Systems of two equations with a non-linear equation

This lesson will show you the algebraic way to solve a pair of equations where one is a linear equation and the other is a non-linear equation in which at least one of the variables is squared.

Remember that an equation of a circle or an ellipse has both an x^2 term and a y^2 term. It might look like $x^2 + 4y^2 = 100$. On the other hand, an equation of a line has an x term and a y term. An example of a linear equation would be $y = -(3/2)x - 5$.

If you take the two equations and put them together,

$$\begin{cases} x^2 + 4y^2 = 100 \\ y = -\frac{3}{2}x - 5 \end{cases}$$

then you have a system of equations.

The solutions to a system of equations are the points (x, y) where the graphs of the equations in the system intersect.

Let's look at how to solve the system that was given above.

Example

Solve the system for x and y.

$$\begin{cases} x^2 + 4y^2 = 100 \\ y = -\frac{3}{2}x - 5 \end{cases}$$

In this case the second equation is already solved for y, so we can begin by substituting that expression for y into the first equation.

$$x^2 + 4y^2 = 100$$

$$x^2 + 4\left(-\frac{3}{2}x - 5\right)^2 = 100$$

Expand the square.

$$x^2 + 4\left(-\frac{3}{2}x - 5\right)\left(-\frac{3}{2}x - 5\right) = 100$$

$$x^2 + 4\left(\frac{9}{4}x^2 + 15x + 25\right) = 100$$

Distribute the 4 over everything inside the parentheses.

$$x^2 + 9x^2 + 60x + 100 = 100$$

$$x^2 + 9x^2 + 60x + 100 - 100 = 100 - 100$$

$$10x^2 + 60x = 0$$

Factor out a $10x$ to help solve for x.

$$10x(x + 6) = 0$$

To solve this equation, we set the factors, $10x$ and $x + 6$, to 0 separately, and then solve the resulting equations.

$10x = 0$ gives $x = 0$

$x + 6 = 0$ gives $x = -6$

Plug these x-values into the equation $y = -(3/2)x - 5$, to find the y-values that go with them.

For $x = 0$:

$$y = -\frac{3}{2}x - 5$$

$$y = -\frac{3}{2}(0) - 5$$

$$y = 0 - 5$$

$$y = -5$$

So we have the solution $(0, -5)$.

For $x = -6$:

$$y = -\frac{3}{2}x - 5$$

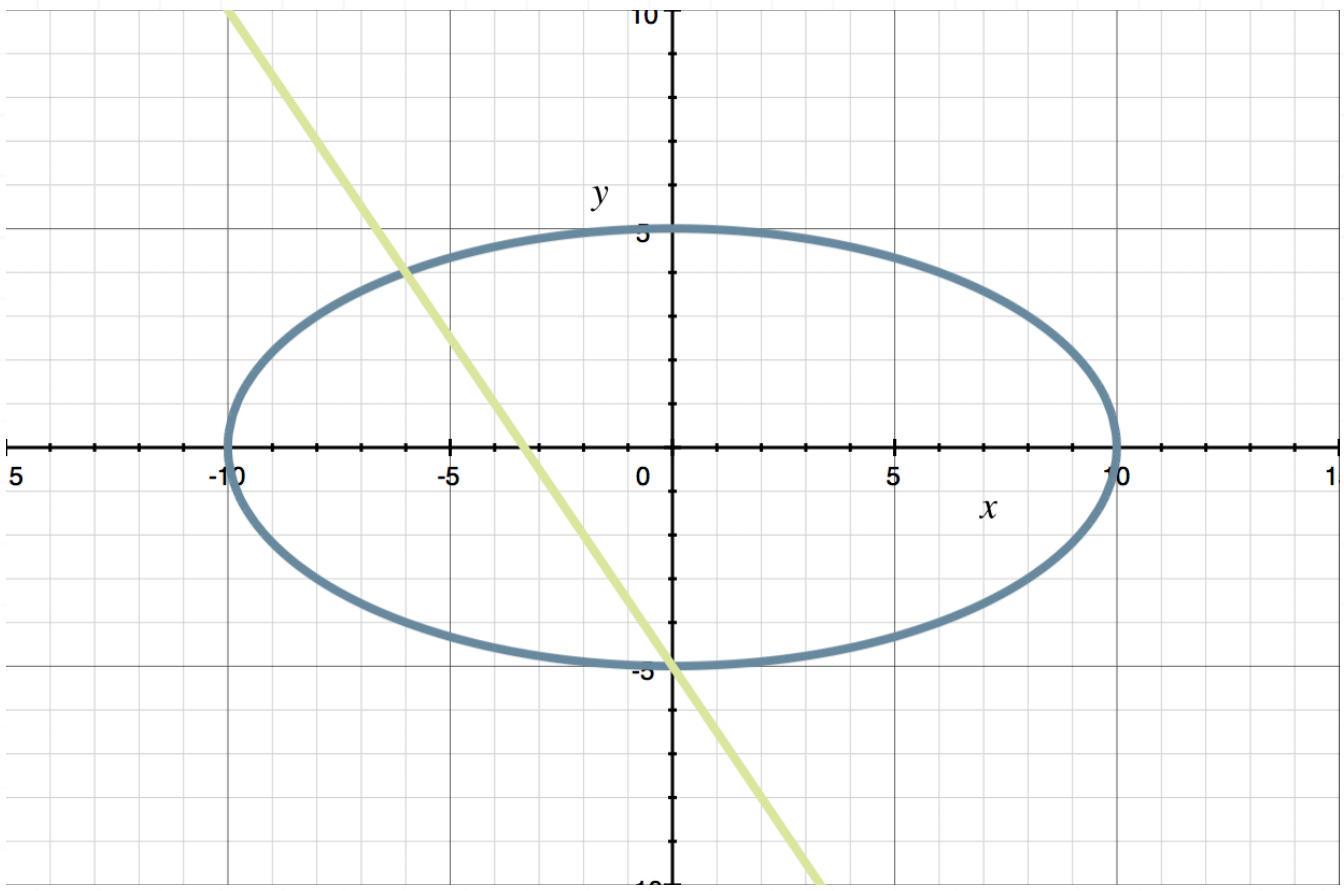
$$y = -\frac{3}{2}(-6) - 5$$

$$y = 9 - 5$$

$$y = 4$$

So we have the solution $(-6,4)$.

The non-linear equation in this system is the equation of an ellipse, and (as always) the linear equation is the equation of a line. You can look at this picture of the system to see that the solutions are the points (x, y) where the ellipse and line intersect.



Let's do an example that involves a few more steps.

Example

Solve the system for x and y.

$$3x^2 + 2y^2 - 54y = 143$$

$$x - 3y = 3$$

Let's solve this system by solving the second equation for y, and then substituting the resulting expression for y into the first equation.

$$x - 3y = 3$$

$$-3y = -x + 3$$

$$\frac{-3}{-3}y = \frac{-x}{-3} + \frac{3}{-3}$$

$$y = \frac{1}{3}x - 1$$

Plug this expression for y into the first equation, and then solve for x.

$$3x^2 + 2y^2 - 54y = 143$$

$$3x^2 + 2\left(\frac{1}{3}x - 1\right)^2 - 54\left(\frac{1}{3}x - 1\right) = 143$$

Expand the square.

$$3x^2 + 2\left(\frac{1}{3}x - 1\right)\left(\frac{1}{3}x - 1\right) - 54\left(\frac{1}{3}x - 1\right) = 143$$

$$3x^2 + 2\left(\frac{1}{9}x^2 - \frac{2}{3}x + 1\right) - 18x + 54 = 143$$

$$3x^2 + \frac{2}{9}x^2 - \frac{4}{3}x + 2 - 18x + 54 = 143$$

Move the 143 to the left-hand side, and then combine like terms.

$$3x^2 + \frac{2}{9}x^2 - \frac{4}{3}x - 18x + 2 + 54 - 143 = 0$$

$$\frac{29}{9}x^2 - \frac{58}{3}x - 87 = 0$$

Clear the fractions by multiplying both sides by 9.

$$9\left(\frac{29}{9}x^2-\frac{58}{3}x-87\right)=9\cdot 0$$

$$9\cdot\frac{29}{9}x^2-9\cdot\frac{58}{3}x-9\cdot 87=0$$

$$29x^2-174x-783=0$$

We can simplify more by dividing everything by 29.

$$\frac{29}{29}x^2-\frac{174}{29}x-\frac{783}{29}=\frac{0}{29}$$

$$x^2-6x-27=0$$

Factor, and then solve for x.

$$(x-9)(x+3)=0$$

To solve this equation, we set the factors, $x-9$ and $x+3$, to 0 separately, and then solve the resulting equations.

$x-9=0$ gives $x=9$

$x+3=0$ gives $x=-3$

Plug these values of x into the expression we found for y, to get the corresponding y-values.

For $x=9$:

$$y=\frac{1}{3}x-1$$

$$y = \frac{1}{3}(9) - 1$$

$$y = 3 - 1$$

$$y = 2$$

So one solution is $(9,2)$.

For $x = -3$:

$$y = \frac{1}{3}x - 1$$

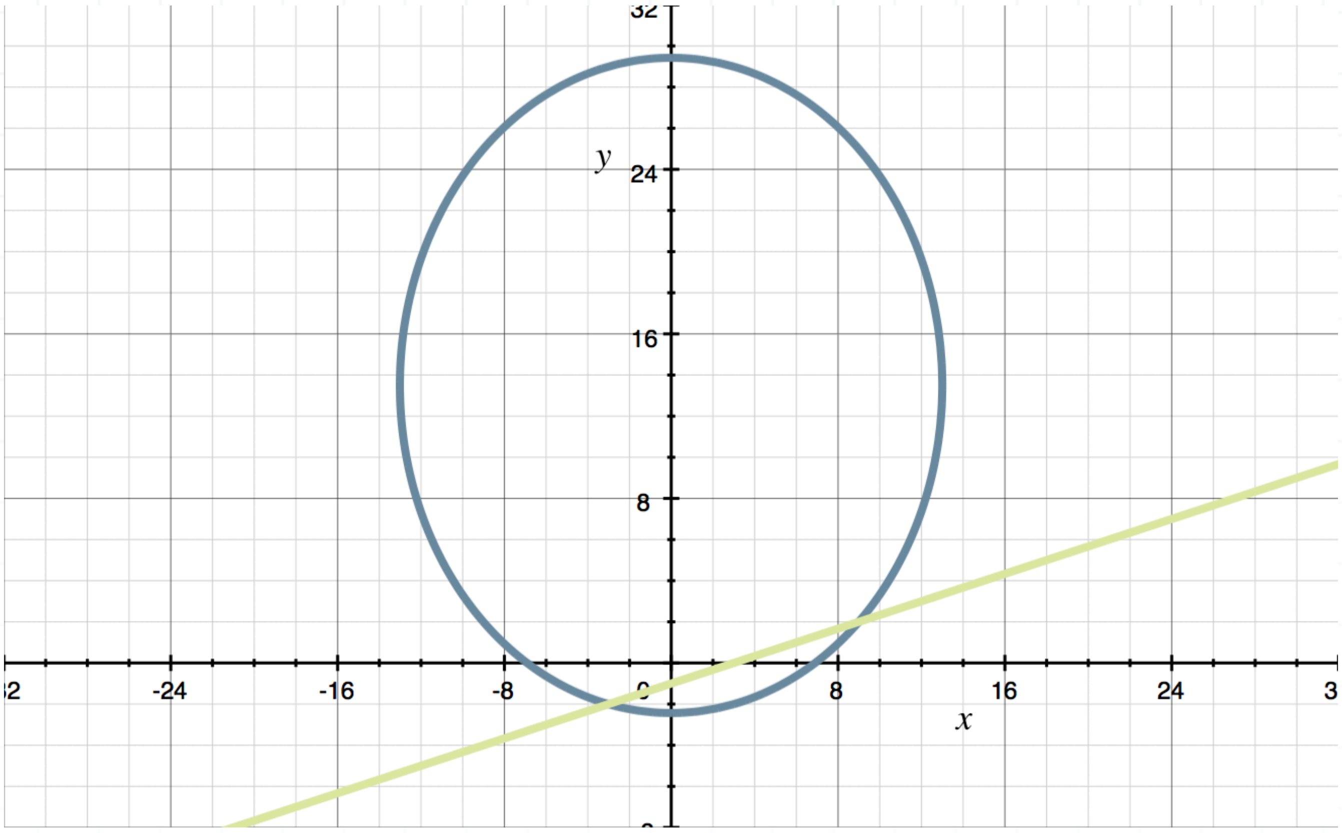
$$y = \frac{1}{3}(-3) - 1$$

$$y = -1 - 1$$

$$y = -2$$

So the other solution is $(-3, -2)$.

Sometimes it's nice to have a visual of what we did algebraically. Here are the graphs of the non-linear equation in this system (which is the equation of an ellipse) and the linear equation (which is the equation of a line). Notice that they intersect at the solution points $(-3, -2)$ and $(9,2)$.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