

A NOTE OF TOPOLOGICAL METHODS IN GROUP THEORY

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ABSTRACT. This is a course note of 2026 Spring MATH 569, taught by Daniel Groves at UIC, on the topic of Topological Methods in Group Theory. Major reference textbooks are: [Hat01], [Bro82], [Geo08].

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1. ASPHERICAL SPACES AND SAMPLE EARLY THEOREMS

In this course we will explore the relationship between aspherical spaces and (discrete) groups. A convention: unless otherwise stated, all spaces in this course will be CW complexes.

Definition 1.1 ($K(G, 1)$). Let G be a group. A $K(G, 1)$ is a connected space such that $\pi_1(K(G, 1)) \cong G$ and whose universal covering space $\widehat{K}(G, 1)$ is contractible.

It is also called *the classifying space* or *Eilenberg-MacLane space*.

Definition 1.2 (Aspherical Space). If X is a connected space such that \widetilde{X} is contractible, then we say X is a $K(\pi, 1)$ (i.e. $K(\pi_1(X), 1)$). We call such X *aspherical*.

Theorem 1.3 ([Hat01, Example 1.B.7, Theorem 1.B.8, Proposition 1.B.9]). *For any group G , there exists a (CW) $K(G, 1)$ which is unique up to homotopy equivalence. In fact, for any two groups G, H , there exists a bijection*

$$\mathrm{Hom}(G, H) \leftrightarrow \{\text{based maps } K(G, 1) \rightarrow K(H, 1)\} / \text{based homotopy}$$

Let X be a space and x_0 a point in X . Recall that $\pi_0(X)$ represents the path components of space X . For $n \geq 1$, $\pi_n(X, x_0)$ is the group of based homotopy classes of maps $(\mathbb{S}^n, \star) \rightarrow (X, x_0)$.

For $n \geq 2$, any map $(\mathbb{S}^n, \star) \rightarrow (X, x_0)$ lifts to $(\mathbb{S}^n, \star) \rightarrow (\widetilde{X}, \widetilde{x}_0)$ (Since $\pi_1(\mathbb{S}^n, \star)$ is trivial). If X is aspherical, then $(\mathbb{S}^n, \star) \rightarrow (\widetilde{X}, \widetilde{x}_0)$ is null-homotopy. Hence $\pi_n(X, x_0) = 0$. This proves one direction of Theorem 1.4. The other direction is harder, and we leave it for future discussion.

Theorem 1.4 (Whitehead's Theorem, [Hat01, Theorem 4.5]). *A CW complex X is aspherical if and only if it's connected and for all $n \geq 2$, $\pi_n(X, x_0) = 0$.*

Example 1.5 (Aspherical Spaces). (1) A point.

(2) \mathbb{S}^1 .

(3) Products of aspherical spaces (corresponding to direct products of groups).

(4) Wedges of aspherical spaces (corresponding to free products of groups).

(5) Graphs (1-dimensional CW complexes).

(6) Closed orientable surfaces with genus ≥ 1 .

(7) Complete Riemannian manifolds of non-positive sectional curvature (Cartan-Hadamard theorem).

(8) Complete locally CAT(0) spaces (A variant of Cartan-Hadamard theorem. See for example [BH99, Theorem II.4.1]).

Example 1.6 (Non-aspherical Spaces). Connect sums of aspherical spaces typically produce non-aspherical spaces.

Question 1.7. • What kind of $K(G, 1)$ spaces does [insert any interesting group] have?

- For [insert any interesting group], which $K(G, 1)$ is “better”?
- What do properties of $K(G, 1)$ say about G ?

Below we list some sample early theorems without proof.

Theorem 1.8. *A group G is finitely generated if and only if G has a $K(G, 1)$ with a finite 1-skeleton.*

Theorem 1.9. *A group G is finitely presented if and only if G has a $K(G, 1)$ with a finite 2-skeleton.*

Theorem 1.10. *If G has non-trivial torsion, then there does not exist any finite-dimensional $K(G, 1)$.*

Another way to phrase Theorem 1.10 is: if G has a finite-dimensional $K(G, 1)$, then it's torsion free.

Observe that those theorems all related to some sort of finiteness of $K(G, 1)$. We introduce some notions of finiteness below.

Definition 1.11. A group G is of *type F_n* if it has a $K(G, 1)$ with finite n -skeleton.

A group G is of *type F_∞* if it's of type F_n for all n .

A group G is of *type F* if it has a finite $K(G, 1)$.

A group G has *finite geometric dimension* if it has a finite-dimensional $K(G, 1)$. The *geometric dimension* is the minimal dimension of all of its $K(G, 1)$.

It is sometimes difficult to understand the $K(G, 1)$ of a group. We thus use multiple invariance to help determine those types. If G is a group and X is its $K(G, 1)$, then the algebro-topological invariance of X is invariant of G . In particular, we look at the cohomology of groups (with trivial coefficients):

$$H^*(G, \mathbb{Z}) := H^*(X, \mathbb{Z})$$

2. COHOMOLOGY OF GROUPS

Definition 2.1. We say a space Y, y_0 is *n -connected* if $\pi_i(Y, y_0) = 0$ for all $0 \leq i \leq n$.

Other than the trivial coefficient \mathbb{Z} , we use also group rings to calculate the cohomology of groups to get different properties.

Definition 2.2. Let G be a group. The group ring $\mathbb{Z}G$ is the set of finite \mathbb{Z} -linear combinations of elements of G .

Remark 2.3. We may also use \mathbb{R} , fields, or PIDs, etc. instead of \mathbb{Z} to get different group rings.

Example 2.4. Let $G = \mathbb{Z} = \langle t \rangle$. Then $\mathbb{Z}G = \mathbb{Z}[t, t^{-1}]$.

Example 2.5. Let $G = \mathbb{Z}/n\mathbb{Z} = \langle t \rangle$. Then $\mathbb{Z}G = \mathbb{Z}[t]/(t^n - 1)$.

Notice that $g^n = 1$ for some non-trivial element $g \in G$, i.e. G has a non-trivial torsion, then we have:

$$0 = g^n - 1 = (g - 1)(g^{n-1} + \cdots + g + 1)$$

which implies that $g - 1$ is a non-zero zero-divisor. The other direction of this fact is an open question.

Question 2.6 (Kaplansky's Conjecture). If G is torsion-free, then $\mathbb{Z}G$ has no non-zero zero-divisor.

A motivation for Question 2.6 is that we may embed $\mathbb{Z}G$ into its division ring, so we wish to have no non-zero zero-divisors in $\mathbb{Z}G$.

Typically G is not abelian, so $\mathbb{Z}G$ is not a commutative ring. Left and right $\mathbb{Z}G$ -modules are typically different. By default, we use left $\mathbb{Z}G$ -modules.

Remark 2.7. If M is a left $\mathbb{Z}G$ -module, it can be turned into a right $\mathbb{Z}G$ -module by defining a right G -action $mg = g^{-1}m$.

Definition 2.8. A free resolution of \mathbb{Z} (the trivial $\mathbb{Z}G$ -module) by free $\mathbb{Z}G$ -module is a resolution

$$\cdots \rightarrow F_1 \rightarrow F_0 \rightarrow \mathbb{Z} \rightarrow 0$$

that is exact, and every F_i is a free $\mathbb{Z}G$ -module (i.e. $(\mathbb{Z}G)^n$ for some n).

We sometimes use also resolutions by projective $\mathbb{Z}G$ -modules.

Remark 2.9. The free resolution arises naturally from $K(G, 1)$ in the following manner. Say X is a CW complex $K(G, 1)$ (recall Definition 1.1), then any i -cell in \tilde{X} is $g\sigma$ for some $g \in G$ and σ an i -cell in X . So we can view the cellular i -chain $C_i(\tilde{X}; \mathbb{Z})$ as a free $\mathbb{Z}G$ -module with basis in bijective with i -cells in X .

Since \tilde{X} is contractible, $H_*(\tilde{X}) = 0$. That is, the complex

$$\cdots \rightarrow C_{n+1}(\tilde{X}) \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} \cdots \xrightarrow{\partial_1} C_0(\tilde{X}) \xrightarrow{\epsilon} \mathbb{Z} \rightarrow 0$$

is exact. The boundary maps are $\mathbb{Z}G$ -module homomorphisms. This gives a free resolution of \mathbb{Z} by free $\mathbb{Z}G$ -modules.

TODO picture example

Theorem 2.10. *There is a unique resolution of \mathbb{Z} by free (projective) $\mathbb{Z}G$ -module.*

Definition 2.11. Let M be a left $\mathbb{Z}G$ -module. Let $F = \{(F_i, d_i)\}$ be a free resolution of \mathbb{Z} . We truncate F to get a (non-exact) complex:

$$\cdots \rightarrow F_i \xrightarrow{d_i} F_{i-1} \xrightarrow{d_{i-1}} \cdots \xrightarrow{d_1} F_0 \rightarrow 0.$$

Apply $\text{Hom}_{\mathbb{Z}G}(-, M)$ to the complex. Denote $F^i = F_i^* = \text{Hom}_{\mathbb{Z}G}(F_i, M)$. We have:

$$\cdots \leftarrow F^i \xleftarrow{d^{i-1}} F^{i-1} \xleftarrow{d^{i-2}} \cdots \xleftarrow{d^0} F^0 \leftarrow 0.$$

The *cohomology* of G with coefficients in M , denoted as $H^*(G; M)$, is the homology of the complex above:

$$H^i(G; M) = \ker d^i / \text{im } d^{i-1}.$$

If we do not truncate the resolution, then we get the exact sequence:

$$\cdots \leftarrow \text{Hom}_{\mathbb{Z}G}(F_1, M) \xleftarrow{d^0=d_1^*} \text{Hom}_{\mathbb{Z}G}(F_0, M) \xleftarrow{\epsilon^*} \text{Hom}_{\mathbb{Z}G}(\mathbb{Z}, M) \xleftarrow{0}.$$

Denote $H^0(G; M)$ as M^G . Notice that $M^G = \ker d^0 = \text{im } \epsilon^* \cong \text{Hom}_{\mathbb{Z}G}(\mathbb{Z}, M)$. Recall that G acts on \mathbb{Z} trivially, so elements in $\text{Hom}_{\mathbb{Z}G}(\mathbb{Z}, M)$ has to be elements in M that are invariant under G , i.e. $M^G = \{m \in M \mid \forall g \in G, gm = m\}$.

Since free resolutions are chain homotopic, $H^*(G; M)$ is independent of the choices of the free resolution.

Example 2.12. Let $G = \mathbb{Z}/n\mathbb{Z}$, $M = \mathbb{Z}$. Then $\mathbb{Z}G = \mathbb{Z}[t]/(t^n - 1)$. Consider $\mathbb{Z}G \xrightarrow{\epsilon} \mathbb{Z} \rightarrow 0$, $\ker \epsilon = (t - 1)$. Let $N = t^{n-1} + \cdots + t + 1$. We have the free resolution:

$$\cdots \rightarrow \mathbb{Z}G \xrightarrow{\cdot N} \mathbb{Z}G \xrightarrow{\cdot (t-1)} \mathbb{Z}G \xrightarrow{\cdot N} \mathbb{Z}G \xrightarrow{\cdot (t-1)} \mathbb{Z}G \xrightarrow{\epsilon} \mathbb{Z} \rightarrow 0.$$

Apply $\text{Hom}_{\mathbb{Z}G}(-, \mathbb{Z})$ to the resolution. For any $f \in \text{Hom}_{\mathbb{Z}G}(\mathbb{Z}G, \mathbb{Z})$, $f(\sum_i \alpha_i t^i)$ has to be $k \sum_i \alpha_i$ where $k \in \mathbb{Z}$, i.e. $\text{Hom}_{\mathbb{Z}G}(\mathbb{Z}G, \mathbb{Z}) = \mathbb{Z}$. Thus the cochain complex we obtained is:

$$\cdots \xleftarrow{\cdot n} \mathbb{Z} \xleftarrow{\cdot 0} \mathbb{Z} \xleftarrow{\cdot n} \mathbb{Z} \xleftarrow{\cdot 0} \mathbb{Z} \leftarrow 0.$$

We conclude that

$$H^i(\mathbb{Z}/n\mathbb{Z}; \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } i = 0, \\ \mathbb{Z}/n\mathbb{Z} & \text{if } i \geq 1 \text{ is odd,} \\ 0 & \text{if } i \geq 1 \text{ is even.} \end{cases}$$

Proposition 2.13. *If X is an n -dimensional $K(G, 1)$, M is any $\mathbb{Z}G$ -module, then $\forall m > n$, $H^m(G; M) = 0$.*

That's because the free resolution is trivial for $m > n$.

Corollary 2.14. *There does not exist any finite-dimensional $K(\mathbb{Z}/n\mathbb{Z}, 1)$ for $n \geq 2$.*

We are now ready to prove Theorem 1.10. Recall the statement:

Theorem 1.10. *If G has non-trivial torsion, then there does not exist any finite-dimensional $K(G, 1)$.*

Proof. Let X be a $K(G, 1)$. For any subgroup $H \leq G$, let X^H be the cover of X corresponding to H . Then X^H is a $K(H, 1)$. Observe that X and X^H have the same dimension.

If G has non-trivial torsion, then there exists $\mathbb{Z}/n\mathbb{Z} \leq G$ with $n \geq 2$. By Corollary 2.14, X^H is infinite-dimensional, hence so is X . \square

3. COMPACTLY SUPPORTED CELLULAR COHOMOLOGY

In this course we focus on $\mathbb{Z}G$ -module with trivial coefficient. For coefficients other than \mathbb{Z} , a key phrase is “local systems of coefficients on X ”.

Theorem 3.1. *If X is a $K(G, 1)$, then for all $i \geq 0$, $H^i(G; \mathbb{Z}) \cong H^i(X; \mathbb{Z})$.*

Proof. Recall that X gives a free $\mathbb{Z}g$ resolution of \mathbb{Z} (See Remark 2.9):

$$\cdots \rightarrow C_1(\tilde{X}) \xrightarrow{d_1} C_0(\tilde{X}) \xrightarrow{\epsilon} \mathbb{Z} \rightarrow 0.$$

Apply $\text{Hom}_{\mathbb{Z}G}(-, \mathbb{Z})$ to the resolution and consider $\text{Hom}_{\mathbb{Z}G}(C_i(\tilde{X}), \mathbb{Z})$. Recall that $\mathbb{Z}G$ homomorphism has to be invariant under G , so the map has to assign a same number to each i -cell in the same G -orbit. It is naturally equivalent to assign numbers to i -cells in X i.e. cellular chains in X . \square

Definition 3.2. Let M be a $\mathbb{Z}G$ -module. Define $\text{Hom}_c(M, \mathbb{Z}) \subseteq \text{Hom}(M, \mathbb{Z})$ to be

$$\{f : M \rightarrow \mathbb{Z} \mid \forall m \in M, \text{ for all but finitely many } g \in G, f(gm) = 0\}.$$

Lemma 3.3 ([Bro82, Lemma VIII.7.4]). *Let Γ be a $\mathbb{Z}G$ -module, then*

$$\text{Hom}_{\mathbb{Z}G}(M, \mathbb{Z}G) \cong \text{Hom}_c(M, \mathbb{Z}).$$

Proof. Take $F \in \text{Hom}_{\mathbb{Z}G}(M, \mathbb{Z}G)$. For $m \in M$, $F(m)$ can be written as $\sum_{g \in G} f_g(m)g$ where $f_g(m) = 0$ for all but finitely many g .

Verify that $F \in \text{Hom}_{\mathbb{Z}G}(M, \mathbb{Z}G)$ if and only if $f_g(m) = f_1(g^{-1}m)$. So $F \mapsto f_1$ gives a map whose inverse is:

$$f \mapsto \{m \mapsto \sum_{g \in G} f(g^{-1}m)g\}.$$

\square

Definition 3.4. Let Y be a locally compact CW complex. That is for all n -cell σ , there are finitely many $n+1$ cells whose boundary meets σ . Let $C_c^i(Y; \mathbb{Z}) \subseteq C^i(Y; \mathbb{Z})$ be the set of finitely supported cellular cochains with

$$\delta_c^i : C_c^i(Y; \mathbb{Z}) \rightarrow C_c^{i+1}(Y; \mathbb{Z}).$$

The homology of this chain complex, denoted as $H_c^i(Y; \mathbb{Z})$, is the *compactly supported cohomology*.

Remark 3.5. The requirement of locally compactness is to ensure that finitely supported cochains and compactly supported cochains are the same set.

Theorem 3.6. *If X is a finite $K(G, 1)$, then*

$$H^i(G; \mathbb{Z}G) \cong H_c^i(\tilde{X}; \mathbb{Z}).$$

Proof. Recall that by Lemma 3.3,

$$\text{Hom}_{\mathbb{Z}G}(C_i(\tilde{X}), \mathbb{Z}G) \cong \text{Hom}_c(C_i(\tilde{X}), \mathbb{Z}) = C_c^i(\tilde{X}; \mathbb{Z}).$$

So they induce the same cohomology (See Definitions 2.11 and 3.4). \square

Remark 3.7. A more natural definition of compactly supported cohomology arises from the direct limit of relative cohomology groups. Observe that a finitely supported i -cochain is 0 outside some compact set K , so it lies in $C^i(Y, Y \setminus K; \mathbb{Z})$. We define:

$$H_c^*(Y; \mathbb{Z}) = \varinjlim_{K \text{ compact}} H^*(Y, Y \setminus K; \mathbb{Z}).$$

Example 3.8 (Compactly supported cohomology of \mathbb{R}). Observe that for a connected infinite CW complex Y , $H_c^0(Y; \mathbb{Z}) = \ker \delta_c^0$, which is the set of finitely supported 1-cochains that are constant. So it has to be trivial. (It would be \mathbb{Z} if Y is finite.)

We calculate the compactly supported cohomology for \mathbb{R} . It's cellulated as an infinite graph, which is contractible, so $H_c^0(\mathbb{R}; \mathbb{Z}) = 0$. For $n > 2$, since there are no n -cells, $H_c^n(\mathbb{R}; \mathbb{Z}) = 0$. Consider now $H_c^1(\mathbb{R}; \mathbb{Z})$. Since there are no 2-cells, all 1-cochains are 1-cocycles. Therefore $H_c^1(\mathbb{R}; \mathbb{Z}) = C_c^1(\mathbb{R}; \mathbb{Z}) / \text{im } \delta_c^0$.

Consider $\text{im } \delta_c^0$. Recall that $\delta_c^0(\sigma)(e) = \sigma(v_o) - \sigma(v_t)$ for any edge v with endpoints v_o, v_t . Observe that each vertex serves as the starting endpoint of one edge and the terminal endpoint of another edge. So $\sum_e \delta_c^0(\sigma)e = 0$, i.e. $\text{im } \delta_c^0 = 0$. It follows that $H_c^1(\mathbb{R}; \mathbb{Z}) \cong C_c^1(\mathbb{R}; \mathbb{Z})$.

Consider now the homomorphisms $C_c^1(\mathbb{R}; \mathbb{Z}) \rightarrow \mathbb{Z}$, they have the form:

$$\sum_i m_i e_i^* \mapsto \sum_i m_i,$$

where $e_i^*(e_i) = 0$ and it sends everything else to 0.

We need to show that if $\sum_i m_i e_i^*$ has $\sum_i m_i = 0$ then $\sum_i m_i e_i^*$ is a coboundary. The idea is one can modify a cochain by a coboundary consecutively until one gets something supported on a single edge whose coefficient sum is zero. So it's a zero coboundary. For example, applying $\delta(v_i^*)$ to $e_1 + e_2$ gives $(1 - 1)e_1 + 0e_2$.

TODO not quite so sure

Corollary 3.9.

$$H_c^i(\mathbb{Z}; \mathbb{Z}[\mathbb{Z}]) = \begin{cases} \mathbb{Z} & i = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Example 3.10 (Compactly supported cohomology of infinite 4-valence tree). Consider the infinite 4-valence tree T (the universal cover of the wedge of two circles). Similarly, $H_c^i(T; \mathbb{Z}) = 0$ when $i \neq 1$. For $H_c^1(T; \mathbb{Z})$, by Remark 3.7 it is the direct limit of $H^1(T, T \setminus K; \mathbb{Z})$ over all compact sets. Consider the long exact sequence of reduced cohomology groups:

$$\tilde{H}^0(T; \mathbb{Z}) \rightarrow \tilde{H}^0(T \setminus K; \mathbb{Z}) \rightarrow \tilde{H}^1(T, T \setminus K; \mathbb{Z}) \rightarrow \tilde{H}^1(T; \mathbb{Z}).$$

Since T is contractible, $\tilde{H}^i(T; \mathbb{Z}) = 0$, hence $\tilde{H}^0(T \setminus K; \mathbb{Z}) \cong \tilde{H}^1(T, T \setminus K; \mathbb{Z})$. Now $\tilde{H}^0(T \setminus K; \mathbb{Z})$ counts the components of $T \setminus K$, which increases infinitely as the size of K increases.

We conclude that $\tilde{H}^1(T, T \setminus K; \mathbb{Z})$ is infinitely generated. In fact, it's an infinitely generated free abelian group.

Corollary 3.11.

$$H_c^i(\mathbb{F}_2; \mathbb{Z}[\mathbb{F}_2]) = \begin{cases} D & i = 1, \\ 0 & \text{otherwise,} \end{cases}$$

where D is infinitely generated.

Observe that the argument of Example 3.10 works for any contractible space. That is, in general, 1-cohomology of a group with group-ring coefficients has rank of numbers of ends of the group plus 1.

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