"At least" and "at most," and mean, variance, and standard deviation

We can do more than just calculate the probability of pulling exactly 3 red marbles in 5 total pulls. For any binomial random variable, we can also calculate something like the probability of pulling at least 3 red marbles, or the probability of pulling no more than 3 marbles.

What we want to know is that the probability of pulling at least 3 red marbles is the probability that we pull 3, or 4, or 5 red marbles, which is simply the probability of each of these, all added together.

$$P(\text{at least } 3 \text{ reds in } 5 \text{ pulls}) = P(3 \text{ reds}) + P(4 \text{ reds}) + P(5 \text{ reds})$$

$$P(\text{at least } 3 \text{ reds in } 5 \text{ pulls}) = \binom{5}{3}\left(\frac{1}{3}\right)^3\left(\frac{2}{3}\right)^2$$

$$+\binom{5}{4}\left(\frac{1}{3}\right)^4\left(\frac{2}{3}\right)^1+\binom{5}{5}\left(\frac{1}{3}\right)^5\left(\frac{2}{3}\right)^0$$

$$P(\text{at least } 3 \text{ reds in } 5 \text{ pulls}) = (10)\left(\frac{1}{3}\right)^3\left(\frac{2}{3}\right)^2$$

$$+(5)\left(\frac{1}{3}\right)^4\left(\frac{2}{3}\right)^1+(1)\left(\frac{1}{3}\right)^5\left(\frac{2}{3}\right)^0$$

$$P(\text{at least } 3 \text{ reds in } 5 \text{ pulls}) \approx 0.1646 + 0.0412 + 0.0041$$

$$P(\text{at least } 3 \text{ reds in } 5 \text{ pulls}) \approx 0.2099$$

$$P(\text{at least } 3 \text{ reds in } 5 \text{ pulls}) \approx 21\,\%$$

In the same way, the probability of pulling at most 3 red marbles would be the probability of pulling 0, 1, 2, or 3 red marbles, all added together.

If we're calculating the probability of at least one success or at least one failure, we can use these formulas:

$$P(\text{at least 1 success}) = 1 - P(\text{all failures})$$

$$P(\text{at least 1 failure}) = 1 - P(\text{all successes})$$

This is because all probability distribution functions must add up to 1.

Example

Find the probability that we get at least 1 heads on 5 coin flips.

We can actually simplify this problem a lot by realizing that every single set of 5 coin flips will have at least one heads, unless every one of the 5 flips is tails: $TTTTT$. The probability of getting 5 tails in a row is

$$P(TTTTT) = \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{1}{32}$$

The probability of getting "at least one heads" is the same as the probability of <u>not</u> getting "all tails." Therefore, since total probability is always equal to 1, we can say that the probability of at least one heads is

$$P(\text{at least 1 heads}) = 1 - \frac{1}{32} = \frac{31}{32}$$

Let's do another example where we find an "at most" probability for a binomial random variable.

Example

Let X be a binomial random variable with $n=10$ and $p=0.30$. Find $P(X \leq 5)$.

The variable X follows a binomial distribution, but instead of finding the probability of exactly k successes in n trials, we're asked to find the probability of k or fewer successes in n trials. Specifically, find the chance of 5 or fewer successes in 10 trials, where the probability of success on any one trial is $p=0.30$.

Find the probability of 0 successes, 1 success, 2 successes, etc., up to 5 successes, and then find the sum of those probabilities.

$$P(X \leq 5) = P(X=0) + P(X=1) + P(X=2) + P(X=3) + P(X=4) + P(X=5)$$

To find the probability for each value of k, we use the binomial probability formula.

$$P(k \text{ successes in } n \text{ trials}) = \binom{n}{k} p^k (1-p)^{n-k}$$

So the probability $P(X \leq 5)$ is

$$P(X \leq 5) = \binom{10}{0}(0.30)^0(1-0.30)^{10} + \binom{10}{1}(0.30)^1(1-0.30)^9$$

$$+\binom{10}{2}(0.30)^2(1-0.30)^8+\binom{10}{3}(0.30)^3(1-0.30)^7$$

$$+\binom{10}{4}(0.30)^4(1-0.30)^6+\binom{10}{5}(0.30)^5(1-0.30)^5$$

$$P(X \le 5) = 0.9527$$

Mean, variance, and standard deviation

The mean of a binomial random variable X can be expressed as μ_X. The mean is also called the expected value, and that's indicated as $E(X)$. Either way, the mean is given by

$$\mu_X = E(X) = np$$

where n is the fixed number of independent trials, and p is the probability of a success. The variance of a binomial random variable X is given by

$$\sigma_X^2 = np(1-p)$$

Standard deviation is the square root of the variance and is therefore given by

$$\sqrt{\sigma_X^2} = \sqrt{np(1-p)}$$

$$\sigma_X = \sqrt{np(1-p)}$$

If we continue with our example of the number of heads we get on 5 coin flips, we can say that the number of trials n is 5, and the probability of success (getting heads) is $p = 0.5$. Therefore, the mean is

$$\mu_X = np$$

$$\mu_X = 5(0.5)$$

$$\mu_X = 2.5$$

The variance is

$$\sigma_X^2 = np(1 - p)$$

$$\sigma_X^2 = 5(0.5)(1 - 0.5)$$

$$\sigma_X^2 = 2.5(1 - 0.5)$$

$$\sigma_X^2 = 2.5(0.5)$$

$$\sigma_X^2 = 1.25$$

And the standard deviation is

$$\sigma_X = \sqrt{np(1 - p)}$$

$$\sigma_X = \sqrt{1.25}$$

$$\sigma_X \approx 1.12$$