# A NOTE OF TOPOLOGICAL METHODS IN GROUP THEORY

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Abstract. This is a course note of 2026 Spring MATH 569, taught by Daniel Groves at UIC, on the topic of Topological Methods in Group Theory. Reference textbooks are: Topological Methods in Group Theory, Cohomology of Groups.

## Contents

## 1. Aspherical Spaces and Sample Early Theorems

In this course we will explore the relationship between aspherical spaces and (discrete) groups. A convention: unless otherwise stated, all spaces in this course will be CW complexes.

**Definition 1.1** ($K(G,1)$)**.** Let $G$ be a group. A $K(G,1)$ is a connected space such that $\pi_1(K(G,1)) \cong G$ and whose universal covering space $\widehat{K}(G,1)$ is contractible.

It is also called *the classifying space* or *Eilenberg-Maclane space*.

**Definition 1.2** (Aspherical Space)**.** If $X$ is a connected space such that $\widetilde{X}$ is contractible, then we say $X$ is a $K(\pi,1)$ (i.e. $K(\pi_1(X),1)$). We call such $X$ *aspherical*.

**Theorem 1.3** ([Hat01, Theorem 1.B.7-9])**.** *For any group $G$, there exists a (CW) $K(G,1)$ which is unique up to homotopy equivalence. In fact, for any two groups $G, H$, there exists a bijection*

$$\hom(G,H) \leftrightarrow \{\textit{based maps } K(G,1) \to K(H,1)\}/\textit{based homotopy}$$

Let $X$ be a space and $x_0$ a point in $X$. Recall that $\pi_0(X)$ represents the path components of space $X$. For $n \geq 1$, $\pi_n(X,x_0)$ is the group of based homotopy classes of maps $(\mathbb{S}^n,\star) \to (X,x_0)$.

For $n \geq 2$, any map $(\mathbb{S}^n,\star) \to (X,x_0)$ lifts to $(\mathbb{S}^n,\star) \to (\widetilde{X},\widetilde{x}_0)$ (Since $pi_1(\mathbb{S}^n,\star)$ is trivial). If $X$ is aspherical, then $(\mathbb{S}^n,\star) \to (\widetilde{X},\widetilde{x}_0)$ is null-homotopy. Hence $\pi_n(X,x_0) = 0$.

**Theorem 1.4** ([Hat01, Theorem 4.5])**.** *A CW complex $X$ is aspherical if and only if it's connected and for all $n \geq 2$, $\pi_n(X,x_0) = 0$.*

**Example 1.5** (Aspherical Spaces)**.**
1. A point.
2. $\mathbb{S}^1$.
3. Products of aspherical spaces (corresponding to direct products of groups).
4. Wedges of aspherical spaces (corresponding to free products of groups).
5. Graphs (1-dimensional CW complexes).
6. Closed orientable surfaces if genus $\geq 1$.
7. Complete Riemannian manifolds of non-positive sectional curvature (Cartan-Hadamard theorem).
8. Complete locally CAT(0) spaces.

**Example 1.6** (Non-aspherical Spaces)**.** Connect sums typically produce non-aspherical spaces.

**Question 1.7.**
- What kind of $K(G,1)$ spaces does [insert any interesting group] have?
- For [insert any interesting group], which $K(G,1)$ is "better"?

- What do properties of $K(G,1)$ say about $G$?

Below we list some sample early theorems without proof.

**Theorem 1.8.** *A group $G$ is finitely generated if and only if $G$ has a $K(G,1)$ with a finite 1-skeleton.*

**Theorem 1.9.** *A group $G$ is finitely presented if and only if $G$ has a $K(G,1)$ with a finite 2-skeleton.*

**Theorem 1.10.** *If $G$ has a finite-dimensional $K(G,1)$, then it's torsion free.*

Observe that those theorems all related to some sort of finiteness of $K(G,1)$. We introduce some notions of finiteness below.

**Definition 1.11.** A group $G$ is of *type* $F_n$ if it has a $K(G,1)$ with finite $n$-skeleton.
A group $G$ is of *type* $F_\infty$ if it's of type $F_n$ for all $n$.
A group $G$ is of *type* $F$ if it has a finite $K(G,1)$.
A group $G$ has *finite geometric dimension* if it has a finite-dimensional $K(G,1)$. The *geometric dimension* is the minimal dimension of all of its $K(G,1)$.

It is sometimes difficult to understand the $K(G,1)$ of a group. We thus use multiple invariance to help determine those types. If $G$ is a group and $X$ is its $K(G,1)$, then the algebraic topological invariance of $X$ is invariant of $G$. In particular, we look at the cohomology groups (with trivial coefficients):

$$H^*(G,\mathbb{Z}) := H^*(X,\mathbb{Z})$$

## References

[Hat01] Allen Hatcher. *Algebraic Topology.* Cambridge university press, New York, 2001.

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