Completing the square with complex roots

As you saw in the previous lesson, when you have a quadratic equation of the form $ax^2 + bx + c = 0$ with $a = 1$ and you solve it by completing the square, you eventually get an equation of the following form:

$$\left[x + \left(\frac{b}{2}\right)\right]^2 = -c + \left(\frac{b}{2}\right)^2$$

In that lesson, we set out to find real numbers x that are roots of that equation, and we said that if the expression on the right side,

$$-c + \left(\frac{b}{2}\right)^2$$

is negative, then the equation has no (real) roots, so we stopped.

However, there are numbers, called **complex numbers**, that are roots of a quadratic equation like that. So now you're going to learn how to solve quadratic equations that have complex numbers as roots.

The roots of a polynomial equation that are real numbers are also called **real zeroes** of the corresponding polynomial. Similarly, the roots of a polynomial equation that are complex numbers are also called **complex zeroes** of the corresponding polynomial.

Complex numbers are based on the numbers i, which is defined as $\sqrt{-1}$ (something that you've always been told doesn't exist!). We call i an imaginary number. A complex number is a number that can be written in the form $a+bi$, where a and b are real numbers. If $a=0$, the complex number $a+bi$ is equal to bi, which is said to be a **pure imaginary number**.

If d is any negative real number, we can write $\sqrt{d}$ as $i\sqrt{-d}$. Notice that $-d$ is a positive real number. So we "fix" the square root of a negative number by factoring out an i. You may think, "What's the point of learning about imaginary numbers?" Imaginary numbers are incredibly useful in the real world and are most often used in electrical engineering.

When we solve a quadratic equation that has complex roots, we'll follow the same steps as before, except that we won't stop when we get a negative number under the radical sign. Here are the steps, but recall that we first divide the polynomial by a (the coefficient of the x^2 term) if $a \neq 1$. So just as before, we'll be starting with a quadratic equation of the form $x^2+bx+c=0$.

1. Move the constant term, c, to the right side of the equation by subtracting c from both sides.

$$x^2+bx+c-c=0-c$$

$$x^2+bx=-c$$

2. Find $(b/2)^2$. Take the coefficient of the x term, divide it by 2, and then square the result.

3. Add $(b/2)^2$ to both sides of the equation.

$$x^2+bx+\left(\frac{b}{2}\right)^2=-c+\left(\frac{b}{2}\right)^2$$

4. Factor the left side, which is now

$$x^2 + bx + \left(\frac{b}{2}\right)^2$$

$$\left(x + \frac{b}{2}\right)\left(x + \frac{b}{2}\right)$$

$$\left(x + \frac{b}{2}\right)^2$$

So the equation we have to solve is

$$\left(x + \frac{b}{2}\right)^2 = -c + \left(\frac{b}{2}\right)^2$$

5. Square root both sides of the equation. Remember that the right side will now include a ± sign.

$$\sqrt{\left(x + \frac{b}{2}\right)^2} = \sqrt{-c + \left(\frac{b}{2}\right)^2}$$

$$x + \frac{b}{2} = \pm\sqrt{-c + \left(\frac{b}{2}\right)^2}$$

$$x + \frac{b}{2} = \pm i\sqrt{c - \left(\frac{b}{2}\right)^2}$$

6. Solve for x to get the roots of the original quadratic equation, by subtracting $b/2$ from both sides.

$$x = -\frac{b}{2} \pm i\sqrt{c - \left(\frac{b}{2}\right)^2}$$

Example

Solve for x by completing the square.

$$x^2 + 4x + 7 = 0$$

There is no pair of factors of 7 whose sum is 4, so we'll need to solve by completing the square. Start by moving the constant term, 7, to the right side of the equation by subtracting 7 from both sides.

$$x^2 + 4x + 7 - 7 = 0 - 7$$

$$x^2 + 4x = -7$$

Find $(b/2)^2$. In this case, $b = 4$.

$$\left(\frac{b}{2}\right)^2 = \left(\frac{4}{2}\right)^2 = (2)^2 = 4$$

Add 4 to both sides of the equation.

$$x^2 + 4x + 4 = -7 + 4$$

$$x^2 + 4x + 4 = -3$$

Notice that $x^2 + 4x + 4$ factors as the square of a binomial, $(x+2)^2$, so our equation becomes

$$(x+2)^2 = -3$$

Take the square root of each side of the equation.

$$\sqrt{(x+2)^2} = \sqrt{-3}$$

Factor out an i on the right side, so that the -3 under the radical sign becomes 3.

$$\sqrt{(x+2)^2} = i\sqrt{3}$$

So we get

$$x+2 = \pm i\sqrt{3}$$

Solve for x by subtracting 2 from both sides. To avoid confusion, put the -2 in front of the $\pm i\sqrt{3}$.

$$x+2-2 = -2 \pm i\sqrt{3}$$

$$x = -2 \pm i\sqrt{3}$$

Let's try another example of completing the square with complex roots.

Example

Solve for the variable by completing the square.

$$2x^2 - 6x + 9 = 0$$

In this case, $a \neq 1$, so we'll first divide everything by a, which in this problem is 2.

$$\frac{2x^2}{2} - \frac{6x}{2} + \frac{9}{2} = \frac{0}{2}$$

$$x^2 - 3x + \frac{9}{2} = 0$$

Move the constant term, $9/2$, to the right side of the equation by subtracting $9/2$ from both sides.

$$x^2 - 3x + \frac{9}{2} - \frac{9}{2} = 0 - \frac{9}{2}$$

$$x^2 - 3x = -\frac{9}{2}$$

Find $(b/2)^2$. In this case, $b = -3$.

$$\left(\frac{b}{2}\right)^2 = \left(\frac{-3}{2}\right)^2 = \frac{9}{4}$$

Add $9/4$ to both sides of the equation.

$$x^2 - 3x + \frac{9}{4} = -\frac{9}{2} + \frac{9}{4}$$

$$x^2 - 3x + \frac{9}{4} = -\frac{9}{4}$$

Now we'll factor

$$x^2 - 3x + \frac{9}{4}$$

as the square of a binomial, $[x - (3/2)]^2$, so our equation becomes

$$\left(x - \frac{3}{2}\right)^2 = -\frac{9}{4}$$

Take the square root of each side of the equation.

$$\sqrt{\left(x - \frac{3}{2}\right)^2} = \sqrt{-\frac{9}{4}}$$

Factor out an i on the right side of the equation, so that the $-9/4$ under the radical sign becomes $9/4$.

$$\sqrt{\left(x - \frac{3}{2}\right)^2} = \sqrt{\frac{9}{4}}i$$

$$\sqrt{\left(x - \frac{3}{2}\right)^2} = \frac{3}{2}i$$

Therefore,

$$x - \frac{3}{2} = \pm\frac{3}{2}i$$

Solve for x by adding $3/2$ to both sides. To avoid confusion, put the $3/2$ in front of the $\pm(3/2)i$.

$$x - \frac{3}{2} + \frac{3}{2} = \frac{3}{2} \pm \frac{3}{2}i$$

$$x = \frac{3}{2} \pm \frac{3}{2}i$$

$$x = \frac{3 \pm 3i}{2}$$