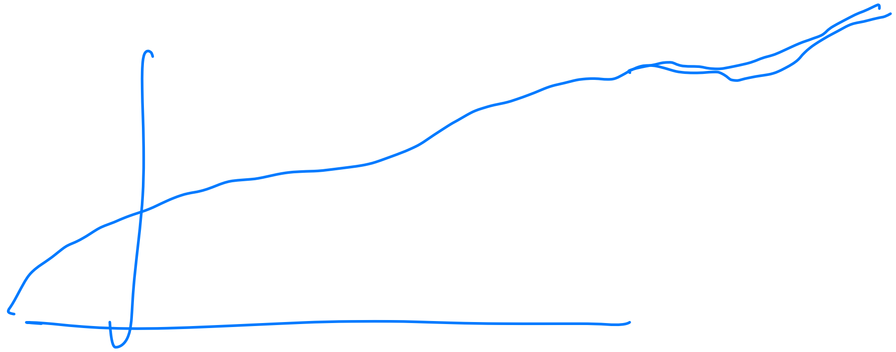
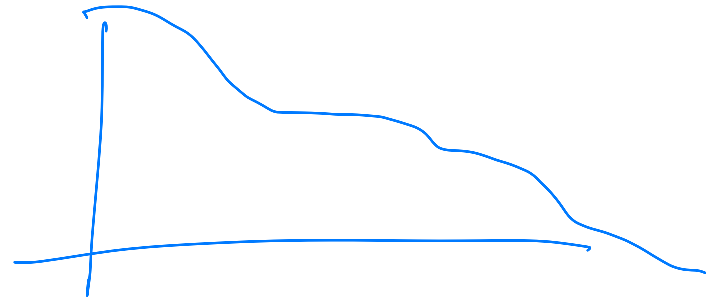
# FUNCTIONS OF RANDOM VARIABLES

Method of transformations and method of mgfs





# Method of transformations

- This method is useful for finding the pdf of $U = h(Y)$ if $h(\cdot)$ is a *monotone function* (strictly increasing or strictly decreasing).
- If so, the inverse function $h^{-1}(\cdot)$ exists.
- Suppose $h(\cdot)$ is increasing (then so is $h^{-1}(\cdot)$). Then

method of cdfs

$$F_U(u) = P(U \leq u) = P(h(Y) \leq u) = P(h^{-1}(h(Y)) \leq h^{-1}(u))$$
$$= P(Y \leq h^{-1}(u)) = F_Y(h^{-1}(u))$$

- And so

$$f_U(u) = \frac{d}{du} F_U(u) = \frac{d}{du} F_Y(h^{-1}(u)) = f_Y(h^{-1}(u)) \frac{d}{du} h^{-1}(u)$$

def

subst.

chain rule

# Method of transformations

- If $h(\cdot)$ is *decreasing* then so is $h^{-1}(\cdot)$ and we could go through a similar derivation. But $\frac{d}{du}h^{-1}(u)$ will be negative.
- The general formula for the pdf of of $U = h(Y)$ for either increasing or decreasing $h$ is thus

$$f_U(u) = f_Y(h^{-1}(u))\left|\frac{d}{du}h^{-1}(u)\right|$$

pdf of $Y$ $\quad f_Y(y) = \frac{1}{\beta} e^{-y/\beta}, \quad y > 0$

# Method of transformations: Example

- Example 6: If $Y \sim \exp(\beta)$, find the pdf of $\boxed{U = \sqrt{Y}}$.

$U = h(Y)$

$h(y) = \sqrt{y}$

Find $h^{-1}$ $\rightarrow$ $\quad h(y) = \sqrt{y} \overset{\text{set}}{=} u$ $\quad$ solve for $y$

$\boxed{y = u^2 = h^{-1}(u)}$

$\left| \frac{d}{du} h^{-1}(u) \right|$

$|2u| = 2u$

$f_U(u) = f_Y(h^{-1}(u)) \left| \frac{d}{du} h^{-1}(u) \right| = \frac{1}{\beta} e^{-\frac{u^2}{\beta}} \cdot 2u = \boxed{\frac{2u}{\beta} e^{-\frac{u^2}{\beta}}, \quad u > 0}$

$\frac{1}{b-a}$

$$f_Y(y) = \begin{cases} \frac{1}{\pi} & -\frac{\pi}{2} < y < \frac{\pi}{2} \\ 0 & \text{o.w.} \end{cases}$$

# Method of transformations: Another example

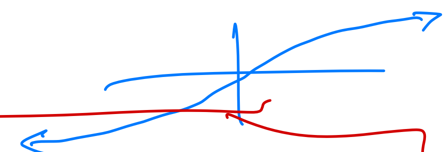
- Example 7: If $Y \sim U\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, find the pdf of $U = \tan(Y)$.

$h(y) = \tan(y)$

Find $h^{-1}$: $h(y) = \tan(y) \overset{\text{set}}{=} u$ solve for $y$

$y = \tan^{-1}(u) = \text{Arctan}(u) = h^{-1}(u)$

$\frac{d}{du} h^{-1}(u) = \frac{d}{du} \text{Arctan}(u) = \frac{1}{1+u^2}$



$f_U(u) = f_Y(h^{-1}(u)) \left| \frac{d}{du} h^{-1}(u) \right| = \boxed{\frac{1}{\pi} \frac{1}{1+u^2}, \quad -\infty < u < \infty}$

CAUCHY DISTRIBUTION

# Method of moment generating functions

- This method is based on the fact that mgfs uniquely characterize distributions.
- Theorem 1: Suppose $X$ and $Y$ are random variables with mgfs $m_X(t)$ and $m_Y(t)$, respectively. If $m_X(t) = m_Y(t)$ for all $t$ then $X$ and $Y$ have the same distribution.
- Main idea: It may be straightforward to calculate $m_U(t) = E[e^{tU}]$ for some function of random variables. If $m_U(t)$ matches the mgf of some known distribution, then we know $U$ must have that distribution.

Gamma$(\alpha, \beta)$

Book: $m(t) = E[e^{tY}] = \frac{1}{(1-\beta t)^{\alpha}}$

$m(s) = E[e^{sY}]$

# Method of mgfs: Example

- Example 8: If $Y \sim gamma(\alpha, \beta)$, what is the distribution of $U = 2Y/\beta$?

Find the mgf of $U$

$E[e^{tU}] = E[e^{t\frac{2Y}{\beta}}] = E[e^{\frac{2t}{\beta}Y}] = \int_0^\infty e^{\frac{2t}{\beta}y} \, p(y)\,dy$

takes too long

$= m_Y\left(\frac{2t}{\beta}\right) = \frac{1}{\left(1-\cancel{\beta}\left(\frac{2t}{\cancel{\beta}}\right)\right)^{\alpha}}$

$= \frac{1}{(1-2t)^{\alpha}} \leftarrow$ mgf of Gamma$(\alpha, 2)$

$\boxed{U \sim \text{Gamma}(\alpha, 2)}$

# Method of mgfs: Sums of independent random variables

- The method of mgfs can be especially useful for finding the distribution of sums of independent random variables.
- Theorem 2: If $Y_1, Y_2, \dots, Y_n$ are independent random variables with respective mgfs $m_1(\cdot), m_2(\cdot), \dots, m_n(\cdot)$, then the mgf of $\boxed{U = \sum_{i=1}^{n} Y_i}$ is $m_U(t) = \prod_{i=1}^{n} m_i(t)$

Proof

$$m_U(t) = E[e^{tU}] = E[e^{t\sum_{i=1}^{n} Y_i}]$$

$$= E[e^{tY_1 + tY_2 + \dots + tY_n}] = E[e^{tY_1} \cdot e^{tY_2} \cdot \dots \cdot e^{tY_n}]$$

$$= E[e^{tY_1}]\, E[e^{tY_2}] \cdot \dots \cdot E[e^{tY_n}]$$

b/c indep

$$= m_1(t) \times m_2(t) \times \dots \times m_n(t) = \prod_{i=1}^{n} m_i(t)$$

$\square$

# Method of mgfs: Example, sum of independent exponentials

- Example 9: If $Y_1, Y_2, \ldots, Y_n$ are independent exponential$(\beta)$, find the distribution of $U = \sum_{i=1}^{n} Y_i$.

mgf of exp $\quad m(t) = \dfrac{1}{1-\beta t}$

$$m_u(t) = \underbrace{\frac{1}{1-\beta t}}_{\text{mgf of } Y_1} \times \underbrace{\frac{1}{1-\beta t}}_{\text{mgf of } Y_2} \times \cdots \times \frac{1}{1-\beta t} = \frac{1}{(1-\beta t)^n}$$

$\boxed{\text{mgf of Gamma}(n, \beta)}$

# Method of mgfs: Sum of independent Poisson rvs

- Example 10: If $Y_1, Y_2, \dots, Y_n$ are independent Poisson random variables with respective rate parameters $\lambda_1, \lambda_2, \dots, \lambda_n$, find the distribution of $U = \sum_{i=1}^{n} Y_i$.

Book: mgf of Poisson $m(t) = e^{\lambda(e^t - 1)}$

$$m_U(t) = e^{\lambda_1(e^t-1)} \times e^{\lambda_2(e^t-1)} \times \dots \times e^{\lambda_n(e^t-1)}$$

$$= e^{(\lambda_1+\lambda_2+\dots+\lambda_n)(e^t-1)} = e^{(\Sigma\lambda_i)(e^t-1)}$$

mgf of Poisson$(\Sigma\lambda_i)$

$$U \sim \text{Poisson}\left(\sum_{i=1}^{n}\lambda_i\right)$$

Book mgf of normal$(\mu, \sigma^2)$ $m(t) = e^{\mu t + \frac{1}{2}\sigma^2 t^2}$

# Method of mgfs: Sum of independent normal rvs

- Example 11: If $Y_1, Y_2, \dots, Y_n$ are independent normal random variables with respective means $\mu_1, \mu_2, \dots, \mu_n$ and variances $\sigma_1^2, \sigma_2^2, \dots, \sigma_n^2$, find the distribution of $L = a_1Y_1 + a_2Y_2 + \cdots + a_nY_n$.

First look at $B = Y_1 + Y_2 + \dots + Y_n$ $\quad a_i = 1$

$$m_B(t) = \prod_{i=1}^{n}\left(e^{\mu_i t + \frac{1}{2}\sigma_i^2 t^2}\right) = e^{\mu_1 t + \frac{1}{2}\sigma_1^2 t^2} \times e^{\mu_2 t + \frac{1}{2}\sigma_2^2 t^2} \times \dots$$

$$e^{(\mu_1 + \mu_2 + \dots + \mu_n)t + \frac{1}{2}(\sigma_1^2 + \sigma_2^2 + \dots + \sigma_n^2)t^2} = e^{(\sum \mu_i)t + \frac{1}{2}(\sum \sigma_i^2)t^2}$$

$$\left(B \sim N\left(\sum_{i=1}^{n} \mu_i, \sum_{i=1}^{n} \sigma_i^2\right)\right)$$

$Z_i = a_i Y_i$ mgf of $Z_i$ $E[e^{tZ_i}] = E[e^{t a_i Y_i}] = m_{Y_i}(t a_i)$

$$m_Z(t) = E[e^{t(Z_1 + Z_2 + \dots + Z_n)}] = m_{Y_1}(t a_1)\, m_{Y_2}(t a_2) \times \dots \times m_{Y_n}(t a_n)$$

$$\boxed{L \sim N\left(\sum a_i \mu_i, \sum a_i^2 \sigma_i^2\right)}$$