Quadratic formula

The quadratic formula is another way to solve quadratics that we can't easily factor. You can think of the quadratic formula as a short-cut for completing the square. In fact, it was discovered by completing the square.

So let's look at completing the square. Spoiler: the result will be the quadratic formula! Remember that quadratic equations are formally written as

$$ax^2 + bx + c = 0$$

The only thing we'll assume is that $a > 0$. If that isn't the case, we can just multiply both sides by -1 to get a quadratic equation of the form $ax^2 + bx + c = 0$ with $a > 0$. First, divide everything by a.

$$\frac{ax^2}{a} + \frac{bx}{a} + \frac{c}{a} = \frac{0}{a}$$

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

Move c/a to the right side by subtracting c/a from both sides.

$$x^2 + \frac{b}{a}x + \frac{c}{a} - \frac{c}{a} = 0 - \frac{c}{a}$$

$$x^2 + \frac{b}{a}x = -\frac{c}{a}$$

Divide the coefficient of the x term, b/a, by 2 and then square the result.

$$\left[\frac{\frac{b}{a}}{2}\right]^2=\left(\frac{b}{a}\cdot\frac{1}{2}\right)^2=\left(\frac{b}{2a}\right)^2=\frac{b^2}{4a^2}$$

Add $b^2/4a^2$ to both sides.

$$x^2+\frac{b}{a}x+\frac{b^2}{4a^2}=\frac{b^2}{4a^2}-\frac{c}{a}$$

Get a common denominator on the right side so that we can do the indicated subtraction.

$$x^2+\frac{b}{a}x+\frac{b^2}{4a^2}=\frac{b^2}{4a^2}-\frac{4ac}{4a^2}$$

$$x^2+\frac{b}{a}x+\frac{b^2}{4a^2}=\frac{b^2-4ac}{4a^2}$$

Factor the left side of the equation and solve for x.

$$\left(x+\frac{b}{2a}\right)^2=\frac{b^2-4ac}{4a^2}$$

$$\sqrt{\left(x+\frac{b}{2a}\right)^2}=\sqrt{\frac{b^2-4ac}{4a^2}}$$

$$\sqrt{\left(x+\frac{b}{2a}\right)^2}=\frac{\sqrt{b^2-4ac}}{\sqrt{4a^2}}$$

$$\sqrt{\left(x+\frac{b}{2a}\right)^2}=\frac{\sqrt{b^2-4ac}}{2a}$$

So our equation becomes

$$x + \frac{b}{2a} = \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

$$x + \frac{b}{2a} - \frac{b}{2a} = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

The two fractions on the right side of this equation have the same denominator ($2a$), so we can easily combine them.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The quadratic formula is what we just got:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

You can use it to find the roots (whether they're real or complex) of any quadratic equation.

- When $b^2 - 4ac = 0$, the solution is one real number
- When $b^2 - 4ac > 0$, the solutions are two real numbers
- When $b^2 - 4ac < 0$, the solutions are two real complex numbers

After you use the quadratic formula a number of times, you may get to the point where you know it from memory. Until then, one way that might help you remember the quadratic formula is to sing the following to the tune of Pop Goes the Weasel:

> "x equals opposite b, plus or minus the square root of b^2 minus $4ac$, all over $2a$."

Example

Solve for x using the quadratic formula.

$$x^2 + 2x - 8 = 0$$

In this problem $a = 1$, $b = 2$, and $c = -8$.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-2 \pm \sqrt{2^2 - 4(1)(-8)}}{2(1)}$$

$$x = \frac{-2 \pm \sqrt{4 + 32}}{2}$$

$$x = \frac{-2 \pm \sqrt{36}}{2}$$

$$x = \frac{-2 \pm 6}{2}$$

We can split this apart into addition and subtraction and find the roots to be $x = 2$ and $x = -4$.

$$x = \frac{-2+6}{2} = \frac{4}{2} = 2$$

$$x = \frac{-2-6}{2} = \frac{-8}{2} = -4$$

Let’s try another example of solving using the quadratic formula.

Example

Solve for the variable using the quadratic formula.

$$3x^2 + 6x + 2 = 0$$

In this problem $a = 3$, $b = 6$, and $c = 2$.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-6 \pm \sqrt{6^2 - 4(3)(2)}}{2(3)}$$

$$x = \frac{-6 \pm \sqrt{36 - 24}}{6}$$

$$x = \frac{-6 \pm \sqrt{12}}{6}$$

$$x = \frac{-6 \pm \sqrt{4 \cdot 3}}{6}$$

$$x = \frac{-6 \pm 2\sqrt{3}}{6}$$

Factor out a 2 in the numerator and a 2 in the denominator, and then cancel those 2's.

$$x = \frac{2(-3 \pm \sqrt{3})}{2(3)}$$

$$x = \frac{-3 \pm \sqrt{3}}{3}$$