(B \cap C) = P(\{5\}) = \frac{1}{6}$

$P(B) \times P(C) = \frac{1}{2} \times \frac{1}{3} = \frac{1}{6}$

Yes! independent.

Multiplicative law of probability

- The probability of the intersection of two events is

$$P(A \cap B) = \boxed{P(A)P(B|A)}$$
$$= P(B)P(A|B)$$

- If A and B are independent

$$P(A \cap B) = P(A)P(B)$$

- These follow directly from the definitions of conditional probability and independence.

$$P(A|B) = \frac{\boxed{P(A \cap B)}}{P(B)}$$

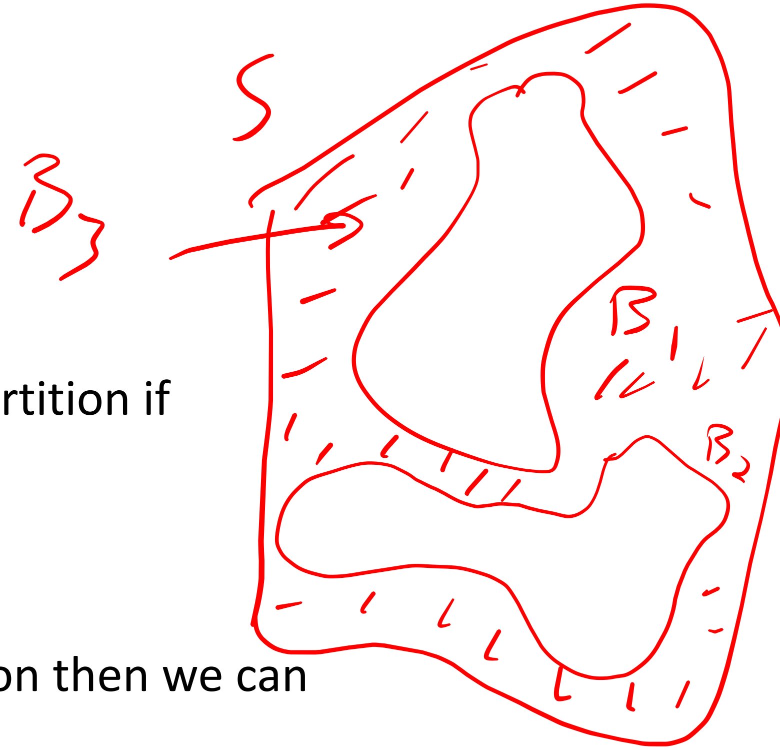
$$P(A \cap B) = P(A|B)P(B)$$

if indep.

$$P(B|A) = P(B)$$

Partition

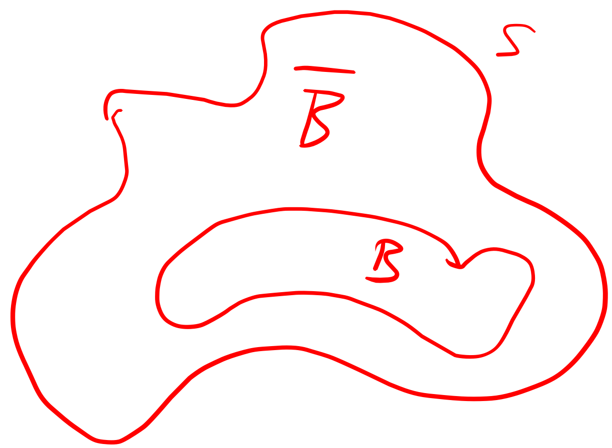
- PARTITION: A collection of sets $\{B_1, B_2, \ldots, B_k\}$ is a partition if
    1. $B_1 \cup B_2 \cup \cdots \cup B_k = S$ fill space S
    2. $B_i \cap B_j = \emptyset$ for $i \neq j$ no overlap



Note: if A is any set in S and $\{B_1, B_2, \ldots, B_k\}$ is a partition then we can decompose A as

$$A = (A \cap B_1) \cup (A \cap B_2) \cup \ldots \cup (A \cap B_k)$$

A very simple partition is simply $\{B, \bar{B}\}$. $k=2$

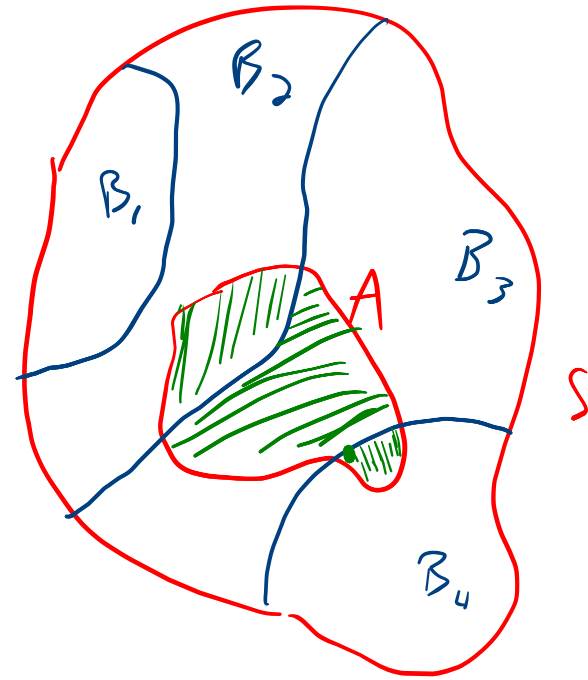


Law of total probability

- If $\{B_1, B_2, \ldots, B_k\}$ is a partition of S then for any event A,

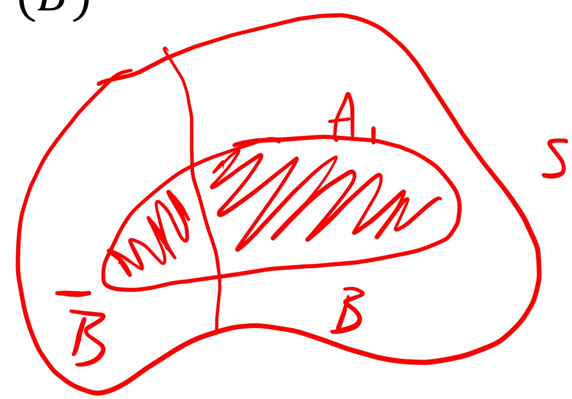
$$P(A) = \sum_{i=1}^{k} P(A|B_i)P(B_i)$$

$$= \sum_{i=1}^{k} P(A \cap B_i)$$



- The simplest case:

$$P(A) = P(A|B)P(B) + P(A|\bar{B})P(\bar{B})$$



Example 2

- A diagnostic test gives a positive result 90% of the time for patients who have the condition and gives a negative result 90% of the time for patients who do not. The condition is present in 1% of the population. If a person is randomly selected and tested, what is the probability the result will be positive?

A = event positive test result

B = event person has condition

want: $P(A)$

know: $P(A|B) = 0.9$ $\quad P(A|\bar{B}) = 0.1$ $\quad P(B) = 0.01$

LOTP: $P(A) = P(A|B)P(B) + P(A|\bar{B})P(\bar{B})$

$= 0.9 \times 0.01 + 0.1 \times 0.99$

$= 0.009 + 0.099 = 0.108 = 10.8\%$