

MULTIVARIATE DISTRIBUTIONS

Independence of random variables

Independent random variables

- Definition 5: Two random variables Y_1 with cdf $F_1(y_1)$ and Y_2 with cdf $F_2(y_2)$ and with joint cdf $F(y_1, y_2)$ are INDEPENDENT if $F(y_1, y_2) = F_1(y_1)F_2(y_2)$ for all pairs (y_1, y_2) . (Otherwise the random variables are DEPENDENT.)

if A & B indep.

$$P(A \text{ and } B) = P(A \cap B) = P(A) \cdot P(B)$$

if Y_1 & Y_2 indep.

$$F(y_1, y_2) = P(Y_1 \leq y_1, \overset{\text{and}}{\downarrow} Y_2 \leq y_2) = P(Y_1 \leq y_1) P(Y_2 \leq y_2) = F_1(y_1) F_2(y_2)$$

Independence of discrete random variables

Theorem 2: If Y_1 and Y_2 are discrete random variables with pmfs $p_1(y_1)$ and $p_2(y_2)$, respectively, and with joint pmf $p(y_1, y_2)$, then Y_1 and Y_2 are independent if and only if

$$p(y_1, y_2) = p_1(y_1)p_2(y_2)$$

for all pairs (y_1, y_2) .

$$P(Y_1 = y_1, Y_2 = y_2) = P(Y_1 = y_1) P(Y_2 = y_2)$$

Independence of continuous random variables

Theorem 3: If Y_1 and Y_2 are continuous random variables with pdfs $f_1(y_1)$ and $f_2(y_2)$, respectively, and with joint pdf $f(y_1, y_2)$, then Y_1 and Y_2 are independent if and only if

$$f(y_1, y_2) = f_1(y_1)f_2(y_2)$$

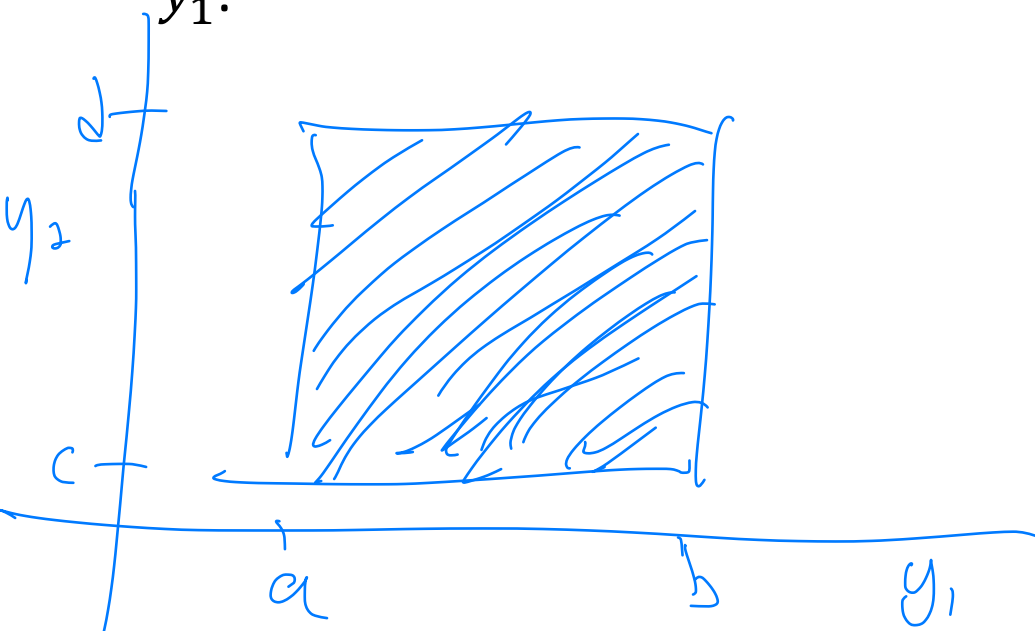
for all pairs (y_1, y_2) .

Factorization theorem

Theorem 4: For random variables Y_1 and Y_2 with joint pdf $f(y_1, y_2)$ that is positive for $a \leq y_1 \leq b$ and $c \leq y_2 \leq d$ for some constants a, b, c , and d (and zero elsewhere), Y_1 and Y_2 are independent if and only if the joint pdf can be factored into two nonnegative functions, i.e.,

$$f(y_1, y_2) = g(y_1)h(y_2)$$

where $g(y_1)$ does not depend on y_2 and $h(y_2)$ does not depend on y_1 .

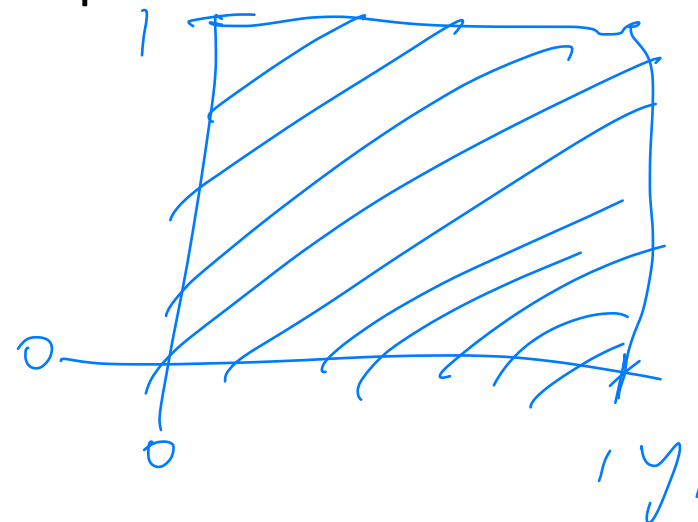


Example 6: Random variables Y_1 and Y_2 have joint pdf

$$f(y_1, y_2) = 2y_1, 0 \leq y_1 \leq 1, 0 \leq y_2 \leq 1.$$

Are they independent?

YES



$$f(y_1, y_2) = 2y_1$$

$$\begin{array}{c} \uparrow \\ h(y_2) \end{array} \quad \begin{array}{c} \nwarrow \\ g(y_1) \end{array}$$

$$\begin{aligned} &\Rightarrow g(y_1) = y_1 \leftarrow \dots y_2 \\ &\Rightarrow h(y_2) = 2 \leftarrow \text{does not depend on } y_1 \end{aligned}$$

true for all $0 \leq y_1 \leq 1, 0 \leq y_2 \leq 1$

$\Rightarrow Y_1$ & Y_2 are independent.

$$h(y_2) = 2 \quad g(y_1) = \frac{1}{10} y_1 \quad g(y_1)h(y_2) = 2y_1 \quad \checkmark$$

Example 7: Random variables Y_1 and Y_2 have joint pdf

$$f(y_1, y_2) = 6y_1 \text{ for } 0 < y_1 < y_2 < 1.$$

Are Y_1 and Y_2 independent?

Suppose $y_1 = 0.75$

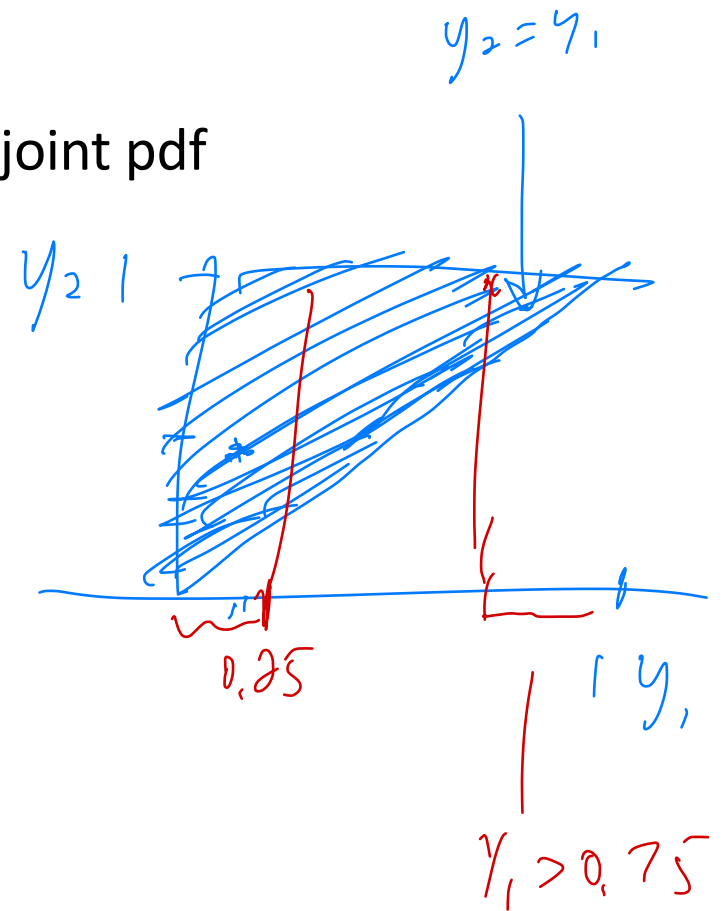
y_2 must be in $(0.75, 1]$

Suppose $y_1 = 0.25$

y_2 must be in $(0.25, 1]$

Possible values for y_2 depend on y_1 !

y_1 and y_2 not independent \Rightarrow dependent



Helpful tip

- If the support of a joint pdf involves y_1 being constrained by y_2 (or y_2 being constrained by y_1), then the random variables cannot be independent.

General factorization theorem

Theorem 5 (Independence of n random variables): Random variables Y_1, Y_2, \dots, Y_n are independent if and only if

$$f(y_1, y_2, \dots, y_n) = f_1(y_1)f_2(y_2) \dots f_n(y_n),$$

where $f(y_1, y_2, \dots, y_n)$ is the joint pdf of Y_1, Y_2, \dots, Y_n .

Example 7: The lifetime of each of five components follows an exponential(100) distribution. If the lifetimes are independent, find the joint pdf.