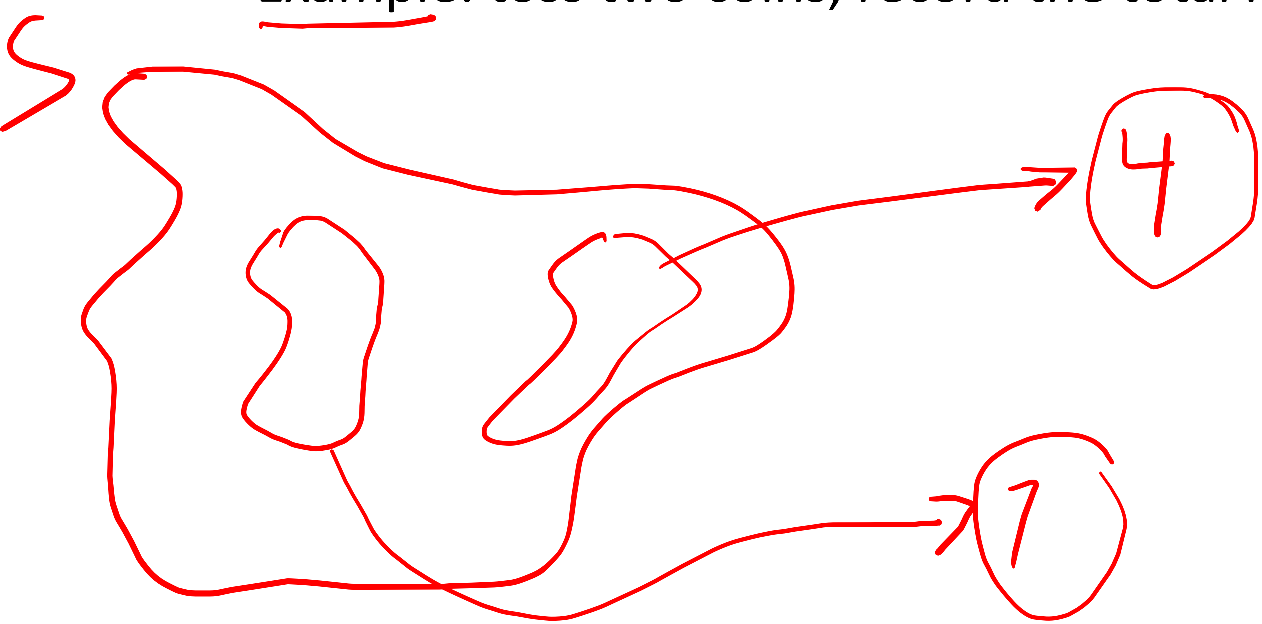
DISCRETE RANDOM VARIABLES

Distributions and expected values

Random variables

- RANDOM VARIABLE: a function that assigns a real number to each sample point in S.
- The random variable X maps the sample space (domain) to the real line (range), i.e., $X\colon S \to \mathbb{R}$.
- Example: toss two coins, record the total number of "heads"



0 1 2

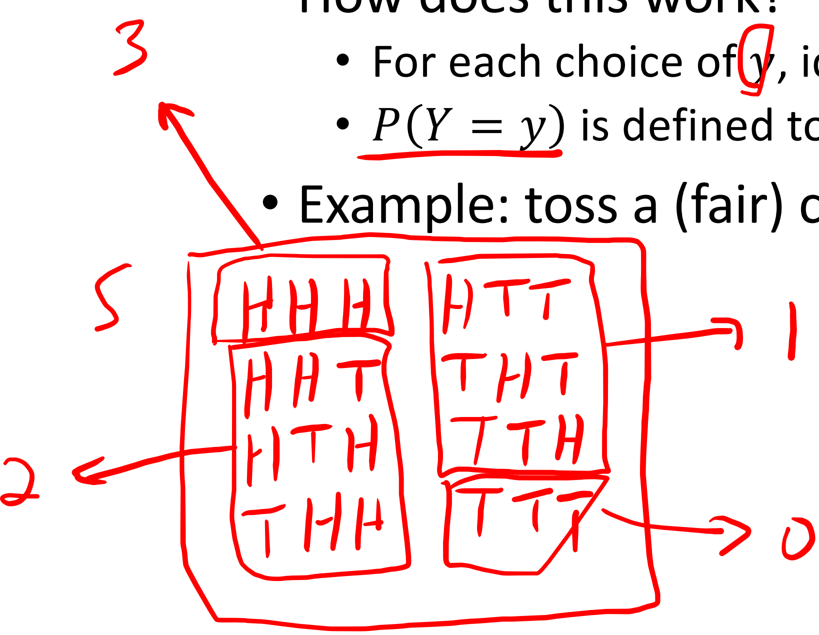
Discrete random variables

- A random variable is <u>discrete</u> if it can take on a finite or countably infinite number of different values.
- Since each sample point has an associated probability, it is possible to find the probability of each possible value of *X*.
- Notation:
  - we will use capital letters (*X, Y, Z*) to denote random variables
  - We will use lower-case letters (*x, y, z*) to represent any particular value
- *P(Y=0), P(Y=1), …, P(Y=y),* etc.

function of y

Probability distributions for discrete random variables

- The probability distribution of a random variable *Y* is defined by *P(Y=y)* for all possible values of *y*.
- Sometimes we write $p(y) = P(\boxed{Y = y})$.
- How does this work?
  - For each choice of y, identify the elements of S that map to $Y = y$.
  - $P(Y = y)$ is defined to be the sum of those probabilities.
- Example: toss a (fair) coin, 3X record the number of "heads" = Y

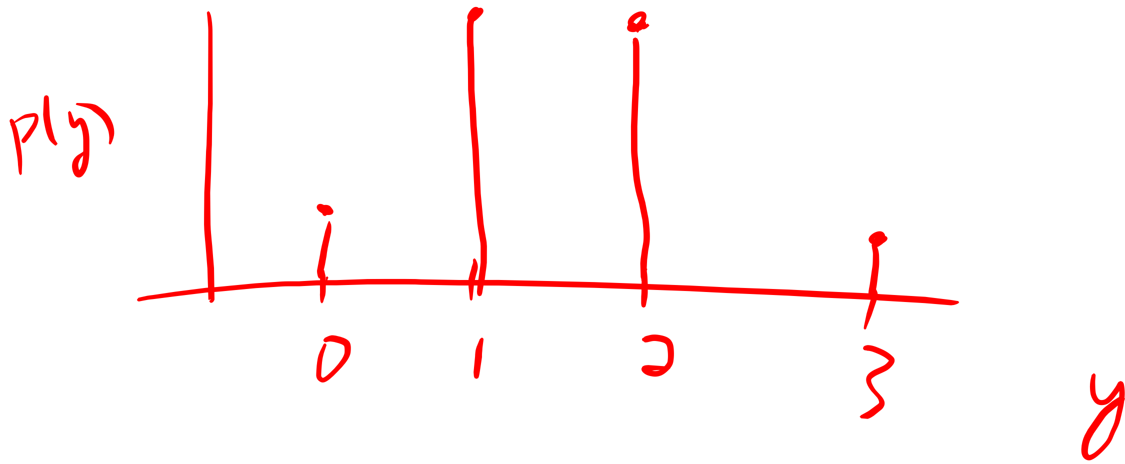


$$P(Y = 2)$$
$$= P(\{HHT, HTH, THH\})$$
$$= \frac{3}{8}$$

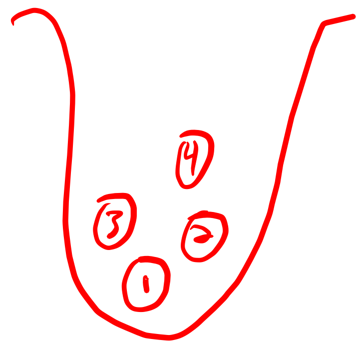
Representing probability distributions of discrete random variables

- The distribution of a discrete random variable can be represented by a table, graph, or formula.

y	$p(y)$
0	1/4
1	3/8
2	3/8
3	1/8



Example 1



- An urn holds four balls that are labeled 1, 2, 3, and 4. Draw two of them at random (without replacement). Find the probability distribution of Y, the sum of the numbers of the selected balls.

S	y
1, 2	3
1, 3	4
1, 4	5
2, 3	5
2, 4	6
3, 4	7

eq. likely, prob = $\frac{1}{6}$

y	$p(y)$
3	1/6
4	1/6
5	2/6
6	1/6
7	1/6

Two additional points about distributions of discrete random variables

- Theorem 1: If Y is a discrete random variable with $p(y) = P(Y = y)$. Then
  - $0 \leq P(Y = y) \leq 1$ for all y.
  - $\sum_y p(y) = 1$, where the sum is over all possible y.
- Probability distributions are used as models (approximations) of the behavior of processes in the real world that generate data.
  - Q: Is any particular model useful?
  - A: If it fits the observed data it might be.

Expected value of a random variable

- EXPECTED VALUE: If Y is a discrete random variable with distribution $p(y)$, the expected value of Y is defined to be

$$E[Y] = \sum_y yp(y).$$

sum over all possible y

- Example 2: Toss a fair coin three times, what is the expected value of the number of "heads" observed?

y	$p(y)$
0	1/8
1	3/8
2	3/8
3	1/8

$$E[Y] = 0 \times \frac{1}{8} + 1 \times \frac{3}{8} + 2 \times \frac{3}{8} + 3 * \frac{1}{8}$$

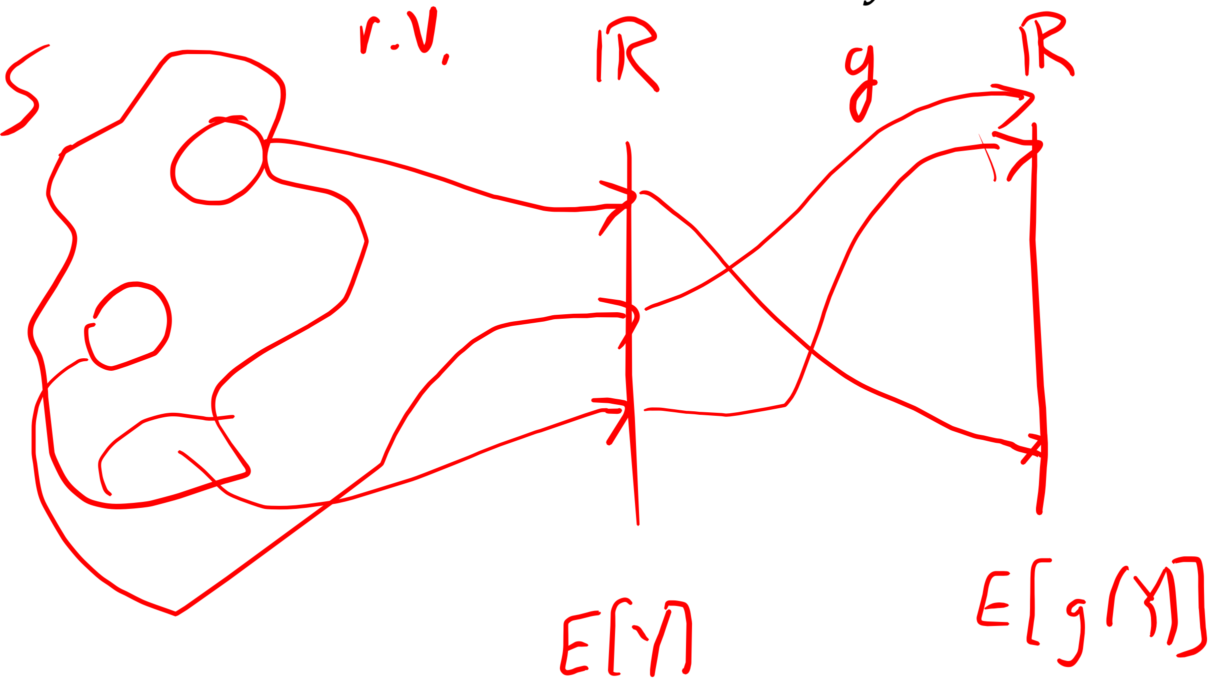
$$= \frac{3}{8} + \frac{6}{8} + \frac{3}{8} = \frac{12}{8} = 1.5$$

long-run average

Functions of random variables

- A function of a random variable is also a random variable!
- Theorem 2: If Y is a random variable with distribution $p(y)$ and g is a function, the expected value of $g(Y)$ is

$$E[g(Y)] = \sum_y g(y)p(y).$$



Example 3

- Toss a fair coin three times. What is the expected value of the *square* of the number of "heads" observed?

y	$p(y)$	y^2
0	1/8	0
1	3/8	1
2	3/8	4
3	1/8	9

$$E[Y^2] = 0 \times \frac{1}{8} + 1 \times \frac{3}{8} + 4 \times \frac{3}{8} + 9 \times \frac{1}{8}$$

$$= \frac{3}{8} + \frac{12}{8} + \frac{9}{8} = \frac{24}{8}$$