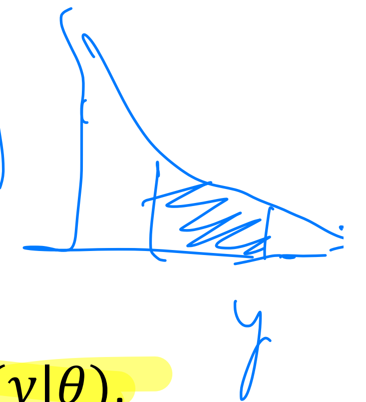
PROPERTIES OF POINT ESTIMATORS AND METHODS OF ESTIMATION

Maximum likelihood estimation I

pdf of exp(θ): $\frac{1}{\theta} e^{-\frac{y}{\theta}}$

$f_Y(y)$



Likelihood function

iid

- If $Y_1, Y_2, \ldots, Y_n$ are a random sample from a distribution with pdf $f_Y(y|\theta)$, then the joint density of $Y_1, Y_2, \ldots, Y_n$ is

$$f(y_1, y_2, \ldots, y_n|\theta) = \prod_{i=1}^{n} f_Y(y_i|\theta).$$

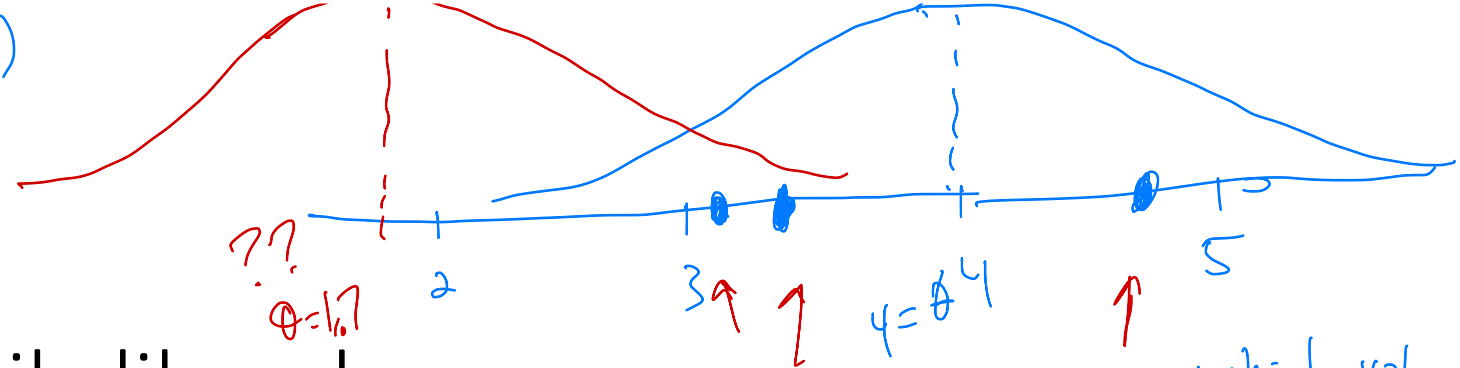
$f_Y(y_1|\theta) \times f_Y(y_2|\theta) \times \ldots f_Y(y_n|\theta$

- When doing probability we think of this as being a function of $y_1, y_2, \ldots, y_n$.
- When doing inference we think of this as being a function of θ.
- Definition 3: For iid observations $Y_1, Y_2, \ldots, Y_n$ with pdf $f_Y(y|\theta)$, the LIKELIHOOD FUNCTION is

$$L(\theta) = L(y_1, y_2, \ldots, y_n|\theta) = \prod_{i=1}^{n} f_Y(y_i|\theta)$$

$Y_1, Y_2, Y_3 \overset{iid}{\sim} N(\theta, 1)$

DATA: 3.4, 3.1, 4.7



Maximum likelihood

which value of θ is most likely to result in these data?

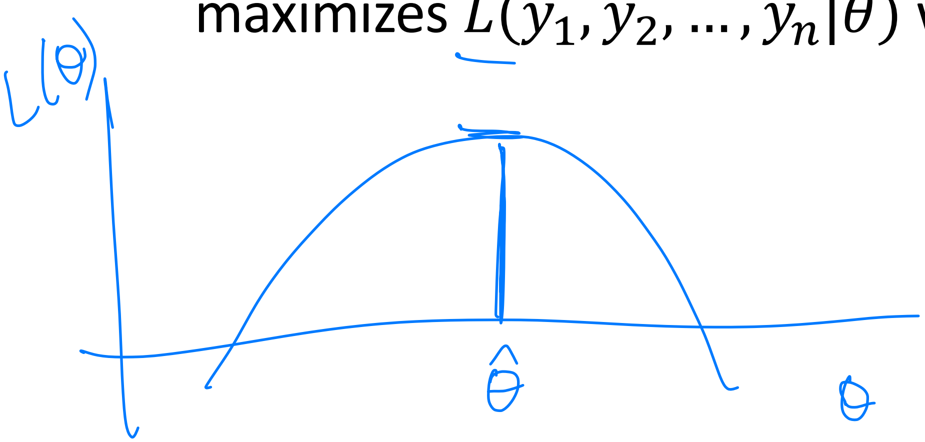
- The idea of maximizing the likelihood $L(y_1, y_2, \ldots, y_n|\theta)$ (loosely speaking) is to choose the value of θ that is most likely to result in outcomes $Y_1 = y_1, Y_2 = y_2, \ldots, Y_n = y_n$.

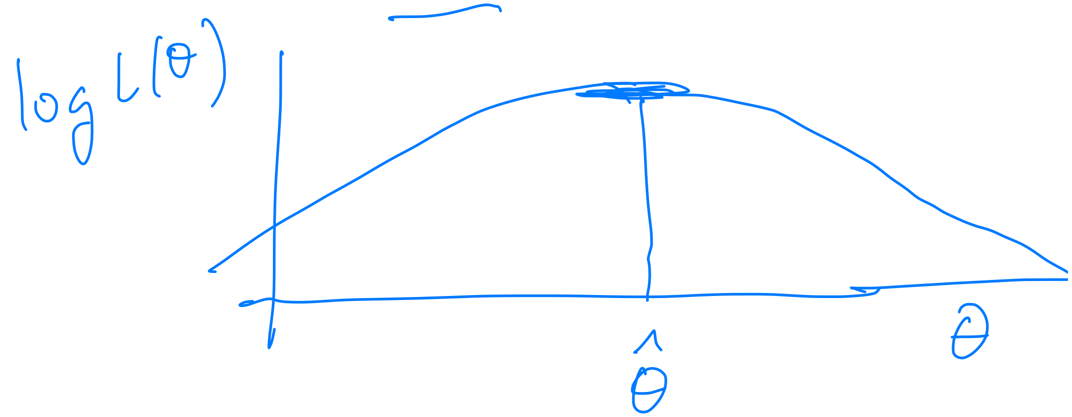
DATA

- Definition 4: The MAXIMUM LIKELIHOOD ESTIMATOR (MLE) of a parameter θ given data $y_1, y_2, \ldots, y_n$ is

$$\hat{\theta} = \arg\max_{\theta} L(y_1, y_2, \ldots, y_n|\theta)$$

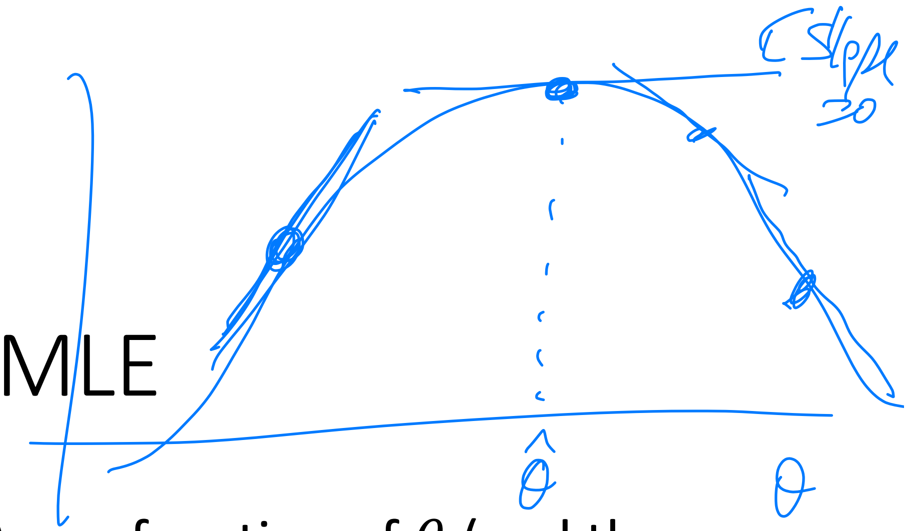
- Note: Since log is a strictly increasing function, the same θ that maximizes $L(y_1, y_2, \ldots, y_n|\theta)$ will also maximize $\log L(y_1, y_2, \ldots, y_n|\theta)$.





Typical steps to finding an MLE

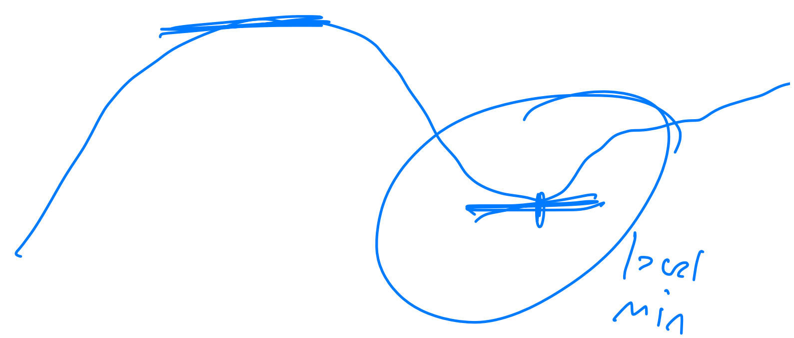
$\log L(\theta)$



1. Write the likelihood (or log likelihood) as a function of θ (and the data). $Y_1, Y_2, \ldots, Y_n$
2. Calculate the derivative w.r.t. θ.
3. Set the derivative equal to zero – solve for θ.
4. To ensure that the solution is a maximum, check to see whether the second derivative is negative.

$L''(\theta) > 0$

$L''(\theta) = 0$



Bernoulli(p) Binomial$(1,p)$ $\boxed{f_Y(y) = p^y (1-p)^{1-y}}$

$P(Y=1) = p$ $y \in \{0, 1\}$

$P(Y=0) = 1-p$

MLE: An example

0's & 1's

- Example 4: $Y_1, Y_2, \ldots, Y_n \sim iid\ Bernoulli(p)$. Find the MLE of p.

$$L(y_1, y_2, \ldots, y_n \mid p) = \prod_{i=1}^{n} f(y_i \mid p) = \prod_{i=1}^{n} p^{y_i}(1-p)^{1-y_i}$$

$$= p^{y_1} p^{y_2} \cdots p^{y_n} (1-p)^{1-y_1} \cdot (1-p)^{1-y_2} \cdots (1-p)^{1-y_n}$$

$$= p^{\sum y_i}(1-p)^{\sum(1-y_i)} = \boxed{p^{\sum y_i}(1-p)^{n-\sum y_i}}$$

$$\log L(y_1, \ldots, y_n \mid p) = \left(\sum y_i\right)\log p + \left(n - \sum y_i\right)\log(1-p)$$

$$\frac{d}{dp}\log L(p) = \frac{\sum y_i}{p} - \frac{(n-\sum y_i)}{1-p} \overset{set}{=} 0 \quad \text{solve for } p$$

$$\frac{\sum y_i}{p} - \frac{n - \sum y_i}{1-p} = 0$$

$$\frac{\sum y_i}{p} = \frac{n - \sum y_i}{1-p}$$

$$(1-p)\sum y_i = p(n - \sum y_i)$$

$$\sum y_i - p\sum y_i = p(n - \sum y_i)$$

$$\sum y_i = p(n - \cancel{\sum y_i} + \cancel{\sum y_i})$$

1/0

$$\sum y_i = pn$$

$$p = \underbrace{\frac{\sum y_i}{n} = \bar{Y}}_{\text{candidate MLE}}$$

1, 0, 0, 1, 1, 1, 1, 0, 1, 1, 0, 0?

#1's/n

$$\frac{d^2}{dp^2}\log(p) = \underbrace{-\frac{\sum y_i}{p^2}}_{<0} \underbrace{-\frac{(n-\sum y_i)}{(1-p)^2}}_{<0}$$

< 0

Since 2nd deriv. is always negative

we know that 2nd deriv. at $p = \bar{Y}$ is also neg.

$$\rightarrow \hat{p} = \frac{\sum y_i}{n} = \bar{Y}$$

is THE MLE!

BATHTUB

MLE: Another example

- Example 5: $Y_1, Y_2, \ldots, Y_n \sim iid$ with pdf $f_Y(y) = \frac{1}{\theta} y^{(1-\theta)/\theta}$ for $0 < y < 1$. Find the MLE of θ.

① $L(\theta) = \prod_{i=1}^{n} \frac{1}{\theta} y_i^{\frac{1-\theta}{\theta}} = \frac{1}{\theta^n} \prod_{i=1}^{n} y_i^{\frac{1-\theta}{\theta}}$

$\log L(\theta) = \ell(\theta) = -n \log \theta + \frac{1-\theta}{\theta} \sum \log y_i$

② $\ell'(\theta) = -\frac{n}{\theta} + \sum \log y_i \frac{(1-\theta)\cdot 1 - \theta(-1)}{\theta^2} = -\frac{n}{\theta} + \sum \log y_i \left(\frac{1}{\theta^2}\right)$

~~for~~ 0

③ $-\frac{n}{\theta} + \sum \log y_i \left(\frac{1}{\theta^2}\right) = 0$

$\sum \log y_i \frac{1}{\theta} = n$ candidate

$\boxed{\theta = \frac{\sum \log y_i}{n}}$

④ $\ell''(\theta) = n\theta^{-2} - 2\theta^{-3} \sum \log y_i$

plug in candidate MLE $\frac{\sum \log y_i}{n}$

$$\frac{n}{\left(\frac{\sum \log y_i}{n}\right)^2} - \frac{2 \sum \log y_i}{\left(\frac{\sum \log y_i}{n}\right)^3}$$

$$\frac{n}{\left(\frac{\sum \log y_i}{n}\right)^2} - \frac{2 \cancel{\sum \log y_i} \left(\frac{n}{\cancel{\sum \log y_i}}\right)}{\left(\frac{\sum \log y_i}{n}\right)^2} = \frac{-n}{\underbrace{\left(\quad\right)^2}} < 0$$

den > 0

$-n < 0$

therefore

$$\boxed{\hat{\theta} = \frac{\sum \log y_i}{n} \text{ is the MLE of } \theta!}$$