solve : $2y'''-6y''+6y'-2=2x^{\frac{1}{2}}e^{x}$

$y(0)=1, y'(0)=2, y''(0)=1$

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variation of parameter

$$2y''' - 6y'' + 6y' - 2 = 2x^{\frac{1}{2}}e^x \Rightarrow y''' - 3y'' + 3y' - 1 = x^{\frac{1}{2}}e^x$$

characteristic equation:

$$(\lambda - 1)^3 = 0 \Rightarrow \lambda = 1, 1, 1$$

$$y_h = Ax^2e^x + Bxe^x + Ce^x$$

variation of parameter

initial assumption

$$\begin{cases} u_1' y_1 + u_2' y_2 & = & 0 \\ u_1' y_1' + u_2' y_2' & = & f(x) \end{cases}$$

promote to 3

$$\Rightarrow \begin{cases} u_1' y_1 + u_2' y_2 + u_3' y_3 & = & 0 \\ u_1' y_1' + u_2' y_2' + u_3' y_3' & = & 0 \\ u_1' y_1'' + u_2' y_2'' + u_3' y_3'' & = & f(x) \end{cases}$$

$$u_1 = \int \frac{\begin{bmatrix} 0 & xe^x & (x^2)e^x \\ 0 & (1+x)e^x & (2x+x^2)e^x \\ x^{\frac{1}{2}}e^x & (2+x)e^x & (2+4x+x^2)e^x \end{bmatrix}}{\begin{bmatrix} e^x & xe^x & (x^2)e^x \\ e^x & (1+x)e^x & 2(2x+x^2)e^x \\ e^x & (2+x)e^x & (2+4x+x^2)e^x \end{bmatrix}} dx = \int \frac{\begin{bmatrix} 0 & x & x^2 \\ 0 & 1 & 2x \\ x^{\frac{1}{2}} & 2 & 2+4x \end{bmatrix}}{\begin{bmatrix} 1 & x & x^2 \\ 0 & 1 & 2x \\ 0 & 2 & 2+4x \end{bmatrix}} dx$$

$$= \int \frac{x^{\frac{5}{2}}}{\begin{bmatrix} 1 & x & x^2 \\ 0 & 1 & 2x \\ 0 & 0 & 2 \end{bmatrix}} dx = \int \frac{x^{\frac{5}{2}}}{2} dx = \frac{1}{7}x^{\frac{7}{2}}$$

$$u_2 = \int \frac{\begin{bmatrix} 1 & 0 & x^2 \\ 1 & 0 & 2x+x^2 \\ 1 & x^{\frac{1}{2}} & 2+4x+x^2 \end{bmatrix}}{2} dx = \int \frac{-2x^{\frac{3}{2}}}{2} dx = -\frac{2}{5}x^{\frac{5}{2}}$$

$$u_3 = \int \frac{\begin{bmatrix} 1 & x & 0 \\ 1 & 1+x & 0 \\ 1 & 2+x & x^{\frac{1}{2}} \end{bmatrix}}{2} dx = \int \frac{x^{\frac{1}{2}}}{2} dx = \frac{1}{3}x^{\frac{3}{2}}$$

variation of parameter

$$y_p = y_1 u_1 + y_2 u_2 + y_3 u_3 = (\frac{1}{7} - \frac{2}{5} + \frac{1}{3}) x^{\frac{7}{2}} e^x$$

$$\Rightarrow y = y_h + y_p = Ax^2 e^x + Bxe^x + Ce^x + (\frac{1}{7} - \frac{2}{5} + \frac{1}{3}) x^{\frac{7}{2}} e^x$$

$$\begin{cases} y(0) & = 1 & = & C \\ y'(0) & = 2 & = & B + C \\ y''(0) & = 1 & = & 2A + 2B + C \end{cases} \Rightarrow A = -1, B = 1, C = 1$$

$$y = y_h + y_p = -x^2 e^x + xe^x + e^x + \frac{8}{105} x^{\frac{7}{2}} e^x$$

method of convolution

initial formula

$$y = y_k(t) + y_s * f(t) = \mathcal{L}^{-1}\{\frac{(as+b)Y_0 + aY_1}{as^2+bs+c}\} + \mathcal{L}^{-1}\{\frac{F(s)}{as^2+bs+c}\}$$

promote to 3

$$\Rightarrow \mathcal{L}^{-1}\{\frac{(as^2+bs+c)Y_0 + (as+b)Y_1 + aY_2}{as^3+bs^2+cs+d}\} + \mathcal{L}^{-1}\{\frac{F(s)}{as^3+bs^2+cs+d}\}$$

method of convolution

a=1, b=-3,c=3,d=-1

$$\Rightarrow y_s = \mathcal{L}^{-1}\{\frac{1}{(s-1)^3}\} = \frac{e^x x^2}{2}$$

$$y_s * f(x) = \int_0^x \frac{e^{x-t}(x-t)^2}{2} t^{\frac{1}{2}} e^t dt = \frac{e^x}{2}\int_0^x x^2 t^{\frac{1}{2}} - 2xt^{\frac{3}{2}} + x^{\frac{5}{2}} dt$$

$$= \frac{e^x}{2}[\frac{2}{3}x^{\frac{7}{2}} - \frac{4}{5}x^{\frac{7}{2}} + \frac{2}{7}x^{\frac{7}{2}}] = \frac{8}{105}x^{\frac{7}{2}}e^x$$

method of convolution

$$y_k = \mathcal{L}^{-1}\{\frac{(s^2-3s+3)(1)+(s-3)(2)+(1)}{(s-1)^3}\} = \mathcal{L}^{-1}\{\frac{(s^2-s-2)}{(s-1)^3}\}$$

$$= \mathrm{L}^{-1}\{\tfrac{-2}{(s-1)^3} + \tfrac{1}{(s-1)^2} + \tfrac{1}{s-1}\} = -x^2e^x + xe^x + e^x$$

$$y = y_s * f(x) + y_k = -x^2e^x + xe^x + e^x + \frac{8}{105}x^{\frac{7}{2}}e^x$$

class outline

First-order Differential Equations

- *Separable Equations*
- *Linear Equations*
- *Exact Equations*
- *Bernoulli Equations*

Second-order Differential Equations

- *Homogeneous Solution*
- *Particular Solution*
- *Variation Of Parameter*

Laplace Transform

- *Transform And Inverse Transform*
- *Method Of Convolution*

First-order Differential Equations

Separable Equations

$$y' = \frac{f(x)}{g(y)} \Rightarrow \int g(y)dy = \int f(x)dx$$

Linear Equations

$$y' + p(x)y = g(x), let\ u(x) = e^{\int p(x)dx}$$

Exact Equations

$$M(x,y)dx + N(x,y)dy = 0$$

Bernoulli Equations

$$y' + p(x)y = q(x)y^n, let\ u(x) = y^{1-n}$$

Second-order Differential Equations

homogeneous solution:

$$a_k\lambda^k + a_{k-1}\lambda^{k-1}\dots$$

$$1.repeat\ root : A_{k-1}x^{k-1}e^{ax} + A_{k-2}x^{k-2}e^{ax}\dots$$

$$2.imagnary\ root : e^{(a+bi)x} \Rightarrow e^{ax}(A\cos(bx) + iB\sin(bx))$$

$$3.other : Ae^{ax} + Be^{bx}$$

particular solution

$$e^x, \sin(x), \cos(x)$$

variation of parameter

Laplace Transform

Transform and inverse Transform

$$F(s) = \int_0^\infty f(t)e^{-st}dt$$

$$t^k \Rightarrow \frac{k!}{s^{k+1}}$$

$$e^{at} \Rightarrow F(s) \to F(s-a)$$

$$e^{it} = \cos(t) + i\sin(t) \Rightarrow \frac{as+bi}{1+s^2}$$

$$f^{(n)}(t) \Rightarrow s^n F(s) - s^{(n-1)} f(0) - s^{(n-2)} f'(0) \cdots$$

$$\delta(t - t_0) \Rightarrow e^{-st_0}$$

$$u(t-a) = \frac{e^{-as}}{s}$$

method of convolution

my favorite part

laplace transform