CONTINUOUS RANDOM VARIABLES

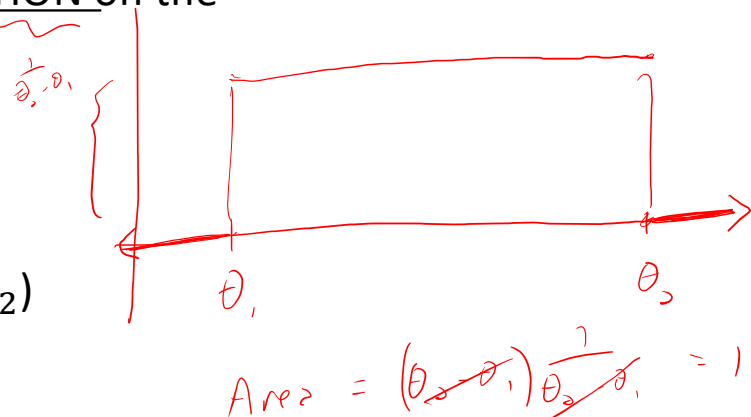The uniform distribution

The uniform distribution

- Example 6: A bus is scheduled to arrive between 8:00am and 8:10am. The probability that it will arrive in any time interval is proportional to the length of the interval. If you arrive at the bus stop at 8:00am, let Y be the time you wait until the bus arrives.

- This is an example of a continuous distribution.

- Definition 4: A random variable Y has a UNIFORM DISTRIBUTION on the interval (θ_1, θ_2) if its pdf is

$$f(y) = \begin{cases} \dfrac{1}{\theta_2 - \theta_1}, & \theta_1 \leq y \leq \theta_2, \\ 0, & \text{otherwise} \end{cases}$$

- Shorthand notation: $Y \sim U(\theta_1, \theta_2)$. (Note: must have $\theta_1 < \theta_2$)
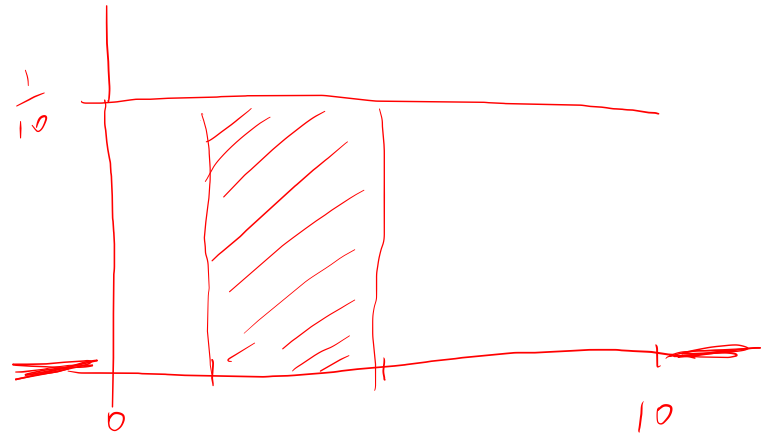
$U(0,10)$

$\theta_1 = 0$

$\theta_2 = 10$

$\frac{1}{10-0} = \frac{1}{10}$

The uniform distribution

$\frac{1}{10}$

0   10
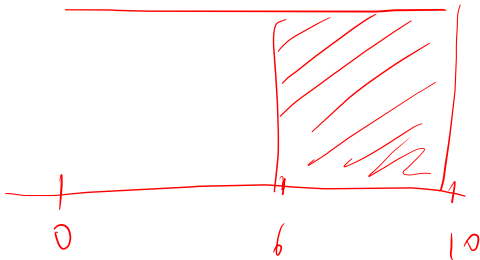
- Example 7: For the bus example, calculate the probability that the bus arrives (a) between 8:02 and 8:05; (b) after 8:06.

(a) $P(2 < Y < 5) = \int_2^5 \frac{1}{10} \, dy = \frac{1}{10} y \Big|_2^5 = \frac{1}{10}(5-2) = \frac{3}{10}$
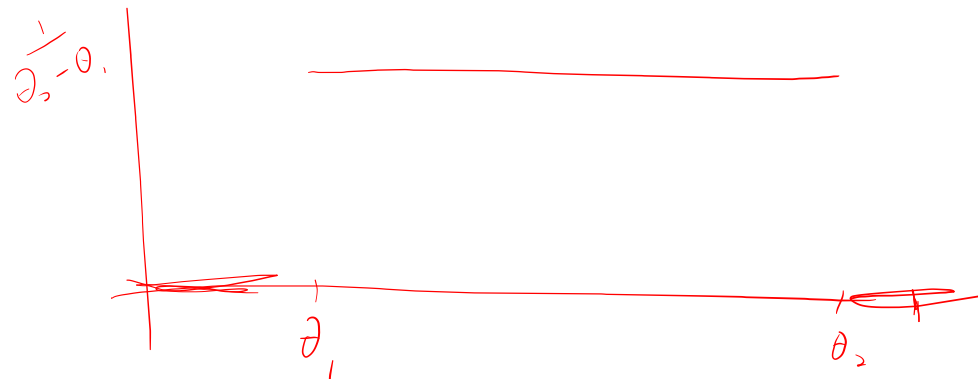
Rectangle $(5-2)\frac{1}{10} = \frac{3}{10}$

(b)

0   6   10

$P(6 < Y < 10) =$ $(10-6)\frac{1}{10} = \frac{4}{10}$

$$f(y) = \frac{1}{\theta_2 - \theta_1}$$

for $\theta_1 < y < \theta_2$

$(0$ $o.w.)$

$\frac{1}{\theta_2 - \theta_1}$

θ_1 $\qquad$ θ_2

The uniform distribution

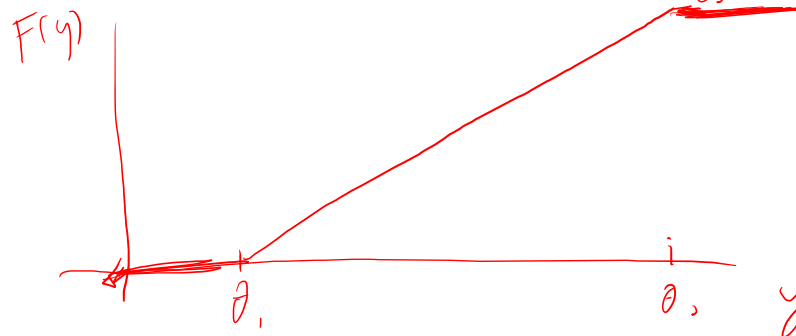- Example 8: Find the cdf of Y if $Y \sim U(\theta_1, \theta_2)$.

$$F(y) = P(Y \leq y)$$

CASE I $\quad y < \theta_1 \quad\quad P(Y \leq y) = 0$
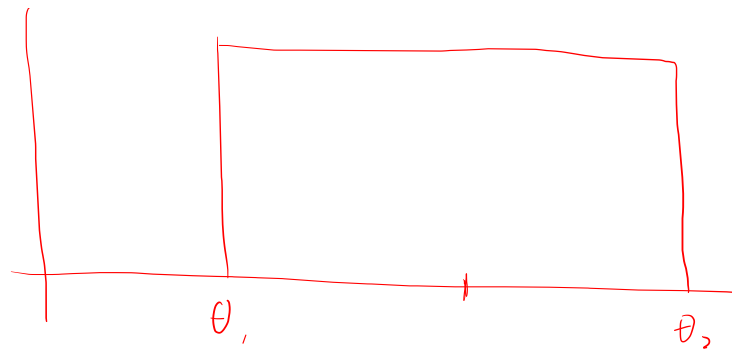
CASE III $\quad y > \theta_2 \quad\quad P(Y \leq y) = 1$

CASE II $\quad \theta_1 < y < \theta_2$

$$P(Y \leq y) = \int_{-\infty}^{y} f(t)\, dt = \int_{\theta_1}^{y} \frac{1}{\theta_2 - \theta_1}\, dt = \frac{1}{\theta_2 - \theta_1} t \Big|_{\theta_1}^{y}$$

$$= \frac{1}{\theta_2 - \theta_1}(y - \theta_1) = \frac{y - \theta_1}{\theta_2 - \theta_1}$$

$$F(y) = \begin{cases} 0 & , \quad y < \theta_1 \\ \frac{y - \theta_1}{\theta_2 - \theta_1} & , \quad \theta_1 \leq y \leq \theta_2 \\ 1 & \quad y > \theta_2 \end{cases}$$

$F(y)$

$\theta_1 \qquad\qquad \theta_2 \quad y$

The uniform distribution

- Theorem 3: If $Y \sim U(\theta_1, \theta_2)$ then

$$\mu = E[Y] = \frac{\theta_1 + \theta_2}{2} \text{ and } \sigma^2 = Var(Y) = \frac{(\theta_1 - \theta_2)^2}{12}.$$

$$E[Y] = \int_{-\infty}^{\infty} y \, f(y) \, dy = \int_{\theta_1}^{\theta_2} y \frac{1}{\theta_2 - \theta_1} \, dy$$

factoring $= \frac{1}{\theta_2 - \theta_1} \left(\frac{1}{2} y^2 \Big|_{\theta_1}^{\theta_2} \right) = \frac{1}{\theta_2 - \theta_1} \frac{1}{2} (\theta_2^2 - \theta_1^2)$

$$E[Y^2] = \frac{1}{\theta_2 - \theta_1} \int_{\theta_1}^{\theta_2} y^2 \, dy = \frac{1}{\theta_2 - \theta_1} \frac{1}{3} y^3 \Big|_{\theta_1}^{\theta_2} = \frac{1}{3} (\theta_2 - \theta_1)^2$$

$$= \frac{1}{\theta_2 - \theta_1} \frac{1}{2} (\theta_2 - \theta_1)(\theta_2 + \theta_1)$$

$$\boxed{= \frac{1}{2} (\theta_1 + \theta_2)}$$

$$Var(Y) = E[Y^2] - (E[Y])^2 = \frac{1}{3}(\theta_2 - \theta_1)^2 - \frac{1}{4}(\theta_1 + \theta_2)^2 = \dots = \frac{1}{12}(\theta_1 - \theta_2)^2$$

The uniform distribution

- Any probability distribution is determined by its *parameters*.

- The *parameters* of the $U(\theta_1, \theta_2)$ distribution are θ_1 and θ_2.

- If a uniform random variable U has $\theta_1 = 0$ and $\theta_2 = 1$ (i.e., $U \sim U(0,1)$) then we say that U has the standard uniform distribution.

$$f(y) = \begin{cases} 1 & 0 < y < 1 \\ 0 & o.w. \end{cases}$$

Moment generating functions for continuous random variables

- Moments and mgfs for continuous random variables are defined just as for discrete random variables.

- The kth moment [taken about the origin] of a random variable is $\mu'_k = E[Y^k]$.

- The kth central moment (or kth moment about the mean) of a random variable is $\mu_k = E[(Y - \mu)^2]$.

- The moment generating function for a random variable Y is $m(t) = E[e^{tY}]$. (For this to exist there must be some constant $b > 0$ such that $m(t)$ is finite for $-b < t < b$.)

Moment generating function for the uniform distribution

$Y \sim U(\theta_1, \theta_2)$

- Example 9: Find the mgf for the ~~standard~~ uniform random variable $Y \sim U(\theta_1, \theta_2)$ and determine whether it exists.

$$m(t) = E\left[e^{tY}\right] = \int_{\theta_1}^{\theta_2} \frac{1}{\theta_2 - \theta_1} e^{ty} \, dy$$

$$= \frac{1}{\theta_2 - \theta_1} \left(\left. \frac{1}{t} e^{ty} \right|_{\theta_1}^{\theta_2} \right) = \frac{e^{\theta_2 t} - e^{\theta_1 t}}{(\theta_2 - \theta_1) t}$$

$$\frac{t}{-b} \quad \frac{t}{b}$$

Any $t \neq 0$ finite

$$\frac{e^0 - e^0}{(\theta_2 - \theta_1) 0}$$

$$\frac{1 - 1}{0} = \frac{0}{0} \quad !!!$$

$a\,\theta_2$

$t = 0$?

indeterminate

$$\lim_{t \to 0} \quad \frac{L'Hospital}{} \quad \frac{e^{\theta_2 t} - e^{\theta_1 t}}{(\theta_2 - \theta_1) t} = \lim_{t \to 0} \frac{\frac{d}{dt} e^{\theta_2 t} - e^{\theta_1 t}}{\frac{d}{dt} (\theta_2 - \theta_1) t} = \lim_{t \to 0} \frac{\theta_2 e^{\theta_2 t} - \theta_1 e^{\theta_1 t}}{\theta_2 - \theta_1}$$

$$\lim_{t \to 0} \text{ finite} \implies mgf \text{ exists}$$

$$\frac{\theta_2 e^0 - \theta_1 e^0}{\theta_2 - \theta_1} = \frac{\theta_2 - \theta_1}{\theta_2 - \theta_1} = 1$$