ESTIMATION

Confidence intervals

Confidence intervals

- Definition 6: A CONFIDENCE INTERVAL (or INTERVAL ESTIMATE) consists of a set of numbers thought to be "reasonable values" for an unknown parameter θ.
- Ideally, an interval estimator will
  - Have a high probability of including the "true" value of θ;
  - Be relatively narrow (precise).

Interval estimators

- Definition 7: A random interval $[\hat{\theta}_L, \hat{\theta}_U]$ is a $100(1-\alpha)\%$ CONFIDENCE INTERVAL ESTIMATOR for θ if
$$P\left(\theta \in [\hat{\theta}_L, \hat{\theta}_U]\right) = P\left(\hat{\theta}_L \leq \theta \leq \hat{\theta}_U\right) = 1 - \alpha$$
- Common choices for α are 0.10, 0.05 and 0.01.
- Note: sometimes we are interested in one-sided confidence intervals, i.e., $[\hat{\theta}_L, \infty)$ or $(-\infty, \hat{\theta}_U]$

Pivotal quantity

- A useful method for deriving confidence intervals involves using a pivotal quantity.
- Definition 8: A PIVOTAL QUANTITY
  1. Is a function of the sample data, the unknown parameter and no other unknown quantities.
  2. Has a distribution that does not depend on the unknown parameter.

Pivotal method for constructing confidence intervals: 3 steps

1. Find a pivotal quantity
2. Make an interval statement
3. Pivot!

Let's look at some examples . . .

Pivotal method: Example

$Y \sim \exp(\theta)$

$f_Y(y) = \frac{1}{\theta} e^{-\frac{1}{\theta} y}, \quad y > 0$

$F_Y(y) = 1 - e^{-\frac{1}{\theta} y} \quad y > 0$

- Example 2: We observe one random variable Y from an exponential distribution with unknown parameter θ. Find a formula for a 90% confidence interval for θ.

① Find a P.Q.

What is the dist. of $U = \frac{Y}{\theta}$?

cdf method: $F_U(u) = P(U \le u) = P\left(\frac{Y}{\theta} \le u\right)$

$= P(Y \le \theta u) = 1 - e^{-\frac{1}{\cancel{\theta}} \cancel{\theta} u} = 1 - e^{-u}$ cdf of exp(1)

$U \sim \frac{Y}{\theta} \sim \exp(1)$. Is this a P.Q.?

1. ✓ 2. ✓

Yes

② Find a & b so that $P(a < U < b) = 0.9$

want ⊛ + ⊛⊛ $= 0.1$

$.09 \quad .01$

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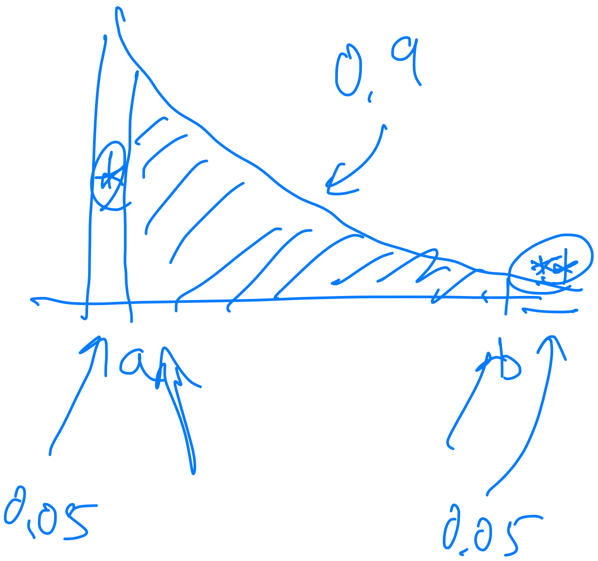
want 0.05 for each (other things possible)



⊛ $P(U < a) = 0.05$

$1 - e^{-a} = 0.05$ solve for a

$e^{-a} = 0.95$

$a = -\log(0.95) = 0.0513$

$P(0.0513 < U < 2.996)$
$= 0.9$

⊛⊛ $P(U > b) = 0.05$

$P(U \le b) = 0.95$

$1 - e^{-b} = 0.95$ solve for b

$e^{-b} = 0.05$ $\quad b = -\log(0.05) = 2.996$

$$P(a < U < b) = 0.90$$

$$P\left(a < \frac{Y}{\theta} < b\right) = 0.90$$

$$P\left(\frac{a}{Y} < \frac{1}{\theta} < \frac{b}{Y}\right) = 0.90$$

$$P\left(\frac{Y}{b} < \theta < \frac{Y}{a}\right) = 0.90$$

$$P\left(\frac{Y}{2.996} < \theta < \frac{Y}{0.0513}\right) = 0.9$$

So a 90% c.i. for θ is $\boxed{\left(\frac{Y}{2.996}, \frac{Y}{0.0513}\right)}$

Pivotal method: Another example

- Example 3: We observe one random variable Y from a uniform$(0, \theta)$ distribution where θ is unknown. Find a 95% lower confidence bound for θ.

$Y \sim U(0, \theta)$

$f_Y(y) = \frac{1}{\theta} \quad 0 < y < \theta$

$\qquad\quad\; 0 \quad y < 0$

$F_Y(y) = \begin{cases} \frac{1}{\theta} y & 0 \le y \le \theta \\ 1 & y > \theta \end{cases}$

① Try $U = \frac{Y}{\theta}$. Is this pivotal?

$P(U \le u) = P\left(\frac{Y}{\theta} \le u\right) = P(Y \le \theta u) = F_Y(u\theta)$

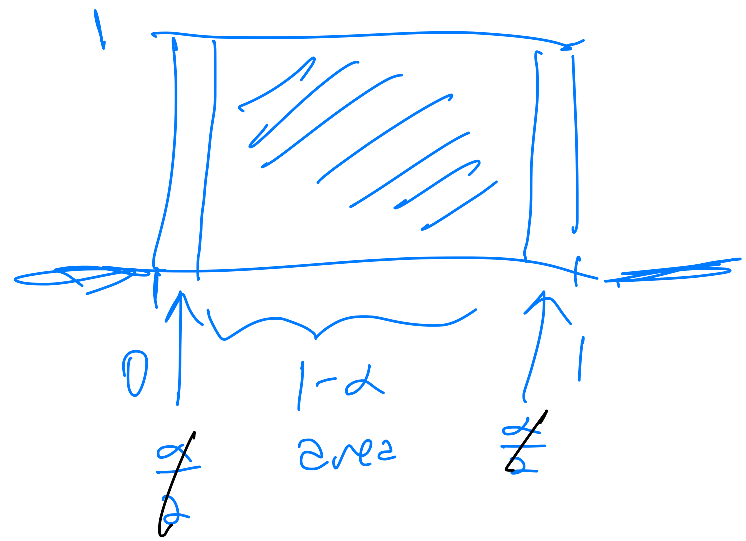
cdf of $U = \frac{Y}{\theta} = \begin{cases} 0 & u < 0 \\ u & 0 \le u \le 1 \\ 1 & u \ge 1 \end{cases} = \begin{cases} 0 & u\cancel{\theta} < 0 \\ \frac{1}{\cancel{\theta}} u\cancel{\theta} & 0 \le u\cancel{\theta} \le \cancel{\theta}\,1 \\ 1 & u\cancel{\theta} > \cancel{\theta}\,1 \end{cases}$

$U \sim U(0, 1)$

$f_U(u) = 1, \quad 0 < u < 1$

Is U P.Q.? 1 ✓ 2 ✓ Yes!

② $P(a < \frac{Y}{\theta} < b) = 1-\alpha$



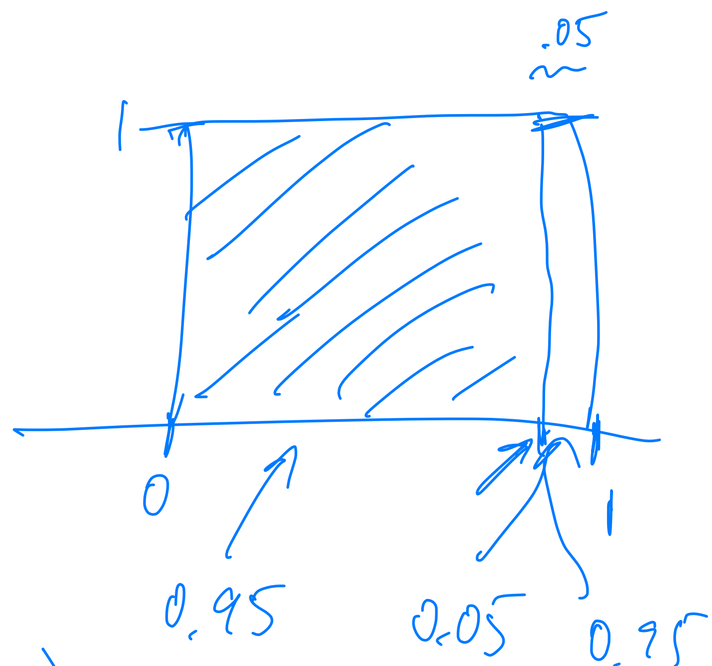
③ Pivot!

$$P\left(a < \frac{Y}{\theta} < b\right) = 1-\alpha$$

$$P\left(\frac{a}{Y} < \frac{1}{\theta} < \frac{b}{Y}\right) = 1-\alpha$$

$$P\left(\frac{Y}{b} < \theta < \frac{Y}{a}\right) = 1-\alpha$$

(we want lower c.i.
set this to be ∞)



$$P\left(\frac{Y}{b} < \theta < \infty\right) = 1-\alpha = 0.95$$

$$P\left(\frac{Y}{0.95} < \theta\right) = 0.95$$

95% lower c.i. for θ

$$\left(\frac{Y}{0.95}, \infty\right)$$