

BIST P8110: Applied Regression II

24. Random Slope Models

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Random intercept model

- ▶ A random intercept model with one covariate for continuous responses is given by

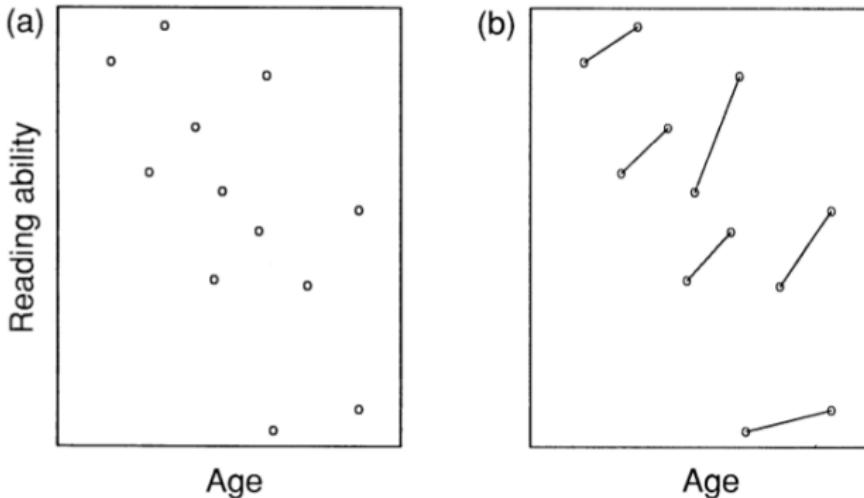
$$y_{ij} = \beta_0 + b_i + \beta_1 x_{ij} + \epsilon_{ij}$$

where

- ▶ $b_i \stackrel{iid}{\sim} N(0, \tau^2)$
- ▶ $\epsilon_{ij} \stackrel{iid}{\sim} N(0, \nu^2)$
- ▶ b_i and ϵ_{ij} are independent

- ▶ Assumptions in the random intercept model
 - ▶ the subject-specific lines have the same slope as the overall regression line
 - ▶ the effect of X is the same for all subjects

Hypothetical data



Random slope model

- ▶ A random slope model with one covariate for continuous responses is given by

$$y_{ij} = \beta_0 + (\beta_1 + b_{1i})x_{ij} + \epsilon_{ij}$$

where

- ▶ $b_{1i} \stackrel{iid}{\sim} N(0, \tau^2)$
- ▶ $\epsilon_{ij} \stackrel{iid}{\sim} N(0, \nu^2)$

- ▶ A random slope model allows
 - ▶ each subject line to have a different slope ($\beta_1 + b_{1i}$) but the same intercept (β_0)

Fixed and random part

- ▶ The random slope model has two parts:

$$y_{ij} = \beta_0 + \beta_1 x_{ij} + b_{1i} X_{ij} + \epsilon_{ij}$$

- ▶ “Fixed” part: parameters that we estimate are the coefficients

$$\beta_0, \beta_1$$

- ▶ “random” part: parameters that we estimate are the variances

$$\tau^2, \nu^2$$

Random intercept and slope model

- ▶ A random intercept and slope model with one covariate for continuous responses is given by

$$y_{ij} = (\beta_0 + b_{0i}) + (\beta_1 + b_{1i})x_{ij} + \epsilon_{ij}$$

where

- ▶ $\begin{bmatrix} b_{0i} \\ b_{1i} \end{bmatrix} \sim N(0, \Omega)$, $\Omega = \begin{bmatrix} \tau_1^2 & \tau_{12} \\ \tau_{12} & \tau_2^2 \end{bmatrix}$
- ▶ $\epsilon_{ij} \stackrel{iid}{\sim} N(0, \nu^2)$

- ▶ A random intercept and slope model allows
 - ▶ each subject line to have a different slope ($\beta_1 + b_{1i}$) and different intercept ($\beta_0 + b_{0i}$)

Fixed and random part

- The random slope model has two parts:

$$y_{ij} = \beta_0 + \beta_1 x_{ij} + b_{0i} + b_{1i} X_{ij} + \epsilon_{ij}$$

- “Fixed” part: parameters that we estimate are the coefficients

$$\beta_0, \beta_1$$

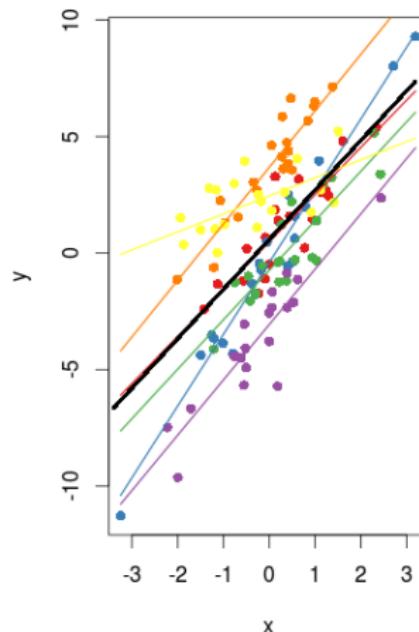
- “random” part: parameters that we estimate are the variances or covariance

$$\tau_1^2, \tau_2^2, \tau_{12}, \nu^2$$

- Compared to the random intercept model, although we only introduce one extra term, b_{1i} , we have 2 extra parameters, τ_2^2 and τ_{12} .

Regression lines

- ▶ For a random intercept and slope model
 - ▶ The slope for the overall regression line is still β_1
 - ▶ For each subject the slope is $\beta_1 + b_{1i}$
 - ▶ The overall average line has equation $\beta_0 + \beta_1 x_{ij}$
 - ▶ Each subject has its own line
$$(\beta_0 + b_{0i}) + (\beta_1 + b_{1i})x_{ij}$$

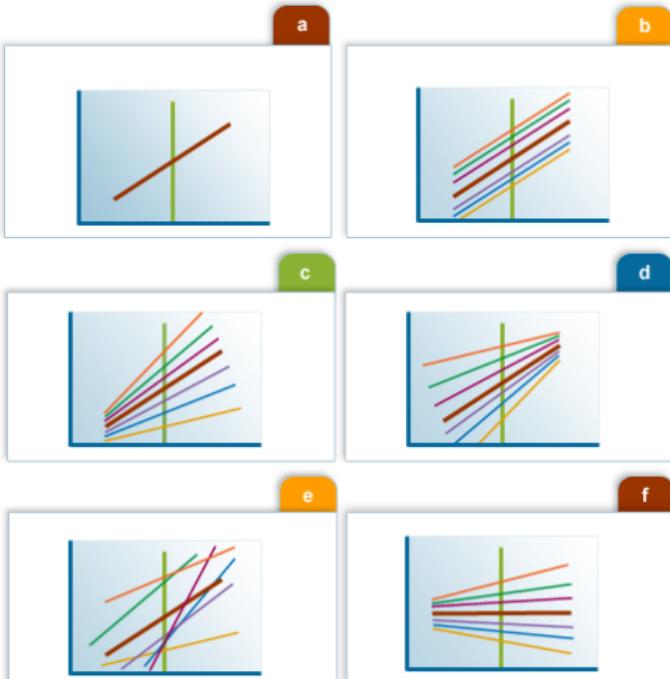


Interpretation

- ▶ “Fixed” part
 - ▶ β_1 is the mean increase in the response for each unit increase in X
 - ▶ the same as that in linear regression models
- ▶ “Random” part
 - ▶ τ_1^2 is the variance in intercepts between subjects
 - ▶ τ_2^2 is the variance in slopes between subjects
 - ▶ τ_{12} is the covariance between intercepts and slopes

Covariance between intercepts and slopes

- a. Fixed intercept, fixed slope
- b. Random intercepts, fixed slope
- c. Random intercepts, random slopes ($\tau_{12} > 0$)
- d. Random intercepts, random slopes ($\tau_{12} < 0$)
- e. Random intercepts, random slopes ($\tau_{12} = 0$)
- f. Random intercepts, random slopes ($\tau_{12} > 0$)



Source: <http://www.esourceresearch.org/tabcid/334/Default.aspx>

The total variance

- ▶ Observation level (level 1)
 - ▶ only one term: ϵ_{ij}
 - ▶ the level 1 variance is ν^2
- ▶ Subject level (level 2)
 - ▶ two random terms: b_{0i} and $b_{1i}x_{ij}$
 - ▶ the level 2 variance is
$$\begin{aligned}\text{Var}(b_{0i} + b_{1i}x_{ij}) &= \text{Var}(b_{0i}) + 2\text{Cov}(b_{0i}, b_{1i}x_{ij}) + \text{Var}(b_{1i}x_{ij}) \\ &= \tau_1^2 + 2\tau_{12}x_{ij} + \tau_2^2x_{ij}^2\end{aligned}$$

Hypothesis testing

- ▶ “Fixed” part
 - ▶ $H_0 : \beta_1 = 0$ vs. $H_\alpha : \beta_1 \neq 0$
 - ▶ t-test or F-test
- ▶ “Random” part: comparing to the random intercept model
 - ▶ $H_0 : \tau_2^2 = \tau_{12} = 0$ vs. $H_\alpha : \text{not } H_0$
 - ▶ Likelihood ratio test:
$$G = -2(l_{\text{random intercept model}} - l_{\text{random intercept & slope model}})$$
 - ▶ p-value: $Pr(\chi_2^2 \geq G)$

Adding more covariates

- ▶ The random intercept and slope model with one covariate can be easily extended to allow multiple covariates
 - ▶ a random slope on just one of the covariates

$$y_{ij} = \beta_0 + \beta_1 x_{1ij} + \beta_2 x_{2ij} + \beta_3 x_{3ij} + b_{0i} + b_{1i} x_{1ij} + \epsilon_{ij}$$

- ▶ random slopes on several of the covariates

$$y_{ij} = \beta_0 + \beta_1 x_{1ij} + \beta_2 x_{2ij} + \beta_3 x_{3ij} + b_{0i} + b_{1i} x_{1ij} + b_{2i} x_{2ij} + \epsilon_{ij}$$

Some final notes

- ▶ In a random slope model
 - ▶ we often also assume a random intercept
 - ▶ if there is a good reason to believe all the subject lines cross at $x = 0$, we can fit a random slope model without random intercept
- ▶ They are also called
 - ▶ random coefficients model
 - ▶ multilevel model
 - ▶ hierarchical model

PROC MIXED

Sample SAS code:

```
proc mixed data=A;
  class ID;
  model Y = X1 X2 /solution;
  random int X1 /type=un subject=ID;
run;
```

- ▶ Most commonly used "type=" options

UN Unstructured variance-covariance matrix with $\begin{bmatrix} \tau_1^2 & \tau_{12} \\ \tau_{12} & \tau_2^2 \end{bmatrix}$

VC Variance components with $\begin{bmatrix} \tau_1^2 & 0 \\ 0 & \tau_2^2 \end{bmatrix}$, the default option

- ▶ Other options can be found in Table 56.13 in the link below

- ▶ https://support.sas.com/documentation/cdl/en/statug/63033/HTML/default/viewer.htm#statug_mixed_sect019.htm#statug.mixed.mixedcovstruct

Summary: Key Points

- ▶ What is random slope model?
- ▶ What is random intercept and slope model?
- ▶ What are the fixed and random parts in a random intercept and slope model?
- ▶ How to interpret the parameters in a random intercept and slope model?
- ▶ Covariance between intercepts and slopes?
- ▶ Model selection: random intercept model vs. random intercept and slope model?
- ▶ How to use SAS to code a random intercept and slope model?

Suggested Reading

- ▶ Chapter 9 (Davidian)