

PROPERTIES OF POINT ESTIMATORS AND METHODS OF ESTIMATION

Consistency

$$\bar{Y}_n$$

$$S_n^2$$

Consistency

- Another property that we would like an estimator to have is for it to get closer and closer to the “truth” as the sample size n increases.
- Sometimes we will put a subscript n on an estimator when we are thinking about its properties as a function of n . 
- Definition 2: An estimator $\hat{\theta}_n$ is a CONSISTENT ESTIMATOR of θ if, for any positive constant ϵ , 

$$\lim_{n \rightarrow \infty} P(|\hat{\theta}_n - \theta| > \epsilon) = 0$$



$$\text{Unb: } E(\hat{\theta}_n) = \theta$$

Consistency of unbiased estimators

- Theorem 1: If $\hat{\theta}_n$ is unbiased for θ , then it is consistent for θ if

$$\lim_{n \rightarrow \infty} \text{Var}(\hat{\theta}_n) = 0$$

$$\text{Var}(\hat{\theta}_n) \rightarrow 0 \quad \text{as } n \rightarrow \infty$$

Consistency: Example

- Example 3: If $Y_1, Y_2, \dots, Y_n \sim iid N(\mu, \sigma^2)$, is the sample variance $s_n^2 = \frac{1}{n-1} \sum_{i=1}^n (Y_i - \bar{Y})^2$ consistent? What about the variance with a divisor of n instead?

Know: $\frac{(n-1) \sum_n^2}{\sigma^2} \sim \chi^2_{n-1}$

Know: $E\left[\frac{(n-1) \sum_n^2}{\sigma^2}\right] = n-1$

$\frac{n-1}{\sigma^2} E[\sum_n^2] \approx n-1$

$E[\sum_n^2] = \sigma^2$

Unbiased!

for σ^2
prop. of χ^2

$$Var\left(\frac{(n-1) \sum_n^2}{\sigma^2}\right) = 2(n-1)$$

$$\frac{(n-1)^2}{\sigma^4} Var(\sum_n^2) = 2(n-1)$$

$$Var(\sum_n^2) = \frac{2\sigma^4}{n-1}$$

$$\lim_{n \rightarrow \infty} Var(\sum_n^2) = 0$$

So $\hat{\sigma}_n^2$ is unb. $\text{Var}(\hat{\sigma}_n^2) \rightarrow 0$ as $n \rightarrow \infty$

So by Thm 1, $\hat{\sigma}_n^2$ is consistent for σ^2 .

$$\frac{1}{n} \sum_{i=1}^n (Y_i - \bar{Y})^2 = \frac{n-1}{n} \hat{\sigma}_n^2$$

Asymptotically unbiased $\xrightarrow[n \rightarrow \infty]{\text{as}} \sigma^2$

$$E\left[\frac{n-1}{n} \hat{\sigma}_n^2\right] = \frac{n-1}{n} E[\hat{\sigma}_n^2] = \frac{n-1}{n} \sigma^2 \neq \sigma^2$$

This est. is biased for σ^2 !!

$$\begin{aligned} \text{Var}\left(\frac{n-1}{n} \hat{\sigma}_n^2\right) &= \frac{(n-1)^2}{n^2} \text{Var}(\hat{\sigma}_n^2) = \frac{(n-1)^2}{n^2} \frac{2\sigma^4}{n-1} \\ &= \frac{2\sigma^4(n-1)}{n^2} \xrightarrow[n \rightarrow \infty]{\text{as}} 0 \end{aligned}$$

Is this consistent
for σ^2 ?

Consistency of asymptotically unbiased estimators

- Theorem 2: An estimator $\hat{\theta}_n$ for θ is consistent if both

$$\lim_{n \rightarrow \infty} \text{bias}(\hat{\theta}_n) = 0 \quad \checkmark$$

$$\lim_{n \rightarrow \infty} \text{Var}(\hat{\theta}_n) = 0$$

$$\text{bias} \left(\frac{1}{n} \sum_{i=1}^n (Y_i - \bar{Y})^2 \right) = \frac{n-1}{n} \sigma^2 - \frac{1}{n} \sigma^2 = \sigma^2 \left(\frac{-1}{n} \right) \rightarrow 0$$

$\text{Var} (\quad) \rightarrow 0$ as $n \rightarrow \infty$

This estimate is also consistent for σ^2 .

Consistency of asymptotically unbiased estimators

- Theorem 2: An estimator $\hat{\theta}_n$ for θ is consistent if both

$$\lim_{n \rightarrow \infty} \text{bias}(\hat{\theta}_n) = 0$$

$$\lim_{n \rightarrow \infty} \text{Var}(\hat{\theta}_n) = 0$$