Inverse functions

In this lesson we'll look at the definition of an inverse function and how to find a function's inverse.

If you remember from the last lesson, a function is invertible (has an inverse) if and only if it's one-to-one. Now let's look a little more into how to find an inverse and what an inverse does.

When you have a function with points $(x, f(x))$, the inverse function will have points $(f(x), x)$. You could think of the inverse of a function f as the function that "undoes" f. If you first evaluate $f(x)$ at some x in the domain of f, and then evaluate the inverse of f at that value of $f(x)$, what you get is just x (the input you started with). The inverse of a function $f(x)$ is written as $f^{-1}(x)$. Because f^{-1} "undoes" f, you could think of the function $g(x) = x$ as the composite of $f^{-1}(x)$ and $f(x)$, because

$$g(x) = x = f^{-1}(f(x))$$

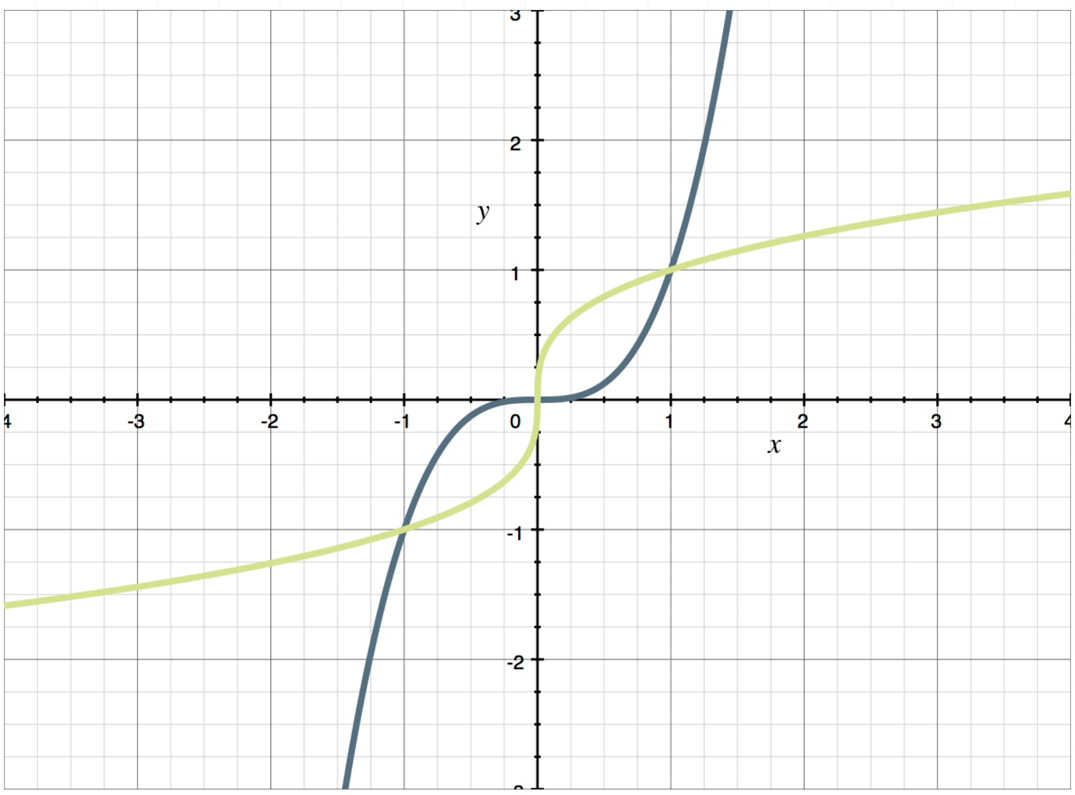
For example, if $g(x)$ and $g^{-1}(x)$ are inverses of one another, then the tables below would give sets of points from each.

x	g(x)
1	4
4	8
10	12
16	2

x	$g^{-1}(x)$
4	1
8	4
12	10
2	16

Now let's look at the graphs of a function and its inverse. Look at the graph of the function $f(x) = x^3$ (in blue) and the graph of its inverse (in green). Notice that in order to "get back to x" from $f(x)$ (to get back to x from x^3), you have to take the cube root of $f(x)$, because

$$x = (x^3)^{\frac{1}{3}} = \sqrt[3]{x^3}$$



Notice that the x- and y-coordinates of the points on the blue curve are the y- and x-coordinates, respectively, of the points on the green curve, that is, the coordinates of the points of the graph of $f^{-1}(x)$ have switched places with the coordinates of the points of the graph of $f(x)$. Now let's look at how to calculate an inverse algebraically.

Example

What is the inverse of the function?

$$f(x) = \frac{2}{3}x - 4$$

First, notice that this function is invertible, because its graph is a line that's neither vertical nor horizontal (so its graph passes both the Vertical Line Test and the Horizontal Line Test, which means that the function is one-to-one).

To find the inverse of this function, first replace $f(x)$ with the variable y.

$$y = \frac{2}{3}x - 4$$

Next, switch x with y.

$$x = \frac{2}{3}y - 4$$

Now solve for y.

$$x + 4 = \frac{2}{3}y$$

$$\frac{3}{2}(x + 4) = \frac{3}{2}\left(\frac{2}{3}y\right)$$

$$\frac{3}{2} \cdot x + \frac{3}{2} \cdot 4 = \frac{3}{2} \cdot \frac{2}{3}y$$

$$\frac{3}{2}x + 6 = y$$

Now you can write the inverse function by replacing y with $f^{-1}(x)$ (and then turning the equation around so that $f^{-1}(x)$ is on the left side).

$$f^{-1}(x) = \frac{3}{2}x + 6$$

Let's do one more example.

Example

Find the inverse of the function.

$$g(x) = \frac{x}{x-3}$$

First replace $g(x)$ with y.

$$y = \frac{x}{x-3}$$

At this point in finding the inverse of the function in the other example, we first switched x with y, and then solved for y. When we use algebra to get the inverse of a function, we could just as well first solve for x, and then switch x with y, so we'll do it that way here.

$$y(x-3) = x$$

$$xy - 3y = x$$

$$xy - x = 3y$$

$$x(y-1) = 3y$$

$$x = \frac{3y}{y-1}$$

Now switch x with y.

$$y = \frac{3x}{x-1}$$

Finally, write the inverse function by replacing y with $g^{-1}(x)$.

$$g^{-1}(x) = \frac{3x}{x-1}$$