A NOTE OF TOPOLOGICAL METHODS IN GROUP THEORY

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Abstract. This is a course note of 2026 Spring MATH 569, taught by Daniel Groves at UIC, on the topic of Topological Methods in Group Theory. Major reference textbooks are: [Hat01], [Bro82], [Geo08].

Contents

1. Aspherical Spaces and Sample Early Theorems

In this course we will explore the relationship between aspherical spaces and (discrete) groups. A convention: unless otherwise stated, all spaces in this course will be CW complexes.

Definition 1.1 ($K(G,1)$)**.** Let G be a group. A $K(G,1)$ is a connected space such that $\pi_1(K(G,1)) \cong G$ and whose universal covering space $\widehat{K}(G,1)$ is contractible.

It is also called *the classifying space* or *Eilenberg-Maclane space*.

Definition 1.2 (Aspherical Space)**.** If X is a connected space such that $\widetilde{X}$ is contractible, then we say X is a $K(\pi,1)$ (i.e. $K(\pi_1(X),1)$). We call such X *aspherical*.

Theorem 1.3 ([Hat01, Example 1.B.7, Theorem 1.B.8, Proposition 1.B.9])**.** *For any group G, there exists a (CW) $K(G,1)$ which is unique up to homotopy equivalence. In fact, for any two groups G, H, there exists a bijection*

$$\mathrm{Hom}(G,H) \leftrightarrow \{\textit{based maps } K(G,1) \to K(H,1)\}/\textit{based homotopy}$$

Let X be a space and x_0 a point in X. Recall that $\pi_0(X)$ represents the path components of space X. For $n \geq 1$, $\pi_n(X, x_0)$ is the group of based homotopy classes of maps $(\mathbb{S}^n, \star) \to (X, x_0)$.

For $n \geq 2$, any map $(\mathbb{S}^n, \star) \to (X, x_0)$ lifts to $(\mathbb{S}^n, \star) \to (\widetilde{X}, \widetilde{x}_0)$ (Since $\pi_1(\mathbb{S}^n, \star)$ is trivial). If X is aspherical, then $(\mathbb{S}^n, \star) \to (\widetilde{X}, \widetilde{x}_0)$ is null-homotopy. Hence $\pi_n(X, x_0) = 0$. This proves one direction of Theorem 1.4. The other direction is harder, and we leave it for future discussion.

Theorem 1.4 (Whitehead's Theorem, [Hat01, Theorem 4.5])**.** *A CW complex X is aspherical if and only if it's connected and for all $n \geq 2$, $\pi_n(X, x_0) = 0$.*

Example 1.5 (Aspherical Spaces)**.**
(1) A point.
(2) $\mathbb{S}^1$.
(3) Products of aspherical spaces (corresponding to direct products of groups).
(4) Wedges of aspherical spaces (corresponding to free products of groups).
(5) Graphs (1-dimensional CW complexes).
(6) Closed orientable surfaces with genus ≥ 1.
(7) Complete Riemannian manifolds of non-positive sectional curvature (Cartan-Hadamard theorem).
(8) Complete locally CAT(0) spaces (A variant of Cartan-Hadamard theorem. See for example [BH99, Theorem II.4.1]).

Example 1.6 (Non-aspherical Spaces)**.** Connect sums of aspherical spaces typically produce non-aspherical spaces.

Question 1.7.
- What kind of $K(G,1)$ spaces does [insert any interesting group] have?
- For [insert any interesting group], which $K(G,1)$ is "better"?
- What do properties of $K(G,1)$ say about G?

Below we list some sample early theorems without proof.

Theorem 1.8. *A group G is finitely generated if and only if G has a $K(G,1)$ with a finite 1-skeleton.*

Theorem 1.9. *A group G is finitely presented if and only if G has a $K(G,1)$ with a finite 2-skeleton.*

Theorem 1.10. *If G has non-trivial torsion, then there does not exist any finite-dimensional $K(G,1)$.*

Another way to phrase Theorem 1.10 is: if G has a finite-dimensional $K(G,1)$, then it's torsion free.

Observe that those theorems all related to some sort of finiteness of $K(G,1)$. We introduce some notions of finiteness below.

Definition 1.11. A group G is of *type* F_n if it has a $K(G,1)$ with finite n-skeleton.

A group G is of *type* F_∞ if it's of type F_n for all n.

A group G is of *type* F if it has a finite $K(G,1)$.

A group G has *finite geometric dimension* if it has a finite-dimensional $K(G,1)$. The *geometric dimension* is the minimal dimension of all of its $K(G,1)$.

It is sometimes difficult to understand the $K(G,1)$ of a group. We thus use multiple invariance to help determine those types. If G is a group and X is its $K(G,1)$, then the algebro-topological invariance of X is invariant of G. In particular, we look at the cohomology of groups (with trivial coefficients):

$$H^*(G,\mathbb{Z}) := H^*(X,\mathbb{Z})$$

2. Cohomology of Groups

Definition 2.1. We say a space Y, y_0 is *n-connected* if $\pi_i(Y, y_0) = 0$ for all $0 \leq i \leq n$.

Other than the trivial coefficient $\mathbb{Z}$, we use also group rings to calculate the cohomology of groups to get different properties.

Definition 2.2. Let G be a group. The group ring $\mathbb{Z}G$ is the set of finite Z-linear combinations of elements of G.

Remark 2.3. We may also use $\mathbb{R}$, fields, or PIDs, etc. instead of $\mathbb{Z}$ to get different group rings.

Example 2.4. Let $G = \mathbb{Z} = \langle t \rangle$. Then $\mathbb{Z}G = \mathbb{Z}[t, t^{-1}]$.

Example 2.5. Let $G = \mathbb{Z}/n\mathbb{Z} = \langle t \rangle$. Then $\mathbb{Z}G = \mathbb{Z}[t]/(t^n - 1)$.

Notice that $g^n = 1$ for some non-trivial element $g \in G$, i.e. G has a non-trivial torsion, then we have:

$$0 = g^n - 1 = (g-1)(g^{n-1} + \cdots + g + 1)$$

which implies that $g - 1$ is a non-zero zero-divisor. The other direction of this fact is an open question.

Question 2.6 (Kaplansky's Conjecture)**.** If G is torsion-free, then $\mathbb{Z}G$ has no non-zero zero-divisor.

A motivation for Question 2.6 is that we may embed $\mathbb{Z}G$ into its division ring, so we wish to have no non-zero zero-divisors in $\mathbb{Z}G$.

Typically G is not abelian, so $\mathbb{Z}G$is not a commutative ring. Left and right $\mathbb{Z}G$-modules are typically different. By default, we use left $\mathbb{Z}G$-modules.

Remark 2.7. If M is a left $\mathbb{Z}G$-module, it can be turned into a right $\mathbb{Z}G$-module by defining a right G-action $mg = g^{-1}m$.

Definition 2.8. A free resolution of $\mathbb{Z}$ (the trivial $\mathbb{Z}G$-module) by free $\mathbb{Z}G$-module is a resolution

$$\cdots \to F_1 \to F_0 \to \mathbb{Z} \to 0$$

that is exact, and every F_i is a free $\mathbb{Z}G$-module (i.e. $(\mathbb{Z}G)^n$ for some n).

We sometimes use also resolutions by projective $\mathbb{Z}G$-modules.

Remark 2.9. The free resolution arises naturally from $K(G, 1)$ in the following manner. Say X is a CW complex $K(G, 1)$ (recall Definition 1.1), then any i-cell in $\widetilde{X}$ is $g\sigma$ for some $g \in G$ and σ an i-cell in X. So we can view the cellular i-chain $C_i(\widetilde{X}; \mathbb{Z})$ as a free $\mathbb{Z}G$-module with basis in bijective with i-cells in X.

Since $\widetilde{X}$ is contractible, $H_*(\widetilde{X}) = 0$. That is, the complex

$$\cdots \to C_{n+1}(\widetilde{X}) \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} \cdots \xrightarrow{\partial_1} C_0(\widetilde{X}) \xrightarrow{\epsilon} \mathbb{Z} \to 0$$

is exact. The boundary maps are $\mathbb{Z}G$-module homomorphisms. This gives a free resolution of $\mathbb{Z}$ by free $\mathbb{Z}G$-modules.

TODO picture example

Theorem 2.10. *There is a unique resolution of $\mathbb{Z}$ by free (projective) $\mathbb{Z}G$-module.*

Definition 2.11. Let M be a left $\mathbb{Z}G$-module. Let $F = \{(F_i, d_i)\}$ be a free resolution of $\mathbb{Z}$. We truncate F to get a (non-exact) complex:

$$\cdots \to F_i \xrightarrow{d_i} F_{i-1} \xrightarrow{d_{i-1}} \cdots \xrightarrow{d_1} F_0 \to 0.$$

Apply $\mathrm{Hom}_{\mathbb{Z}G}(\_, M)$ to the complex. Denote $F^i = F_i^* = \mathrm{Hom}_{\mathbb{Z}G}(F_i, M)$. We have:

$$\cdots \leftarrow F^i \xleftarrow{d^{i-1}} F^{i-1} \xleftarrow{d^{i-2}} \cdots \xleftarrow{d^0} F^0 \leftarrow 0.$$

The *cohomology of* G with coefficients in M, denoted as $H^*(G; M)$, is the homology of the complex above:

$$H^i(G; M) = \ker d^i / \operatorname{im} d^{i-1}.$$

If we do not truncate the resolution, then we get the exact sequence:

$$\cdots \leftarrow \mathrm{Hom}_{\mathbb{Z}G}(F_1, M) \xleftarrow{d^0 = d_1^*} \mathrm{Hom}_{\mathbb{Z}G}(F_0, M) \xleftarrow{\epsilon^*} \mathrm{Hom}_{\mathbb{Z}G}(\mathbb{Z}, M) \xleftarrow{0} .$$

Denote $H^0(G; M)$ as M^G. Notice that $M^G = \ker d^0 = \operatorname{im} \epsilon^* \cong \mathrm{Hom}_{\mathbb{Z}G}(\mathbb{Z}, M)$. Recall that G acts on $\mathbb{Z}$ trivially, so elements in $\mathrm{Hom}_{\mathbb{Z}G}(\mathbb{Z}, M)$ has to be elements in M that are invariant under G, i.e. $M^G = \{m \in M \mid \forall g \in G, gm = m\}$.

Since free resolutions are chain homotopic, $H^*(G; M)$ is independent of the choices of the free resolution.

Example 2.12. Let $G = \mathbb{Z}/n\mathbb{Z}$, $M = \mathbb{Z}$. Then $\mathbb{Z}G = \mathbb{Z}[t]/(t^n - 1)$. Consider $\mathbb{Z}G \xrightarrow{\epsilon} \mathbb{Z} \to 0$, $\ker \epsilon = (t-1)$. Let $N = t^{n-1} + \cdots + t + 1$. We have the free resolution:

$$\cdots \to \mathbb{Z}G \xrightarrow{\cdot N} \mathbb{Z}G \xrightarrow{\cdot (t-1)} \mathbb{Z}G \xrightarrow{\cdot N} \mathbb{Z}G \xrightarrow{\cdot (t-1)} \mathbb{Z}G \xrightarrow{\epsilon} \mathbb{Z} \to 0.$$

Apply $\mathrm{Hom}_{\mathbb{Z}G}(_, \mathbb{Z})$to the resolution. For any $f \in \mathrm{Hom}_{\mathbb{Z}G}(\mathbb{Z}G, \mathbb{Z})$, $f(\sum_i \alpha_i t^i)$ has to be $k \sum_i \alpha_i$ where $k \in \mathbb{Z}$, i.e. $\mathrm{Hom}_{\mathbb{Z}G}(\mathbb{Z}G, \mathbb{Z}) = \mathbb{Z}$. Thus the cochain complex we obtained is:

$$\cdots \xleftarrow{\cdot n} \mathbb{Z} \xleftarrow{\cdot 0} \mathbb{Z} \xleftarrow{\cdot n} \mathbb{Z} \xleftarrow{\cdot 0} \mathbb{Z} \leftarrow 0.$$

We conclude that

$$H^i(\mathbb{Z}/n\mathbb{Z}; \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } i = 0, \\ \mathbb{Z}/n\mathbb{Z} & \text{if } i \geq 1 \text{ is odd}, \\ 0 & \text{if } i \geq 1 \text{ is even}. \end{cases}$$

Proposition 2.13. *If X is an n-dimensional $K(G, 1)$, M is any $\mathbb{Z}G$-module, then $\forall m > n$, $H^m(G; M) = 0$.*

That's because the free resolution is trivial for $m > n$.

Corollary 2.14. *There does not exist any finite-dimensional $K(\mathbb{Z}/n\mathbb{Z}, 1)$ for $n \geq 2$.*

We are now ready to prove Theorem 1.10. Recall the statement:

Theorem 1.10. *If G has non-trivial torsion, then there does not exist any finite-dimensional $K(G, 1)$.*

Proof. Let X be a $K(G,1)$. For any subgroup $H \leq G$, let X^H be the cover of X corresponding to H. Then X^H is a $K(H,1)$. Observe that X and X^H have the same dimension.

If G has non-trivial torsion, then there exists $\mathbb{Z}/n\mathbb{Z} \leq G$ with $n \geq 2$. By Corollary 2.14, X^H is infinite-dimensional, hence so is X. □

3. Compactly supported cellular cohomology

In this course we focus on $\mathbb{Z}G$-module with trivial coefficient. For coefficients other than $\mathbb{Z}$, a key phrase is "local systems of coefficients on X".

Theorem 3.1. *If X is a $K(G,1)$, then for all $i \geq 0$, $H^i(G;\mathbb{Z}) \cong H^i(X;\mathbb{Z})$.*

Proof. Recall that X gives a free $\mathbb{Z}g$ resolution of Z (See Remark 2.9):

$$\cdots \to C_1(\widetilde{X}) \xrightarrow{d_1} C_0(\widetilde{X}) \xrightarrow{\epsilon} \mathbb{Z} \to 0.$$

Apply $\mathrm{Hom}_{\mathbb{Z}G}(\_,\mathbb{Z})$ to the resolution and consider $\mathrm{Hom}_{\mathbb{Z}G}(C_i(\widetilde{X}),\mathbb{Z})$. Recall that $\mathbb{Z}G$ homomorphism has to be invariant under G, so the map has to assign a same number to each i-cell in the same G-orbit. It is naturally equivalent to assign numbers to i-cells in X i.e. cellular chains in X. □

Definition 3.2. Let M be a $\mathbb{Z}G$-module. Define $\mathrm{Hom}_c(M,\mathbb{Z}) \subseteq \mathrm{Hom}(M,\mathbb{Z})$ to be

$$\{f : M \to \mathbb{Z} \mid \forall m \in M, \text{ for all but finitely many } g \in G, f(gm) = 0\}.$$

Lemma 3.3 ([Bro82, Lemma VIII.7.4])**.** *Let Γ be a $\mathbb{Z}G$-module, then*

$$\mathrm{Hom}_{\mathbb{Z}G}(M,\mathbb{Z}G) \cong \mathrm{Hom}_c(M,\mathbb{Z}).$$

Proof. Take $F \in \mathrm{Hom}_{\mathbb{Z}G}(M,\mathbb{Z}G)$. For $m \in M$, $F(m)$ can be written as $\sum_{g\in G} f_g(m)g$ where $f_g(m) = 0$ for all but finitely many g.

Verify that $F \in \mathrm{Hom}_{\mathbb{Z}G}(M,\mathbb{Z}G)$ if and only if $f_g(m) = f_1(g^{-1}m)$. So $F \mapsto f_1$ gives a map whose inverse is:

$$f \mapsto \{m \mapsto \sum_{g\in G} f(g^{-1}m)g\}.$$

□

Definition 3.4. Let Y be a locally compact CW complex. That is for all n-cell σ, there are finitely many $n+1$ cells whose boundary meets σ. Let $C^i_c(Y;\mathbb{Z}) \subseteq C^i(Y;\mathbb{Z})$ be the set of finitely supported cellular cochains with

$$\delta^i_c : C^i_c(Y;\mathbb{Z}) \to C^{i+1}_c(Y;\mathbb{Z}).$$

The homology of this chain complex, denoted as $H^i_c(Y;\mathbb{Z})$, is the *compactly supported cohomology.*

Remark 3.5. The requirement of locally compactness is to ensure that finitely supported cochains and compactly supported cochains are the same set.

Theorem 3.6. *If X is a finite $K(G,1)$, then*

$$H^i(G;\mathbb{Z}G) \cong H^i_c(\widetilde{X};\mathbb{Z}).$$

Proof. Recall that by Lemma 3.3,

$$\mathrm{Hom}_{\mathbb{Z}G}(C_i(\widetilde{X}),\mathbb{Z}G) \cong \mathrm{Hom}_c(C_i(\widetilde{X}),\mathbb{Z}) = C^i_c(\widetilde{X};\mathbb{Z}).$$

So they induce they same cohomology (See Definitions 2.11 and 3.4). □

Remark 3.7. A more natural definition of compactly supported cohomology arises from the direct limit of relative cohomology groups. Observe that a finitely supported i-cochain is 0 outside some compact set K, so it lies in $C^i(Y, Y\backslash K;\mathbb{Z})$. We define:

$$H^*_c(Y;\mathbb{Z}) = \varinjlim_{K \text{ compact}} H^*(Y, Y\backslash K;\mathbb{Z}).$$

Example 3.8 (Compactly supported cohomology of $\mathbb{R}$)**.** Observe that for a connected infinite CW complex Y, $H^0_c(Y;\mathbb{Z}) = \ker \delta^0_c$, which is the set of finitely supported 1-cochains that are constant. So it has to be trivial. (It would be $\mathbb{Z}$ if Y is finite.)

We calculate the compactly supported cohomology for $\mathbb{R}$. It's cellulated as an infinite graph, which is contractible, so $H^0_c(\mathbb{R};\mathbb{Z}) = 0$. For $n > 2$, since there are no n-cells, $H^n_c(\mathbb{R};\mathbb{Z}) = 0$. Consider now $H^1_c(\mathbb{R};\mathbb{Z})$. Since there are no 2-cells, all 1-cochains are 1-cocycles. Therefore $H^1_c(\mathbb{R};\mathbb{Z}) = C^1_c(\mathbb{R};\mathbb{Z})/\operatorname{im}\delta^0_c$.

Consider $\operatorname{im}\delta^0_c$. Recall that $\delta^0_c(\sigma)(e) = \sigma(v_o) - \sigma(v_t)$ for any edge v with endpoints v_o, v_t. Observe that each vertex serves as the starting endpoint of one edge and the terminal endpoint of another edge. So $sum_e \delta^0_c(\sigma)e = 0$, i.e. $\operatorname{im}\delta^0_c = 0$. It follows that $H^1_c(\mathbb{R};\mathbb{Z}) \cong C^1_c(\mathbb{R};\mathbb{Z})$.

Consider now the homomorphisms $C^1_c(\mathbb{R};\mathbb{Z}) \to \mathbb{Z}$, they have the form:

$$\sum_i m_i e_i^* \mapsto \sum_i m_i,$$

where $e_i^*(e_i) = 0$ and it sends everything else to 0.

We need to show that if $\sum_i m_i e_i^*$ has $\sum_i m_i = 0$ then $\sum_i m_i e_i^*$ is a coboundary. The idea is one can modify a cochain by a coboundary consecutively until one get something supported on a single edge whose coefficient sum is zero. So it's a zero coboundary. For example, applying $\delta(v_i^*)$ to $e_1 + e_2$ gives $(1-1)e_1 + 0e_2$.

TODO not quite so sure

Corollary 3.9.

$$H_c^i(\mathbb{Z};\mathbb{Z}[\mathbb{Z}]) = \begin{cases} \mathbb{Z} & i = 1, \\ 0 & \textit{otherwise}. \end{cases}$$

Example 3.10 (Compactly supported cohomology of $\mathbb{F}_2$)**.** Consider the infinite 4-valence tree T (the universal cover of the wedge of two circles). Similarly, $H_c^i(T;\mathbb{Z}) = 0$ when $i \neq 1$. For $H_c^1(T;\mathbb{Z})$, by Remark 3.7 it is the direct limit of $H^1(T, T\backslash K;\mathbb{Z})$ over all compact sets. Consider the long exact sequence of reduced cohomology groups:

$$\widetilde{H}^0(T;\mathbb{Z}) \to \widetilde{H}^0(T\backslash K;\mathbb{Z}) \to \widetilde{H}^1(T, T\backslash K;\mathbb{Z}) \to \widetilde{H}^1(T;\mathbb{Z}).$$

Since T is contractible, $\widetilde{H}^i(T;\mathbb{Z}) = 0$, hence $\widetilde{H}^0(T\backslash K;\mathbb{Z}) \cong \widetilde{H}^1(T, T\backslash K;\mathbb{Z})$. Now $\widetilde{H}^0(T\backslash K;\mathbb{Z})$ counts the components of $T\backslash K$, which increases infinitely as the size of K increases.

We conclude that $\widetilde{H}^1(T, T\backslash K;\mathbb{Z})$ is infinitely generated. In fact, it's an infinitely generated free abelian group.

Corollary 3.11.

$$H_c^i(\mathbb{F}_2;\mathbb{Z}[\mathbb{F}_2]) = \begin{cases} D & i = 1, \\ 0 & \textit{otherwise}, \end{cases}$$

where D is infinitely generated.

The example above shows that 1-cohomology of the group with group-ring coefficients has rank of numbers of ends of the group plus 1.

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