A NOTE OF TOPOLOGICAL METHODS IN GROUP THEORY

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Abstract. This is a course note of 2026 Spring MATH 569, taught by Daniel Groves at UIC, on the topic of Topological Methods in Group Theory. Reference textbooks are: Topological Methods in Group Theory, Cohomology of Groups.

Contents

1. Aspherical Spaces and Sample Early Theorems

In this course we will explore the relationship between aspherical spaces and (discrete) groups. A convention: unless otherwise stated, all spaces in this course will be CW complexes.

Definition 1.1 ($K(G,1)$)**.** Let G be a group. A $K(G,1)$ is a connected space such that $\pi_1(K(G,1)) \cong G$ and whose universal covering space $\widehat{K}(G,1)$ is contractible.

It is also called *the classifying space* or *Eilenberg-Maclane space*.

Definition 1.2 (Aspherical Space)**.** If X is a connected space such that $\widetilde{X}$ is contractible, then we say X is a $K(\pi,1)$ (i.e. $K(\pi_1(X),1)$). We call such X *aspherical*.

Theorem 1.3 ([Hat01, Example 1.B.7, Theorem 1.B.8, Proposition 1.B.9])**.** *For any group G, there exists a (CW) $K(G,1)$ which is unique up to homotopy equivalence. In fact, for any two groups G, H, there exists a bijection*

$$\hom(G,H) \leftrightarrow \{\textit{based maps } K(G,1) \to K(H,1)\}/\textit{based homotopy}$$

Let X be a space and x_0 a point in X. Recall that $\pi_0(X)$ represents the path components of space X. For $n \geq 1$, $\pi_n(X,x_0)$ is the group of based homotopy classes of maps $(\mathbb{S}^n,\star) \to (X,x_0)$.

For $n \geq 2$, any map $(\mathbb{S}^n,\star) \to (X,x_0)$ lifts to $(\mathbb{S}^n,\star) \to (\widetilde{X},\widetilde{x}_0)$ (Since $\pi_1(\mathbb{S}^n,\star)$ is trivial). If X is aspherical, then $(\mathbb{S}^n,\star) \to (\widetilde{X},\widetilde{x}_0)$ is null-homotopy. Hence $\pi_n(X,x_0) = 0$. This proves one direction of Theorem 1.4. The other direction is harder, and we leave it for future discussion.

Theorem 1.4 (Whitehead's Theorem, [Hat01, Theorem 4.5])**.** *A CW complex X is aspherical if and only if it's connected and for all $n \geq 2$, $\pi_n(X,x_0) = 0$.*

Example 1.5 (Aspherical Spaces)**.**
(1) A point.
(2) $\mathbb{S}^1$.
(3) Products of aspherical spaces (corresponding to direct products of groups).
(4) Wedges of aspherical spaces (corresponding to free products of groups).
(5) Graphs (1-dimensional CW complexes).
(6) Closed orientable surfaces with genus ≥ 1.
(7) Complete Riemannian manifolds of non-positive sectional curvature (Cartan-Hadamard theorem).
(8) Complete locally CAT(0) spaces (A variant of Cartan-Hadamard theorem. See for example [BH99, Theorem II.4.1]).

Example 1.6 (Non-aspherical Spaces)**.** Connect sums typically produce non-aspherical spaces.

Question 1.7.
- What kind of $K(G,1)$ spaces does [insert any interesting group] have?
- For [insert any interesting group], which $K(G,1)$ is "better"?
- What do properties of $K(G,1)$ say about G?

Below we list some sample early theorems without proof.

Theorem 1.8. *A group G is finitely generated if and only if G has a $K(G,1)$ with a finite 1-skeleton.*

Theorem 1.9. *A group G is finitely presented if and only if G has a $K(G,1)$ with a finite 2-skeleton.*

Theorem 1.10. *If G has a finite-dimensional $K(G,1)$, then it's torsion free.*

Observe that those theorems all related to some sort of finiteness of $K(G,1)$. We introduce some notions of finiteness below.

Definition 1.11. A group G is of *type* F_n if it has a $K(G,1)$ with finite n-skeleton.

A group G is of *type* F_∞ if it's of type F_n for all n.

A group G is of *type* F if it has a finite $K(G,1)$.

A group G has *finite geometric dimension* if it has a finite-dimensional $K(G,1)$. The *geometric dimension* is the minimal dimension of all of its $K(G,1)$.

It is sometimes difficult to understand the $K(G,1)$ of a group. We thus use multiple invariance to help determine those types. If G is a group and X is its $K(G,1)$, then the algebraic topological invariance of X is invariant of G. In particular, we look at the cohomology groups (with trivial coefficients):

$$H^*(G,\mathbb{Z}) := H^*(X,\mathbb{Z})$$

References

[BH99] Martin R. Bridson and André Haefliger. *Metric Spaces of Non-Positive Curvature*, volume 319 of *Grundlehren Der Mathematischen Wissenschaften.* Springer Berlin Heidelberg, Berlin, Heidelberg, 1999.

[Hat01] Allen Hatcher. *Algebraic Topology.* Cambridge university press, New York, 2001.

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