Tema – Analiza Algoritmilor

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Problema 1

1.1 Clase de complexitate

1. Verificati valoarea de adevar a relatiilor, argumentand pentru fiecare raspunsul.
    - $\Theta(n\log n) \cup o(n\log n) \neq O(n\log n)$

$\Theta(n \log n) \cup o(n \log n) \neq O(n \log n)$ Fals

$\Theta(g(n)) = \{ f: \mathbb{N}^* \to \mathbb{R}^*_+ \mid \exists\, c_1, c_2 \in \mathbb{R}^*_+, \exists\, n_0 \in \mathbb{N}^*$ a.î. $c_1 \cdot g(n) \leq f(n) \leq c_2 \cdot g(n), \forall\, n \geq 0 \}$

$o(g(n)) = \{ f: \mathbb{N}^* \to \mathbb{R}^*_+ \mid \forall\, c \in \mathbb{R}^*_+, \exists\, n_0 \in \mathbb{N}^*$ a.î. $f(n) < c \cdot g(n), \forall\, n \geq n_0 \}$

$O(g(n)) = \{ f: \mathbb{N}^* \to \mathbb{R}^*_+ \mid \exists\, c \in \mathbb{R}^*_+, \exists\, n_0 \in \mathbb{N}^*$ a.î. $f(n) \leq c \cdot g(n), \forall\, n \geq n_0 \}$

Din propoziția 1, din curs 3, știm că $o(g(n)) \subseteq O(g(n))$

Având în vedere definițiile funcțiilor, putem deduce următoarele relații: Data:

1. Știm că $0 \leq c_1 * g(n) \leq f(n) \leq c_2 * g(n)$,
$0 \leq f(n) \leq c * g(n)$.

Mulțimea $\Theta(g(n))$ poate fi exprimată în formă de interval astfel: $[c_1 * g(n); c_2 * g(n)]$, unde $c_1 < c_2$, $c_1, c_2 \in \mathbb{R}^*_+$.

Mulțimea $O(g(n))$ poate fi exprimată în formă de interval astfel: $[0; c * g(n)]$, unde $c \in \mathbb{R}^*_+$.

Astfel, se observă că mulțimea $\Theta(g(n))$ este inclusă în mulțimea $O(g(n))$, condiția fiind să avem aceeași funcție $g(n)$ (în cazul de față fiind, pentru ambele, $n \log n$).

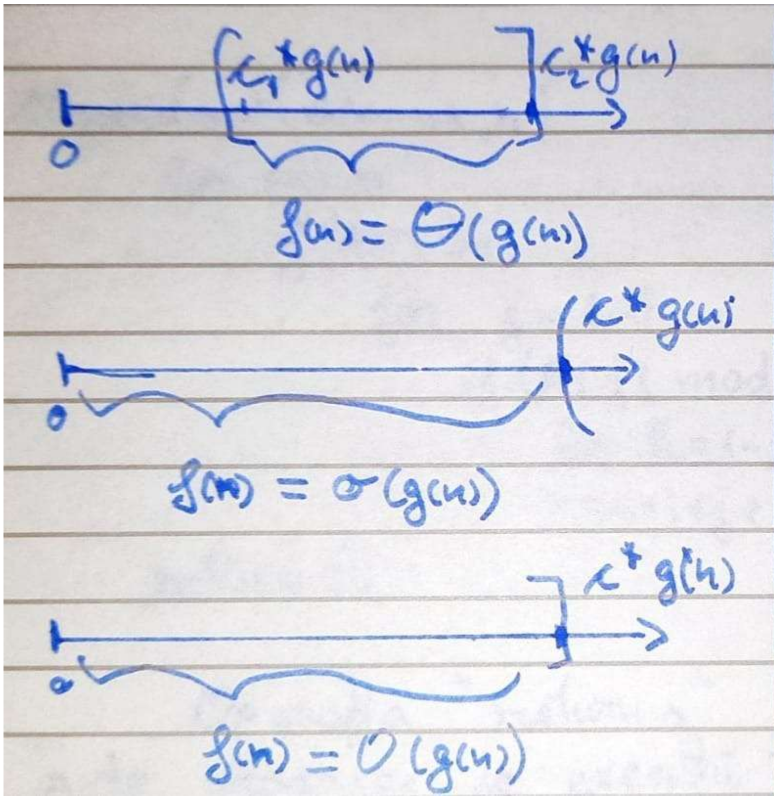
$$\cancel{\Theta(g(n)) \subset O(g(n))} \quad \Theta(n \log n) \subset O(n \log n)$$

2. ~~Dacă~~ Din cursul 3, propoziția 1, știm că $\cancel{o(g(n)) \le}$ $o(g(n)) \le O(g(n))$. Acest lucru este echivalent cu a spune

$o(g(n)) \subset O(g(n))$. Pentru $g(n) = n \log n$, $\Rightarrow$

$\Rightarrow$ $o(n \log n) \subset O(n \log n)$.

Din 1 și 2 $\Rightarrow$ $\Theta(n \log n) \cup o(n \log n) \subset O(n \log n)$



Data:

Având în vedere cele spuse anterior și graficele din partea stângă, putem afirma că

$\Theta(n \log n) \cup o(n \log) \neq O(n \log n)$

Fals.

- $\sqrt{n}logn! = \theta(n^{\frac{3}{2}}logn)$

$\sqrt{n}\log(n!) = \Theta(n^{\frac{3}{2}}\log n)$ Adevărat

Data:

$\Theta(g(n)) = \{ f: \mathbb{N}^* \to \mathbb{R}^*_+ \mid \exists\, c_1, c_2 \in \mathbb{R}^*_+, \exists\, n_0 \in \mathbb{N}^*$ a.î. $c_1 \cdot g(n) \le f(n) \le c_2 \cdot g(n), \forall n \ge n_0 \}$

$\Leftrightarrow \lim\limits_{n\to\infty} \frac{f(n)}{g(n)} = c, \; c = \mathbb{R}^*$

$\lim\limits_{n\to\infty} \frac{\sqrt{n}\cdot\log(n!)}{n^{\frac{3}{2}}\cdot\log n} = \lim\limits_{n\to\infty} \frac{\cancel{\sqrt{n}}\cdot\log(n!)}{n\cdot\cancel{\sqrt{n}}\cdot\log n} = \lim\limits_{n\to\infty} \frac{\log(n!)}{n\cdot\log n}$

I) $n^n > n! \mid \log \Rightarrow \log n^n > \log n! \Rightarrow n\log n > \log(n!) \;\}$

~~Trecem la limită~~ $1 > \frac{\log n!}{n\log n}$; ~~Trecem la limită~~ $\Rightarrow$

~~$\Rightarrow 1 \ge \lim\limits_{n\to\infty} \frac{\log n!}{n\log n}$~~

II

~~n!~~ $\frac{n^n}{2^n} < n!$

$\log\frac{n^n}{2^n} < \log n!$

$n\log n - \cancel{\log}\, n < \log n! \quad \mid : n\log n$

$1 - \frac{1}{\log n} < \frac{\log(n!)}{n\log n}$ ~~Trecem la limită~~

$\log(n!)$

I, II

$$\Longrightarrow 1 - \frac{1}{\log n} < \frac{\log(n!)}{n \log n} < 1 \qquad \text{Criteriul Cleștelui}$$

$$1$$

$$\Rightarrow \lim_{n \to \infty} \frac{\log(n!)}{n \log n} = 1 > 0$$

$$\Rightarrow \sqrt{n} \log(n!) = \Theta(n^{\frac{3}{2}} \log n) \quad \text{Adevărat}$$

- $n^3 log^4 n = O(n^4)$

$n^3 \log^4 n = O(n^4)$ Adevărat

Data:

$O(g(n)) = \{ f: \mathbb{N}^* \to \mathbb{R}^*_+ \mid \exists\, c \in \mathbb{R}^*_+ , \exists\, n_0 \in \mathbb{N}^*$ aî $f(n) \le c \cdot g(n) , \forall\, n \ge n_0 \}$

$$\Leftrightarrow \lim_{n \to \infty} \frac{f(n)}{g(n)} \neq \infty$$

$$\lim_{n \to \infty} \frac{n^3 \log^4 n}{n^4} = \lim_{n \to \infty} \frac{\log^4 n}{n}$$

Presupunem că limita este egală cu 0. Vom folosi metoda inducției pentru a demonstra $\lim_{n \to \infty} \frac{\log^k n}{n} = 0$

Caz de bază

$k=1$: $\lim_{n \to \infty} \frac{\log n}{n} \underset{\text{L'Hospital}}{\overset{\frac{\infty}{\infty}}{=}} \lim_{n \to \infty} \frac{(\log n)'}{(n)'} = \lim_{n \to \infty} \frac{\frac{1}{\ln 2 \cdot n}}{1} =$

$$= \lim_{n \to \infty} \frac{1}{n \cdot \ln 2} = 0$$

Ipoteza de inducție

Presupunem $P(k)$: $\lim_{n \to \infty} \frac{\log^k n}{n} = 0$

Demonstrăm $P(k+1)$: $\lim_{n \to \infty} \frac{\log^{k+1} n}{n} = 0$

$\lim_{n \to \infty} \frac{\log^{k+1} n}{n} \overset{\frac{\infty}{\infty}}{=} \lim_{n \to \infty} (k+1) \cdot \frac{\log^k n}{n \cdot \ln 2} = \frac{(k+1)}{\ln 2} \lim_{n \to \infty} \frac{\log^k n}{n} =$

$\underset{\text{inducție}}{\overset{\text{ip. de}}{=}} 0 \Rightarrow$ ~~ipotez~~ Presupunerea făcută este adevărată

$\Rightarrow \lim_{n \to \infty} \frac{\log^4 n}{n} \neq \infty \Rightarrow n^3 \log^4 n = O(n^4)$ Adevărat

- $log^{2020}n = o(n)$

$\log^{2020} n = o(n)$ Adevărat

Data:

~~$o(n) =$~~

$o(g(n)) = \{ f: \mathbb{N}^* \to \mathbb{R}_+^* \mid \forall\, c \in \mathbb{R}_+^*, \exists\, n_0 \in \mathbb{N}^*$ aî ~~lo~~

$f(n) < c \cdot g(n), \forall\, n \geq n_0 \}$

$$\Leftrightarrow \lim_{n\to\infty} \frac{f(n)}{g(n)} < c$$

$$\Rightarrow \lim_{n\to\infty} \frac{\log^{2020} n}{n} < c$$

Presupunem că limita aceasta tinde la 0. Vom demonstra prin metoda inducției.

Caz de bază

$$k=1 \Rightarrow \lim_{n\to\infty} \frac{\log n}{n} \underset{\text{L'Hospital}}{\overset{\frac{\infty}{\infty}}{=}} \lim_{n\to\infty} \frac{(\log n)'}{n'} = \lim_{n\to\infty} \frac{\frac{1}{\ln 2 \cdot n}}{1} =$$

$$= \lim_{n\to\infty} \frac{1}{\ln(2)\cdot n} = 0 \quad \text{Adev}$$

Ipoteza de inducție

Presupunem $P(k): \lim_{n\to\infty} \frac{\log^k n}{n} = 0, \forall\, k \in \mathbb{N}$

Demonstrăm $P(k+1): \lim_{n\to\infty} \frac{\log^{k+1} n}{n} = 0, \forall\, k \in \mathbb{N}$

$$\lim_{n\to\infty} \frac{\log^{k+1} n}{n} \overset{\frac{\infty}{\infty}}{=} \lim_{n\to\infty} \frac{(k+1)\cdot \log^k n}{n} = (k+1)\cdot \lim_{n\to\infty} \frac{\log^k n}{n} \underset{\text{ind}}{\overset{\text{ip. de}}{=}} 0$$

$\Rightarrow$ Presupunerea făcută este adevărată.

$$\Rightarrow \lim_{n\to\infty} \frac{\log^{2020} n}{n} = 0 \Rightarrow \log^{2020} n = o(n) \text{ Adevărat}$$

2. Calculati estimativ, oferind explicatii, complexitatea temporala pentru urmatorii algoritmi, pentru cele trei cazuri(favorabil, defavorabil si mediu):

Data:

```
Algoritm1(A[0..n], n)
    for i=1..n                          c1   n+1
        if (A[i] > 0)                   c2   Σ_{i=1}^{n} 1
            for j=1..i                  c3   Σ_{i=1}^{n} (t_i + 1)
                if (A[j] mod 2 == 0)    c4   Σ_{i=1}^{n} t_i
                    for k=1..j          c5   Σ_{1}^{n} Σ_{1}^{t_i} (t_j + 1)
                    s=s+i+j+k           c6   Σ_{1}^{n} Σ_{1}^{t_i} (t_j)
    return s
```

Operatia "return s" nu va fi luată în considerate deoarece se execută în $O(1)$ și nu afectează complexitatea algoritmului.

$$T(n) = c_1(n+1) + c_2 \cdot \sum_{i=1}^{n} 1 + c_3 \cdot \sum_{i=1}^{n} (t_i+1) + c_4 \cdot \sum_{i=1}^{n} t_i +$$

$$c_5 \cdot \sum_{1}^{n} \cdot \sum_{1}^{t_i} (t_j+1) + c_6 \cdot \sum_{1}^{n} \sum_{1}^{t_i} t_j$$

a) Cazul favorabil: Elementele vectorului A vor fi negative. Costurile $c_3 \div c_6$ nu se vor executa: $t_i = 0$, $t_j = 0$

$$\Rightarrow T(n) = c_1(n+1) + c_2 \cdot \sum_{i=1}^{n} 1 = c_1(n+1) + n = n(c_1+c_2) + c_1 = O(n)$$

b) Cazul nefavorabil: Toate elementele sunt pozitive și pare. $t_i = i$; $t_j = j$

$$T(n) = c_1(n+1) + c_2 \sum_{i=1}^{n} 1 + c_3 \cdot \sum_{i=1}^{n} (i+1) + c_4 \cdot \sum_{i=1}^{n} i +$$

$$+ c_5 \cdot \sum_{1}^{n} \sum_{1}^{i} (j+1) + c_6 \cdot \sum_{1}^{n} \sum_{1}^{i} j$$

Data:

$$T(n) = c_1(n+1) + c_2 \cdot n + c_3 \cdot \left(\frac{n(n+1)}{2} + n\right) + c_4 * n + c_5 \cdot \sum_1^n \left(\frac{i(i+1)}{2} + i\right) + $$
$$+ c_6 * \sum_1^n \frac{i(i+1)}{2}$$

$$T(n) = c_1(n+1) + c_2 \cdot n + \cancel{c_3\left(\frac{n(n+1)}{2} + n\right)}\, c_3\left(\frac{n^2+n}{2} + n\right) + c_4 n +$$
$$+ c_5 \cdot \sum_1^n \frac{i^2+3i}{2} + c_6 \cdot \sum_1^n \frac{i^2+i}{2}$$

$$T(n) = c_1(n+1) + c_2 \cdot n + c_3 \cdot \frac{n^2+3n}{2} + c_4 \cdot n + c_5 \cdot \frac{\frac{n(n+1)(2n+1)}{6} + 3\frac{n(n+1)}{2}}{2}$$
$$+ c_6 \cdot \frac{\frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2}}{2}$$

$$T(n) = c_1(n+1) + c_2 \cdot n + c_3 \cdot \frac{n^2+3n}{2} + c_4 n + c_5 \cdot \frac{n(n+1)(2n+10)}{12} +$$
$$+ c_6 \cdot \frac{n(n+1)(2n+4)}{12}$$

$$T(n) = \frac{1}{12}\Big[12 c_1 (n+1) + 12 c_2 n + 6 \cdot c_3 (n^2+3n) + 12 c_4 n + c_5 n(n+1) \cdot$$
$$\cdot (2n+10) + c_6 n(n+1)(2n+4)\Big]$$

$$T(n) = \frac{1}{12}\Big[12 c_1 n + 12 c_1 + 12 c_2 n + 6 c_3 n^2 + 18 c_3 n + 12 c_4 n +$$
$$+ 2 c_5 n^3 + 12 c_5 n^2 + 10 c_5 n + 2n^3 \cdot c_6 + 6 c_6 n^2 + 4n c_6\Big] =$$

$$T(n) = \frac{1}{12}\Big[n^3(2c_5 + 2c_6) + n^2(6 c_3 + 12 c_5 + 6 c_6) + n(12 c_1 +$$
$$+ 12 c_2 + 18 c_3 + 12 c_4 + 10 c_5 + 4 c_6) + 12 c_1\Big] =$$

$$T(n) = n^3 \frac{c_5 + c_6}{6} + n^2 \cdot \frac{c_3 + 2c_5 + c_6}{2} + n \cdot \frac{6c_1 + 6c_2 + 9c_3 + 6c_4 + 5c_5 + 2c_6}{6}$$

$$+ c_1$$

$$T(n) = \Theta(n^3)$$

c) Cazul mediu : $t_i = \frac{i}{2}$; $t_j = \frac{j}{2}$

$$T(n) = c_1 \cdot (n+1) + c_2 \cdot \sum_{i=1}^{n} 1 + c_3 \cdot \sum_{i=1}^{n} \left(\frac{i}{2}+1\right) + c_4 \cdot \sum_{i=1}^{n} \frac{i}{2} + c_5 \cdot \sum_{i=1}^{n} \sum_{j=1}^{\frac{i}{2}} \left(\frac{j}{4}+1\right)$$

$$+ c_6 \cdot \sum_{i=1}^{n} \sum_{1}^{\frac{i}{2}} \frac{j}{2}$$

$$T(n) = c_1(n+1) + c_2 \cdot \frac{n}{2} + c_3 \cdot \left(\frac{1}{2} \sum_{i=1}^{n} i + \sum_{i=1}^{n} 1\right) + \frac{1}{2} c_4 \cdot \sum_{i=1}^{n} i +$$

$$c_5 \cdot \sum_{i=1}^{n} \left(\frac{1}{2} \sum_{j=1}^{\frac{i}{2}} j + \sum_{j=1}^{\frac{i}{2}} 1\right) + c_6 \cdot \sum_{i=1}^{n} \left(\frac{1}{2} \sum_{j=1}^{\frac{i}{2}} j\right) =$$

$$T(n) = c_1 \cdot (n+1) + c_2 \cdot \frac{n}{2} + c_3 \left[\frac{n(n+1)}{4} + n\right] + c_2 \cdot \frac{n(n+1)}{4} +$$

$$+ c_5 \cdot \sum_{i=1}^{n} \left(\frac{1}{2} \frac{i(i+2)}{8} + \frac{i}{2}\right) + c_6 \cdot \sum_{i=1}^{n} \left(\frac{1}{2} \cdot \frac{i(i+2)}{8}\right)$$

$$T(n) = c_1(n+1) + c_2 \cdot \frac{n}{2} + c_3 \cdot \frac{n^2+5n}{4} + c_4 \cdot \frac{n(n+1)}{4} + c_5 \cdot \sum_{i=1}^{n} \frac{i(i+10)}{16} +$$

$$+ c_6 \cdot \sum_{i=1}^{n} \frac{i(i+2)}{16}$$

$$T(n) = c_1(n+1) + c_2 \cdot \frac{n}{2} + c_3 \cdot \frac{n(n+5)}{4} + c_4 \cdot \frac{n(n+1)}{4} +$$

$$+ c_5 \cdot \sum_{i=1}^{n} \frac{i^2+10i}{16} + c_6 \cdot \sum_{i=1}^{n} \frac{i^2+2i}{16}$$

$$T(n) = c_1(n+1) + n\cdot\frac{c_2}{2} + n(n+5)\cdot\frac{c_3}{4} + n(n+1)\,\frac{c_4}{4} +$$

$$+\frac{c_5}{16}\cdot\left(\frac{n(n+1)(2n+1)}{6} + 10\cdot\frac{n(n+1)}{2}\right) + \frac{c_6}{16}\cdot\left(\frac{n(n+1)(2n+1)}{6} + 2\,\frac{n(n+1)}{2}\right)$$

$$T(n) = c_1(n+1) + n\cdot\frac{c_2}{2} + n(n+5)\,\frac{c_3}{4} + n(n+1)\,\frac{c_4}{5} +$$

$$+\frac{c_5}{16}\cdot\frac{n(n+1)(2n+1)+60\,n(n+1)}{6} + \frac{c_6}{16}\cdot\frac{n(n+1)(2n+1)+6n(n+1)}{6}$$

$$T(n) = c_1(n+1) + n\cdot\frac{c_3}{2} + n(n+5)\cdot\frac{c_3}{4} + n(n+1)\cdot\frac{c_4}{4} + \frac{c_5}{16}\cdot\frac{n(n+1)(2n+61)}{6}$$

$$+\frac{c_6}{16}\cdot\frac{n(n+1)(2n+7)}{6}$$

$$T(n) = c_1(n+1) + n\cdot\frac{c_2}{2} + n(n+5)\,\frac{c_3}{4} + n(n+1)\,\frac{c_4}{4} +$$

$$+\frac{c_5}{16}\cdot\frac{2n^3+63n^2+61n}{6} + \frac{c_6}{16}\cdot\frac{2n^3+9n^2+7n}{6}$$

$$T(n) = n^3\cdot\frac{c_5+c_6}{48} + n^2\left(\frac{c_3}{4} + \frac{c_4}{4} + \frac{21c_5}{32} + \frac{3}{36}c_6\right) +$$

$$+ n\left(c_1 + \frac{c_2}{2} + \frac{5c_3}{4} + \frac{c_4}{4} + \frac{61}{96}c_5 + \frac{7c_6}{96}\right) + c_1$$

$$T(n) = O(n^3)$$

Algoritm2 (A[0..n], n)

```
s=0
for i=1..n                      c1    n+1
  for j=i..n                    c2    Σ_{i=1}^{n} t_i +1
    if (A[i] > A[j])            c3    Σ_{i=1}^{n} Σ_{j=1}^{t_i} 1
      for k=1..i                c4    Σ_{i=1}^{n} Σ_{j=1}^{t_i} (t_j+1)
        s = s + A[i]*A[k]       c5    Σ_{i=1}^{n} Σ_{j=1}^{t_i} t_j
return s
```

· Operațiile "s=0" și "return s" nu vor fi luate în considerare deoarece se execută în $\Theta(1)$ și nu afectează complexitatea.

$$T(n) = c_1(n+1) + c_2\sum_{i=1}^{n}(t_i+1) + c_3\cdot\sum_{i=1}^{n}\sum_{j=1}^{t_i} 1 + c_4\sum_{i=1}^{n}\sum_{j=1}^{t_i}(t_j+1) + c_5\sum_{i=1}^{n}\sum_{j=1}^{n} t_j$$

a) Caz favorabil: ~~t_i~~ $t_i = i$, $t_j = 0$

$$T(n) = c_1(n+1) + c_2\sum_{i=1}^{n}(i+1) + c_3\cdot\sum_{i=1}^{n}\sum_{j=1}^{t_i} 1 =$$

$$T(n) = c_1(n+1) + c_2\cdot\sum_{i=1}^{n}(i+1) + c_3\cdot\sum_{i=1}^{n}\sum_{j=}^{n} 1 =$$

$$T(n) = c_1(n+1) + c_2\cdot\frac{n^2+3n}{2} + c_3 n^2$$

$$T(n) = n^2\left(\frac{c_2}{2} + c_3\right) + n\left(c_1 + \frac{3c_2}{2}\right) + c_1$$

$$T(n) = \Theta(n^2)$$

b) Caz nefavorabil: $t_i = i$, $t_j = j$

$$T(n) = c_1(n+1) + c_2\sum_{i=1}^{n}(i+1) + c_3\sum_{i=1}^{n}\sum_{j=1}^{i}1 + c_4\sum_{i=1}^{n}\sum_{j=1}^{i}(j+1) +$$
$$+ c_5\sum_{i=1}^{n}\sum_{j=1}^{i} j$$

$$T(n) = c_1\cdot(n+1) + c_2\,\frac{n^2+3n}{2} + c_3\cdot n^2 + c_4\cdot\sum_{i=1}^{n}\frac{i^2+3i}{2} +$$
$$+ c_5\sum_{i=1}^{n}\frac{i^2+i}{2}$$

$$T(n) = c_1\cdot(n+1) + c_2\cdot\frac{n^2+3n}{2} + c_3 n^2 + c_4\cdot\frac{2n^3+12n^2+10n}{12} +$$
$$+ c_5\cdot\frac{2n^3+6n^2+4n}{12}$$

$$T(n) = n^3\left(\frac{c_4+c_5}{6}\right) + n^2\left(\frac{c_2}{2} + c_3 + c_4 + \frac{c_5}{2}\right) +$$
$$+ n\left(c_1 + \frac{3c_2}{2} + \frac{5c_4}{6} + \frac{c_5}{3}\right) + c_1$$

$$T(n) = O(n^3)$$

c) Caz mediu: $t_i = i$, $t_j = \frac{j}{2}$

$$T(n) = c_1(n+1) + c_2\cdot\sum_{i=1}^{n}(i+1) + c_3\sum_{i=1}^{n}\sum_{j=1}^{i}1 + c_4\cdot\sum_{i=1}^{n}\sum_{j=1}^{i}\left(\frac{j}{2}+1\right) +$$
$$+ c_5\cdot\sum_{i=1}^{n}\sum_{j=1}^{i}\frac{j}{2}$$

Data:

$$T(n) = c_1 \cdot (n+1) + c_2 \cdot \frac{n^2+3n}{2} + c_3 n^2 + c_4 \cdot \sum_{i=1}^{n} \frac{i(i+5)}{4} + c_5 \cdot \sum \frac{i(i+1)}{4}$$

$$T(n) = c_1 \cdot (n+1) + c_2 \cdot \frac{n^2+3n}{2} + c_3 n^2 + c_4 \cdot \frac{2n^3+18n^2+16n}{24} + c_5 \cdot \frac{n^3+3n^2+2n}{3}$$

$$T(n) = n^3\left(\frac{c_4}{12} + \frac{c_5}{3}\right) + n^2\left(\frac{c_2}{2} + c_3 + \frac{3c_4}{4} + c_5\right) + n\left(\cancel{2c}\, c_1 + \frac{3c_2}{2} + \frac{2c_4}{3} + \frac{2c_5}{3}\right) + c_1$$

$$T(n) = O(n^3)$$

Metrica neomogenă

a) Caz favorabil: vectorul nu este sortat crescător, niciun număr nu va fi mai mare decât cele din ~~stânga~~ ~~dre~~ dreapta sa: $c_4 - c_5$ nu se execută; cazuri critice: c_1 și c_2 c_3 ($t_i = i$, $t_j = 0$)

$$T(n) = c_1(n+1) + c_2 \cdot \sum_{i=1}^{n} (t_i+1) + c_3 \cdot \sum_{i=1}^{n} \sum_{j=1}^{t_i} 1$$

$$T(n) = c_1(n+1) + c_2 \cdot \sum_{1}^{n} (i+1) = c_1 \cdot (n+1) + c_2\left(\frac{n(n+1)}{2} + n\right) + c_3 \cdot n^2$$

$$T(n) = c_1(n+1) + c_2 \cdot \frac{n^2+3n}{2} + c_3 \cdot n^2 = \cancel{n^2\left(\frac{c_2}{2} + c_3\right) + n\left(c_1 + \frac{3c_2}{2}\right)} + c_1$$

~~$T(n) = O(n^2)$~~ $T(n) = n^2\left(\frac{c_2}{2} + c_3\right) + n\left(c_1 + \frac{3c_2}{2}\right) + c_1$

$$T(n) = O(n^2)$$

b) Caz nefavorabil: vector sortat crescător strict

$t_i = i$; $t_j = j$. Operații critice : c_4 și c_5

$$T(n) = c_4 \cdot \sum_{i=1}^{n} \sum_{j=1}^{i} (j+1) + c_5 \cdot \sum_{i=1}^{n} \sum_{j=1}^{i} j$$

$$T(n) = c_4 \cdot \sum_{i=1}^{n} \left[\frac{i(i+1)}{2} + i\right] + c_5 \cdot \sum_{i=1}^{n} \frac{i(i+1)}{2}$$

$$T(n) = c_4 \cdot \sum_{i=1}^{n} \frac{i^2+3i}{2} + c_5 \cdot \sum_{i=1}^{n} \frac{i^2+i}{2}$$

$$T(n) = c_4 \cdot \left[\frac{n(n+1)(2n+1)}{12} + 3 \cdot \frac{n(n+1)}{4}\right] + c_5 \left[\frac{n(n+1)(2n+1)}{12} + \frac{n(n+1)}{4}\right]$$

$$T(n) = c_4 \cdot \frac{n(n+1)(2n+10)}{12} + c_5 \cdot \frac{n(n+1)(2n+4)}{12}$$

$$T(n) = c_4 \cdot \frac{2n^3+12n^2+10n}{12} + c_5 \cdot \frac{2n^3+6n^2+4n}{12}$$

$$T(n) = n^3 \cdot \left(\frac{c_4}{6} + \frac{c_5}{6}\right) + n^2 \cdot \left(c_4 + \frac{c_5}{2}\right) + n\left(\frac{5c_4}{6} + \frac{c_5}{3}\right)$$

$$T(n) = O(n^3)$$

c) Caz mediu : $t_i = i$, $t_j = \frac{j}{2}$, Operații critice: c_4 și c_5

$$T(n) = c_4 \cdot \sum_{i=1}^{n} \sum_{j=1}^{i} \left(\frac{j}{2} + 1\right) + c_5 \cdot \sum_{i=1}^{n} \sum_{j=1}^{i} \frac{j}{2}$$

$$T(n) = c_4 \cdot \sum_{i=1}^{n} \frac{i(i+5)}{4} + c_5 \cdot \sum_{i=1}^{n} \frac{i(i+1)}{4}$$

Data:

$$T(n) = c_4 \cdot \sum_{i=1}^{n} \frac{i^2+5i}{4} + c_5 \cdot \sum_{i=1}^{n} \frac{i^2+i}{4}$$

$$T(n) = c_4 \cdot \frac{2n^3+18n^2+16n}{24} + c_5 \cdot \frac{n^3+3n^2+2n}{3}$$

$$T(n) = n^3\left(\frac{c_4}{12} + \frac{c_5}{3}\right) + n^2\left(\frac{3c_4}{4} + c_5\right) + n\left(\frac{2c_4}{3} + \frac{2c_5}{3}\right)$$

$$T(n) = O(n^3)$$

3. Gasiti clasa de complexitate, folosind una dintre cele trei metode predate la curs(iterativa, arbore de recurenta sau Master), pentru:

$$T(n) = \begin{cases} 3\,T\left(\frac{n-2}{2}\right) + k_2 n^2, & n>1,\ k_2 \in \mathbb{R}^*_+ \\ k_1 \text{ dacă } n=1, & k_1 \in \mathbb{R}^*_+ \end{cases}$$

$$S(n) = T(4n-2) \Leftrightarrow S\left(\frac{n}{2}\right) = T(2n-2)$$

$$T(4n-2) = 3\,T\left(\frac{4n-2-2}{2}\right) + k_2(4n-2)^2$$

$$T(4n-2) = 3\,T(2n-2) + k_2(4n-2)^2$$

$$\Rightarrow S(n) = 3\,S\left(\frac{n}{2}\right) + k_2(4n-2)^2$$

$T(1) = \Theta(1)$

$$S(n) = 3\,S\left(\frac{n}{2}\right) + \Theta(n^2) \quad 1 \cdot 3^0$$

$$S\left(\frac{n}{2}\right) = 3\,S\left(\frac{n}{2^2}\right) + \Theta\left(\frac{n^2}{2}\right) \quad 1 \cdot 3^1$$

$$\vdots$$

$$S\left(\frac{n}{2^k}\right) = 3\,S\left(\frac{n}{2^{k+1}}\right) + \Theta\left(\frac{n^2}{2^k}\right) \quad 1 \cdot 3^k$$

$$\Rightarrow S(n) = 3\,S\left(\frac{n}{2^{k+1}}\right) + \sum_{i=0}^{k} \Theta\left(\frac{n^2}{2^k}\right) \quad 1 \cdot 3^k$$

$$S(n) = 3^{k+1} \cdot \Theta(1) + \sum_{i=0}^{k} \Theta\left(\frac{n^2}{2^{2i}}\right) \cdot 3^i$$

~~[illegible]~~ $\frac{n}{2^{k+1}} = 1 \Rightarrow n = 2^{k+1} \Rightarrow k+1 = \log n$

Data:

$$\Rightarrow S(n) = 3^{\log n} \cdot \Theta(1) + \Theta\left(n^2\left(4 - 3^{k+1} \cdot 4^{-\log n+1}\right)\right)$$

$$S(n) = \Theta\left(3^{\log n}\right) + \Theta\left(4n^2 - 3^{\log n} \cdot 4^{-\log n+1} \cdot n^2\right)$$

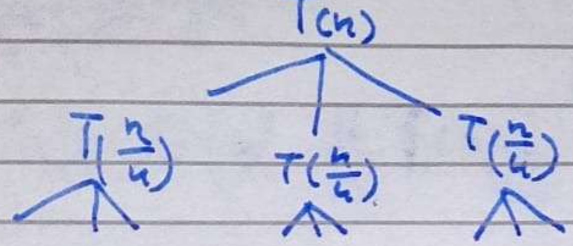
$$3^{\log n} < 3\,n^{\log 3}$$

$$\Rightarrow S(n) = \Theta\left(n^{\log 3} + 4n^2 - \frac{n^{\log 3}}{4}\right)$$

$$\left.\begin{aligned} S(n) &= \Theta(n^2) \\ S(n) &= T(4n-2) \end{aligned}\right| \Rightarrow T(n) = \Theta\left((4n-2)^2\right)$$

$$\Rightarrow T(n) = \Theta(16n^2 - 16n + 4) \Rightarrow T(n) = \Theta(n^2)$$

$$T(n) = \begin{cases} 3\,T\left(\frac{n}{4}\right) + k_2 n^2 & , n > 1 \ , \ k_2 \in \mathbb{R}^*_+ \\ k_1 & , n = 1 \ , \ k_1 \in \mathbb{R}^*_+ \end{cases}$$



$\Theta(n^2)$

$3 * \Theta\left(\frac{n^2}{4^2}\right)$

$T\left(\frac{n}{4^2}\right)$ $\qquad 3^2 * \Theta\left[\left(\frac{n}{4^2}\right)^2\right]$

$\vdots$

$T\left(\frac{n}{4^{k-1}}\right)$ $\qquad 3^{k-1} * \Theta\left[\left(\frac{n}{4^{k-1}}\right)^2\right]$

$$\Rightarrow T(n) = \sum_{0}^{k-1} 3^i \cdot \Theta\left(\left(\frac{n}{4^i}\right)^2\right)$$

$$\frac{n}{4^{k-1}} = 1 \Rightarrow n = 4^{k-1}$$

$$\log_4 n = k-1$$

$$\Rightarrow T(n) = \sum_{0}^{k-1} \left(\frac{3}{16}\right)^i \cdot \Theta(n^2)$$

$$T(n) = \sum_{0}^{\log_4 n} \left(\frac{3}{16}\right)^i \cdot \Theta(n^2)$$

$$T(n) = \frac{\left(\frac{3}{16}\right)^{\log_4 n} - 1}{\frac{3}{16} - 1} \cdot \Theta(n^2)$$

$$T(n) = \frac{16}{13} \cdot \left[1 - \left(\frac{3}{16}\right)^{\log_4 n}\right] \cdot \Theta(n^2)$$

~~$T(n) = 2\Theta\ n^{2}$~~

$$T(n) = \Theta\left(n^2 \cdot \frac{16}{13}\left[1 - \left(\frac{3}{16}\right)^{\log_4 n}\right]\right)$$

$$T(n) = \Theta(n^2)$$

$$T(n) = \begin{cases} 3T\left(\frac{n}{4}\right) + n\log n & , n > 1 \quad , k_2 \in \mathbb{R}_+^* \\ k_1, & n = 1 \quad , k_1 \in \mathbb{R}_+^* \end{cases}$$

$\begin{cases} a = 3 \\ b = 4 \end{cases} \Rightarrow n^{\log_b a} = n^{\log_4 3}$ $\qquad \log_4 3 < 1$

$f(n) = n\log n$

Caz 2: $f(n) = \Theta(n^{\log_4 3}) \Leftrightarrow n\log n = \Theta(n^{\log_4 3})$ Fals

Caz 1: $f(n) = \Theta(n^{\log_4 3 - \varepsilon}) \Leftrightarrow n\log n = \Theta(n^{\log_4 3 - \varepsilon})$ Fals

Caz 3:

a) $f(n) = \Omega(n^{\log_b a + \varepsilon})$

$\exists\ c \in \mathbb{R}_+^*, \exists\ n_0 \in \mathbb{N}^*$ aî ~~$n\log n$~~ $n\log n \geq c \cdot n^{\log_4 3 + \varepsilon}$ $| :n$ $\forall\ n \geq n_0$

$\Rightarrow \log n \geq c \cdot n^{\log_4 3 + \varepsilon - 1}$, $\forall\ n \geq n_0$ ~~$n \geq 1$~~

Pentru $n=1$

$\Rightarrow \log 1 \geq c \Leftrightarrow 0 \geq c$ Fals, $c \in \mathbb{R}_+^*$

$\log_4 3 - 1 + \varepsilon \rightarrow n$ crește mai repede ca $\log n$

$\Rightarrow \varepsilon = 1 - \log_4 3 > 0$

$\log n \geq c \quad , \forall \, n \geq n_c$

$\begin{cases} n_c = 2 \\ c = 1 \end{cases} \Rightarrow \log n \geq 1 \quad , \forall \, n \geq 2$

b) $\exists \, c \in (0,1) \; , \exists \, n_c \in \mathbb{N}^*$ a.î. $a * f\left(\frac{n}{b}\right) \leq c * f(n) \quad , \forall \, n \geq 0$

$\begin{cases} a = 3 \\ b = 4 \\ f(n) = n \log n \end{cases} \Rightarrow 3 \frac{n}{4} \log \frac{n}{4} \leq c * n \log n \quad , \forall \, n \geq n_0$

$\frac{3}{4} (\log n - \log 4) \leq c * \log n \quad , \forall \, n \geq n_c$

$\frac{3}{4} (\log n - 2) \leq c * \log n$

$c \geq \frac{3}{4} \frac{(\log n - 2)}{\log n} \Rightarrow c = \frac{3}{4} \in (0,1) \quad n_c = 2 \quad , \forall \, n \geq n_c$

$\begin{cases} a \text{ adevărat} \\ b \text{ adevărat} \end{cases} \xrightarrow[\cancel{\text{Master}}]{\text{Master}} T(n) = \Theta(n \log n)$

Data:

Recurență	Metodă	Clasa de complexitate obținută
$T(n)=\begin{cases} 3T\left(\frac{n-2}{2}\right)+k_2 n^2, & n>1,\ k_2\in\mathbb{R}_+^* \\ k_1, & n=1,\ k_1\in\mathbb{R}_+^* \end{cases}$	iterativă	$\Theta(16n^2-16n+4)$
$T(n)=\begin{cases} 3T\left(\frac{n}{4}\right)+k_2 n^2, & n>1,\ k_2\in\mathbb{R}_+^* \\ k_1, & n=1,\ k_1\in\mathbb{R}_+^* \end{cases}$	arbore de recurență	$\Theta\left(n^2\cdot\frac{16}{3}\left(1-\left(\frac{3}{16}\right)^{\log_4 n}\right)\right)$
$T(n)=\begin{cases} 3T\left(\frac{n}{4}\right)+n\log n, & n>1,\ k_2\in\mathbb{R}_+^* \\ k_1, & n=1,\ k_1\in\mathbb{R}_+^* \end{cases}$	Master	$\Theta(n\log n)$

4. Gasiti clasa de complexitate, folosind metoda substitutiei si notatia θ, pentru:

$$T(n) = \begin{cases} 10T\left(\frac{n}{3}\right) + k_2 n^2, & n > 1,\ k_2 \in \mathbb{R}_+^* \\ k_1, & n = 1,\ k_1 \in \mathbb{R}_+^* \end{cases}$$

$$T(n) = 10\,T\left(\frac{n}{3}\right) + k_2 n^2 \qquad T(1) = \Theta(1)$$

$$\rightarrow T(n) = \Theta\left(n^{\log_3 10} - n^2\right)$$

$$\Rightarrow \exists\ c_1, c_2 \in \mathbb{R}_+^*,\ n_0 \in \mathbb{N}^* \text{ a.î.}$$

$$c_1\left(n^{\log_3 10} - n^2\right) \le T(n) \le c_2\left(n^{\log_3 10} - n^2\right),\ \forall n \ge n_0$$

Demonstrăm prin metoda inducției

Caz de bază:

$n = 1 \Rightarrow 0 \le T(1) \le 0$ Fals, $T(1) = \Theta(1)$

~~$n = 2 \Rightarrow c$~~

$n = 2 \Rightarrow c_1\left(2^{\log_3 10} - 4\right) \le T(2) \le c_2\left(2^{\log_3 10} - 4\right)$

$T(2) = 10\,T\left(\frac{2}{3}\right) + \Theta(n)$, nu se poate afla $T\left(\frac{2}{3}\right)$

$n = 3 \Rightarrow c_1\left(\underbrace{3^{\log_3 10} - 9}_{1}\right) \le T(3) \le c_2\left(\underbrace{3^{\log_3 10} - 9}_{1}\right)$

$c_1 \le T(3) \le c_2 \Rightarrow n_0 = 3$

Data:

Pas de inducție: $\frac{n}{3} \to n$

Ipoteza de inducție

$$c_1\left(\frac{n^{\log_3 10}}{3^{\log_3 10}} - \frac{n^2}{9}\right) \le T(n) \le c_2\left(\frac{n^{\log_3 10}}{3^{\log_3 10}} - \frac{n^2}{9}\right)$$

$$c_1 \cdot n^{\log_3 10} - \frac{10}{9} c_1 \cdot n^2 + k_2 \cdot n^2 \le T(n) \le c_2\left(n^{\log_3 10} - n^2\right) + \cancel{c}$$
$$+ n^2\left(k_2 - \frac{c_2}{9}\right)$$

$$\left.\begin{aligned} k_2 - \frac{c_1}{9} &\ge 0 \\ k_2 - \frac{c_2}{9} &\le 0 \end{aligned}\right| \Rightarrow c_1 \le 9k_1 \le c_2$$

$$\begin{cases} c_1 \le T(3) \le c_2 \\ c_1 \le 9k_2 \le c_2 \end{cases} \Rightarrow \begin{cases} c_1 \le 10k_1 + 9k_2 \le c_2 \\ c_1 \le 9k_2 \le c_2 \end{cases} \quad k_1, k_2 \in \mathbb{R}_+^*$$

$$\cancel{\Rightarrow \begin{cases} c_1 \le 9k_2 \\ c_2 \ge 10k_1 + c_1 \end{cases}} \Rightarrow \begin{cases} c_1 = \min(10k_1 + 9k_2; 9k_2) \\ c_2 = \max(10k_1 + 9k_2; 9k_2) \end{cases}$$

$$\Rightarrow \begin{cases} c_1 = 9k_2 \\ c_2 = 10k_1 + 9k_2 \end{cases} \Rightarrow \exists\ c_1, c_2 \in \mathbb{R}_+^*,\ n_0 = 3$$

a.î. $T(n) = \Theta\left(n^{\log_3 10} - n^2\right), \forall\ n \ge n_0$

$$T(n) = \begin{cases} 2T\left(\frac{n}{3}\right) + k_2 n^4 & , n > 1 \ , k_2 \in \mathbb{R}^*_+ \\ k_1 & , n = 1 \ , k_1 \in \mathbb{R}^*_+ \end{cases}$$

$$T(n) = 2T\left(\frac{n}{3}\right) + \Theta(n^4) \quad 1 \cdot 2^0$$

$$T\left(\frac{n}{3}\right) = 2T\left(\frac{n}{3^2}\right) + \Theta\left(\left(\frac{n}{3}\right)^4\right) \quad 1 \cdot 2^1$$

$$\vdots$$

$$T\left(\frac{n}{3^k}\right) = 2T\left(\frac{n}{3^{k+1}}\right) + \Theta\left(\left(\frac{n}{3^k}\right)^4\right) \quad 1 \cdot 2^k$$

$$\Rightarrow T(n) = 2^{k+1} T\left(\frac{n}{3^{k+1}}\right) + \sum_{i=0}^{k} \Theta\left(\left(\frac{n}{3^{4i}}\right)^4\right) \cdot 2^i$$

$$\frac{n}{3^{k+1}} = 1 \Rightarrow n = 3^{k+1} \Rightarrow \log_3 n = k+1$$

$$\Rightarrow T(n) = 2^{k+1} \cdot T(1) + \sum_{i=0}^{k} \Theta(n^4) \cdot \frac{2^i}{3^{4i}}$$

$$T(n) = 2^{\log_3 n} \cdot T(1) + \sum_{i=0}^{\log_3 n - 1} \Theta(n^4) \cdot \frac{2^i}{3^{4i}}$$

$$\Rightarrow T(n) = \Theta(n^4)$$

$\exists\ c_1, c_2 \in \mathbb{R}_+^*,\ n_0 \in \mathbb{N}^*$ a.î.

$$c_1 n^4 \leq T(n) \leq c_2 \cdot n^4 \quad , \forall\ n \geq n_0$$

Caz de bază

$n=1 \Rightarrow c_1 \leq T(1) \leq c_2$ este bun

Pas de inducție : $\frac{n}{3} \to n$

Ipoteză de inducție:

$$c_1 \cdot \left(\frac{n}{3}\right)^4 \leq T\left(\frac{n}{3}\right) \leq c_2 \cdot \left(\frac{n}{3}\right)^4 \quad | \cdot 2 \mid + k_2 n^4$$

$$2 \cdot c_1 \cdot \left(\frac{n}{3}\right)^4 + k_2 n^4 \leq T(n) \leq 2 \cdot c_2 \cdot \left(\frac{n}{3}\right)^4 + k_2 n^4$$