

# BIST P8110: Applied Regression II

## 24. Random Slope Models

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# Random intercept model

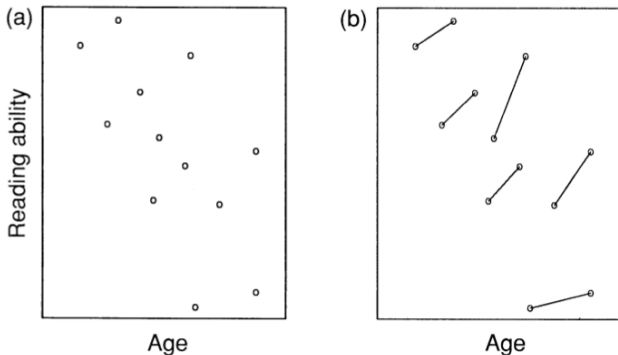
- ▶ A random intercept model with one covariate for continuous responses is given by

$$y_{ij} = \beta_0 + b_i + \beta_1 x_{ij} + \epsilon_{ij}$$

where

- ▶  $b_i \stackrel{iid}{\sim} N(0, \tau^2)$
- ▶  $\epsilon_{ij} \stackrel{iid}{\sim} N(0, \nu^2)$
- ▶  $b_i$  and  $\epsilon_{ij}$  are independent
- ▶ Assumptions in the random intercept model
  - ▶ the subject-specific lines have the same slope as the overall regression line
  - ▶ the effect of  $X$  is the same for all subjects

## Hypothetical data



- ▶ The reading ability improves faster in some kids but slower in other kids as age increases
- ▶ The random intercept model, with parallel subject-specific lines, does not fit this data well

# Random slope model

- ▶ A random slope model with one covariate for continuous responses is given by

$$y_{ij} = \beta_0 + (\beta_1 + b_{1i})x_{ij} + \epsilon_{ij}$$

where

- ▶  $b_{1i} \stackrel{iid}{\sim} N(0, \tau^2)$
- ▶  $\epsilon_{ij} \stackrel{iid}{\sim} N(0, \nu^2)$
- ▶ A random slope model allows
  - ▶ each subject line to have a different slope ( $\beta_1 + b_{1i}$ ) but the same intercept ( $\beta_0$ )

# Fixed and random part

- ▶ The random slope model has two parts:

$$y_{ij} = \beta_0 + \beta_1 x_{ij} + b_{1i} X_{ij} + \epsilon_{ij}$$

- ▶ “Fixed” part: parameters that we estimate are the coefficients

$$\beta_0, \beta_1$$

- ▶ “random” part: parameters that we estimate are the variances

$$\tau^2, \nu^2$$

# Random intercept and slope model

- ▶ A random intercept and slope model with one covariate for continuous responses is given by

$$y_{ij} = (\beta_0 + b_{0i}) + (\beta_1 + b_{1i})x_{ij} + \epsilon_{ij}$$

where

- ▶  $\begin{bmatrix} b_{0i} \\ b_{1i} \end{bmatrix} \sim N(0, \Omega), \quad \Omega = \begin{bmatrix} \tau_1^2 & \tau_{12} \\ \tau_{12} & \tau_2^2 \end{bmatrix}$
- ▶  $\epsilon_{ij} \stackrel{iid}{\sim} N(0, \nu^2)$
- ▶ A random intercept and slope model allows
  - ▶ each subject line to have a different slope  $(\beta_1 + b_{1i})$  and different intercept  $(\beta_0 + b_{0i})$

# Fixed and random part

- ▶ The random slope model has two parts:

$$y_{ij} = \beta_0 + \beta_1 x_{ij} + b_{0i} + b_{1i} x_{ij} + \epsilon_{ij}$$

- ▶ “Fixed” part: parameters that we estimate are the coefficients

$$\beta_0, \beta_1$$

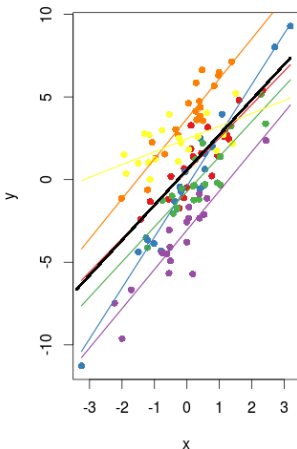
- ▶ “random” part: parameters that we estimate are the variances or covariance

$$\tau_1^2, \tau_2^2, \tau_{12}, \nu^2$$

- ▶ Compared to the random intercept model, although we only introduce one extra term,  $b_{1i}$ , we have 2 extra parameters,  $\tau_2^2$  and  $\tau_{12}$ .

# Regression lines

- ▶ For a random intercept and slope model
  - ▶ The slope for the overall regression line is still  $\beta_1$
  - ▶ For each subject the slope is  $\beta_1 + b_{1i}$
  - ▶ The overall average line has equation  $\beta_0 + \beta_1 x_{ij}$
  - ▶ Each subject has its own line  
 $(\beta_0 + b_{0i}) + (\beta_1 + b_{1i})x_{ij}$



# Interpretation

- ▶ “Fixed” part
  - ▶  $\beta_1$  is the mean increase in the response for each unit increase in  $X$
  - ▶ the same as that in linear regression models
- ▶ “Random” part
  - ▶  $\tau_1^2$  is the variance in intercepts between subjects
  - ▶  $\tau_2^2$  is the variance in slopes between subjects
  - ▶  $\tau_{12}$  is the covariance between intercepts and slopes

# Covariance between intercepts and slopes

- a. Fixed intercept, fixed slope
- b. Random intercepts, fixed slope
- c. Random intercepts, random slopes ( $\tau_{12} > 0$ )
- d. Random intercepts, random slopes ( $\tau_{12} < 0$ )
- e. Random intercepts, random slopes ( $\tau_{12} = 0$ )
- f. Random intercepts, random slopes ( $\tau_{12} > 0$ )



Source: <http://www.esourceresearch.org/tabid/334/Default.aspx>

# The total variance

- ▶ Observation level (level 1)

- ▶ only one term:  $\epsilon_{ij}$
- ▶ the level 1 variance is  $\nu^2$

- ▶ Subject level (level 2)

- ▶ two random terms:  $b_{0i}$  and  $b_{1i}x_{ij}$

- ▶ the level 2 variance is

$$\begin{aligned}\text{Var}(b_{0i} + b_{1i}x_{ij}) &= \text{Var}(b_{0i}) + 2\text{Cov}(b_{0i}, b_{1i}x_{ij}) + \text{Var}(b_{1i}x_{ij}) \\ &= \tau_1^2 + 2\tau_{12}x_{ij} + \tau_2^2x_{ij}^2\end{aligned}$$

# Hypothesis testing

- ▶ “Fixed” part
  - ▶  $H_0 : \beta_1 = 0$  vs.  $H_\alpha : \beta_1 \neq 0$
  - ▶ t-test or F-test
- ▶ “Random” part: comparing to the random intercept model
  - ▶  $H_0 : \tau_2^2 = \tau_{12} = 0$  vs.  $H_\alpha : \text{not } H_0$
  - ▶ Likelihood ratio test:  
$$G = -2(l_{\text{random intercept model}} - l_{\text{random intercept \& slope model}})$$
  - ▶ p-value:  $Pr(\chi_2^2 \geq G)$

## Adding more covariates

- ▶ The random intercept and slope model with one covariate can be easily extended to allow multiple covariates
  - ▶ a random slope on just one of the covariates

$$y_{ij} = \beta_0 + \beta_1 x_{1ij} + \beta_2 x_{2ij} + \beta_3 x_{3ij} + b_{0i} + b_{1i} x_{1ij} + \epsilon_{ij}$$

- ▶ random slopes on several of the covariates

$$y_{ij} = \beta_0 + \beta_1 x_{1ij} + \beta_2 x_{2ij} + \beta_3 x_{3ij} + b_{0i} + b_{1i} x_{1ij} + b_{2i} x_{2ij} + \epsilon_{ij}$$

## Some final notes

- ▶ In a random slope model
  - ▶ we often also assume a random intercept
  - ▶ if there is a good reason to believe all the subject lines cross at  $x = 0$ , we can fit a random slope model without random intercept
- ▶ They are also called
  - ▶ random coefficients model
  - ▶ multilevel model
  - ▶ hierarchical model

# PROC MIXED

## Sample SAS code:

```
proc mixed data=A;  
  class ID;  
  model Y = X1 X2 /solution;  
  random int X1 /type=un subject=ID;  
run;
```

- ▶ Most commonly used “type=” options

**UN** Unstructured variance-covariance matrix with  $\begin{bmatrix} \tau_1^2 & \tau_{12} \\ \tau_{12} & \tau_2^2 \end{bmatrix}$

**VC** Variance components with  $\begin{bmatrix} \tau_1^2 & 0 \\ 0 & \tau_2^2 \end{bmatrix}$ , the default option

- ▶ Other options can be found in Table 56.13 in the link below

- ▶ [https://support.sas.com/documentation/cdl/en/statug/63033/HTML/default/viewer.htm#statug\\_mixed\\_sect019.htm#statug.mixed.mixedcovstruct](https://support.sas.com/documentation/cdl/en/statug/63033/HTML/default/viewer.htm#statug_mixed_sect019.htm#statug.mixed.mixedcovstruct)

## Summary: Key Points

- ▶ What is random slope model?
- ▶ What is random intercept and slope model?
- ▶ What are the fixed and random parts in a random intercept and slope model?
- ▶ How to interpret the parameters in a random intercept and slope model?
- ▶ Covariance between intercepts and slopes?
- ▶ Model selection: random intercept model vs. random intercept and slope model?
- ▶ How to use SAS to code a random intercept and slope model?

# Suggested Reading

- ▶ Chapter 9 (Davidian)