CONTINUOUS RANDOM VARIABLES

A few other distributions

Skewed (to the right) data

- Many real-life datasets involve random variables that are skewed to the right.
- Examples:
  - Survival times of patients after diagnosis
  - Time to relapse after recovery
  - Time to failure of a medical device
  - Time between phone calls to a hotline

The gamma distribution

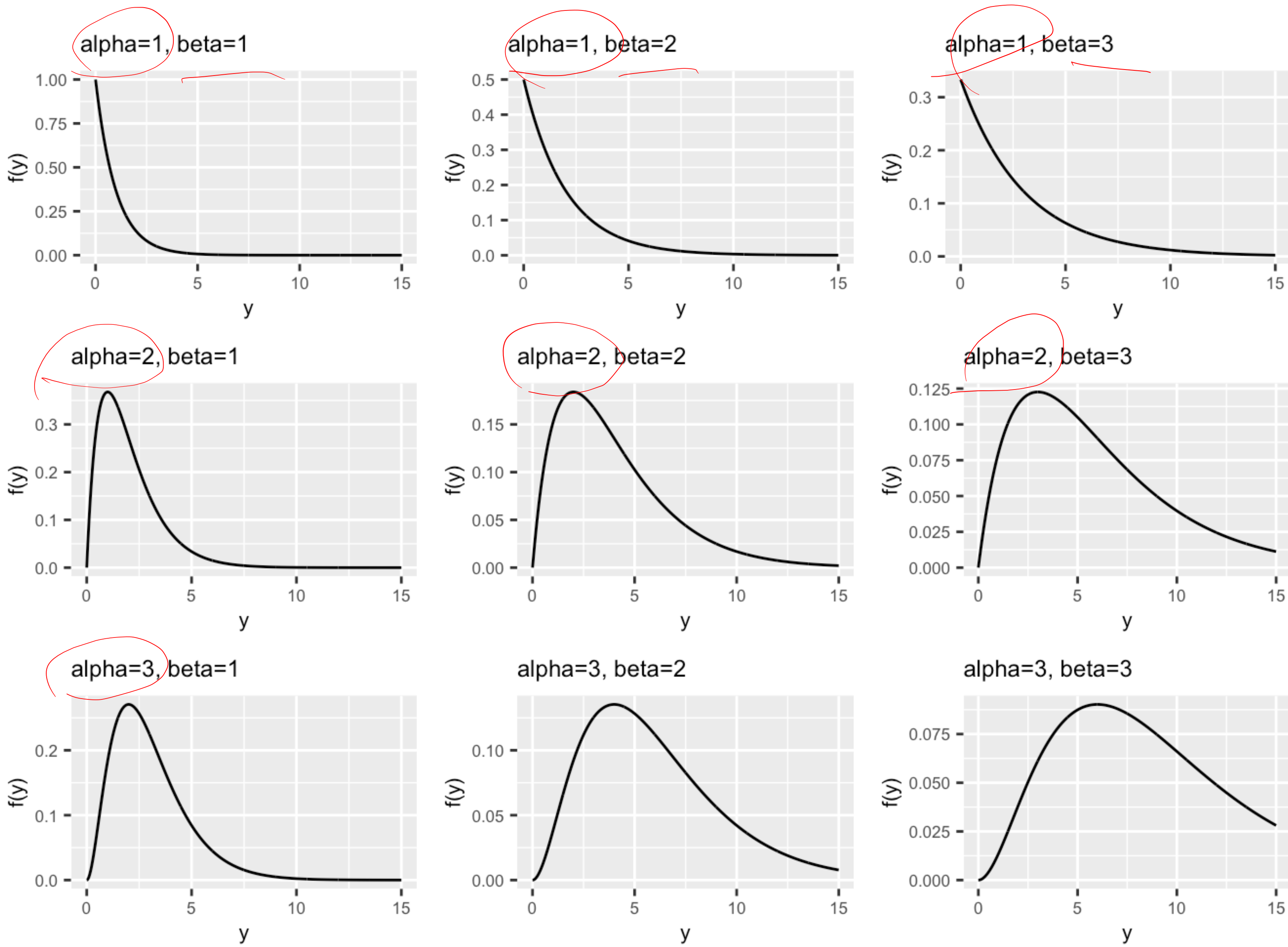
- Definition 6: A random variable Y has a GAMMA DISTRIBUTION with parameters $\alpha > 0$ and $\beta > 0$ if its pdf is

$$f(y) = \frac{y^{\alpha-1}e^{-y/\beta}}{\beta^{\alpha}\Gamma(\alpha)} \text{ for } y \geq 0,$$

where

$$\Gamma(\alpha) = \int_0^{\infty} y^{\alpha-1}e^{-y}dy$$

- $\Gamma(\cdot)$ is the "gamma function":
  - $\Gamma(1) = 1$ (check this!)
  - $\Gamma(\alpha) = (\alpha - 1)\Gamma(\alpha - 1)$ for any $\alpha > 1$ (integration by parts)
  - $\Gamma(n) = (n - 1)!$ If n is a positive integer.
- Shorthand: $Y \sim gamma(\alpha, \beta)$
- α: "shape parameter"; β: "scale parameter"



alpha=1, beta=1
alpha=1, beta=2
alpha=1, beta=3
alpha=2, beta=1
alpha=2, beta=2
alpha=2, beta=3
alpha=3, beta=1
alpha=3, beta=2
alpha=3, beta=3
f(y)
y

The gamma distribution

- Check: Is the gamma pdf a proper pdf? $\rightarrow$ Does it integrate to 1?

$$\int_0^\infty \frac{y^{\alpha-1} e^{-y/\beta}}{\beta^\alpha \Gamma(\alpha)} dy = \frac{1}{\beta^\alpha \Gamma(\alpha)} \int_0^\infty y^{\alpha-1} e^{-y/\beta} dy$$

$$z = \frac{y}{\beta} \Leftrightarrow y = \beta z \qquad dz = \frac{1}{\beta} dy$$

$$dy = \beta dz$$

$$= \frac{1}{\beta^\alpha \Gamma(\alpha)} \int_0^\infty (\beta z)^{\alpha-1} e^{-z} \beta dz$$

$$= \frac{1}{\beta^\alpha \Gamma(\alpha)} \underline{\beta^{\alpha-1} \cdot \beta} \boxed{\int_0^\infty z^{\alpha-1} e^{-z} dz} = \frac{1}{\cancel{\beta^\alpha} \cancel{\Gamma(\alpha)}} \cancel{\beta^\alpha} \cancel{\Gamma(\alpha)} = 1 \checkmark$$

YES!, " $\Gamma(\alpha)$ (defined)!

$f(y) \geq 0 \checkmark$

$y^{\alpha-1} e^{-y/\beta}$

The gamma distribution

- Theorem 7: If $Y \sim gamma(\alpha, \beta)$ then $E[Y] = \alpha\beta$ and $Var(Y) = \alpha\beta^2$.

$$E[Y] = \int_0^\infty y \frac{y^{\alpha-1} e^{-y/\beta}}{\beta^\alpha \Gamma(\alpha)} dy = \int_0^\infty \frac{y^{\alpha} e^{-y/\beta}}{\beta^\alpha \Gamma(\alpha)} dy$$

G.F.F.

$$\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$$

$$\Gamma(\alpha) = \frac{\Gamma(\alpha+1)}{\alpha}$$

$$= \beta \int_0^\infty \frac{y^{\alpha} e^{-y/\beta}}{\beta^{\alpha+1} \Gamma(\alpha+1)/\alpha} dy = \alpha\beta \underbrace{\int_0^\infty \underbrace{\frac{y^{\alpha} e^{-y/\beta}}{\beta^{\alpha+1} \Gamma(\alpha+1)}}_{\text{pdf of } Gamma(\alpha+1,\beta)} dy}_{1}$$

$$= \alpha\beta$$

The gamma distribution

- Theorem 8: If $Y \sim gamma(\alpha, \beta)$ then the mgf of Y is

$$m(t) = \left(\frac{1}{1-\beta t}\right)^{\alpha}, \text{ for } t < \frac{1}{\beta}$$

check book

The gamma distribution

- Example 10: Suppose a random variable Y has pdf

$$f(y) = cy^4 e^{-y/3} \quad , y > 0$$

$$\int_0^\infty c y^4 e^{-y/3} dy$$

What must the constant c be? Find $E[Y]$ and $Var(Y)$.

$\alpha \quad \beta$

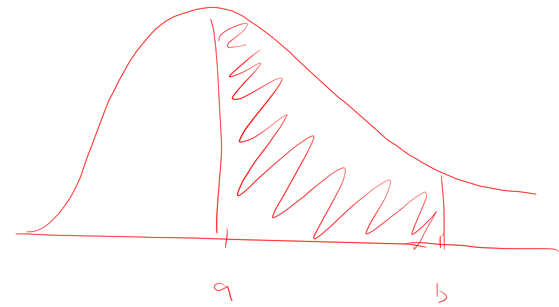
Gamma$(5, 3)$

$$c = \frac{1}{\beta^\alpha \Gamma(\alpha)} = \frac{1}{3^5 \Gamma(5)} = \frac{1}{3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 4 \cdot 3 \cdot 2 \cdot 1}$$

$$E[Y] = \alpha\beta = 5 \cdot 3 = 15$$

$$Var(Y) = \alpha\beta^2 = 5 \cdot 3^2 = 45$$

Calculating probabilities for the gamma distribution



- If $Y \sim gamma(\alpha, \beta)$, there is no closed-form solution for quantities $P(a < Y < b)$ (unless α is an integer).
- Numerical approximation:
  - `pgamma()` function in `R`
  - Online calculators
- WARNING! Not all representations of the gamma function are parameterized the same way! (Sometimes "their β" is the reciprocal of "our β.") It doesn't have to be a problem – just make the substitution – but you have to look out for it!

$e^{-y/\beta}$

Calculating probabilities for the gamma distribution

- Example 11: If $Y \sim gamma(3,5)$, calculate $P(10 < Y < 20)$.
- FIRST: Check expression for pdf!
- It matches our pdf in R!
- From R :

```
> pgamma(20,shape=3,scale=5)
[1] 0.7618967
> pgamma(10,shape=3,scale=5)
[1] 0.3233236
```

$P(10 < Y < 20) = 0.762 - 0.323$

Gamma

$E[Y] = \alpha\beta$ $Var(Y) = \alpha\beta^2$

The chi-square distribution

$\alpha = \frac{\nu}{2}$ $\beta = 2$

$\sim \chi^2(\nu)$

- Definition 7: A random variable Y has a CHI-SQUARE DISTRIBUTION with ν degrees of freedom if $Y \sim gamma(\nu/2, 2)$. (ν must be a positive integer)
- Shorthand: $Y \sim \chi^2(\nu)$
- If $Y \sim \chi^2(\nu)$, then $E[Y] = \nu$ and $Var(Y) = 2\nu$.

$\frac{\nu}{\cancel{2}} \cdot \cancel{2} = \nu$ $\frac{\nu}{\cancel{2}} \cdot \cancel{2} \cdot 2 = 2\nu$

The exponential distribution

- Definition 8: A random variable Y has an EXPONENTIAL DISTRIBUTION with parameter β if $Y \sim gamma(\underset{\alpha = 1}{1}, \beta)$. ($\beta > 0$)
- Shorthand: $Y \sim \exp(\beta)$
- If $Y \sim \exp(\beta)$ then $E[Y] = \beta$ and $Var(Y) = \beta^2$

$E[Y] = \alpha\beta$ $\qquad$ Var $\qquad$ $\alpha\beta^2$

exp $\qquad$ $1\beta = \beta$ $\qquad$ $1 \cdot \beta^2 = \beta^2$

The beta distribution

- Some random variables are constrained to have support only on $[0,1]$.
- This is typical when the random variable is a measured *proportion*.
- Definition 9: A random variable Y has a BETA DISTRIBUTION with parameters α and β if its pdf is

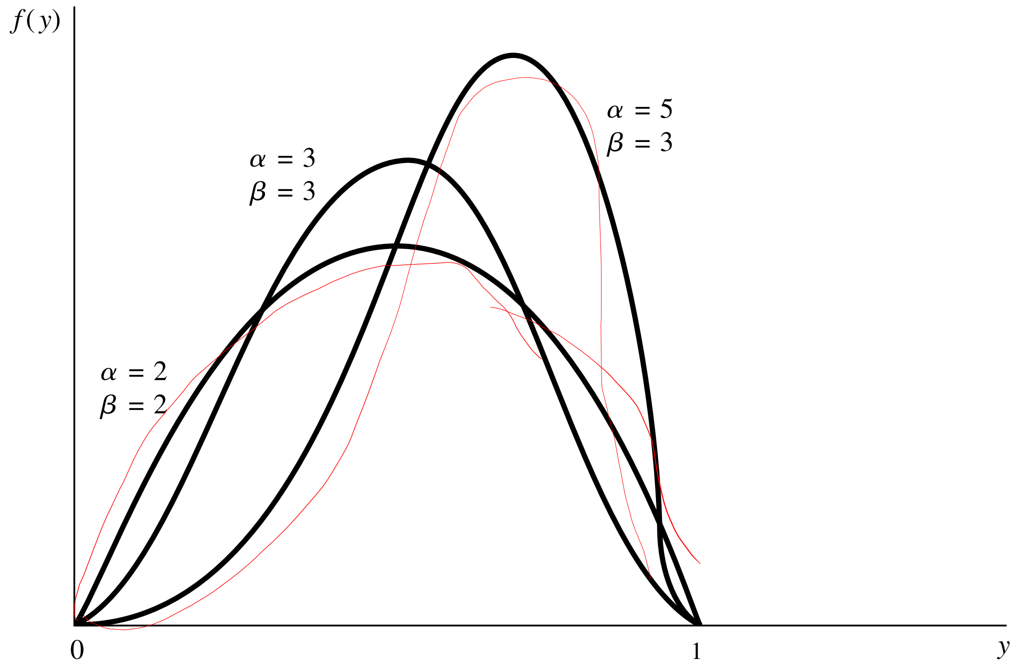
$$f(y) = \frac{y^{\alpha-1}(1-y)^{\beta-1}}{B(\alpha,\beta)} \text{ for } 0 \le y \le 1$$

- Here, $B(\alpha, \beta) = \int_0^1 y^{\alpha-1}(1-y)^{\beta-1}dy = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$
- Shorthand: $Y \sim beta(\alpha, \beta)$

The beta distribution

- The beta distribution, again, constrained to be in $[0,1]$, is quite versatile and flexible.

FIGURE **4.17** Beta density functions



The beta distribution

- If $Y \sim beta(\alpha, \beta)$ then $E[Y] = \frac{\alpha}{\alpha+\beta}$ and $Var(Y) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$
- For computing probabilities $P(a < Y < b)$ (where $0 \le a \le b \le 1$), it is possible to do this analytically if α and β are both integers. (Otherwise, numerical approximations!)