Chebyshev's Theorem

The empirical rule tells us that the percentage of data that falls within 1, 2, and 3 standard deviations of the mean in a normal distribution is 68%, 95%, and 99.7%, respectively.

Of course, the constraint of the empirical rule is that it only applies to normally distributed data. We can't use it when our distribution is left-skewed or right-skewed. But that's where Chebyshev's Theorem comes in.

Chebyshev's Theorem tells us that at least $(1 - 1/k^2)\%$ of our data must be within k standard deviations of the mean, for $k > 1$, and regardless of the shape of the data's distribution. For instance, here's what the theorem can conclude for any distribution when $k = 2$, 3, and 4:

- At least 75% of the data must be within $k = 2$ standard deviations of the mean.

- At least 89% of the data must be within $k = 3$ standard deviations of the mean.

- At least 94% of the data must be within $k = 4$ standard deviations of the mean.

Keep in mind that k doesn't have to be an integer, but it *does* have to be greater than 1. So we could use Chebyshev's Theorem for $k = 1.32$, $k = 2$, or $k = 2.14$, but not for $k = 1$ or for $k = 0.46$.

Notice how these percentages are less than the corresponding percentages for the normal distribution. For instance, in a normal distribution, the Empirical Rule tells us that 95% of the data falls within 2 standard deviations of the mean, but Chebyshev's Theorem only lets us conclude that 75% of the data will fall within 2 standard deviations of the mean. Similarly, the Empirical Rule tells us that 99.7% of the data falls within 3 standard deviations of the mean, but Chebyshev's Theorem only lets us conclude that 89% of the data will fall within 3 standard deviations of the mean.

Because Chebyshev's Theorem has to work for distributions of all shapes, unlike the Empirical Rule which applies only to the normal distribution, Chebyshev's Theorem is required to be more conservative. And that's why we see smaller percentages for Chebyshev's Theorem than we do for the Empirical Rule.

Let's do an example so we can see how to apply Chebyshev's Theorem.

Example

A statistics class of 40 students has a mean final exam score of 86, with a standard deviation of 3. How many students scored between 81 and 91 on the final exam?

We need to determine the distance from the mean of 81 and 91, in terms of standard deviations.

$$k = \frac{81 - 86}{3} = -\frac{5}{3} \approx -1.67$$

$$k = \frac{91 - 86}{3} = \frac{5}{3} \approx 1.67$$

Then we can apply Chebyshev's Theorem, using the value of k that's greater than 1, which is $k = 1.67$.

$$1 - \frac{1}{k^2} = 1 - \frac{1}{1.67^2} \approx 0.64$$

Because we found 0.64, we know at least 64% of the students scored between 81 and 91 on the final exam. Because there are 40 students in the class, 64% of the class is $0.64(40) = 25.6$.

So Chebyshev's Theorem tells us that at least 25.6 students scored between 81 and 91 on the exam. It doesn't make sense to say 25.6 students, but "at least 25.6 students" doesn't meet the threshold of "at least 26 students," so we round down to get our conclusion:

> *"According to Chebyshev's Theorem, at least 25 students scored between 81 and 91 on the final exam."*

We can also use Chebyshev's Theorem to work backwards through probability problems.

Example

A statistics class of 40 students has a mean final exam score of 86, with a standard deviation of 3. Find the score range for the central 80% of test scores.

Using Chebyshev's Theorem,

$$0.8 = 1 - \frac{1}{k^2}$$

$$\frac{1}{k^2} = 1 - 0.8$$

$$1 = 0.2k^2$$

$$k^2 = 5$$

$$k \approx 2.24$$

Approximately 2.24 standard deviations above the mean gives us a test score of

$$86 + 2.24(3)$$

$$86 + 6.72$$

$$92.72$$

And 2.24 standard deviations below the mean gives us a test score of

$$86 - 2.24(3)$$

$$86 - 6.72$$

$$79.28$$

So at least 80% of the final exam scores fell between 79.28 and 92.72. If test scores can only be integers, then at least 80% of the final exam scores must have fallen between 80 and 92.