Rationalizing complex denominators

In this lesson you'll learn how to multiply a fraction that has a complex number in the denominator by the conjugate fraction to make the denominator a rational number.

A **complex number** is a number that can be written in the form $a+bi$, where a and b are real numbers and i is the imaginary number $\sqrt{-1}$; a is called the **real part** of the complex number, and bi is called the **imaginary part** of it.

An **imaginary number** (also called a pure imaginary number) is a complex number whose real part is 0. For example, $-6i$ and $4i$ are imaginary numbers. So every complex number can be written as the sum of a real number and an imaginary number.

There are a few things we want to understand first:

1. The conjugate of a complex number (usually called the complex conjugate of that number) is formed by changing the sign of the imaginary part and leaving the real part unchanged.

   For example, the complex conjugate of $5+3i$ is $5-3i$.

2. You can multiply any number times 1 without changing its value, and 1 can be written as any nonzero number or expression divided by itself.

   For example,

$$\frac{3}{5+3i} = \frac{3}{5+3i} \cdot 1 = \frac{3}{5+3i} \cdot \frac{5-3i}{5-3i}$$

because

$$\frac{5-3i}{5-3i} = 1$$

To rationalize a fraction that has a complex number in the denominator, we multiply it by the fraction in which both the numerator and the denominator are the complex conjugate of that complex number (just like in this last example). This is called the **conjugate method**.

3. Use FOIL or the distributive property to multiply complex numbers, then simplify.

An example of FOIL (multiplying the first terms, the outer terms, the inner terms, and last terms, in that order):

$$(5+3i)(4+2i)$$

$$5 \cdot 4 + 5 \cdot 2i + 3i \cdot 4 + 3i \cdot 2i$$

$$20 + 10i + 12i + 6i^2$$

Combining like terms and replacing i^2 with -1, we get

$$20 + 22i + 6(-1)$$

$$20 - 6 + 22i$$

$$14 + 22i$$

An example of the distributive property:

$$5(3-4i)$$

$$5(3)+5(-4i)$$

$$15+(-20i)$$

$$15-20i$$

Now let's use the conjugate method to simplify a fraction that has a complex number in the denominator.

Example

Use the conjugate method to simplify the expression.

$$\frac{3-4i}{-2+i}$$

We can use the conjugate method to get the imaginary number i, out of the denominator. The complex conjugate of $-2+i$ is $-2-i$.

$$\frac{3-4i}{-2+i}\cdot\frac{-2-i}{-2-i}$$

$$\frac{(3-4i)(-2-i)}{(-2+i)(-2-i)}$$

Now that we have a binomial multiplication problem, we need to make sure that (in the numerator and denominator separately) we multiply the first terms, outer terms, inner terms, and last terms.

$$\frac{-6-3i+8i+4i^2}{4+2i-2i-i^2}$$

$$\frac{4i^2+5i-6}{-i^2+4}$$

Replacing i^2 with -1, and then combining like terms, we get

$$\frac{4(-1)+5i-6}{-(-1)+4}$$

$$\frac{-4+5i-6}{1+4}$$

$$\frac{5i-10}{5}$$

$$\frac{5i}{5}-\frac{10}{5}$$

$$i-2$$

$$-2+i$$

Let's look at another example of rationalizing a complex denominator.

Example

Simplify.

$$\frac{10i^2 - 5i}{-6 + 6i}$$

First, we'll rewrite the expression as

$$\frac{10(-1) - 5i}{-6 + 6i}$$

$$\frac{-10 - 5i}{-6 + 6i}$$

Now we can use the conjugate method to get the imaginary number i out of the denominator. The complex conjugate of $-6 + 6i$ is $-6 - 6i$.

$$\frac{-10 - 5i}{-6 + 6i} \cdot \frac{-6 - 6i}{-6 - 6i}$$

$$\frac{(-10 - 5i)(-6 - 6i)}{(-6 + 6i)(-6 - 6i)}$$

Now that we have a binomial multiplication problem, we need to make sure that (in the numerator and denominator separately) we multiply the first terms, outer terms, inner terms, and last terms.

$$\frac{60 + 60i + 30i + 30i^2}{36 + 36i - 36i - 36i^2}$$

$$\frac{60 + 90i + 30i^2}{36 - 36i^2}$$

Replacing i^2 with -1, we get

$$\frac{60 + 90i + 30(-1)}{36 - 36(-1)}$$

$$\frac{60 + 90i - 30}{36 + 36}$$

$$\frac{30 + 90i}{72}$$

Divide out 6, which goes evenly into 30, $90i$, and 72.

$$\frac{5 + 15i}{12}$$

You can also write this as the sum of two fractions.

$$\frac{5}{12} + \frac{15i}{12}$$

Then you can reduce the second fraction to lowest terms, which gives

$$\frac{5}{12} + \frac{5i}{4}$$

$$\frac{5}{12} + \frac{5}{4}i$$