$(G,\mathscr{P})$–Quasi-convex Subgroups of Acylindrically Hyperbolic Groups

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Summary

We introduce the notion of $(G,\mathscr{P})$–*quasi-convex* subgroups in the setting of acylindrically hyperbolic groups. Such subgroups inherit the structure of hyperbolically embedded subgroups from the ambient group.

We also show that $(G,\mathscr{P})$–quasi-convex subgroups have uniformly quasi-convex constants in coned-off cusped Cayley graphs, which facilitates the generalization of results from relatively hyperbolic groups. As an application, we prove a theorem about CAT(0) cube complexes.

Motivation

Acylindrically hyperbolic groups are groups that admit a non-elementary acylindrical action on some hyperbolic space. This class of groups covers many interesting groups, notably including hyperbolic groups and relatively hyperbolic groups.

Quasi-convex subgroups are important in the study of hyperbolic groups. Analogously, relative quasi-convex subgroups occupy an important place in the study of relatively hyperbolic groups. In particular, relative quasi-convex subgroups inherit relatively hyperbolic structures from the ambient group.

Acylindrically hyperbolic groups admit structures of hyperbolically embedded subgroups. It is then natural to ask whether there is an analogous notion of quasi-convexity in the field of acylindrically hyperbolic groups that preserves the structure of hyperbolically embedded subgroups.

Hyperbolically Embedded Subgroups

Let G be a group, $\mathscr{P}=\{P_\lambda\}_{\lambda\in\Lambda}$ a collection of subgroups of G, X a relative generating set of G with respect to $\mathscr{P}$. Then $\Gamma(P_\lambda,P_\lambda)\subseteq\Gamma(G,\mathscr{P}\sqcup X)$.

For every $h_1,h_2\in P_\lambda$, $\widehat{d}_\lambda(h_1,h_2)$ is defined as the length of the shortest path in $\Gamma(G,\mathscr{P}\sqcup X)$ that connects h_1,h_2 and contains no edge in $\Gamma(P_\lambda,P_\lambda)$. If no such path exists, we set $\widehat{d}_\lambda(h_1,h_2)=\infty$.

Definition. We say $\mathscr{P}$ *is hyperbolically embedded* in G with respect to X and write $\mathscr{P}\hookrightarrow_h(G,X)$, if:

1. the Cayley graph $\Gamma(G,\mathscr{P}\sqcup X)$ is hyperbolic, and
2. $\forall\lambda\in\Lambda$, the metric space $(\Gamma(P_\lambda,P_\lambda),\widehat{d}_\lambda)$ is locally finite.

$(G,\mathscr{P})$–graph

A graph Γ is a $(G,\mathscr{P})$–*graph* if G acts on Γ and

1. Γ is connected and hyperbolic,
2. there are finitely many G-orbits of vertices,
3. G-stabilizers of vertices are finite or conjugates of $P\in\mathscr{P}$, and for every $P\in\mathscr{P}$, there is a vertex with G-stabilizer equal to P,
4. G-stabilizers of edges are finite, and
5. Γ is fine at each vertex with infinite stabilizer. (See [3] for a detailed definition.)

Acylindrically Hyperbolic Groups

Definition. For any group G, it is an *acylindrically hyperbolic group* if any of the following equivalent conditions is true:

- [4, Theorem 1.2] G contains a proper infinite hyperbolically embedded subgroup.
- [3, Proposition 1.4] There exists a proper infinite subgroup $\mathscr{P}$ and a subset X, such that $\widehat{\Gamma}(G,\mathscr{P},X)$, the coned-off Cayley graph, is connected, hyperbolic, and fine at cone vertices.

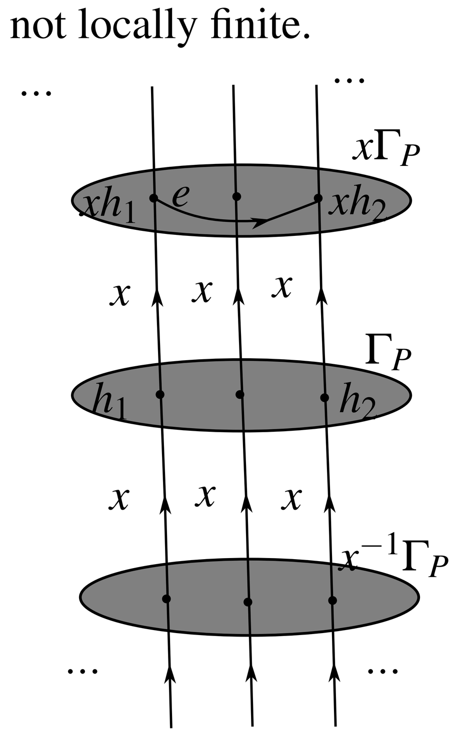
For more equivalent definitions, we refer to [4]. In addition, when G is a finitely generated group, the following condition is equivalent to above conditions:

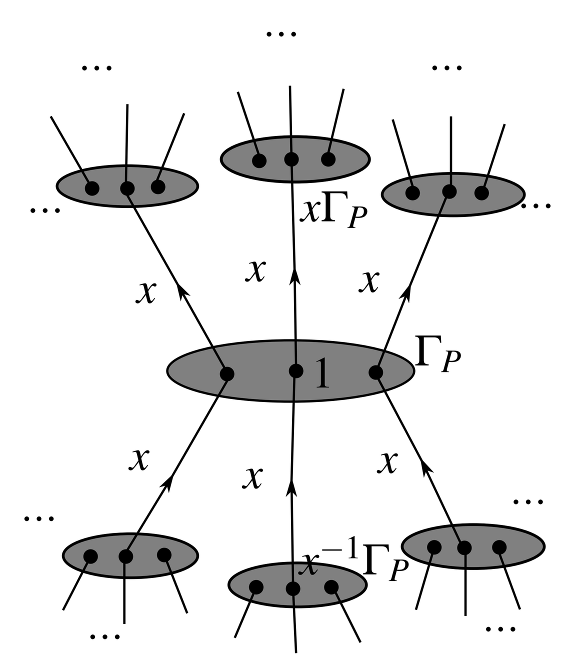
- [3, Theorem 1.2] There exists a proper infinite subgroup $\mathscr{P}$ and a $(G,\mathscr{P})$–graph.

Examples

Below is a non-exhaustive list of examples (and non-example) of acylindrically hyperbolic groups. For a detailed survey, we refer to [4, Section 8].

- **Non-example (Left).** Let $G=P\times\mathbb{Z}$ and $\mathbb{Z}=\langle x\rangle$. Then $\Gamma(G,\mathscr{P}\sqcup X)$ is quasi-isometric to a line and is hence hyperbolic. For every $h_1,h_2\in H$, $\widehat{d}(h_1,h_2)=3$, so $(\Gamma_P,\widehat{d})$ is not locally finite.





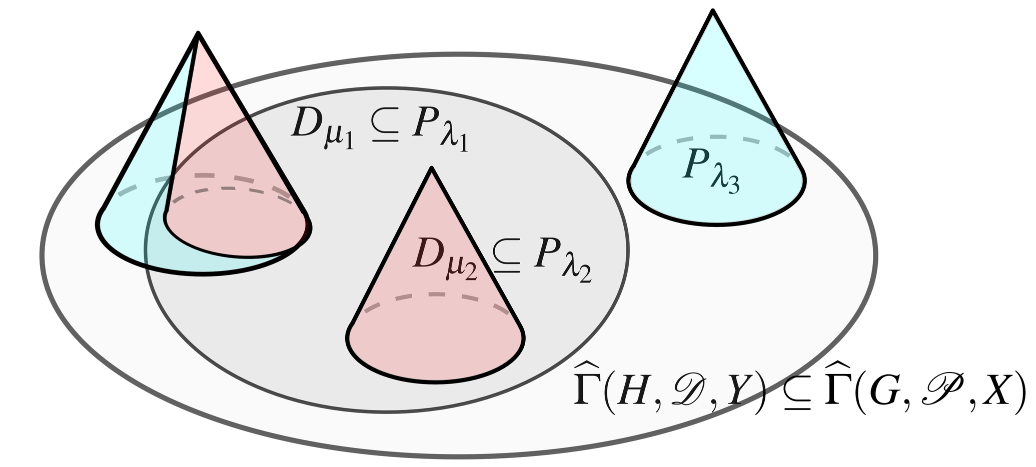
- **First Example (Right).** Let $G=P*\mathbb{Z}$. In this case $\Gamma(G,\mathscr{P}\sqcup X)$ is quasi-isometric to a tree and is hence hyperbolic. It is easy to see that $\widehat{d}(h_1,h_2)=\infty$ unless $h_1=h_2$. This means that every ball of finite radius in the relative metric has cardinality 1. We can conclude that $P\hookrightarrow_h(G,X)$.
- **Hyperbolic groups.**
- **Relatively hyperbolic groups.**
- **Mapping class groups.** All but finitely many mapping class group $MCG(\Sigma_{g,p})$ of a closed surface of finite type.
- **Outer automorphism groups.**
- **RAAGs.** Every RAAG is either cyclic or directly decomposable or acylindrically hyperbolic.
- **Hierarchically hyperbolic groups.** Every HHG admits an acylindrical action on some hyperbolic space. (See [1, Theorem K]).

$(G,\mathscr{P})$–Quasi-convex Subgroup

Convention. In the following results, we assume that G is a finitely generated group, $\mathscr{P}$ is a finite collection of infinite subgroups of G that is hyperbolically embedded in G, and H is a finitely generated subgroup of G.

Definition (Wan). We say H is $(G,\mathscr{P})$–*quasi-convex* if there exists a $(G,\mathscr{P})$–graph K and a nonempty, connected, H-invariant, quasi-isometrically embedded subgraph L in K, such that L has finitely many H-orbits of vertices.

Theorem (Structure of hyperbolically embedded subgroups, Wan). If H is $(G,\mathscr{P})$–quasi-convex, then there exists $X\subseteq G$, $Y\subseteq H$, and a collection $\mathscr{D}$ of infinite subgroups of H such that $\mathscr{P}\hookrightarrow_h(G,X)$, $\mathscr{D}\hookrightarrow_h(H,Y)$, and $\widehat{\Gamma}(H,\mathscr{D},Y)$ is a quasi-isometrically embedded subgraph of $\widehat{\Gamma}(G,\mathscr{P},X)$. In particular, every $D\in\mathscr{D}$ can be conjugated into some $P\in\mathscr{P}$.



Hyperbolically Embedded Dehn Fillings

Given a collection of subgroups $N_\lambda\lhd P_\lambda$, let K be the normal closure of $\cup_{\lambda\in\Lambda}N_\lambda$ in G. The *filling* of G is the quotient group $\overline{G}=G/K$. A property P holds *for all sufficiently long fillings* of $(G,\mathscr{P})$ if there is a collection of finite sets $\{F_\lambda\subseteq P_\lambda\}_{\lambda\in\Lambda}$ so that P holds whenever $N_\lambda\cap F_\lambda=\varnothing$.

For groups with hyperbolically embedded subgroups, [2, Theorem 7.19] states that for all sufficiently long fillings, $\{P_\lambda/N_\lambda\}_{\lambda\in\Lambda}\hookrightarrow_h\overline{G}$.

Coned-off cusped Cayley graphs $\{\mathbb{K}_r(G,\mathscr{P},X)\}_{r\in\mathbb{N}}$ are crucial in the proof of Dehn filling theorem in hyperbolically embedded subgroups. For a detailed construction, we refer to [2, Section 6.5]. For every $r\in\mathbb{N}$, $\mathbb{K}_r$ is $(r+1,1)$–quasi-isometric to $\widehat{\Gamma}(G,\mathscr{P},X)$.

Theorem (Uniform quasi-convexity constants, Wan). If H is $(G,\mathscr{P})$–quasi-convex, then there exists a finite collection of infinite subgroups $\mathscr{D}$ of H such that, $\forall r>0$, there exists a quasi-isometric embedding $\phi_r:\mathbb{K}_r(H,\mathscr{D},Y)\to\mathbb{K}_r(G,\mathscr{P},X)$.

Moreover, there exists $r_0,d_0>0$ such that, for any $r\geq r_0$, the image of $\mathbb{K}_r(H,\mathscr{D},Y)$ in $\mathbb{K}_r(G,\mathscr{P},X)$ under ϕ_r is d_0–quasi-convex. In particular, d_0 does not depend on r.

$$\begin{tikzcd}
\mathbb{K}_r(H,\mathscr{D},Y) \arrow[r, "d_0\text{–QC image}"] \arrow[d, leftrightarrow, "(r+1{,}1)"'] & \mathbb{K}_r(G,\mathscr{P},X) \arrow[d, leftrightarrow, "(r+1{,}1)"] \\
\widehat{\Gamma}(G,\mathscr{D},Y) \arrow[r, hook] & \widehat{\Gamma}(G,\mathscr{P},X)
\end{tikzcd}$$

Applications on CAT(0) Cube Complexes

A subgroup H is *full* if for any $P_\lambda\in\mathscr{P}$, whenever $H\cap P_\lambda$ is infinite, $H\cap P_\lambda$ is a finite indexed subgroup of P_λ.

Theorem (Wan). Suppose G acts cocompactly on a CAT(0) cube complex X. Suppose every parabolic element of G fixes some point of X and that cell-stabilizers are full $(G,\mathscr{P})$–quasi-convex.

For sufficiently long fillings of G that fix each G-orbits of cubes of X pointwise, the quotient X/K is a CAT(0) cube complex.

Future Directions

We expect many properties from a notion of quasi-convexity. In particular, we ask:

- Is the intersection of two $(G,\mathscr{P})$–quasi-convex subgroups still a $(G,\mathscr{P})$–quasi-convex subgroup?
- What are other equivalent definitions? In particular, is there an equivalent definition in the setting of acylindrical action?
- Does $(G,\mathscr{P})$–quasi-convexity depend on the choice of $(G,\mathscr{P})$–graph? In particular, given any X such that $\mathscr{P}\hookrightarrow_h(G,X)$, can we find a subgraph L of $\widehat{\Gamma}(G,\mathscr{P},X)$ satisfying the definition of $(G,\mathscr{P})$–quasi-convex?

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