AXIOMS AND RULES OF PROBABILITY

Probability of events

- In probability, interest often lies in determining how to assign some numerical probability to each event.
- This can be done (in the discrete case) by assigning some probability to each possible outcome. Then the probability of an event will depend on which outcomes it contains.

$$S = \{a_1, a_2, a_3, a_4\}$$

$$a_1 \to 0.4 \qquad a_2 \to 0.1 \qquad a_3 \to 0.2$$

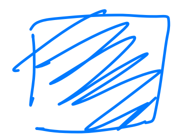
$$P(\{a_2, a_3\})$$

P "set function"

Kolmogorov probability axioms

- Let S be the sample space for a (random) experiment. For every event A in S we assign a number, which we call the probability of A and denote P(A). The following axioms must hold:

  <u>Axiom 1</u>: $P(A) \geq 0$

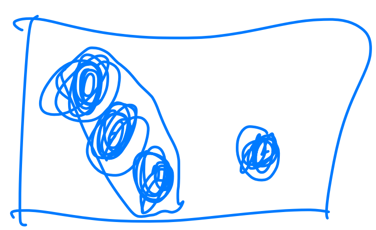
  <u>Axiom 2</u>: $P(S) = 1$

  <u>Axiom 3</u>: If A_1, A_2, A_3, … are all mutually exclusive events, then

  $$P(A_1 \cup A_2 \cup A_3 \cup \cdots) = \sum_{i=1}^{\infty} P(A_i)$$

Corollary: If A_1, A_2,…,A_n are all mutually exclusive events, then

$$P(A_1 \cup A_2 \cup \cdots \cup A_n) = \sum_{i=1}^{n} P(A_i)$$



Some probability rules that follow from the Kolmogorov axioms

Complement rule: $\boxed{P(A) = 1 - P(\bar{A})}$

A and $\bar{A}$ are mut. excl.

$\boxed{P(A \cup \bar{A}) = P(A) + P(\bar{A})}$ K.A. #3

$A \cup \bar{A} = S$

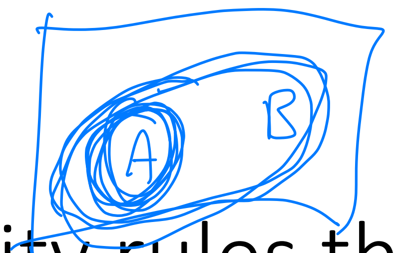
$P(A \cup \bar{A}) = P(S) = 1$ K.A. #2

$1 = P(A) + P(\bar{A})$

$P(A) = 1 - P(\bar{A})$ $\square$

$P(\emptyset) = 0$

If $A = S$ then $\bar{A} = \emptyset$



Some more probability rules that follow from the Kolmogorov axioms

If A is a subset of B then $P(A) \leq P(B)$

mut. excl.

$$P(B) = P(A \cap B) + P(\bar{A} \cap B)$$

K.A. #3

$$P(A \cap B) = P(A)$$

because $A \cap B = A$

because A subset of B

$$P(B) = P(A) + P(\bar{A} \cap B)$$

$$P(\bar{A} \cap B) \geq 0 \quad \text{K.A. \#1}$$

$$P(B) - P(A) = P(\bar{A} \cap B) \geq 0$$

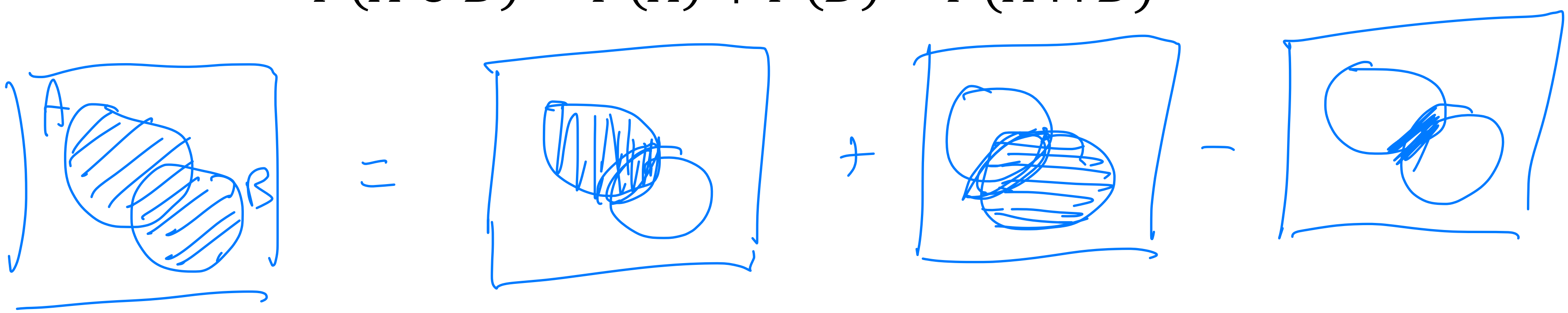
$$P(B) \geq P(A) \quad \square$$

For any subset A of S, $P(A) \leq 1$

Yet another probability rule that follows from the Kolmogorov axioms

- Additive law of probability: for any events A and B (both in S),

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$



Adding mutually exclusive events

K.A. #3

- If A and B are mutually exclusive then $P(A \cup B) = P(A) + P(B)$.
- If A, B, and C are all mutually exclusive then

$$P(A \cup B \cup C) = P(A) + P(B) + P(C)$$

- If we can calculate the probability of every simple event, then we can calculate the probability of any compound event simply by adding up the probabilities.

mut. excl.

Example: Toss a (weighted) coin 3 times

- We have a coin with P(H)=2/3 (and so P(T)=??) 1/3
- Q: What is the sample space?
- Q: Are the (simple) events equally likely? NO

$S = \{H, T\}$

$P(S) = 1$

$S = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$

$P(HHH) = \frac{2}{3} \cdot \frac{2}{3} \cdot \frac{2}{3} = \frac{8}{27}$

$P(TTT) = \frac{1}{3} \cdot \frac{1}{3} \cdot \frac{1}{3} = \frac{1}{27}$

Example: Toss an (unweighted) coin 3 times

- We have a coin with P(H)=1/2 (and so P(T)=??) 1/2
- Q: What is the sample space? same
- Q: Are the (simple) events equally likely?

HHH $\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}$ HTH $\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}$

THT $\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}$

P(any simple event) $= \frac{1}{8}$

$$\underbrace{\frac{1}{8}+\frac{1}{8}+\cdots+\frac{1}{8}}_{8} = 1 = P(S)$$