Dividing rational functions

In this lesson we'll look at how to simplify and divide rational functions by using factoring and cancellation.

Factoring and cancellation

If you can find equivalent values to factor out of both the numerator and denominator of a fraction, they can both be pulled out into a separate fraction and simplified (or canceled) to 1. Here's a simple example with constants.

$$\frac{4+2}{8}=\frac{2(2)+2(1)}{2(4)}=\frac{2(2+1)}{2(4)}=\frac{2}{2}\cdot\frac{(2+1)}{(4)}=1\cdot\frac{(2+1)}{(4)}=\frac{3}{4}$$

We were able to factor a 2 out of every term in both the numerator and denominator, then pull them out into a separate fraction, simplify that fraction to 1, and we're left with a simpler fraction.

Here's an example with variables:

$$\frac{4x^2+2x}{8x^3}=\frac{2x(2x)+2x(1)}{2x(4x^2)}=\frac{2x(2x+1)}{2x(4x^2)}$$

$$=\frac{2x}{2x}\cdot\frac{2x+1}{4x^2}=1\cdot\frac{2x+1}{4x^2}=\frac{2x+1}{4x^2}$$

Dividing rational functions

When we divide one rational function by another, the first fraction is the **dividend**, the second fraction is the **divisor**, and the result of doing the division gives you the **quotient**.

$$\frac{\text{dividend}}{\text{divisor}} = \text{quotient}$$

Just like when we divide one fraction by another, we turn the division problem into a multiplication problem. To do this, we flip the divisor upside down, and change the division into multiplication. So the problem instead becomes

$$\text{dividend} \cdot \frac{1}{\text{divisor}} = \text{quotient}$$

Once we've changed the division into multiplication, we'll multiply the numerators together to get the new numerator, and multiply the denominators together to get the new denominator.

Then, if there are any common factors between the numerator and denominator, we'll cancel those common factors in order to simplify the quotient.

Let's look at an example so that we can see the transition from division to multiplication in action.

Example

Find the quotient of the rational functions.

$$\frac{x+2}{x-1} \div \frac{x+4}{x-1}$$

To find the quotient, we'll flip the divisor (the second fraction) upside down, and simultaneously change the division to multiplication. So we can rewrite the problem as

$$\frac{x+2}{x-1} \cdot \frac{x-1}{x+4}$$

Now that we're doing multiplication instead of division, we can combine the fractions by multiplying across the numerators and across the denominators.

$$\frac{(x+2)(x-1)}{(x-1)(x+4)}$$

We've got a common factor of $x-1$ in the numerator and denominator that can be canceled, leaving us with just

$$\frac{x+2}{x+4}$$

Restrictions on the domain

We've just learned how to divide rational functions, but we left out one really important aspect of these division problems. Whenever we divide rational functions, we have to consider the values for which the quotient will be undefined.

There are three sets of values that will make a quotient undefined:

1. Any values(s) where the **dividend's denominator** is 0

2. Any value(s) where the **divisor's denominator** is 0

3. Any values(s) where the **divisor's numerator** is 0

Remember, any fraction will be undefined whenever its denominator is 0, so these first two value sets make sense. Of course, the dividend will be undefined when its denominator is 0, and the divisor will be undefined when its denominator is 0. We're certainly not going to be able to find the quotient if either the dividend or divisor is undefined, so we need to make sure to state that the quotient won't be defined for any values where either denominator is 0.

The third value set is slightly trickier. If the numerator of a fraction is 0, then the value of the whole fraction will be 0, since 0 divided by anything, is 0. So, when the numerator of the divisor is 0, that makes the entire divisor equal to 0. But if the divisor is 0, then when we try to find the quotient, we'll be taking

$$\frac{\text{dividend}}{0}$$

which we can't do. That's why a zero value in the divisor's numerator will also make the quotient undefined.

So, in actuality, when we divide rational functions, we need to start by considering these restrictions first, and then proceed with the rest of the division problem.

Let's revisit the example we were working with, and this time consider these restrictions on the domain.

Example

Find the quotient of the rational functions.

$$\frac{x+2}{x-1} \div \frac{x+4}{x-1}$$

This time around, we'll look at the restrictions, considering all three value sets that might cause us trouble:

The dividend's denominator will be 0 at

$$x - 1 = 0$$

$$x = 1$$

The divisor's denominator will be 0 at

$$x - 1 = 0$$

$$x = 1$$

The divisor's numerator will be 0 at

$$x + 4 = 0$$

$$x = -4$$

If we put these results together, we can say that the restrictions on the domain are $x \neq -4,1$.

Once we've established this set of values, we can actually do the division problem, following the same steps as the last example.

$$\frac{x+2}{x-1} \div \frac{x+4}{x-1}$$

$$\frac{x+2}{x-1} \cdot \frac{x-1}{x+4}$$

$$\frac{(x+2)(x-1)}{(x-1)(x+4)}$$

$$\frac{x+2}{x+4}$$

The answer to the problem is the combination of the quotient and the restrictions.

$$\frac{x+2}{x+4} \text{ with } x \neq -4,1$$

However, there's just one more thing we always have to check. The quotient itself, given that the denominator is $x+4$, makes it obvious that $x \neq -4$, since $x = -4$ would make the denominator of the quotient equal to 0.

So there's need to list that value as a restriction on the quotient's value, because we can see it just by looking at the quotient. The $x \neq 1$ restriction can't be identified from the quotient (it was lost when we canceled common factors), so we have to keep that restriction. Therefore, the final answer is

$$\frac{x+2}{x+4} \text{ with } x \neq 1$$

Let's do a couple more examples where we follow this same process to find the quotient.

Example

Find the quotient of the rational functions.

$$\frac{6a+27}{18a^2+36a} \div \frac{16a+72}{2a+4}$$

Factor the numerator and denominator of both fractions as completely as possible.

$$\frac{3(2a+9)}{18a(a+2)} \div \frac{8(2a+9)}{2(a+2)}$$

Now we'll consider the restrictions. The dividend's denominator tells us that $a \neq -2,0$, the divisor's denominator tells us that $a \neq -2$, and the divisor's numerator tells us that $a \neq -9/2$. So the full set of restrictions is $a \neq -9/2, -2,0$.

Next we'll find the quotient.

$$\frac{3(2a+9)}{18a(a+2)}\cdot\frac{2(a+2)}{8(2a+9)}$$

$$\frac{6(2a+9)(a+2)}{144a(a+2)(2a+9)}$$

$$\frac{6}{144a}$$

$$\frac{1}{24a}$$

Putting this quotient together with the restrictions gives

$$\frac{1}{24a}\text{ with } a\neq-\frac{9}{2},-2,0$$

But the quotient still shows us that $a\neq 0$, so we don't need to include it in the list of restrictions.

$$\frac{1}{24a}\text{ with } a\neq-\frac{9}{2},-2$$

Let's try another example, this time where none of the restrictions on the domain are identifiable from the quotient.

Example

Find the quotient of the rational functions.

$$\frac{3x^2-25x-18}{27x+18}\div\frac{5x-3}{5x^2-33x+18}$$

Factor the numerator and denominator of both fractions as completely as possible.

$$\frac{(3x+2)(x-9)}{9(3x+2)} \div \frac{5x-3}{(5x-3)(x-6)}$$

Now we'll consider the restrictions. The dividend's denominator tells us that $x \neq -2/3$, the divisor's denominator tells us that $x \neq 3/5, 6$, and the divisor's numerator tells us that $x \neq 3/5$. So the full set of restrictions is $x \neq -2/3, 3/5, 6$.

Next we'll find the quotient.

$$\frac{(3x+2)(x-9)}{9(3x+2)} \cdot \frac{(5x-3)(x-6)}{5x-3}$$

$$\frac{(3x+2)(x-9)(5x-3)(x-6)}{9(3x+2)(5x-3)}$$

$$\frac{(x-9)(x-6)}{9}$$

Putting this quotient together with the restrictions gives

$$\frac{(x-9)(x-6)}{9} \text{ with } x \neq -\frac{2}{3}, \frac{3}{5}, 6$$

None of these restrictions are evident from the quotient, so we keep all of them.

We'll do one more example. This one has a lot of restrictions!

Example

Find the quotient of the rational functions.

$$\frac{x^2-4}{x^2-9x+18} \div \frac{x^2-25}{x^2-8x-20}$$

Factor the numerator and denominator of both fractions as completely as possible.

$$\frac{(x-2)(x+2)}{(x-6)(x-3)} \div \frac{(x-5)(x+5)}{(x-10)(x+2)}$$

Now we'll consider the restrictions. The dividend's denominator tells us that $x \neq 3,6$, the divisor's denominator tells us that $x \neq -2,10$, and the divisor's numerator tells us that $x \neq -5,5$. So the full set of restrictions is $x \neq -5, -2,3,5,6,10$.

Next we'll find the quotient.

$$\frac{(x-2)(x+2)}{(x-6)(x-3)} \cdot \frac{(x-10)(x+2)}{(x-5)(x+5)}$$

$$\frac{(x-2)(x+2)^2(x-10)}{(x-6)(x-3)(x-5)(x+5)}$$

Putting this quotient together with the restrictions gives

$$\frac{(x-2)(x+2)^2(x-10)}{(x-6)(x-3)(x-5)(x+5)} \text{ with } x \neq -5,-2,3,5,6,10$$

But the quotient still shows us that $x \neq -5,3,5,6$, so we don't need to include those in the list of restrictions.

$$\frac{(x-2)(x+2)^2(x-10)}{(x-6)(x-3)(x-5)(x+5)} \text{ with } x \neq -2,10$$