BIST P8110: Applied Regression II

8. Hazard Function and Hazard Ratio

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Hazard Functions

- The **hazard function** at time t

$$h(t) = \frac{Pr\left\{\text{death by time } t + \Delta t | \text{alive at time } t\right\}}{\Delta t}$$

  - $h(t)$ is the instantaneous rate per unit time at which patients are dying at time t.
  - For a large population,

$$h(t)\Delta t \cong \frac{\text{number of deaths in the interval } (t, t + \Delta t)}{\text{number of people alive at time } t}$$

Cumulative Hazard Functions

- The **cumulative hazard** is the integral of the hazard function from time 0 to time t:

$$H(t) = \int_0^t h(u)du$$

  where, $\int_0^t h(u)du$ is the area under the curve $h(t)$ between time 0 and time t.
- The survival function can be written as:

$$S(t) = e^{-H(t)} = e^{-\int_0^t h(u)du}$$

  - $h(t) = 0$: there is no risk of death and $S(t)$ is flat at time t.
  - Large values of $h(t)$ imply a rapid rate of decline in $S(t)$

Proportional Hazards (PH)

- Suppose $h_0(t)$ and $h_1(t)$ are the hazard functions for patients on control and experimental treatments, respectively, then these treatments have **proportional hazards (PH)** if

$$\frac{h_1(t)}{h_0(t)} = r, \text{ for some constant } r$$

  - $h_1(t)/h_0(t)$ is called **hazard ratio (HR)**
  - The proportional hazards assumption places no restrictions on the shape of $h_0(t)$
  - The proportional hazards assumption requires that $h_1(t)/h_0(t) = r$ for all times t

Relative Risks and Hazard Ratios

- Suppose that the risks of death by time $t + \Delta t$ for patients on control and experimental treatments who are alive at time t are $h_0(t)\Delta t$ and $h_1(t)\Delta t$, respectively. Then the risk of experimental subjects at time t relative to control is

$$\frac{h_1(t)\Delta t}{h_0(t)\Delta t} = \frac{h_1(t)}{h_0(t)}$$

  - If $h_1(t) = rh_0(t)$ at all times, then this relative risk is

$$\frac{h_1(t)\Delta t}{h_0(t)\Delta t} = \frac{h_1(t)}{h_0(t)} = r$$

  - thus, HR can be thought of as an instantaneous relative risk. If the PH assumption is true, the HR remains constant over time and equals to the relative risk.

More About the PH Assumption

- What does the PH assumption imply about the survival functions?

$$h_1(t) = rh_0(t) \Rightarrow H_1(t) = rH_0(t) \Rightarrow S_1(t) = [S_0(t)]^r$$

- This assumption is difficult to assess visually via K-M curves. Instead, graphic assessment should be based on plots of $H(t)$.

Survival functions vs. Cumulative hazards