PROPERTIES OF POINT ESTIMATORS AND METHODS OF ESTIMATION

Relative efficiency

Properties of point estimators

function of Y's

- We have discussed before about the characteristics that we would like an estimator $\hat{\theta}$ to have (when estimating an unknown parameter θ).
- These characteristics are related to the sampling distribution of $\hat{\theta}$.
- Given two estimators of θ, how to decide if one is "better" than the other?
- Of all possible estimators of θ, is it possible to say one is "best"?

Relative efficiency

- This is a concept that is used to compare two unbiased estimators.
- If two estimators $\hat{\theta}_1$ and $\hat{\theta}_2$ are both unbiased for θ, then the one with the smaller variance would be preferred.
- Definition 1: The EFFICIENCY of $\hat{\theta}_1$ RELATIVE to $\hat{\theta}_2$ is

$$\text{eff}\left(\hat{\theta}_1, \hat{\theta}_2\right) = \frac{Var\left(\hat{\theta}_2\right)}{Var\left(\hat{\theta}_1\right)}$$

Relative efficiency: Example

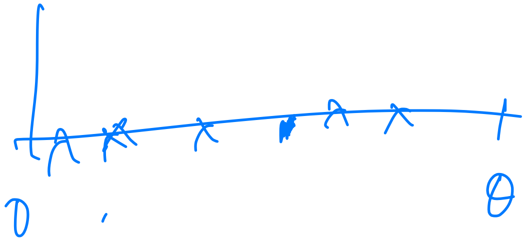
- Example 1: If $Y_1, Y_2, \ldots, Y_n \sim iid\ N(\mu, \sigma^2)$ then the sample mean $\bar{Y}$ and the sample median $\tilde{Y}$ are both unbiased for μ. What is the efficiency of $\bar{Y}$ relative to $\tilde{Y}$? (Hint: The variance of the sample median is $1.5708\sigma^2/n$.) $= Var(\tilde{Y})$

From before $\bar{Y} \sim N(\mu, \frac{\sigma^2}{n})$

$E(\bar{Y}) = \mu$ $\quad Var(\bar{Y}) = \frac{\sigma^2}{n}$

$$eff(\bar{Y}, \tilde{Y}) = \frac{Var(\tilde{Y})}{Var(\bar{Y})} = \frac{1.5708\cancel{\sigma^2/n}}{\cancel{\sigma^2/n}} = 1.5708$$

"57% more efficient"



$$f_Y(y) = \begin{cases} \frac{1}{\theta} & 0 < y < \theta \\ 0 & o.w. \end{cases}$$

Relative efficiency: Another example

- Example 2: If $Y_1, Y_2, \dots, Y_n \overset{iid}{\sim} U(0, \theta)$, then both $\hat{\theta}_1 = 2\bar{Y}$ and $\hat{\theta}_2 = \left(\frac{n+1}{n}\right) Y_{(n)}$ are unbiased for θ. What is the efficiency of $\hat{\theta}_1$ relative to $\hat{\theta}_2$?

If $Y \sim U(0,\theta)$: $E[Y_1] = \int_0^\theta y \cdot \frac{1}{\theta}\, dy = \frac{1}{\theta}\left(\frac{1}{2} y^2 \Big|_0^\theta\right) = \frac{1}{\theta}\left(\frac{1}{2}\theta^2\right) = \frac{\theta}{2}$

$E[\bar{Y}]$ = rules of exp. . . . $= E[Y_1] = \frac{\theta}{2}$

$E[\hat{\theta}_1] = E[2\bar{Y}] = 2E[\bar{Y}] = 2\frac{\theta}{2} = \theta$

$Var(\bar{Y})$ = rules of variance . . . $= \frac{Var(Y_1)}{n}$

$Var(Y) = E[Y^2] - (E[Y])^2 = \int_0^\theta y^2 \frac{1}{\theta}\, dy - \left(\frac{\theta}{2}\right)^2 = \frac{1}{\theta}\left(\frac{1}{3} y^3 \Big|_0^\theta\right) - \frac{\theta^2}{4}$

$= \frac{1}{\theta}\frac{1}{3}\theta^3 - \frac{\theta^2}{4} = \theta^2\left(\frac{1}{3} - \frac{1}{4}\right) = \frac{1}{12}\theta^2$

$Var(\bar{Y}) = \frac{\theta^2}{12n}$

$$Var(\hat{\theta}_1) = Var(2\bar{Y}) = 4\cdot\frac{\theta^2}{12n} = \frac{\theta^2}{3n}$$

$$\hat{\theta}_2 = \frac{n+1}{n}Y_{(n)} \qquad \text{pdf of } Y_{(n)}: \; n(\underbrace{F(y)}_{\text{cdf}})^{n-1}f(y) = n\left(\frac{y}{\theta}\right)^{n-1}\frac{1}{\theta}$$

$$= \frac{n}{\theta^n}y^{n-1}$$

$$E[Y_{(n)}] = \int_0^\theta y\,\frac{n}{\theta^n}y^{n-1}\,dy = \dots = \frac{n}{n+1}\theta$$

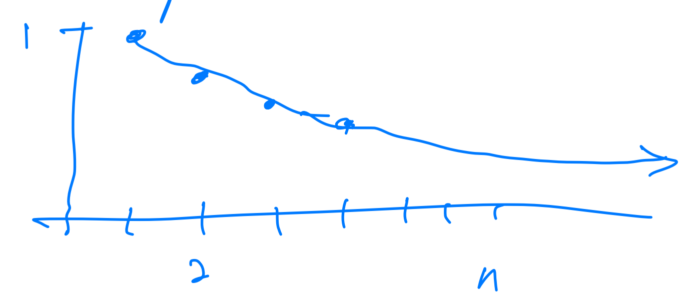
$$Var(Y_{(n)}) = \dots \left(\frac{n}{n+2} - \left(\frac{n}{n+1}\right)^2\right)\theta^2 \quad \text{(check book)}$$

$$E[\hat{\theta}_2] = E\left[\frac{n+1}{n}Y_{(n)}\right] = \frac{n+1}{n}E[Y_{(n)}] = \frac{\cancel{n+1}}{\cancel{n}}\,\frac{\cancel{n}}{\cancel{n+1}}\,\theta = \theta$$

$$Var(\hat{\theta}_2) = Var\left(\frac{n+1}{n}Y_{(n)}\right) = \frac{(n+1)^2}{n^2}Var(Y_{(n)}) = \dots = \frac{\theta^2}{n(n+2)}$$

$$eff(\hat{\theta}_1, \hat{\theta}_2) = \frac{Var(\hat{\theta}_2)}{Var(\hat{\theta}_1)} = \frac{\frac{\cancel{\theta^2}}{\cancel{n}(n+2)}}{\frac{\cancel{\theta^2}}{3\cancel{n}}} = \boxed{\frac{3}{n+2}}$$

depends on n



eff < 1 $\hat{\theta}_1$ is less eff than $\hat{\theta}_2$, increasingly so as $n \to \infty$