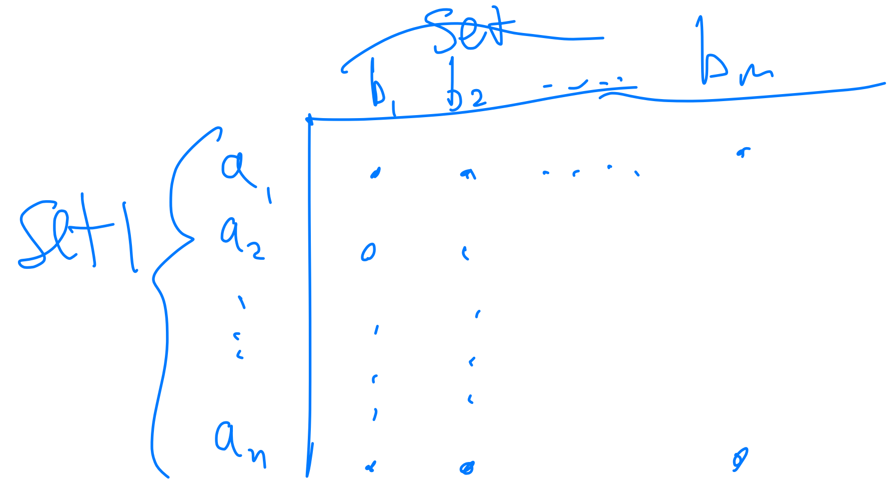
INTRODUCTION TO PROBABILITY:

Counting, Permutations, and Combinations



$n \times m$

Counting

- It can help if we can formalize ”counting” by reviewing combinatorial analysis.
- This is especially useful when many events are *equally likely*.
- Theorem 2.1: If one set has *n* elements and another set has *m* elements then there are $n\times m$ possible pairs with one element from each set.
- Note 1: This can be extended: if $A = \{a_1, a_2, \ldots, a_n\}$, B= $\{b_1, b_2, \ldots, b_m\}$, and C= $\{c_1, c_2, \ldots, c_p\}$, then there are $n\times m\times p$ possible triplets.
- Note 2: The sets could be the same. A & A $n \times n$

Some simple examples

- Example 1: Three coin tosses. How many possible outcomes?

$A = \{H, T\}$ $B = \{H, T\}$ $C = \{H, T\}$

$2 \times 2 \times 2 = 8$

- Example 2: Three questions on the Likert scale: one 5-level question, one 7-level question, one 9-level question. How many unique combinations?

$5 \times 7 \times 9$

$A = \{1, 2, 3, 4, 5\}$

$B = \{1, 2, \ldots, 7\}$

$C = \{1, \ldots, 9\}$

Permutations

- PERMUTATION: Ordered arrangement of a specified number of distinct objects. ("Order matters")
- How many ways are there to order exactly r of n objects?

a b c d e

$n = 5$

$r = 2$

ab, ba, ac . . .

A formula for permutations

- Notation: The number of ways to order exactly r of n objects is P_r^n.
- A formula for P_r^n (Theorem 2.2):

$$P_r^n = \frac{n!}{(n-r)!} = n \times (n-1)\times \ \dots \times (n-r+1) \ \times(n-r)\times\cancel{(n-r-1)}$$

$$(n-r)\times \cancel{(n-r-1)}$$

$A = \{a_1, a_2, \dots, a_n\}$

choose 1 $\quad n$ possibilities ← n

⇒ A except for one chosen $\quad \times(n-1)$

choose 1 $\quad n-1$ possibilities

⇒ A except for two chosen $\quad \times(n-2)$

$n-2$ possibilities $\quad \dots \times(n-r+1)$

Example

- Example 3: Eight patients present at an emergency room. In how many ways (sequences) can the first five patients be triaged?

$$8 \times 7 \times 6 \times 5 \times 4$$

$$P^8_5 = \frac{8!}{(8-5)!} = \frac{8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot \cancel{3!}}{\cancel{3!}}$$

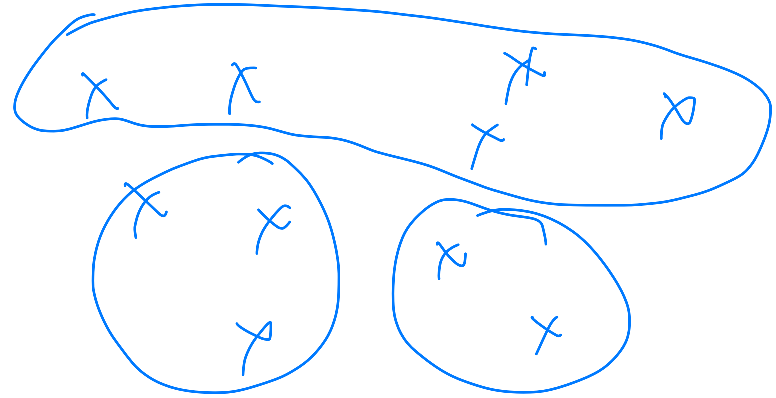
Partitioning

- The number of ways to partition n distinct objects into k distinct groups of sizes $n_1, n_2, \ldots, n_k$, respectively (each object appears in exactly one group) is

$$\binom{n}{n_1 n_2 \ldots n_k} = \frac{n!}{n_1!\, n_2! \ldots n_k!}$$

where $n = n_1 + n_2 + \cdots + n_k$

Note: Counting exercises when order does not matter involves computing <u>combinations</u>.



Example

- Example 4: How many ways can 10 patients be assigned to 3 treatment groups of size 5, 3, and 2?

$$\binom{10}{5\quad 3\quad 2} = \frac{10!}{5!\,3!\,2!} = \frac{\overset{5}{\cancel{10}}\cdot 9\cdot 8\cdot 7\cdot \cancel{6}\cdot \cancel{5!}}{\cancel{5!}\cdot \cancel{3\cdot 2}\cdot \cancel{2}}$$

$$5\times 9\times 8\times 7$$

Combinations with 1 group

- A special case of group partitioning is when there is only 1 group.
- If *k* objects are chosen from among *n* distinct objects to be in a group, the number of possible ways is

$$\binom{n}{k\ \ n-k} = \frac{n!}{k!\,(n-k)!}$$

- (If *k* are in the first group, then the other group must be size *n*-*k*.)
- Notation: $\binom{n}{k} = \frac{n!}{k!(n-k)!}$. "*n* choose *k*"

Example

- Example 5: How many ways are there to choose 4 patients out of 9 to receive an experimental treatment?

$$\binom{9}{4} = \frac{9!}{4!\,5!} = \frac{9 \cdot \overset{2}{\cancel{8}} \cdot 7 \cdot \cancel{6} \cdot \cancel{5!}}{\cancel{4} \cdot \cancel{3} \cdot \cancel{2} \cdot \cancel{5!}}$$

$$9 \cdot 2 \cdot 7$$