One-to-one functions and the horizontal line test

In this section we'll talk about how to determine whether a graph represents a one-to-one function. If a relation is a function, then it has exactly one y-value in its range for each x-value in its domain. If a function is one-to-one, it also has exactly one x-value in its domain for each y-value in its range.

The reason we care about one-to-one functions is that only one-to-one function have an inverse (a concept we'll talk about in the next lesson). If a function is not one-to-one, then some restrictions on its domain will be needed to make it invertible.

The first method we'll use to check whether or not a function is one-to-one is the Horizontal Line Test.

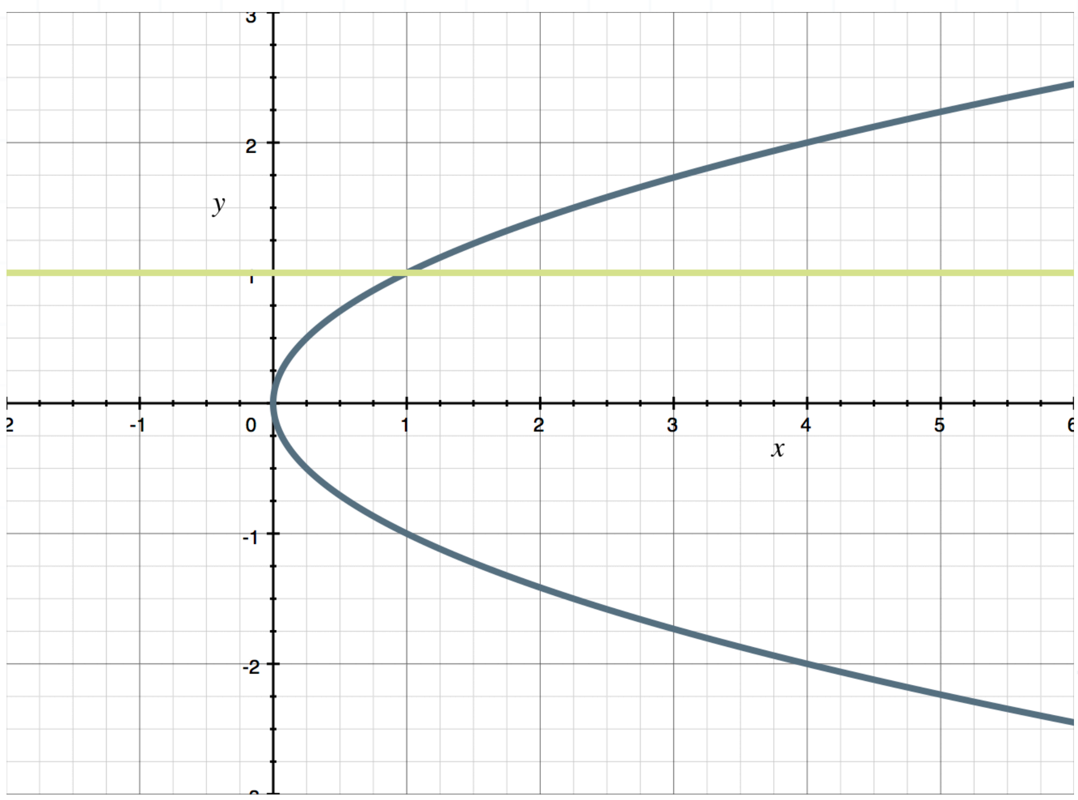
One-to-one functions and the horizontal line test

Remember that we've already talked about the Vertical Line Test, which is a test we use to tell us whether or not a graph represents a function. A graph passes the Vertical Line Test if no two points of the graph have the same x-coordinate (if no vertical line intersects the graph at more than one point).

In the same way that the Vertical Line Test tells us whether or not a graph represents a function, the Horizontal Line Test tells us whether or not the function represented by a graph is one-to-one. A graph passes the Horizontal Line Test if no two points of the graph have the same y-

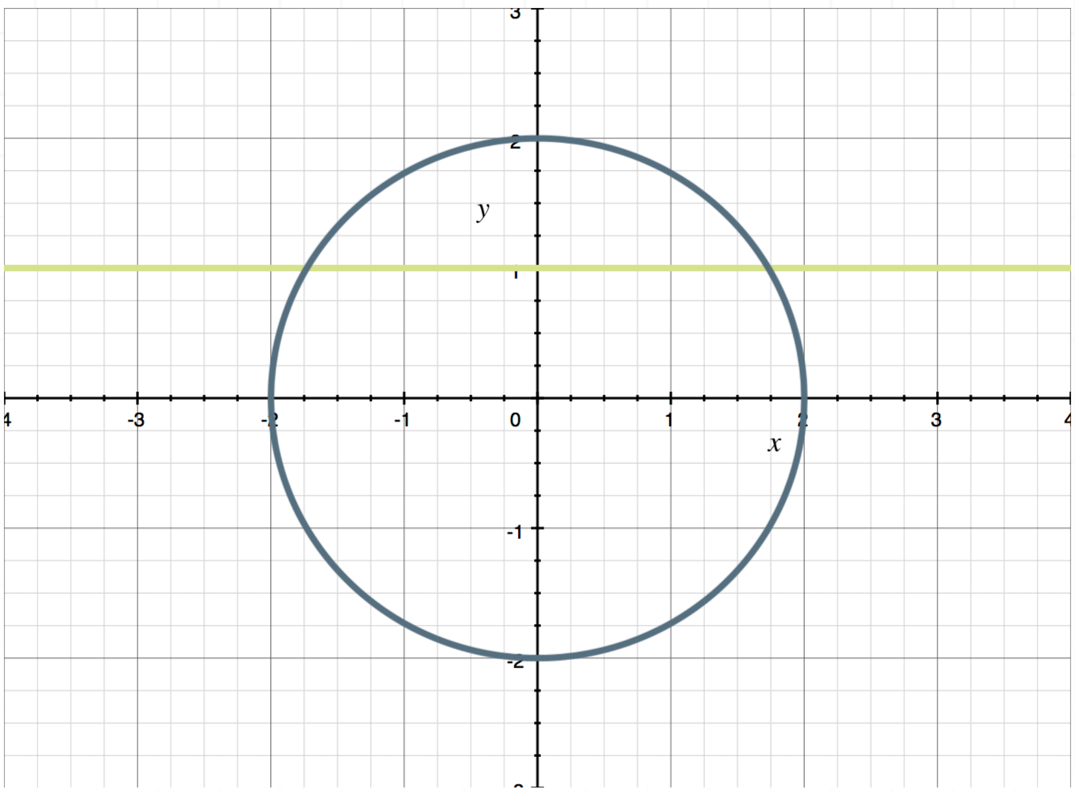
coordinate (if no horizontal line intersects the graph at more than one point).

The graph below passes the Horizontal Line Test, because no horizontal line intersects it at more than one point. Note, however, that this particular graph doesn't represent *any* function (one-to-one or otherwise), because it fails the Vertical Line Test. This shows that even if a graph passes the Horizontal Line Test, it doesn't necessarily represent a one-to-one function.



A graph represents a one-to-one function if and only if it passes the Vertical Line Test *and* the Horizontal Line Test. Passing the Vertical Line Test ensures that the graph represents a *function*, and then also passing the Horizontal Line Test ensures that the function it represents is *one-to-one*.

The next graph doesn't pass the Horizontal Line Test, because any horizontal line between the line $y = -2$ and the line $y = 2$ intersects it at two points.

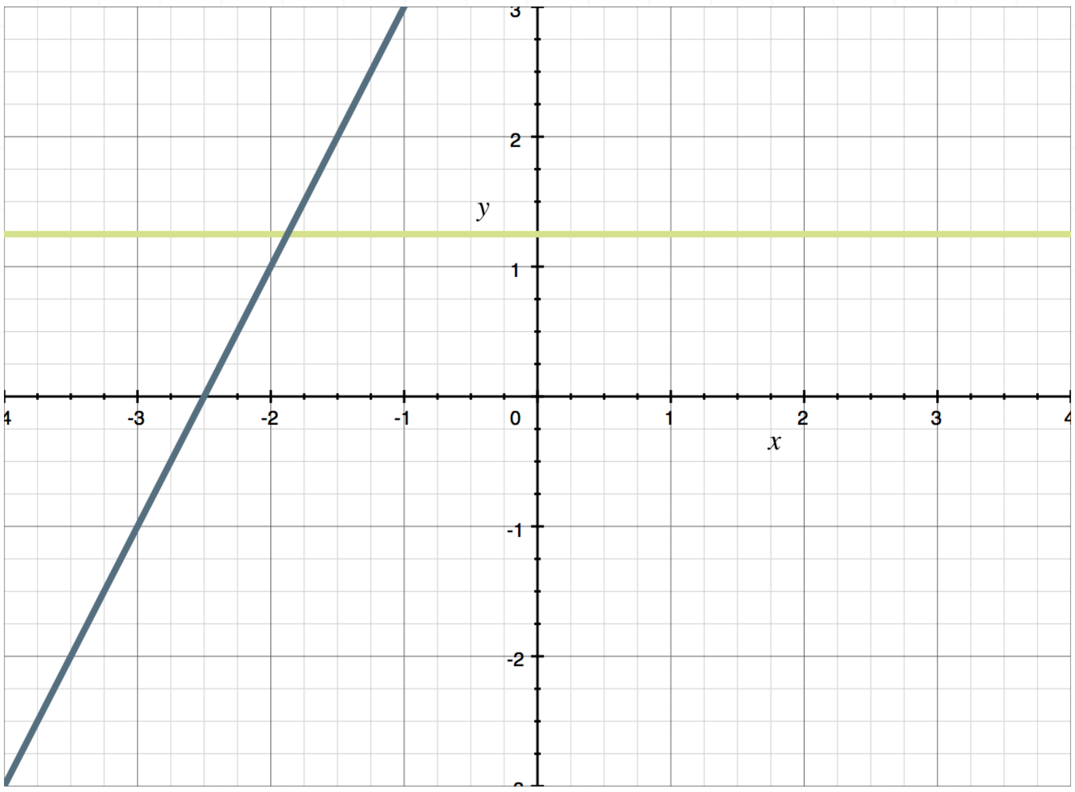


No vertical line is the graph of a function (one-to-one or otherwise), because all the points on a vertical line have the same x-coordinate, which means that a vertical line fails the Vertical Line Test.

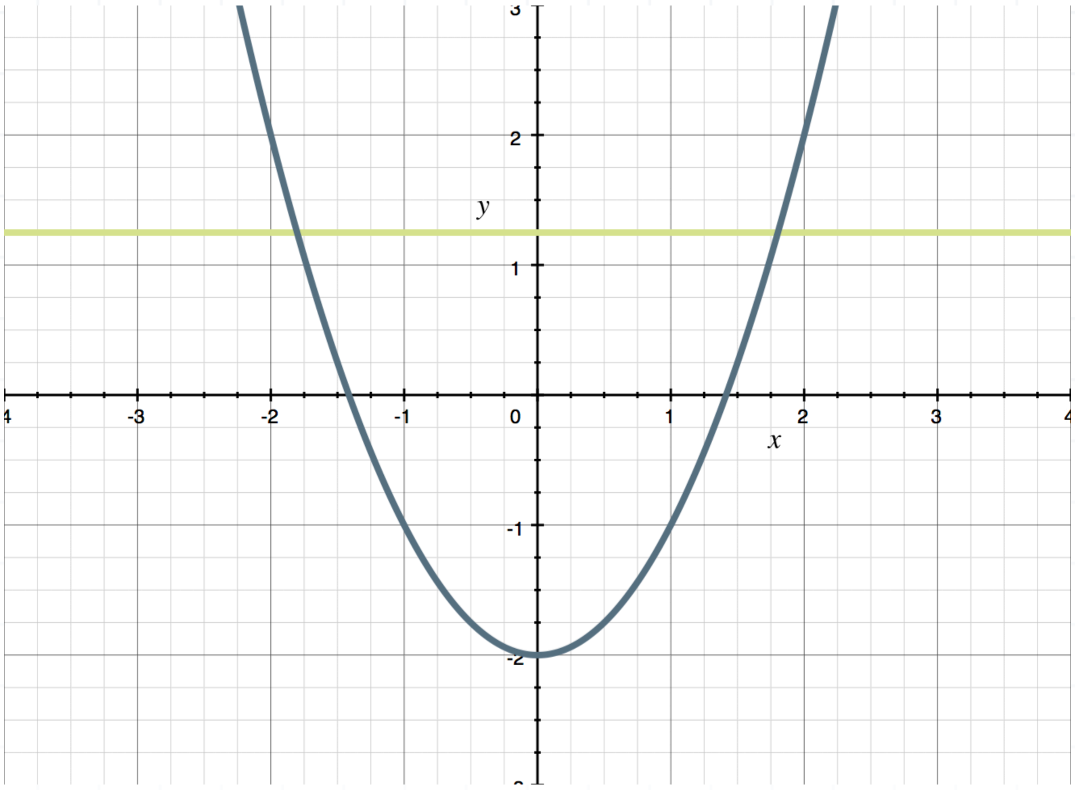
Every horizontal line is the graph of a function, because all the points on a horizontal line have different x-coordinates, which means that it passes the Vertical Line Test. However, no horizontal line is the graph of a one-to-one function, because all the points on a horizontal line have the same y-coordinate, which means that a horizontal line fails the Horizontal Line Test.

Every line that's neither vertical nor horizontal is the graph of a one-to-one function, because no two points on such a line have the same x-coordinate (which means that the line passes the Vertical Line Test) and no two points on such a line have the same y-coordinate (which means that the line passes the Horizontal Line Test as well).

A look at this next graph tells us that there's no horizontal line that intersects the graph at more than one point, so the relation is a function.



On the other hand, no quadratic function is one-to-one. A look at the next graph shows us that it's easy to find a horizontal line that intersects the graph at more than point, thereby proving that the function is not one-to-one.



This is one reason why it's good to have an idea of what the graphs of various "function families" look like. If you're familiar with what the graphs of a certain group of functions look like, then you can think about the graph of a function of that type in your head and decide whether it represents a one-to-one function. For example, the graph of a quadratic function is a parabola so there are horizontal lines that intersect its graph at two points. That's why quadratic functions are never one-to-one.

One-to-one functions algebraically

Another method of checking for a one-to-one function is through the use of algebra to determine whether $f(a)=f(b)$ implies $a=b$, which is true if and only if f is a one-to-one function.

Say we want to know if $f(x)=\sqrt{x-2}$ is one-to-one without drawing or visualizing the graph.

Then we could use algebra to determine whether $f(a)=f(b)$ implies that $a=b$. We know that $f(a)=\sqrt{a-2}$ and $f(b)=\sqrt{b-2}$, so we could say

$$f(a)=f(b)$$

$$\sqrt{a-2}=\sqrt{b-2}$$

$$\left(\sqrt{a-2}\right)^2=\left(\sqrt{b-2}\right)^2$$

$$a-2=b-2$$

$$a=b$$

So $f(x)$ is a one-to-one function.

Let's try another one of those.

Example

Show that the function is one-to-one by showing that $f(a)=f(b)$ leads to $a=b$.

$$f(x)=\frac{x-3}{x+4}$$

We'll start by substituting a for x, and then setting the resulting expression equal to the expression we get when we substitute b for x.

$$\frac{a-3}{a+4}=\frac{b-3}{b+4}$$

$$(a-3)(b+4)=(b-3)(a+4)$$

$$ab+4a-3b-12=ab+4b-3a-12$$

$$4a-3b=4b-3a$$

$$4a+3a=4b+3b$$

$$7a=7b$$

$$a=b$$

This means $f(x)$ is a one-to-one function.

Let's try another example.

Example

Show that $f(x)$ is not one-to-one by showing that $f(a)=f(b)$ does not imply that $a=b$.

$$f(x)=x^2$$

All we need is one case to show that $f(a)=f(b)$ does not imply that $a=b$. That means we can choose one example where $f(a)=f(b)$ but $a\neq b$. Consider the case when $a=2$ and $b=-2$. Then $a\neq b$ but

$$f(a) = f(2) = (2)^2 = 4$$

and

$$f(b) = f(-2) = (-2)^2 = 4$$

Therefore, $f(2) = f(-2)$ but $2 \neq -2$. Since we've found one case, the function is not one-to-one.