DISCRETE RANDOM VARIABLES

Expected values, means, and variances

Some attributes of distributions

- The *expected value* of a random variable Y is a measure of the center of the distribution of Y.
- The mean of the distribution of Y is $\mu = E[Y]$.
- The "spread" or "variability" of a distribution can be measured by its variance:

$$\sigma^2 = E[(Y-\mu)^2] = \sum_y (y-\mu)^2 p(y).$$

- The standard deviation of a distribution is $\sigma = \sqrt{\sigma^2}$.

Linear combinations of expected values: Constant multiplier

- Theorem 3: For a discrete random variable Y with probability function $p(y) = P(Y = y)$, any function g, and any constant c,

$$E[cg(Y)] = cE[g(Y)]$$

proof

$$E[cg(Y)] \underset{\text{Thm 2}}{=} \sum_y c\, g(y)\, p(y)$$

$$= c \underbrace{\sum g(y)\, p(y)}$$

$$\underset{\text{Thm 2}}{=} c\, E[g(Y)]$$

$\square$

Linear combinations of expected values: Sums of functions

- <u>Theorem 4</u>: For a random variable Y and functions $g_1, g_2, \ldots, g_k$,

$$E[g_1(Y) + g_2(Y) + \cdots + g_k(Y)] = E[g_1(Y)] + E[g_2(Y)] + \cdots + E[g_k(Y)]$$

proof (k=2 case)

$$E[g_1(Y) + g_2(Y)] \overset{\text{Thm 2}}{=} \sum_y (g_1(y) + g_2(y))p(y)$$

$$= \sum_y (g_1(y)p(y) + g_2(y)p(y))$$

$$= \sum_y g_1(y)p(y) + \sum_y g_2(y)p(y)$$

$$= E[g_1(Y)] + E[g_2(Y)]$$

$\square$

Linear combinations of expected values: Constant

- Theorem 5: If Y is a random variable and c is a constant, then

$$E[c] = c$$

An alternative expression for variance of a random variable

- Theorem 6: If Y is a random variable with mean $E[Y] = \mu$, then

$$Var(Y) = \underbrace{E[(Y-\mu)^2]}\, = \mathrm{E}[Y^2] - \mu^2$$

proof

$$Var(Y) = E[(Y-\mu)^2] = \sum_y (y-\mu)^2 p(y)$$

$$= \sum_y (y^2 - 2\mu y + \mu^2)\, p(y)$$

$$= \sum_y y^2 p(y) + \sum_y -2\mu y\, p(y) + \sum_y \mu^2 p(y)$$

$$= E[Y^2] - 2\mu E[Y] + E[\mu^2]$$

$$= E[Y^2] - 2\mu\cdot\mu + \mu^2 = E[Y^2] - \mu^2$$

$\square$

$$\mu = E[Y] = 0 \times \frac{1}{125} + 1 \times \frac{12}{125} + 2 \times \frac{48}{125} + 3 \times \frac{64}{125}$$
$$= \ldots = 2.4$$

Example 4

$$E[Y^2] = 0 \frac{1}{125} + 1 \times \frac{12}{125} + 4 \frac{48}{125} + 9 \times \frac{64}{125}$$
$$= \ldots = 6.24$$

- A random variable Y has probability distribution

y	0	1	2	3
p(y)	1/125	12/125	48/125	64/125

Find $E[Y]$, $E[Y^2]$, $\mathrm{Var(Y)}$, and $\mathrm{E}[(3Y+2)^2]$.

$$\mathrm{Var}(Y) = E[Y^2] - \mu^2 = 6.24 - (2.4)^2 = \ldots = 0.48$$

$$E[(3Y+2)^2] = (3 \times 0 + 2)^2 \frac{1}{125} + (3 \times 1 + 2)^2 \frac{12}{125} + \ldots$$

$$= E[9Y^2 + 12Y + 4] = E[9Y^2] + E[12Y] + E[4]$$
$$= 9E[Y^2] + 12E[Y] + 4$$
$$= 9(6.24) + 12(2.4) + 4 = \ldots = 88.96$$