FUNCTIONS OF RANDOM VARIABLES

Method of cdfs

Y, pdf f, cdf F

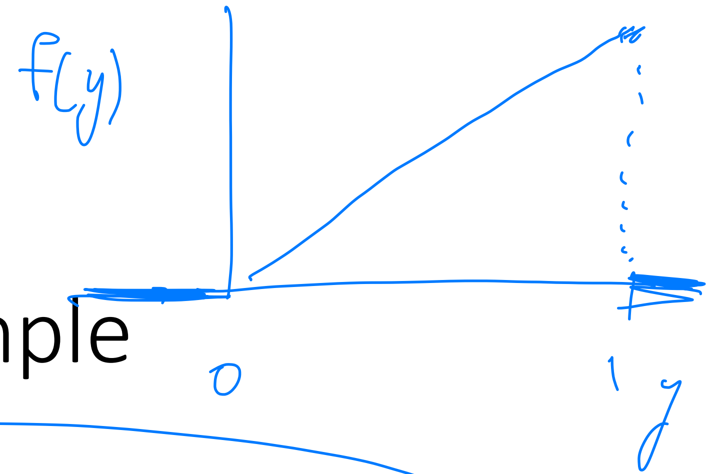
Functions of random variables

- We will now study how to find the distribution of functions of random variables.
  - Functions of a single random variable: Y^2, $\sqrt{Y}$, $sign(Y)$, …, $g(Y)$ for a fixed known function g
  - Functions of multiple random variables: $\frac{1}{n}(Y_1 + Y_2 + \cdots + Y_n)$, $\frac{1}{n-1}\sum_{i=1}^{n}(Y_i - \bar{Y})^2$, $\min(Y_1, Y_2, \ldots, Y_n)$,…, $h(Y_1, Y_2, \ldots, Y_n)$ for a fixed known function h
- Three approaches:
  - Method of cdfs
  - Method of transformation
  - Method of mgfs

Method of cdfs

- We start with a random variable Y and its distribution: either $F(y)$ or $f(y)$ (if Y is continuous) or $p(y)$ (if Y is discrete).
- We want to find the distribution of a "new" random variable U that is a function of Y: $U = U(Y)$.

$$F_U(u) = \begin{cases} 0, & u \le -1 \\ \left(\frac{u+1}{3}\right)^2, & -1 < u \le 2 \\ 1 & u > 2 \end{cases}$$



Method of cdfs: Univariate example

- Example 1: Y has pdf $f(y) = 2y$ for $0 \le y \le 1$. Find the distribution of $3Y - 1$.

$U = 3Y - 1$

① Get cdf of Y $\quad P(Y \le y) = \int_0^y 2x\,dx = x^2 \Big|_0^y = y^2$

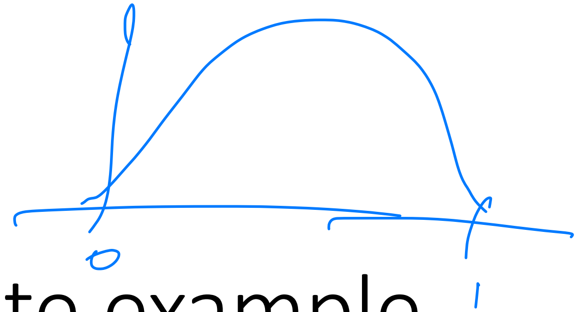
Be careful/clear when expressing cdf (pdf)

$$F_Y(y) = \begin{cases} 0 & y \le 0 \\ y^2 & 0 < y \le 1 \\ 1 & y > 1 \end{cases}$$

② Set $U = 3Y - 1$ $\longrightarrow$ find cdf of U

$$F_U(u) = P(U \le u) = P(3Y - 1 \le u) = P(3Y \le u+1)$$

$$= P\left(Y \le \frac{u+1}{3}\right) = F_Y\left(\frac{u+1}{3}\right) = \begin{cases} 0 & \frac{u+1}{3} \le 0 \\ \left(\frac{u+1}{3}\right)^2, & 0 < \frac{u+1}{3} \le 1 \\ 1, & \frac{u+1}{3} > 1 \end{cases}$$



Method of cdfs: Another univariate example

- Example 2: Y has pdf $f(y) = 6y(1-y)$ for $0 \le y \le 1$. Find the distribution of Y^3.

① Find cdf of Y

$$F_Y(y) = P(Y \le y) = \int_0^y 6x(1-x)\,dx = \int_0^y (6x - 6x^2)\,dx$$

$$= 3x^2 - 2x^3 \Big|_0^y = 3y^2 - 2y^3$$

$$F_Y(y) = \begin{cases} 0 & y < 0 \\ 3y^2 - 2y^3 & 0 \le y < 1 \\ 1 & y \ge 1 \end{cases}$$

② $U = Y^3$

$$F_U(u) = P(U \le u) = P(Y^3 \le u) = P(Y \le u^{1/3})$$

$$= F_Y(u^{1/3})$$

$$\boxed{F_U(u) = \begin{cases} 0 & u < 0 \\ 3u^{2/3} - 2u & 0 \le u < 1 \\ 1 & u \ge 1 \end{cases}}$$

pdf $f_U(u) = 2u^{-1/3} - 2$

$0 \le u \le 1$

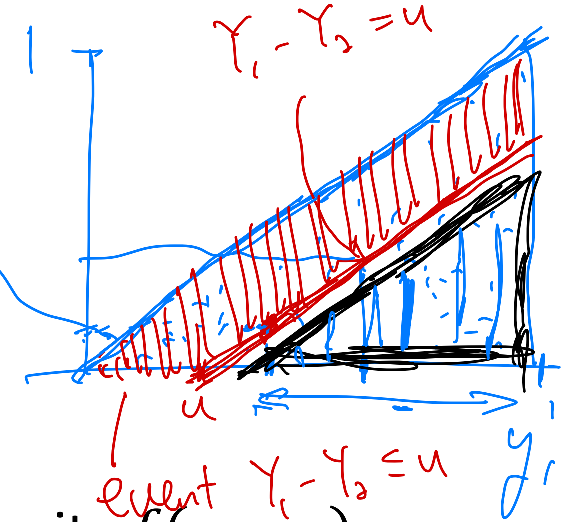
Method of cdfs: Bivariate case

- We start with two random variables Y_1 and Y_2 and their joint density $f(y_1, y_2)$
- We want to find the distribution of a "new" random variable U that is a function of Y: $U = U(Y_1, Y_2)$.
- Method of cdfs in the bivariate case:
  - Determine possible values of U
  - For u in this domain, integrate $f(y_1, y_2)$ to get $\boxed{F_U(u)}$
  - Differentiate $F_U(u)$ to get $f_U(u)$

Method of cdfs: Bivariate example

$Y_2 = Y_1 - u$

$y_1 = y_2$



- Example 3: Random variables Y_1 and Y_2 have joint density $f(y_1, y_2) = 3y_1$ for $0 \le y_2 \le y_1 \le 1$. Find the density of $Y_1 - Y_2$. $= u$

$0 \le u \le 1$

$F_U(u) = P(U \le u) = P(\underbrace{Y_1 - Y_2 \le u}_{\text{event!}})$ ← draw this event in diagram

P(red) = 1 − P(black)

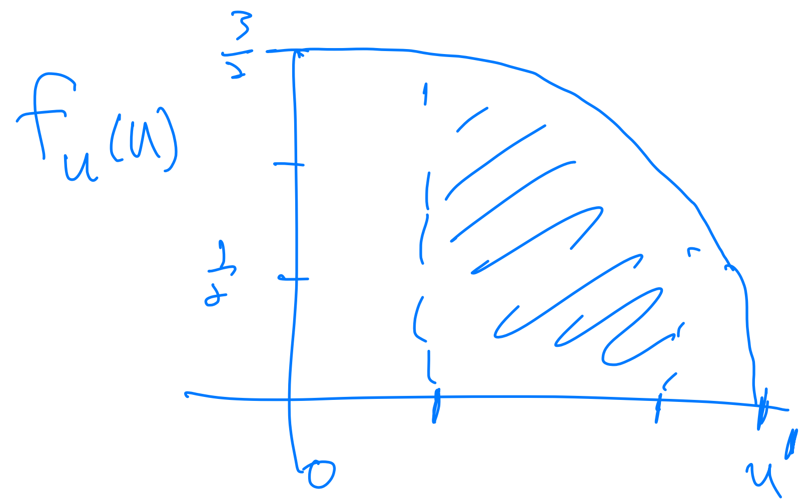
$$= 1 - \int_u^1 \int_0^{y_1-u} 3y_1 \, dy_2 \, dy_1 = 1 - \int_u^1 \left(3y_1 y_2 \Big|_{y_2=0}^{y_1-u} \right) dy_1$$

$$= 1 - \int_u^1 3y_1(y_1 - u)\, dy_1 = 1 - \int_u^1 (3y_1^2 - 3uy_1)\, dy_1 = 1 - \left(y_1^3 - \frac{3}{2}uy_1^2 \right) \Big|_{y_1=u}^{1}$$

$$= 1 - \left(1 - \frac{3}{2}u - u^3 + \frac{3}{2}u^3\right) = \frac{3}{2}u + u^3 - \frac{3}{2}u^3 = \frac{1}{2}(3u - u^3)$$

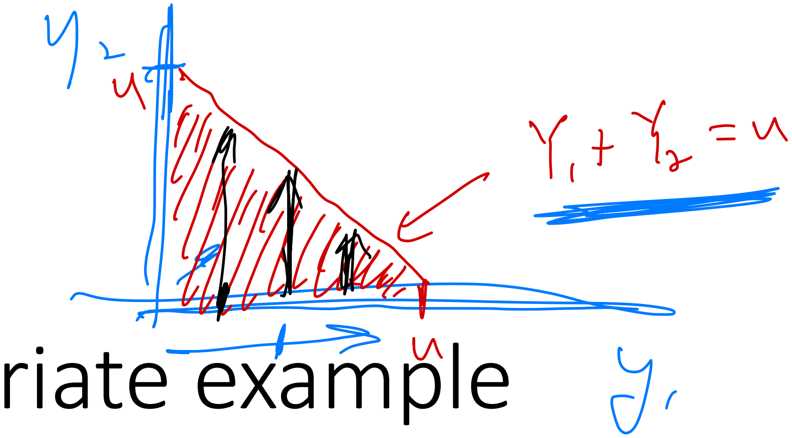
$0 \le u \le 1$

$$f_U(u) = \frac{d}{du} F_U(u) = \frac{1}{2}(3 - 3u^2) = \frac{3}{2}(1-u^2)$$



$$P\left(\frac{1}{4} < U < \frac{3}{4}\right)$$

- For your own amusement: calculate $E[U]$



Method of cdfs: Another bivariate example

- Example 4: Random variables Y_1 and Y_2 have joint density $f(y_1, y_2) = 6e^{-3y_1-2y_2}$ for $y_1 > 0,\ y_2 > 0$. Find the density (pdf) of $Y_1 + Y_2$. $= U$

Possible values for u? $\quad 0 \le U < \infty$

$$F_U(u) = P(U \le u) = P(\underbrace{Y_1 + Y_2 \le u}_{\text{event}})$$

$$= \int_0^u \int_0^{u-y_1} 6e^{-3y_1-2y_1}\, dy_2\, dy_1 =$$

$$= -\int_0^u 6e^{-3y_1}\left(\frac{1}{2}e^{-2y_2}\Big|_0^{u-y_1}\right) dy_1$$

$$= -\int_0^u 3e^{-3y_1}\left(e^{-2u+2y_1} - 1\right) dy_1$$

$$= -3\int_0^u \left(e^{-2u-y_1} - e^{-3y_1}\right) dy_1 = -3\left(-e^{-2u}e^{-y_1} + \frac{1}{3}e^{-3y_1}\Big|_0^u\right)$$

$$= -3\left(-e^{-2u}e^{-u} + \frac{1}{3}e^{-3u} + e^{-2u} - \frac{1}{3}\right)$$

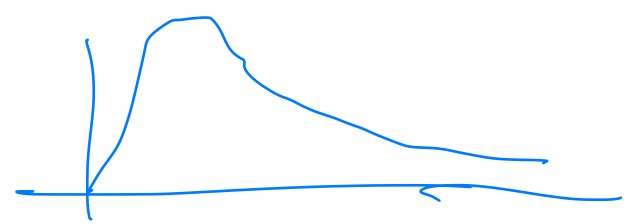
$$= -3\left(-e^{-3u} + \frac{1}{3}e^{-3u} + e^{-2u} - \frac{1}{3}\right)$$

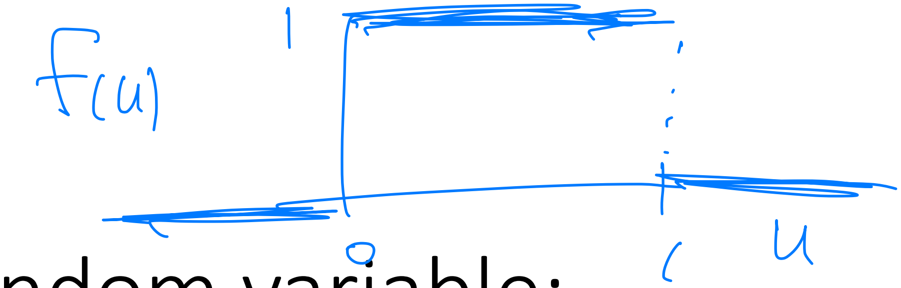
$$= 1 - 3e^{-2u} + 2e^{-3u} = F_U(u),$$

$$\boxed{F_U(u) = \begin{cases} 0 & u < 0 \\ 1 - 3e^{-2u} + 2e^{-3u} & u \geq 0 \end{cases}}$$

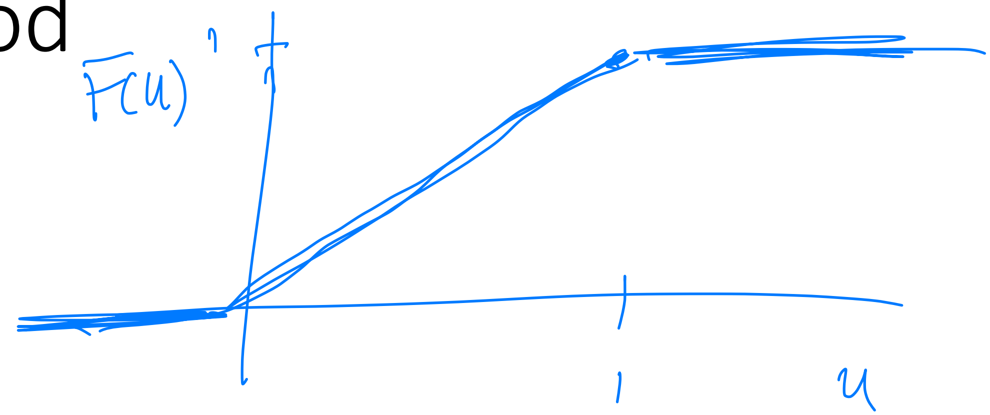
$$f_U(u) = \frac{d}{du}\left(1 - 3e^{-2u} + 2e^{-3u}\right)$$

$$= 6e^{-2u} - 6e^{-3u} = 6\left(e^{-2u} - e^{-3u}\right), \quad \underline{u \geq 0}$$





Transforming a $U(0,1)$ random variable: The inverse cdf method



- If $U \sim U(0,1)$, then the cdf is
- $F(u) = \begin{cases} 0, u < 0 \\ u, 0 \leq u \leq 1 \\ 1, u > 1 \end{cases}$
- From a uniform random variable we can transform to any distribution F_Y by applying the inverse cdf: $F^{-1}(y)$

pdf of exp $f(y) = \frac{1}{\beta} e^{-\frac{1}{\beta} y}$, $y > 0$

cdf $F(y) = 1 - e^{-\frac{1}{\beta} y}$, $y > 0$

Inverse cdf method

- Example 5: Starting with a random variable $U \sim U(0,1)$ find some function h of U so that $Y = h(U) \sim \exp(\beta)$.

TRICK Find the dist of $Y = F^{-1}(U)$

Step 1. Find F^{-1} $\quad 1 - e^{-\frac{1}{\beta} y} \overset{set}{=} u$ solve for y

$$-e^{-\frac{1}{\beta} y} = u - 1$$

$$e^{-\frac{y}{\beta}} = 1 - u$$

$$-\frac{y}{\beta} = \log(1-u)$$

$$y = -\beta \log(1-u) = F^{-1}(u)$$

Step 2 Get dist of $Y = F^{-1}(U)$

$$\boxed{F_Y(y) = P(Y \le y)}$$

$$= P(-\beta \log(1-U) \le y)$$

mult by neg #

$$= P\left(\log(1-U) \ge -\frac{y}{\beta}\right)$$

$$= P(1 - U \ge e^{-y/\beta})$$

$$= P(-U \ge e^{-y/\beta} - 1) = P(U \le 1 - e^{-y/\beta}) = 1 - e^{-y/\beta} \;!!$$

Check: $F^{-1}(F(y))$ should get y

$F(F^{-1}(u))$ s.g. u