MOMENTS AND MOMENT-GENERATING FUNCTIONS

WARNING

CALCULUS AHEAD!!

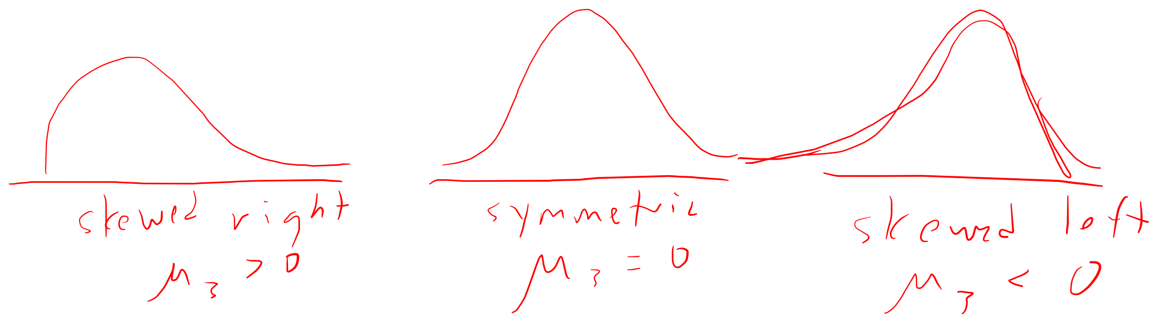
Moments of random variables

- Definition 1: The kth MOMENT [taken about the origin] of a random variable is $\mu'_k = E[Y^k]$.
- The first moment of a random variable is $\mu'_1 = E[Y] = \mu$.
- The second moment is $\mu'_2 = E[Y^2]$

third $E[Y^3]$

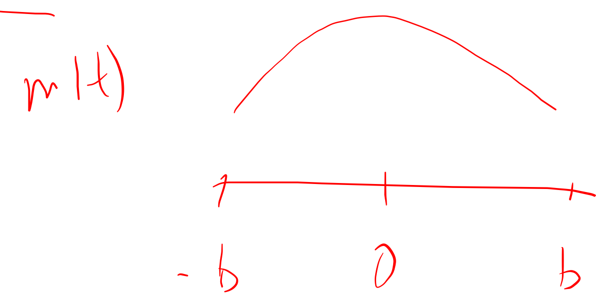
Moments of random variables

- Definition 2: The kth CENTRAL MOMENT (or kth moment about the mean) of a random variable is $\mu_k = E[(Y - \mu)^2]$.
- Most common moments:
  - Mean: $\mu = \mu_1 = E[Y]$
  - Variance: $\sigma^2 = \mu_2 = E[(Y - \mu)^2]$
  - Skewness: $\mu_3 = E[(Y - \mu)^3]$
  - Kurtosis: $\mu_4 = E[(Y - \mu)^4]$



Moment generating functions (mgfs)

- Definition 3: The MOMENT GENERATING FUNCTION for a random variable Y is $m(t) = E[e^{tY}]$.
- NOTE: the mgf exists if there is some $b > 0$ such that $m(t)$ is finite for $|t| < b$.
- This is an "open neighborhood" of 0.
- If there are no open neighborhoods of 0 for which $E[e^{tY}]$ exists (is finite) then we say that the mgf does not exist.



What is so great about moment generating functions?

- First, we like mgfs because they generate moments . . .
- Second, we like mgfs because they characterize a probability distribution. Each unique probability distribution has its own mgf. So if I can show that some random variable has a mgf associated with some distribution, then the random variable must have that distribution!

mgfs same $\Longleftrightarrow$ distributions same

Generating moments with moment generating functions

- Theorem 1: If $m(t)$ exists, then for any positive integer k,

$$\mu'_k = \left.\frac{d^k m(t)}{dt^k}\right|_{t=0} = m^{(k)}(0)$$

mean $E[Y] = \mu'_1 = \left.\frac{d}{dt} m(t)\right|_{t=0} = m'(0) \quad m^{(1)}(0)$

~~variance~~ $E[Y^2] = \mu'_2 = \left.\frac{d^2}{dt} m(t)\right|_{t=0} = m''(0)$

direct $E(Y) = 1\cdot\frac{1}{10} + 2\cdot\frac{2}{10} + 3\frac{3}{10} + 4\cdot\frac{4}{10}$

$= 3$

Generating moments with moment generating functions

y	1	2	3	4
$p(y)$	$\frac{1}{10}$	$\frac{2}{10}$	$\frac{3}{10}$	$\frac{4}{10}$

- Example 1: If Y is a random variable with probability distribution $p(y) = \frac{y}{10}$ for $y = 1, 2, 3, 4$, find $m(t)$ and use $m(t)$ to calculate $E[Y]$.

$$m(t) = E[e^{tY}] = \sum_{y=1}^{4} e^{ty} p(y)$$

$$= e^t \cdot \frac{1}{10} + e^{2t} \cdot \frac{2}{10} + e^{3t} \cdot \frac{3}{10} + e^{4t} \cdot \frac{4}{10}$$

Does this "exist"? $|m(t)| < \infty$ for all $t \in (-b, b)$?

Any finite b will work

$\Rightarrow$ mgf exists!

$$E[Y] = \frac{d}{dt} m(t) \Big|_{t=0}$$

$$= e^t \frac{1}{10} + 2e^{2t}\frac{2}{10} + 3e^{3t}\cdot\frac{3}{10} + 4e^{4t}\cdot\frac{4}{10}\Big|_{t=0}$$

$$= \frac{1}{10}\left(e^t + 4e^{2t} + 9e^{3t} + 16e^{4t}\right)\Big|_{t=0} = \frac{1}{10}(1 + 4 + 9 + 16) = \frac{30}{10}$$

$$= 3$$

FACT $e^x = \sum_{i=0}^{\infty} \frac{x^i}{i!}$ ALL!!

Generating moments with moment generating functions

- Example 2: Find the mgf of a Poisson random variable with rate parameter λ and use it to find the mean and variance.

$m(t) = E[e^{tY}] = \sum_{y=0}^{\infty} e^{ty} p(y) = \sum_{y=0}^{\infty} e^{ty} \frac{\lambda^y e^{-\lambda}}{y!} = e^{-\lambda} \sum_{y=0}^{\infty} \frac{e^{ty} \lambda^y}{y!}$

$= e^{-\lambda} e^{\lambda e^t} = \boxed{e^{\lambda(e^t - 1)}}$

Let $x = \lambda e^t$

FACT: $e^{\lambda e^t} = \sum_{y=0}^{\infty} \frac{(\lambda e^t)^y}{y!} = \sum_{y=0}^{\infty} \frac{e^{ty} \lambda^y}{y!}$

exist? $b > 0$ is $m(t) < \infty$ for $-b < t < b$?

Yes - any finite t

$\boxed{e^{\lambda e^t - \lambda}}$

$$m(t) = e^{\lambda(e^t - 1)}$$

$$E[Y] = \frac{d}{dt} m(t)\Big|_{t=0} = e^{\lambda(e^t-1)} \lambda e^t \Big|_{t=0} = e^{\lambda(1-1)} \lambda \times 1 = e^0 \lambda 1 = \boxed{\lambda}$$

mean

$$E[Y^2] = \frac{d^2}{dt^2} m(t)\Big|_{t=0} = \frac{d}{dt}\left(\underbrace{\lambda e^t}\,\underbrace{e^{\lambda(e^t-1)}}\right)\Big|_{t=0} =$$

$$= \lambda e^t e^{\lambda(e^t-1)} \lambda e^t + e^{\lambda(e^t-1)} \lambda e^t \Big|_{t=0}$$

$$= \lambda e^t e^{\lambda(e^t-1)} (1 + \lambda e^t)\Big|_{t=0} = m''(t)$$

$$= \lambda \cdot 1 \cdot 1 (1+\lambda) = \lambda + \lambda^2$$

$$Var(Y) = E[Y^2] - \mu^2 = \lambda + \lambda^2 - (\lambda)^2 = \boxed{\lambda}$$

Poisson dist

mean = variance

Characterizing distributions with moment generating functions

- If a probability distribution $p(y)$ for a random variable Y has (existing) mgf $m(t)$, then it is the unique mgf for Y.
- If two random variables have the same mgf then they must have the same probability distribution.
- Example 3: If a random variable Y has mgf $e^{7.1e^t - 7.1}$, then what is the distribution of Y?

$m(t) =$

Poisson $\lambda = 7.1$