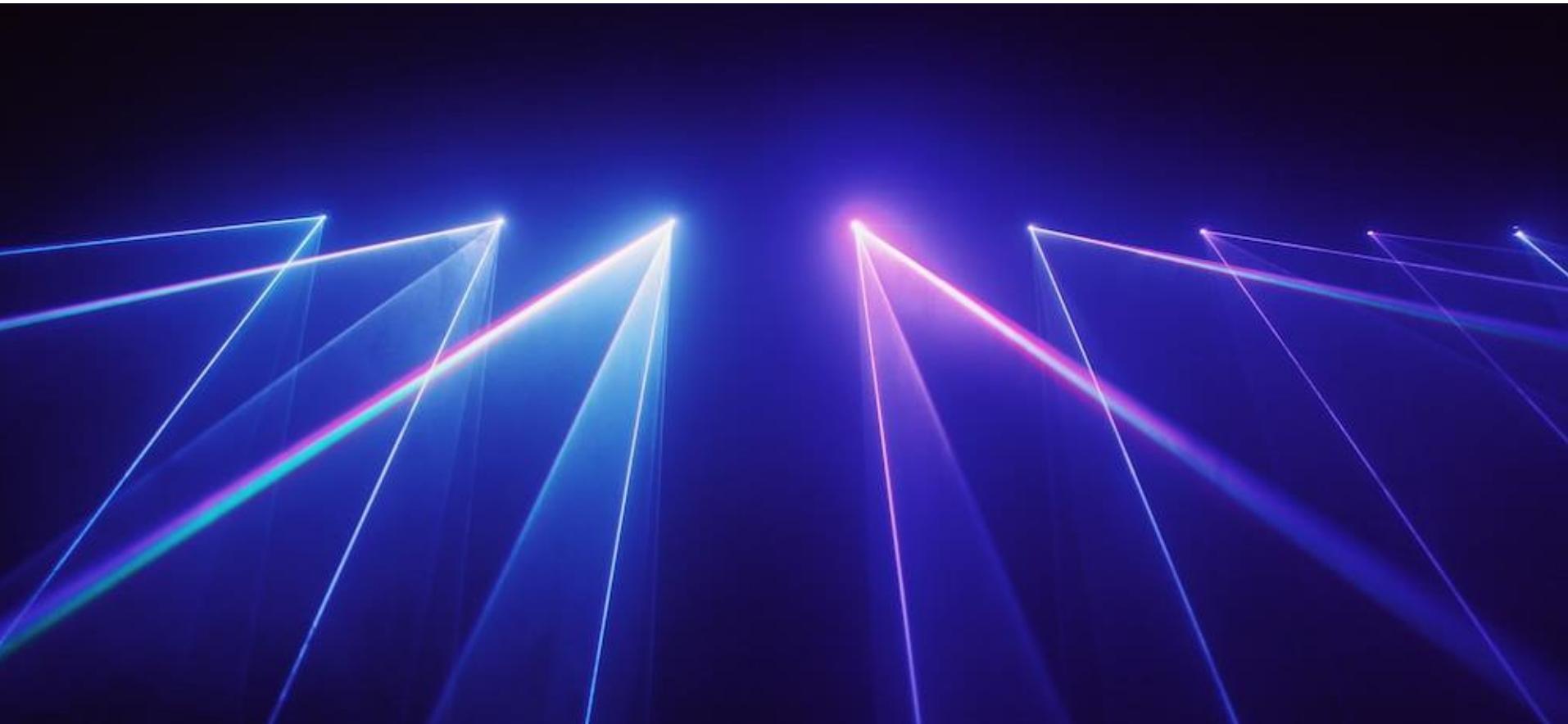


Epipolar Geometry + Calibration

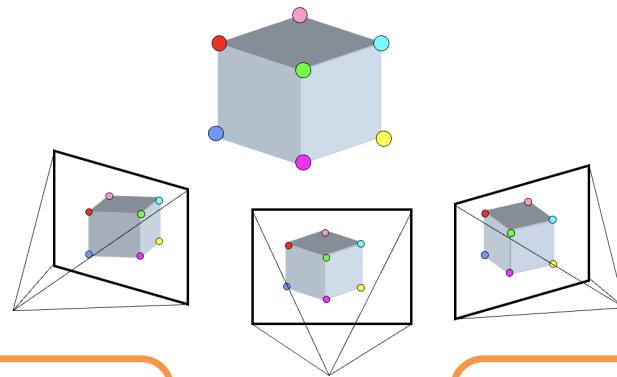


A lot of slides from Noah Snavely +
Shree Nayar's YT series: First principals of Computer Vision

CS180: Intro to Computer Vision and Comp. Photo
Angjoo Kanazawa & Alexei Efros, UC Berkeley, Fall 2025

Many problems in 3D

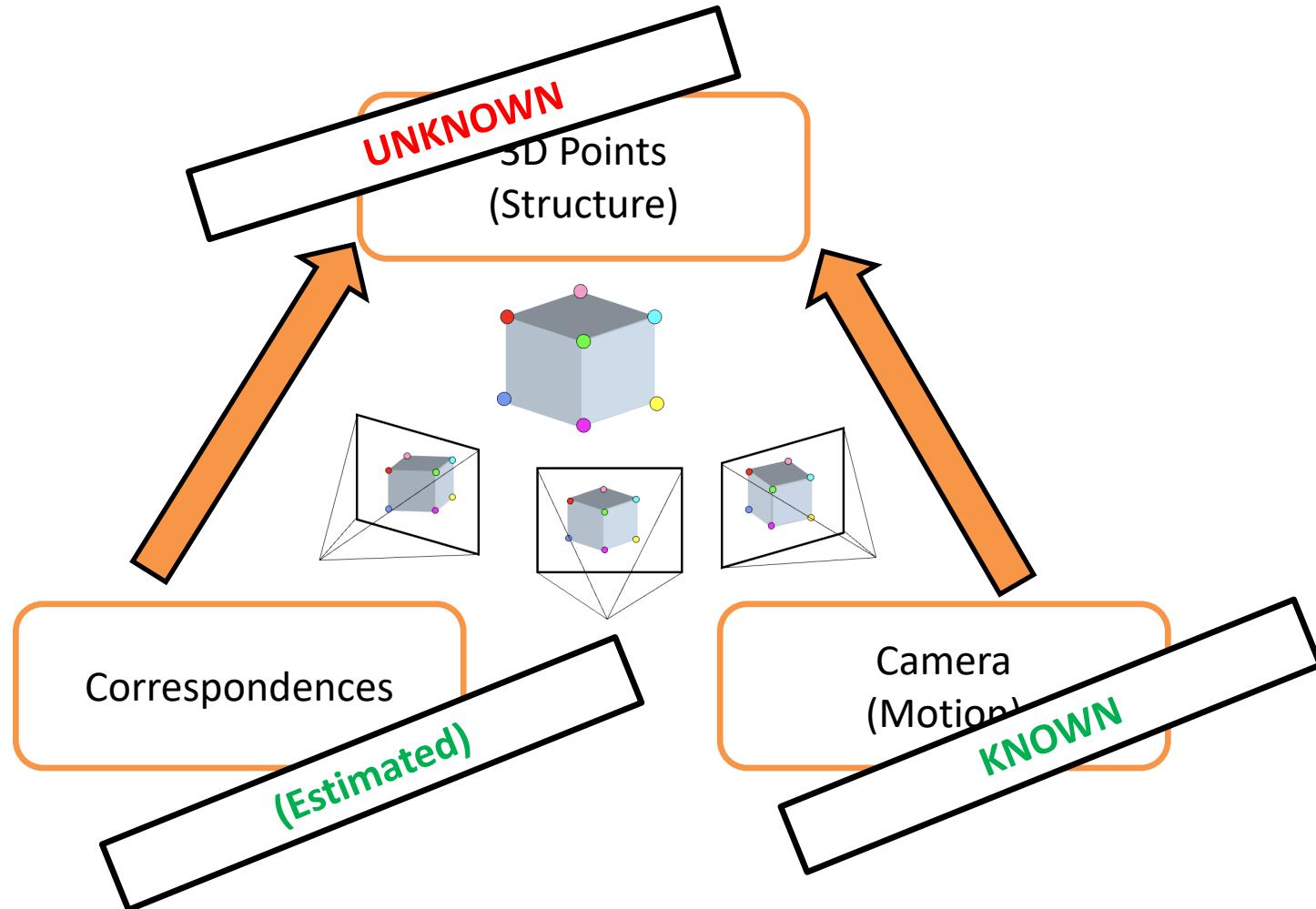
3D Points
(Structure)



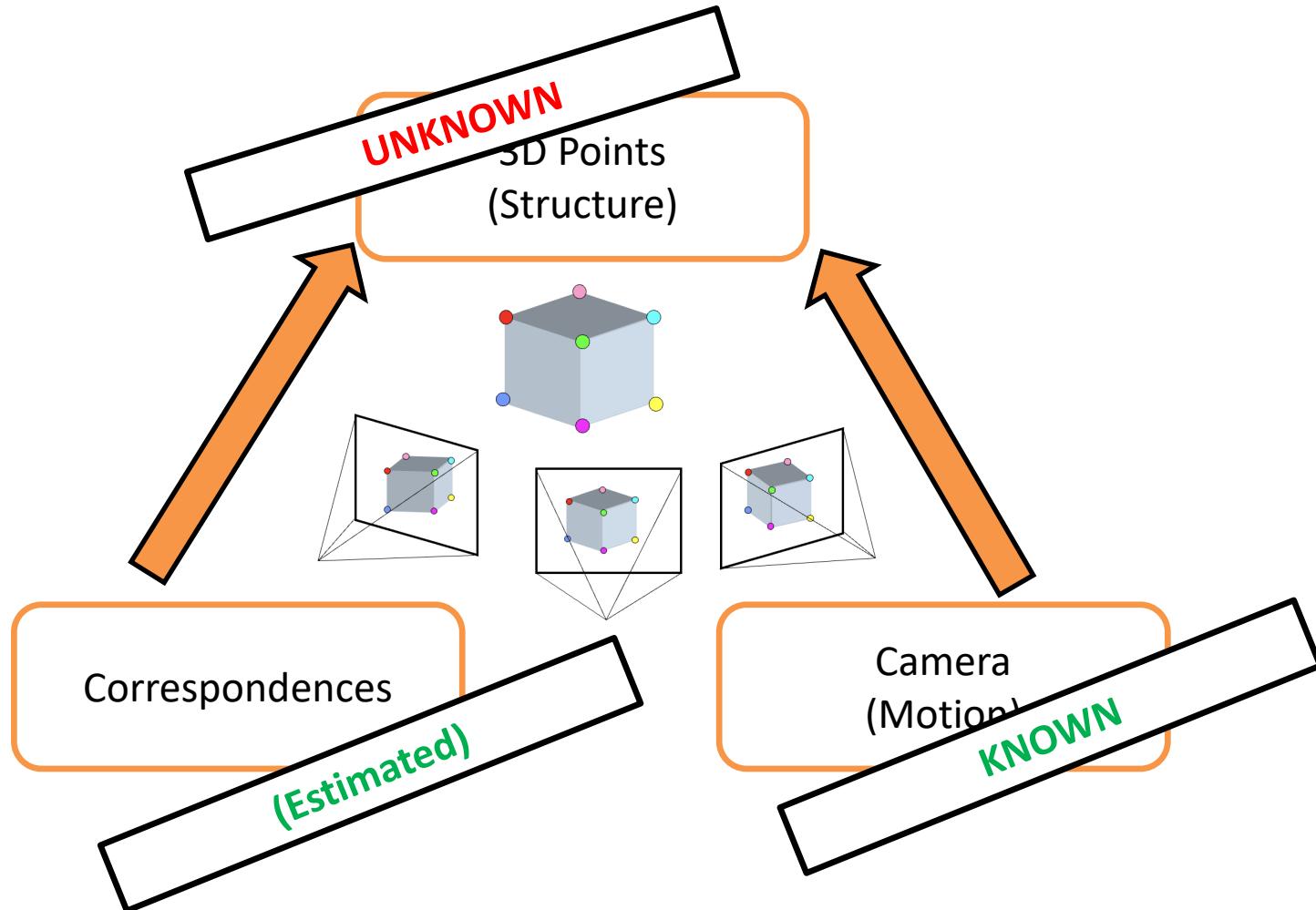
Correspondences

Camera
(Motion)

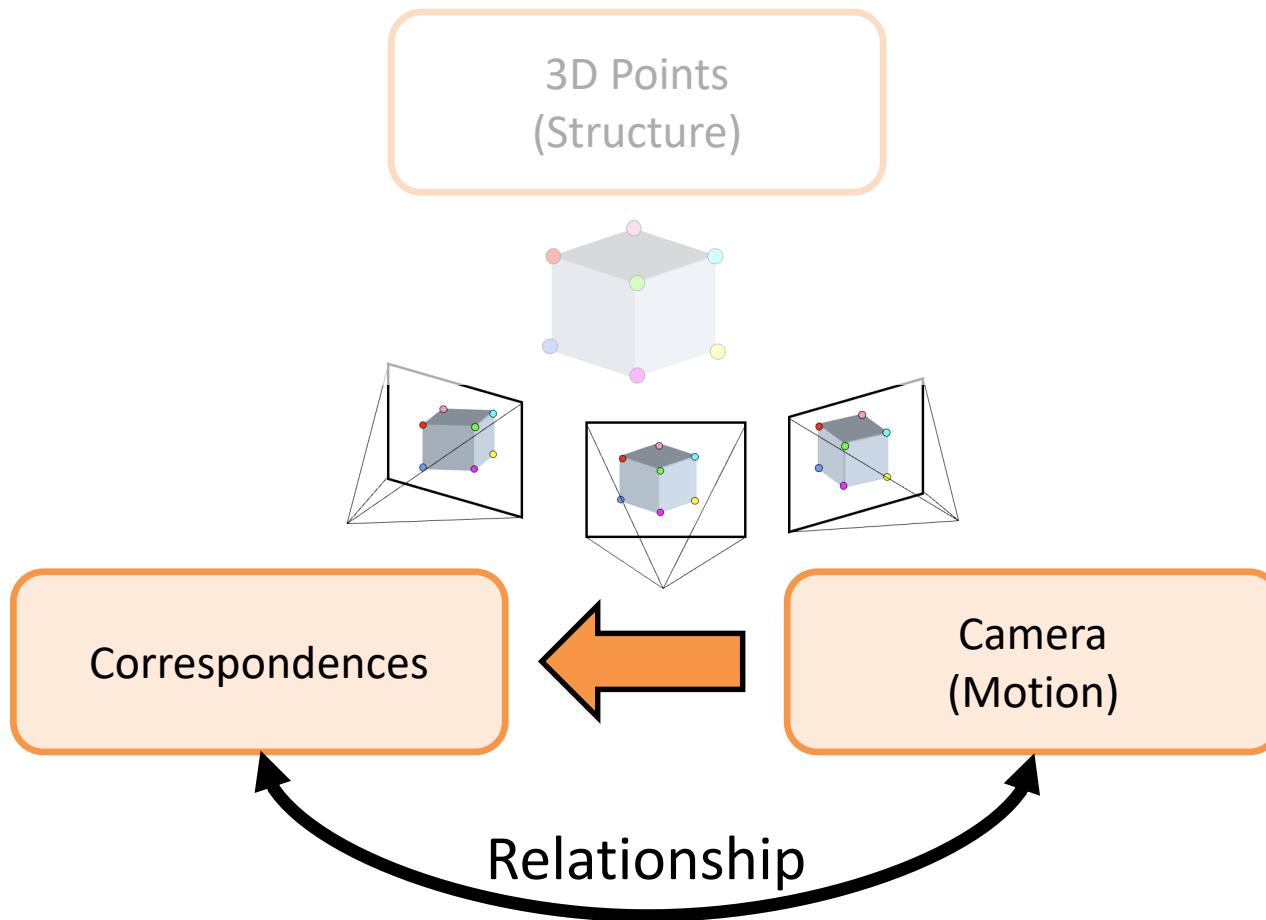
Simple Stereo; Corresp + Camera = Disparity = depth^{-1}



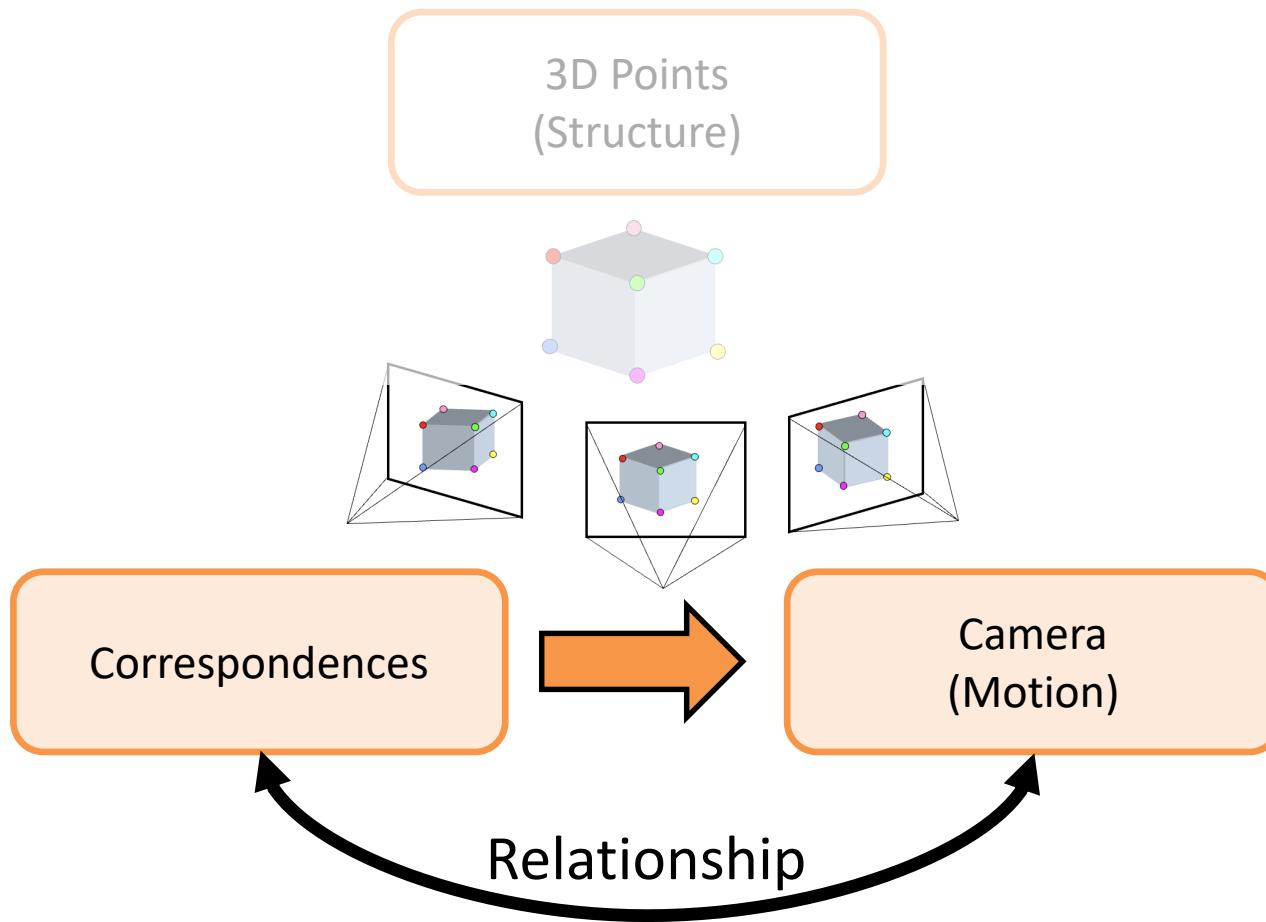
Today: Arbitrary Stereo; Corresp + Camera – Triangulation = depth



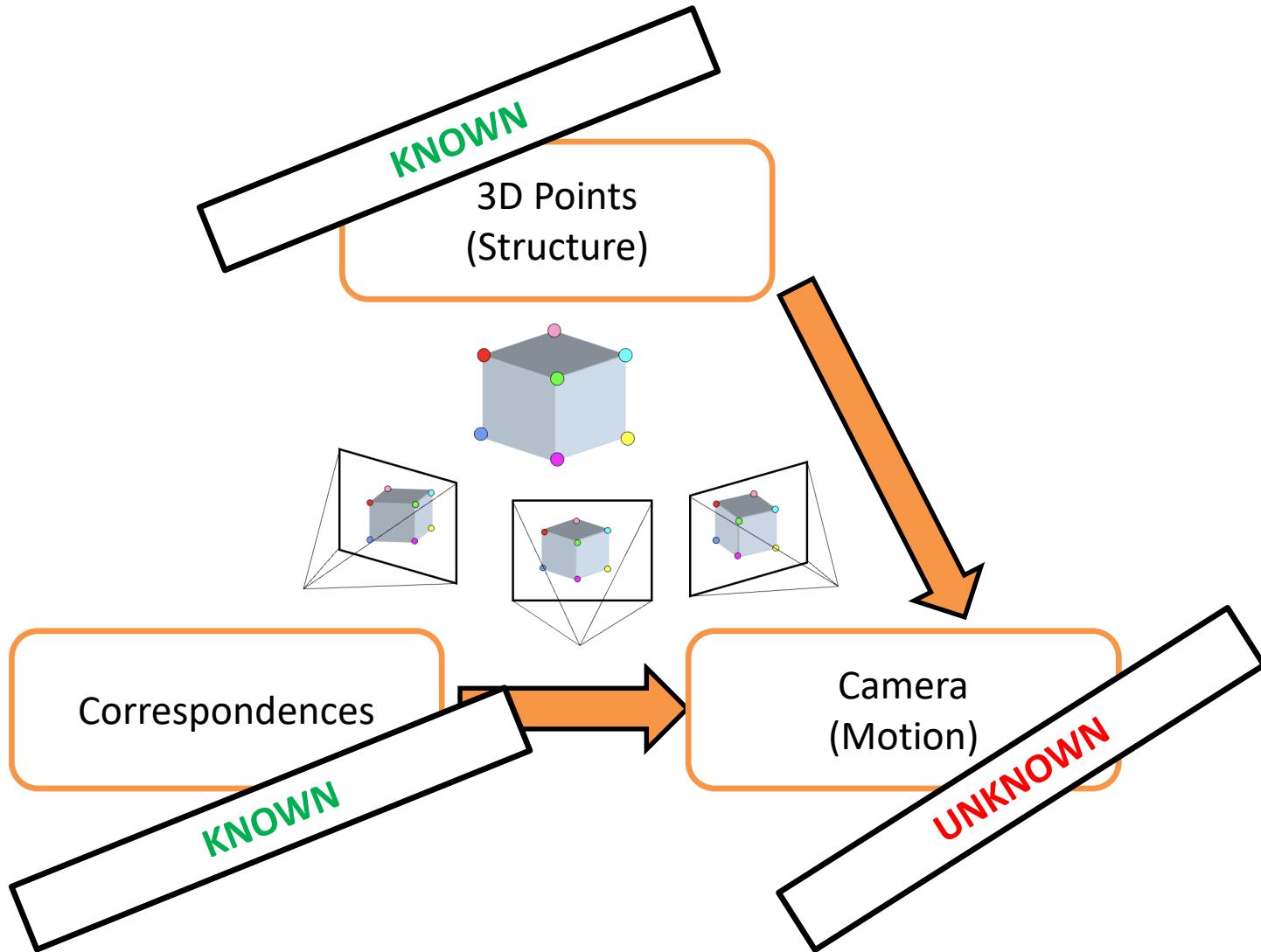
Camera helps Correspondence: Epipolar Geometry



Correspondence gives camera: Epipolar Geometry

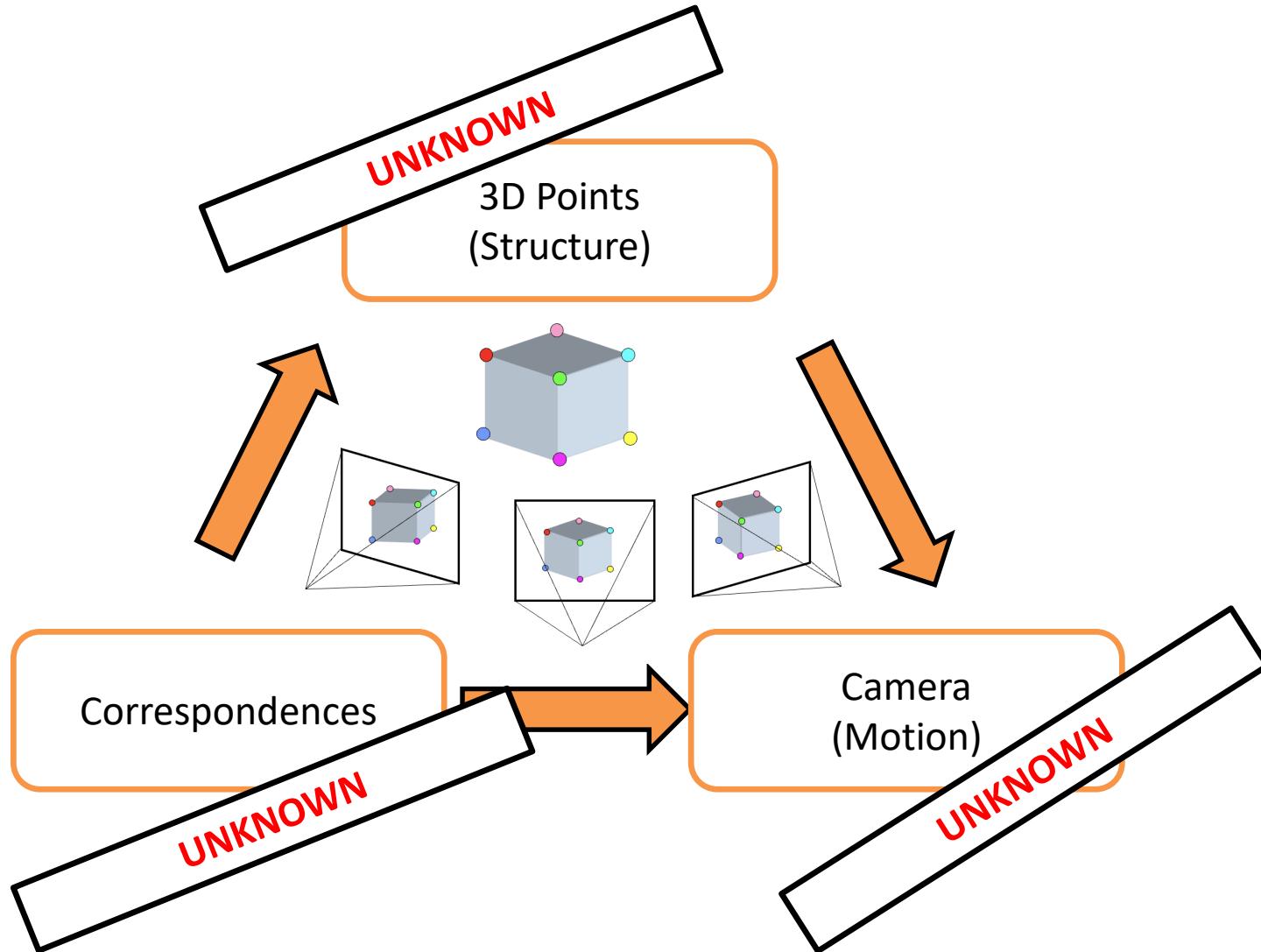


Next: Camera Calibration

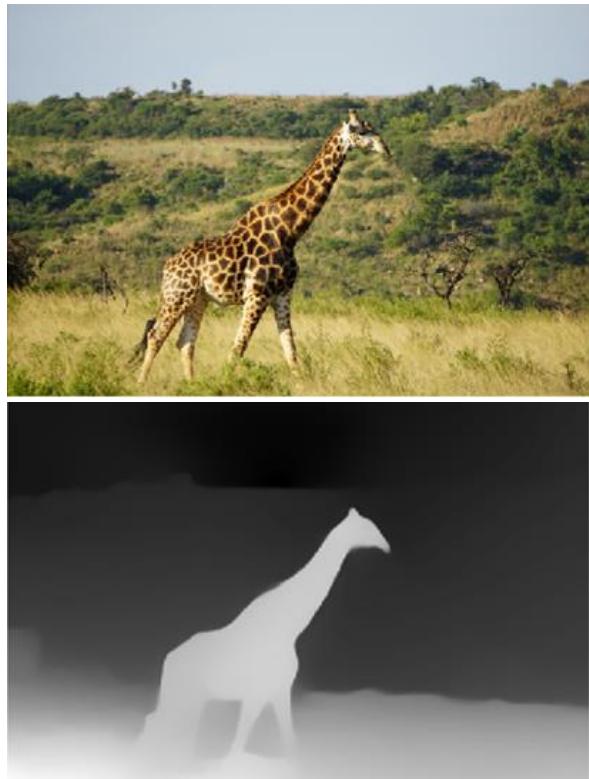


After that: Structure-from-Motion

Everything* Unknown



What Depth Map provides



warping the pixel based on its depth as you change the views



Monocular Depth Prediction [Ranftl et al. PAMI'20]

More cool things with Depth



Niklaus et al. ToG 2019



Shih et al. CVPR 2020

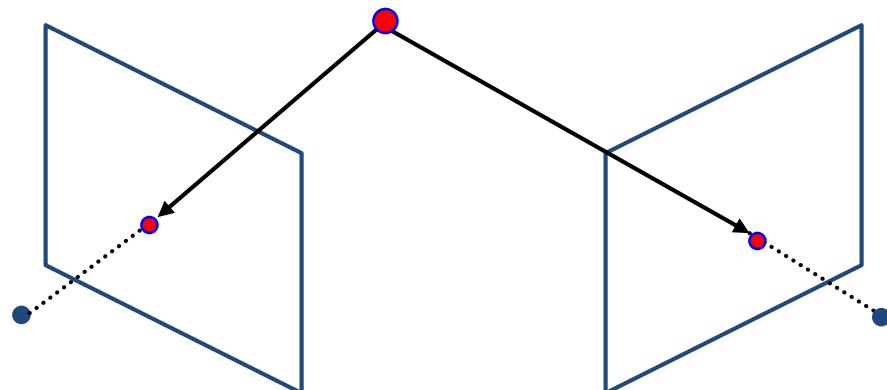
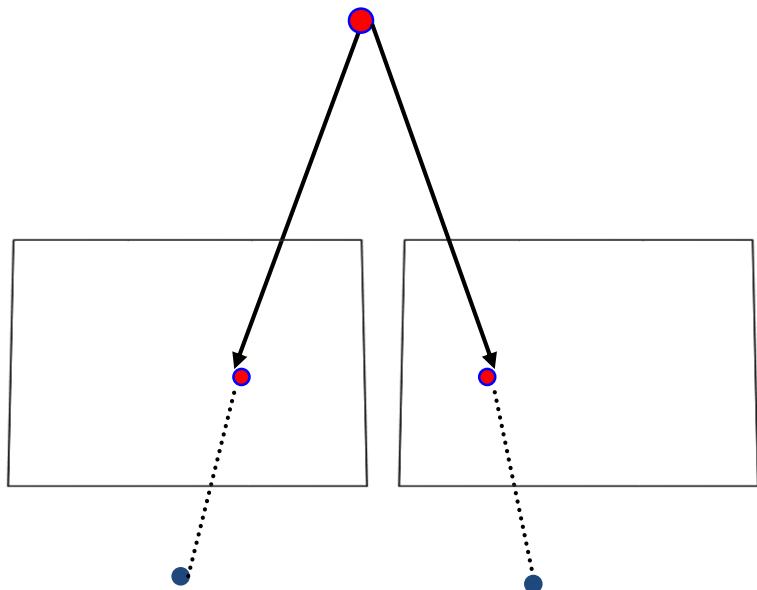
3D photo



AR

Next: General case

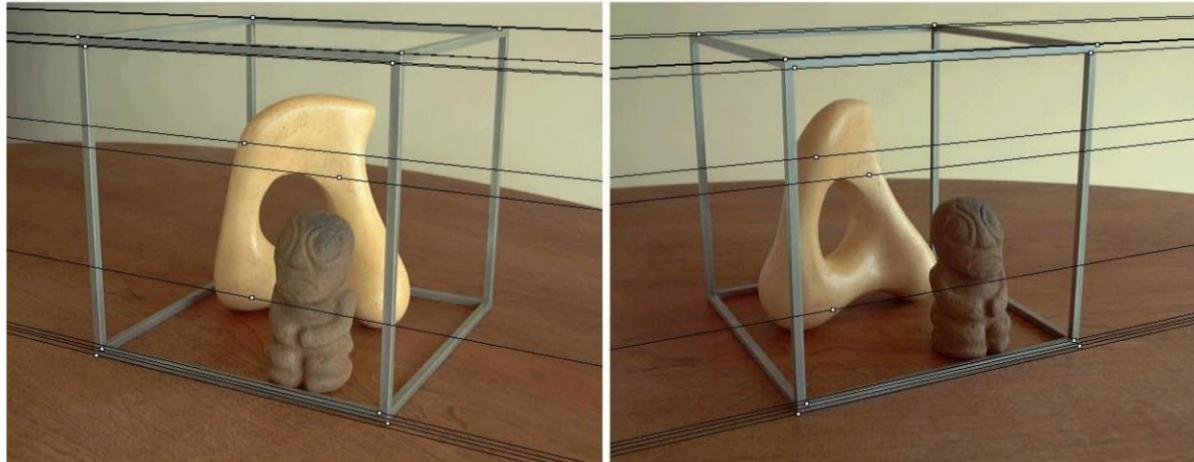
- The two cameras need not have parallel optical axes.
- Assume camera intrinsics are calibrated



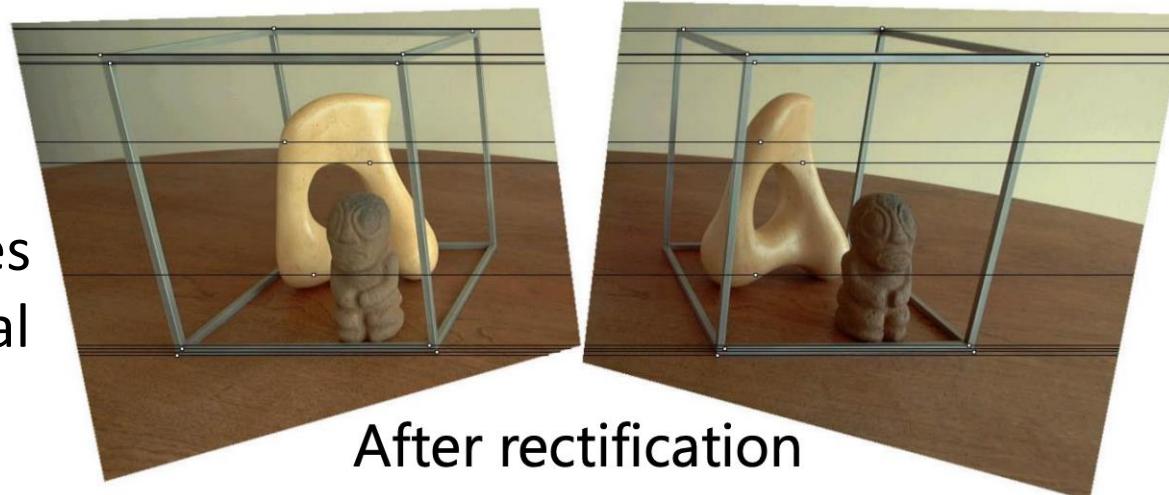
Same hammer:

Find the correspondences, then solve for structure

Option 1: Rectify via homography



Original stereo pair



After rectification

Then find
correspondences
on the horizontal
scan line

General case, known camera, find depth:

Option 2

1. Find correspondences
2. Triangulate

General case, known camera, find depth:

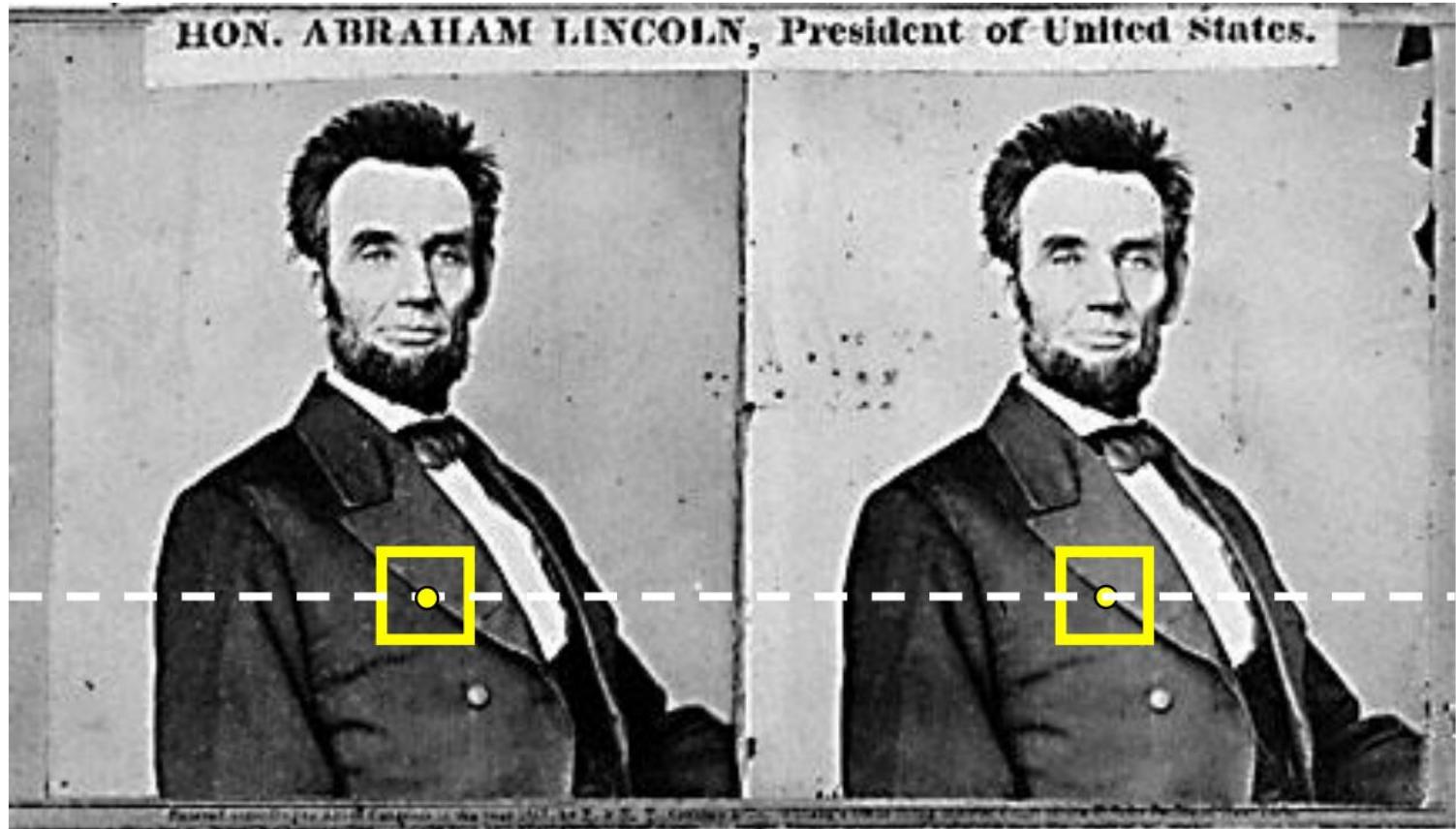
Option 2

- 1. Find correspondences**
- 2. Triangulate**

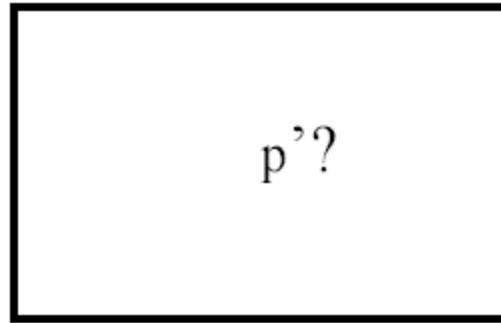
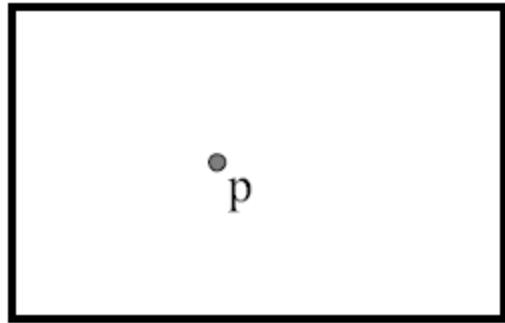
Can we restrict the search space again to 1D?

What is the relationship between the camera +
the corresponding points?

Where do epipolar lines come from?

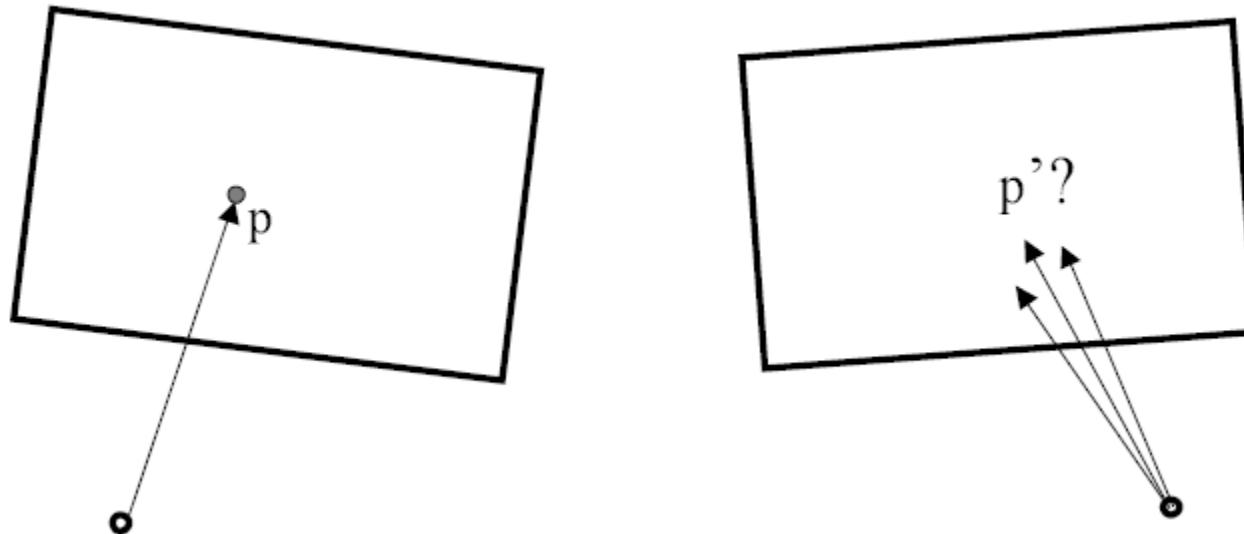


Stereo correspondence constraints



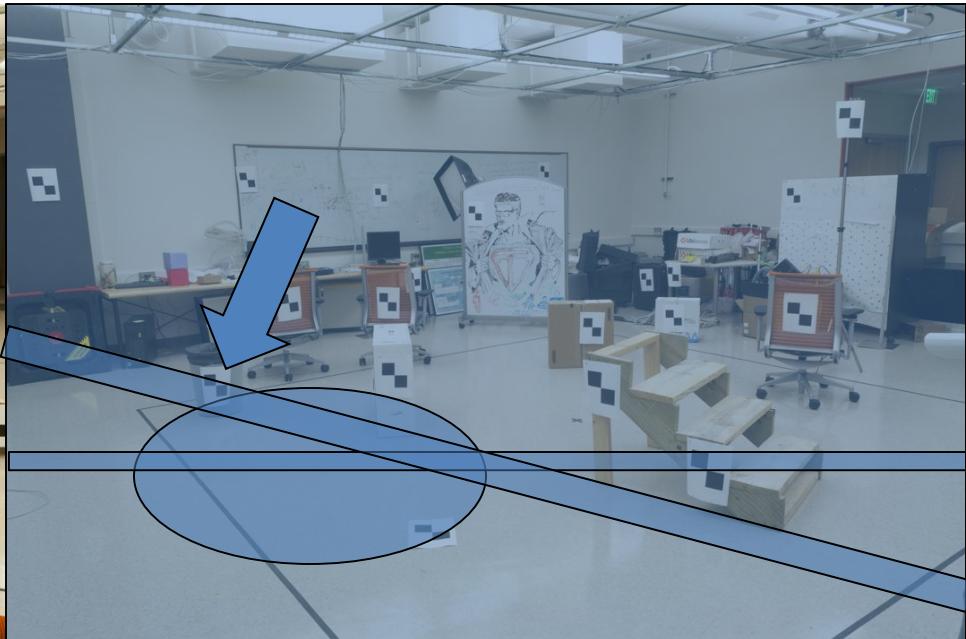
- Given p in left image, where can corresponding point p' be?

Stereo correspondence constraints

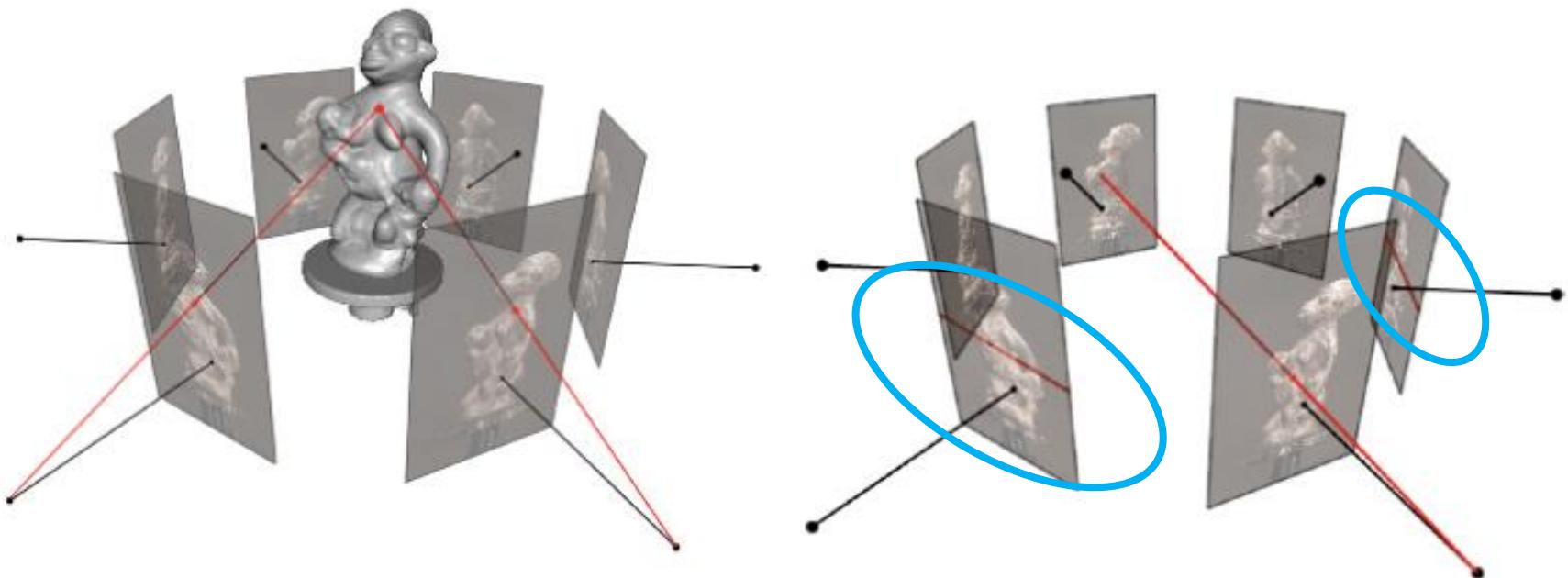


- Given p in left image, where can corresponding point p' be?

Where do we need to search?



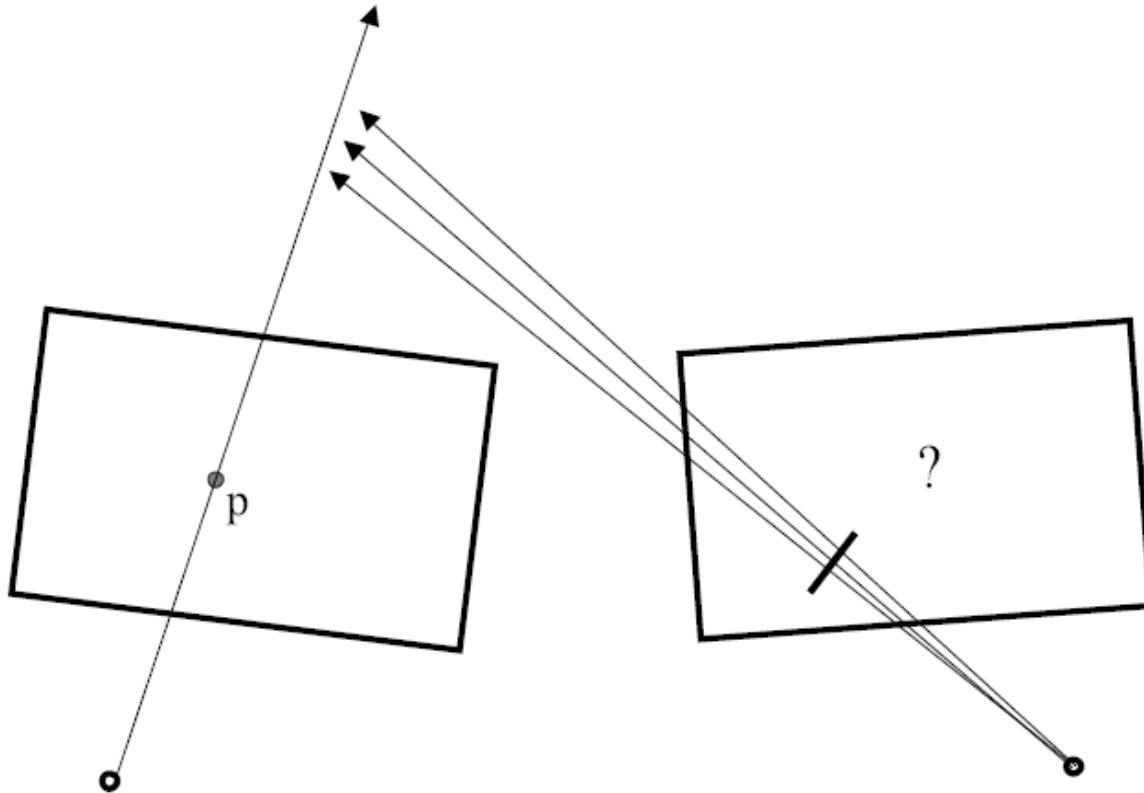
Epipolar Geometry



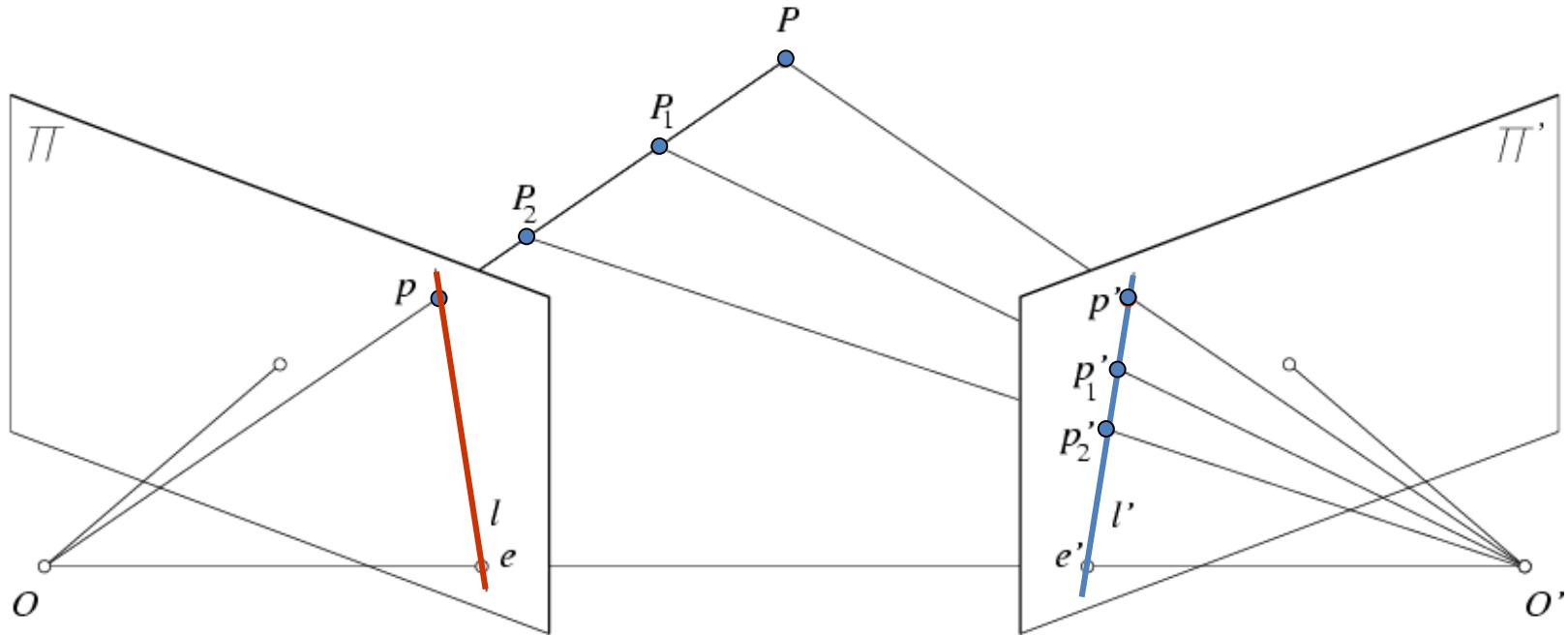
Figures by Carlos Hernandez

If you get confused with the following math,
look at this picture again, it just describes this.

Stereo correspondence constraints

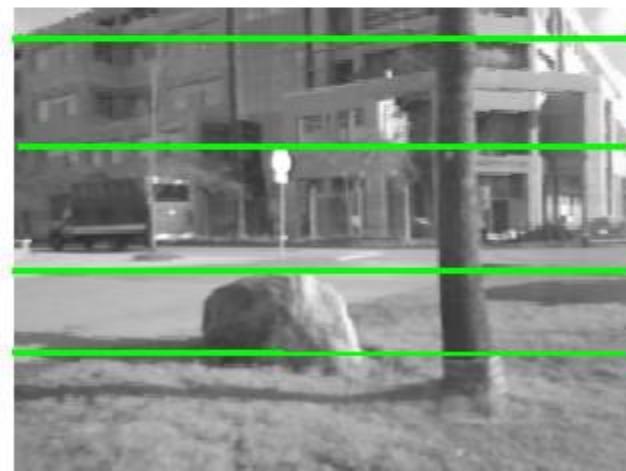


Epipolar constraint

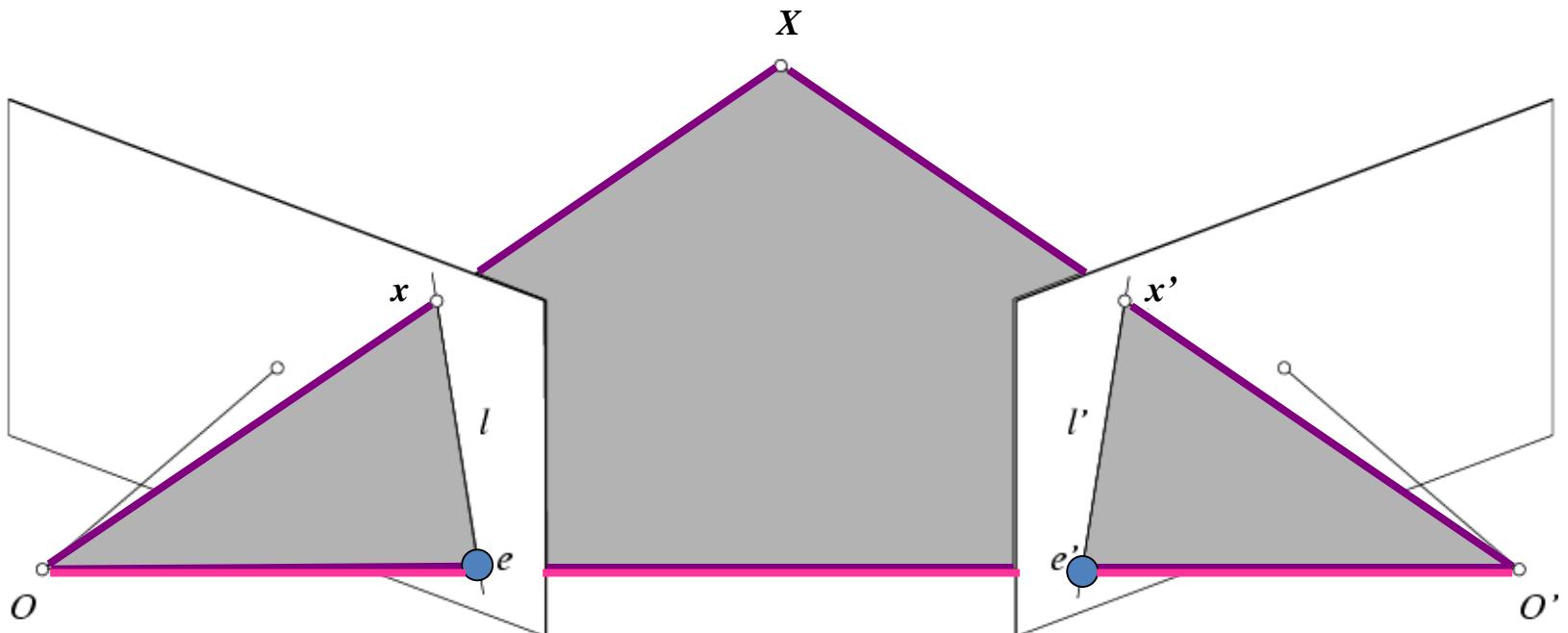


- Potential matches for p have to lie on the corresponding epipolar line l' .
- Potential matches for p' have to lie on the corresponding epipolar line l .

Example



Parts of Epipolar geometry



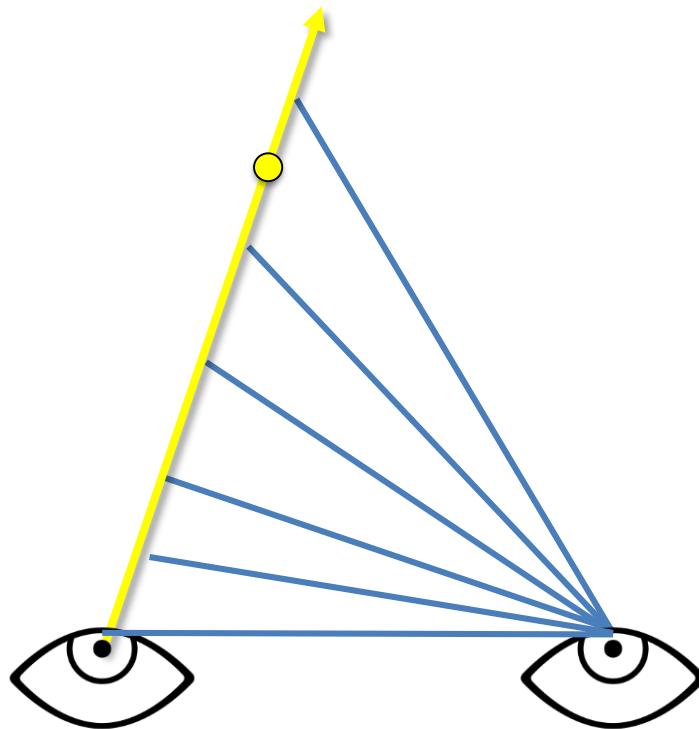
- **Baseline** – line connecting the two camera centers
- **Epipolar Plane** – plane containing baseline (1D family)
- **Epipoles**
 - = intersections of baseline with image planes
 - = projections of the other camera center
 - = vanishing points of the baseline

The Epipole

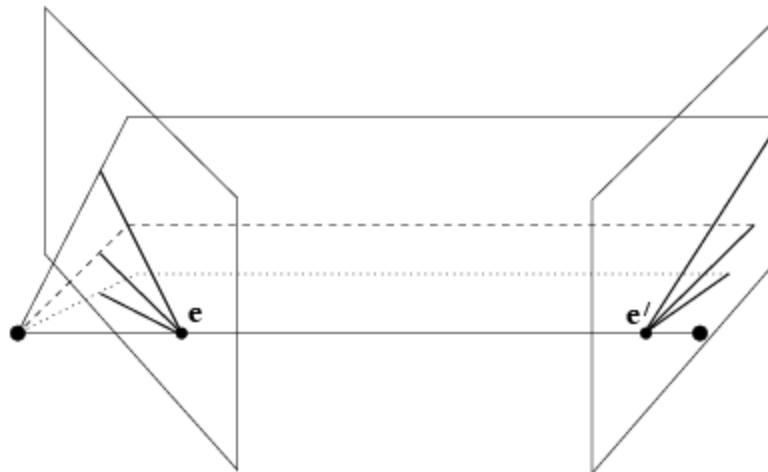


Photo by Frank Dellaert

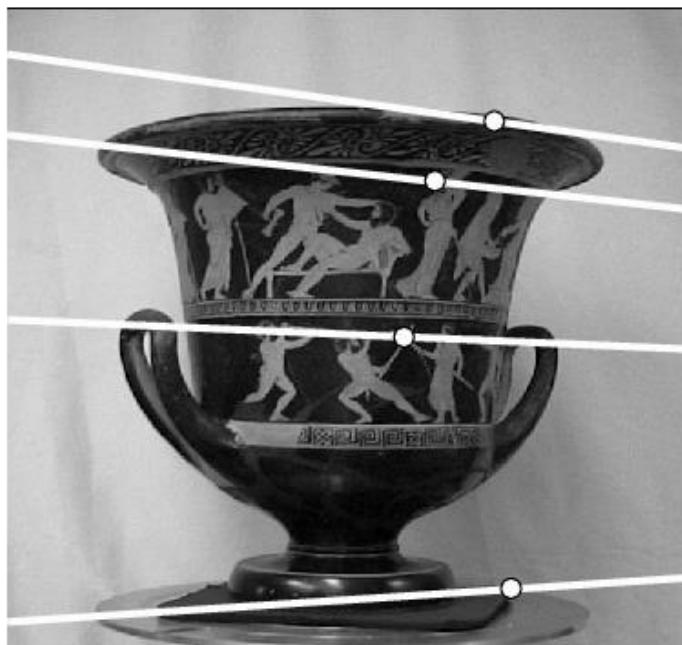
The epipolar line has
to go through the epipole



Example: converging cameras

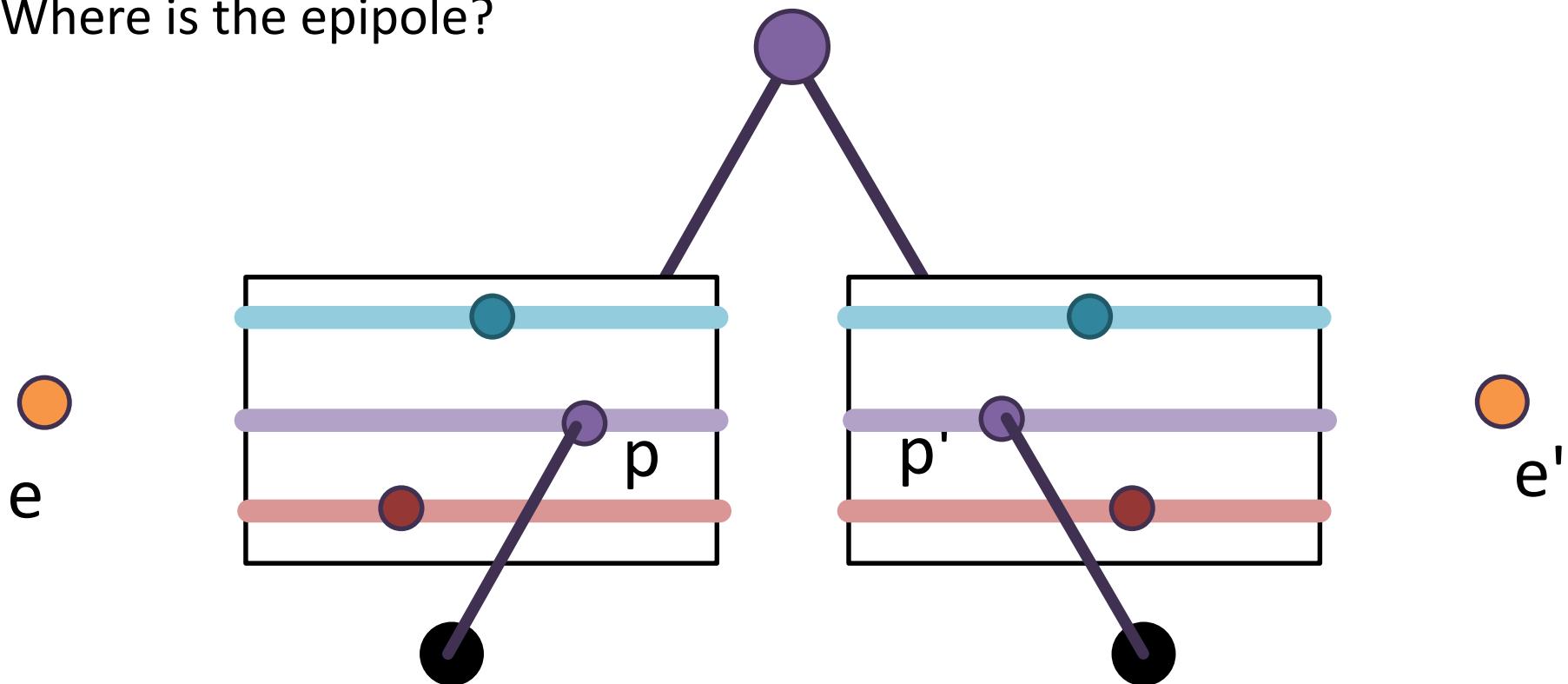


As position of 3d point varies, epipolar lines “rotate” about the baseline



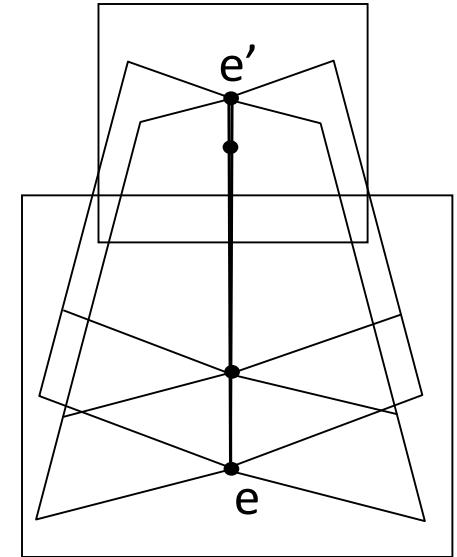
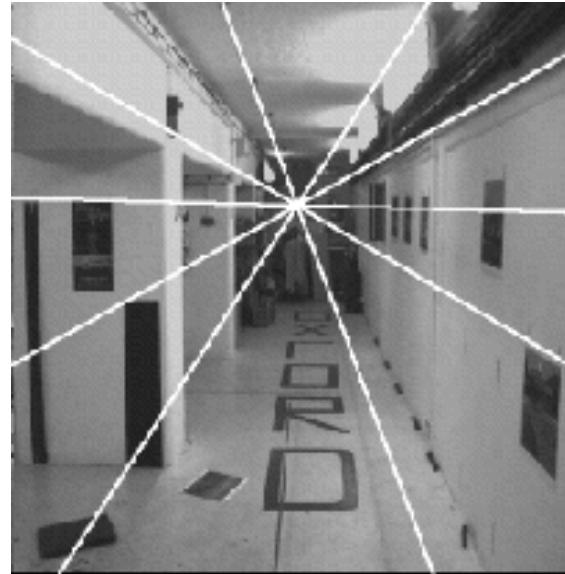
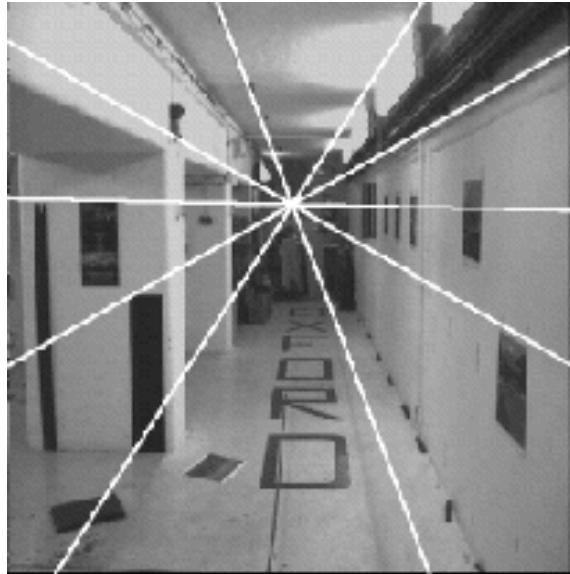
Example: Parallel to Image Plane

Where is the epipole?



Epipoles *infinitely* far away, epipolar lines parallel

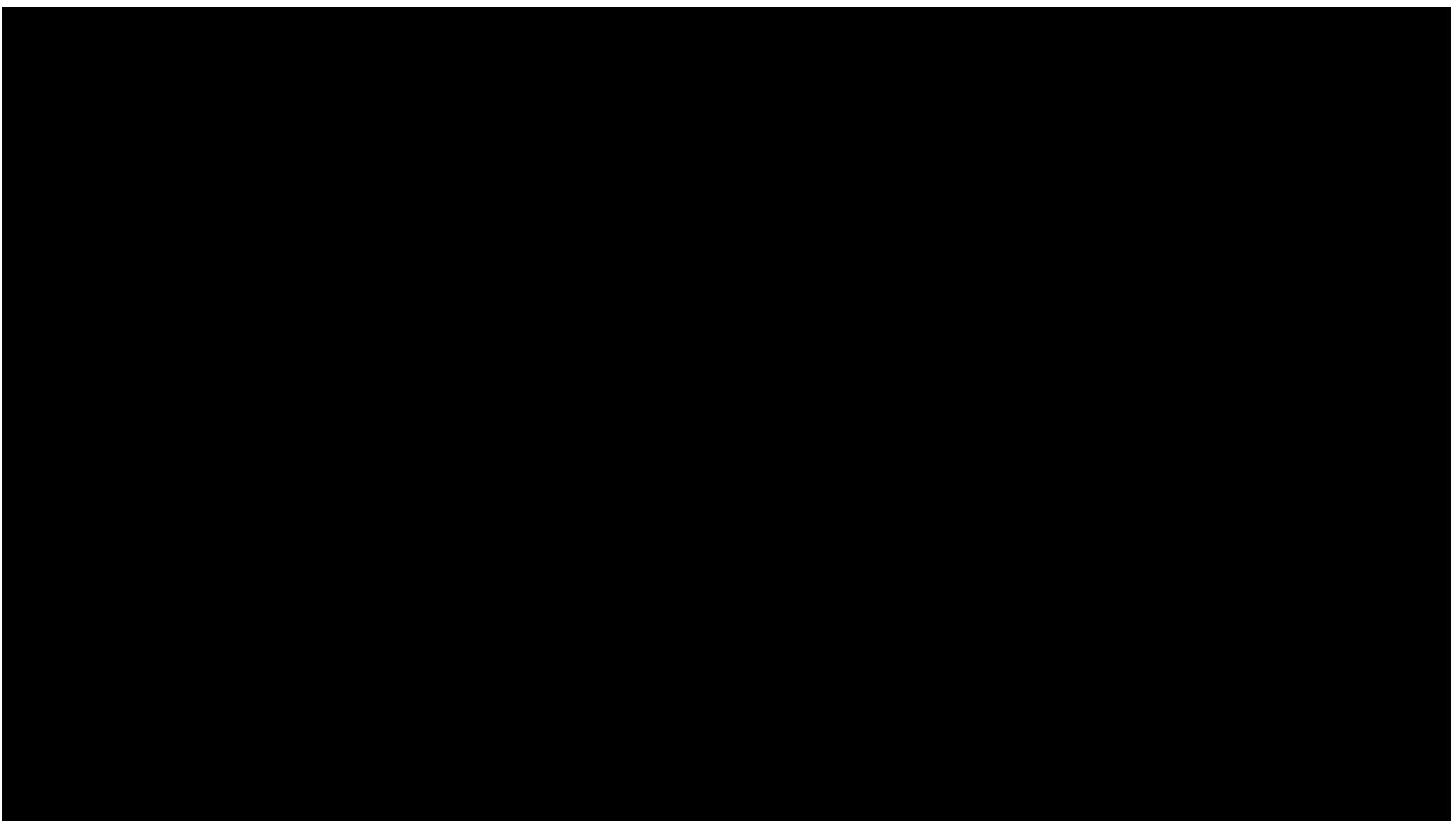
Example: forward motion



Epipole has same coordinates in both images.

Points move along lines radiating from e: “Focus of expansion”

Motion perpendicular to image plane

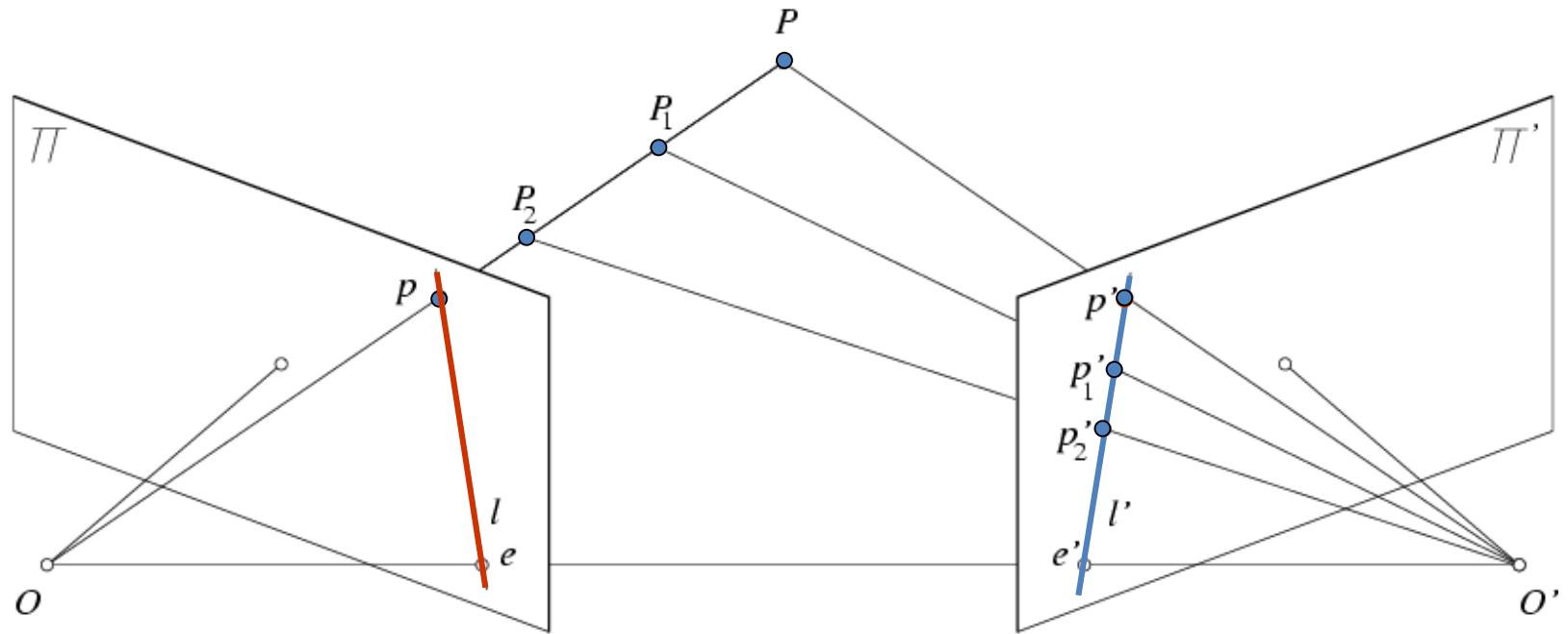


<http://vimeo.com/48425421>

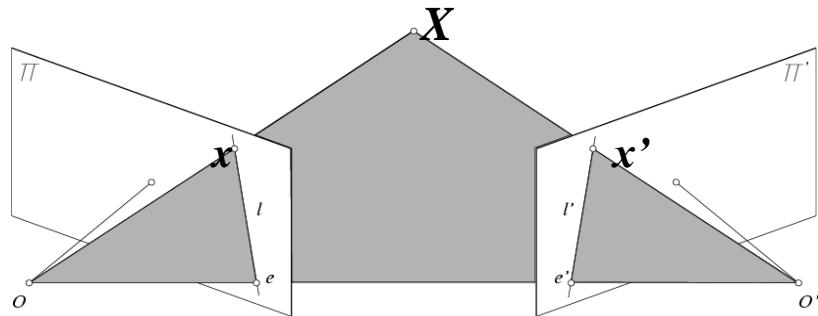
Ok so where were we?

- Setup: Calibrated Camera (both extrinsic & intrinsic)
- Goal: 3D reconstruction of corresponding points in the image
- We need to find correspondences!
 - 1D search along the epipolar line!
 - Need: Compute the epipolar line from camera

Ok so what exactly are l and l'?



Step 0: Factor out intrinsics



$$x = K[R \ t]X$$

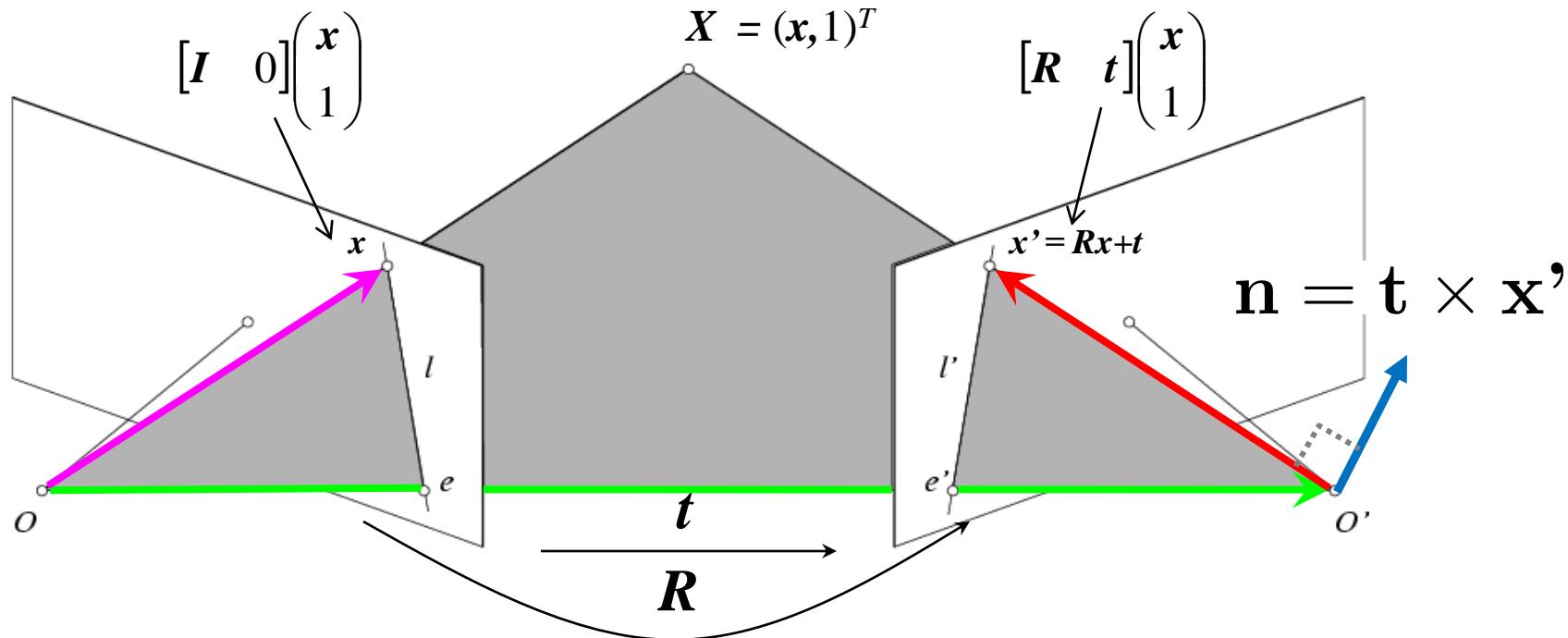
$$K^{-1}x = [R \ t]X$$

- Let's factor out the effect of K (do everything in 3D)
- Make it into a ray with K^{-1} and use depth = 1
- This is called the *normalized* image coordinates. It may be thought of as a set of points with identity K

$$x_{\text{norm}} = K^{-1}x_{\text{pixel}} = [I \ 0]X, \quad x'_{\text{norm}} = K'^{-1}x'_{\text{pixel}} = [R \ t]X$$

- Assume that the points are normalized from here on

Epipolar constraint: Calibrated case



The vectors \mathbf{x} , \mathbf{t} , and \mathbf{x}' are coplanar

What can you say about their relationships, given $\mathbf{n} = \mathbf{t} \times \mathbf{x}'$?

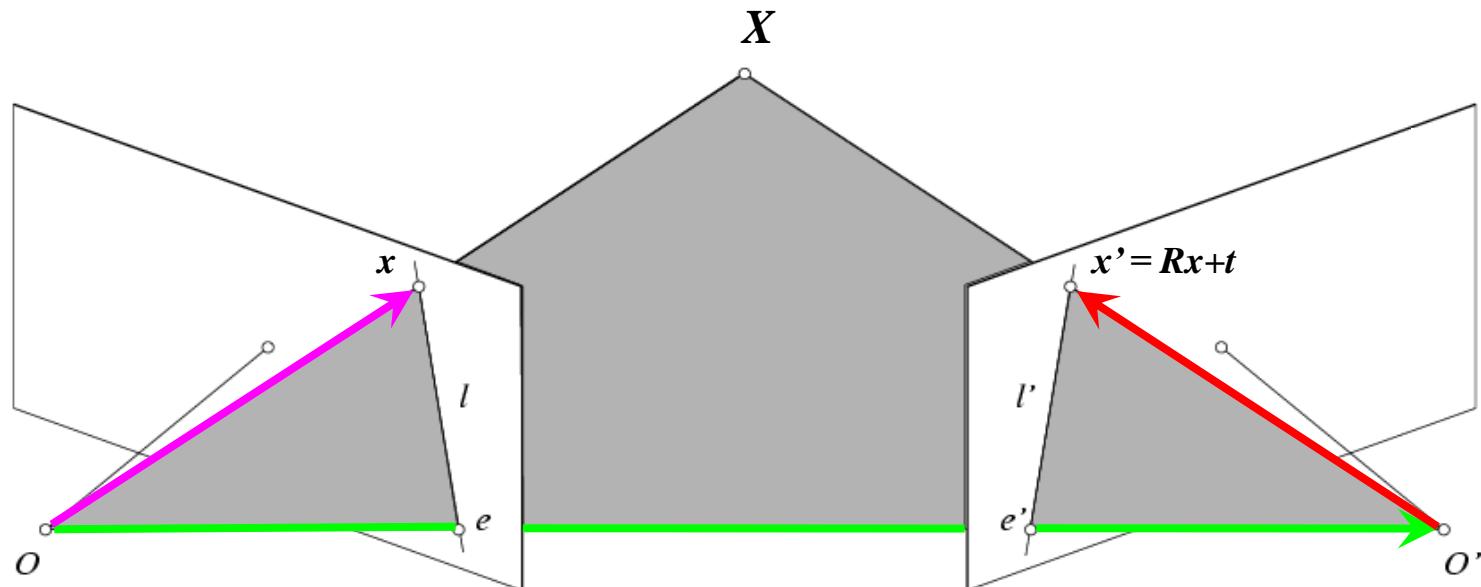
$$\mathbf{x}' \cdot (\mathbf{t} \times \mathbf{x}') = 0$$

$$\mathbf{x}' \cdot (\mathbf{t} \times (R\mathbf{x} + \mathbf{t})) = 0$$

$$\mathbf{x}' \cdot (\mathbf{t} \times R\mathbf{x} + \mathbf{t} \times \mathbf{t}) = 0$$

$$\mathbf{x}' \cdot (\mathbf{t} \times R\mathbf{x}) = 0$$

Epipolar constraint: Calibrated case

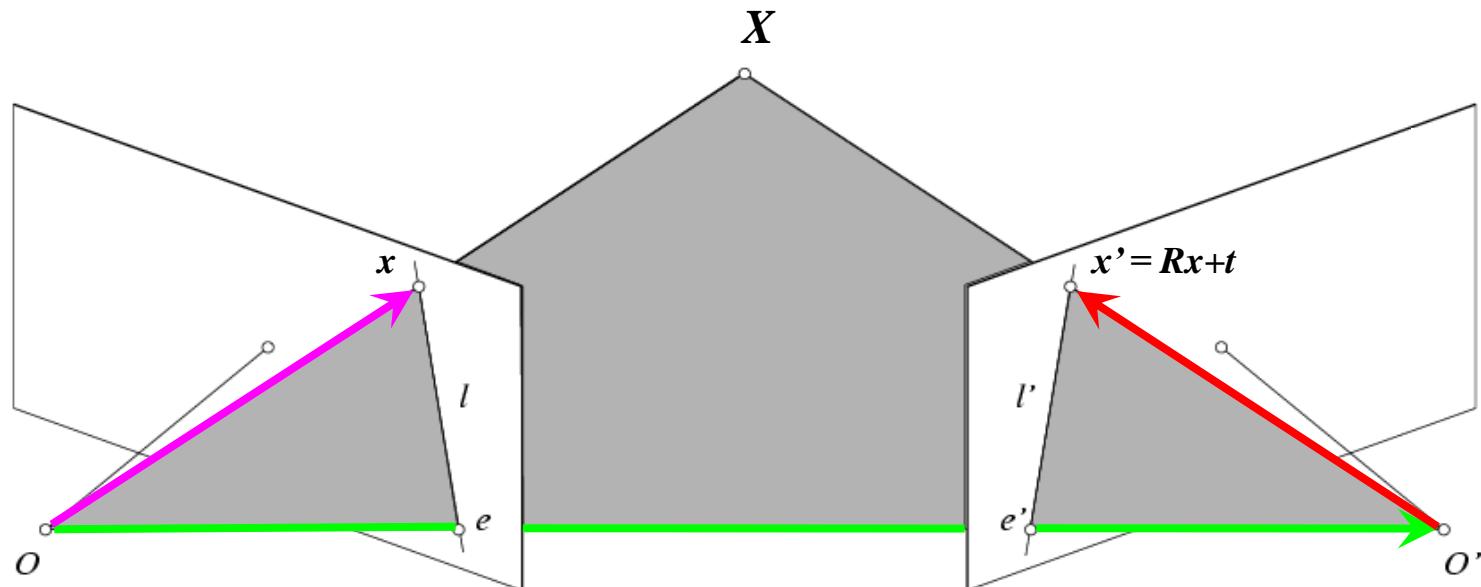


$$x' \cdot [t \times (Rx)] = 0 \quad \rightarrow \quad x^T [t^T R] Rx = 0$$

Recall: $\mathbf{a} \times \mathbf{b} = \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix} \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix} = [\mathbf{a}_x] \mathbf{b}$

The vectors x , t , and x' are coplanar

Epipolar constraint: Calibrated case



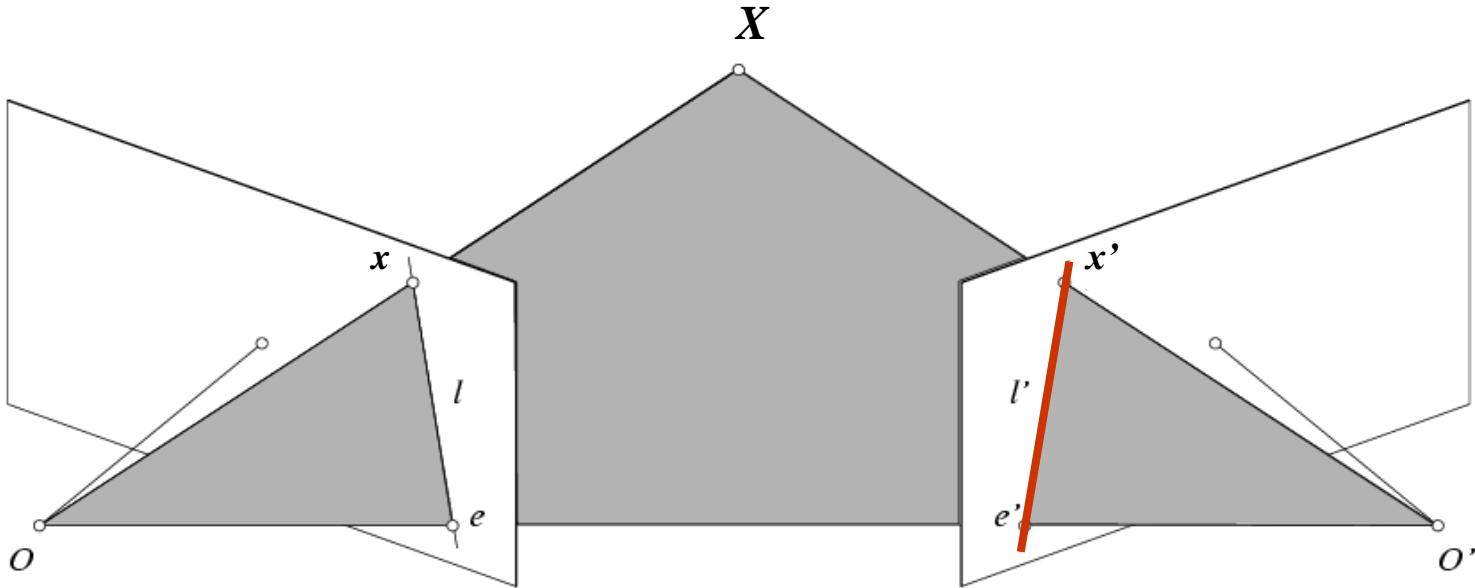
$$x' \cdot [t \times (Rx)] = 0 \quad \rightarrow \quad x^T \underbrace{[t \times]}_E Rx = 0 \quad \rightarrow \quad x^T E x = 0$$

Recall: $\mathbf{a} \times \mathbf{b} = \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix} \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix} = [\mathbf{a}] \mathbf{b}$

Essential Matrix
(Longuet-Higgins, 1981)

The vectors x , t , and x' are coplanar

Epipolar constraint: Calibrated case

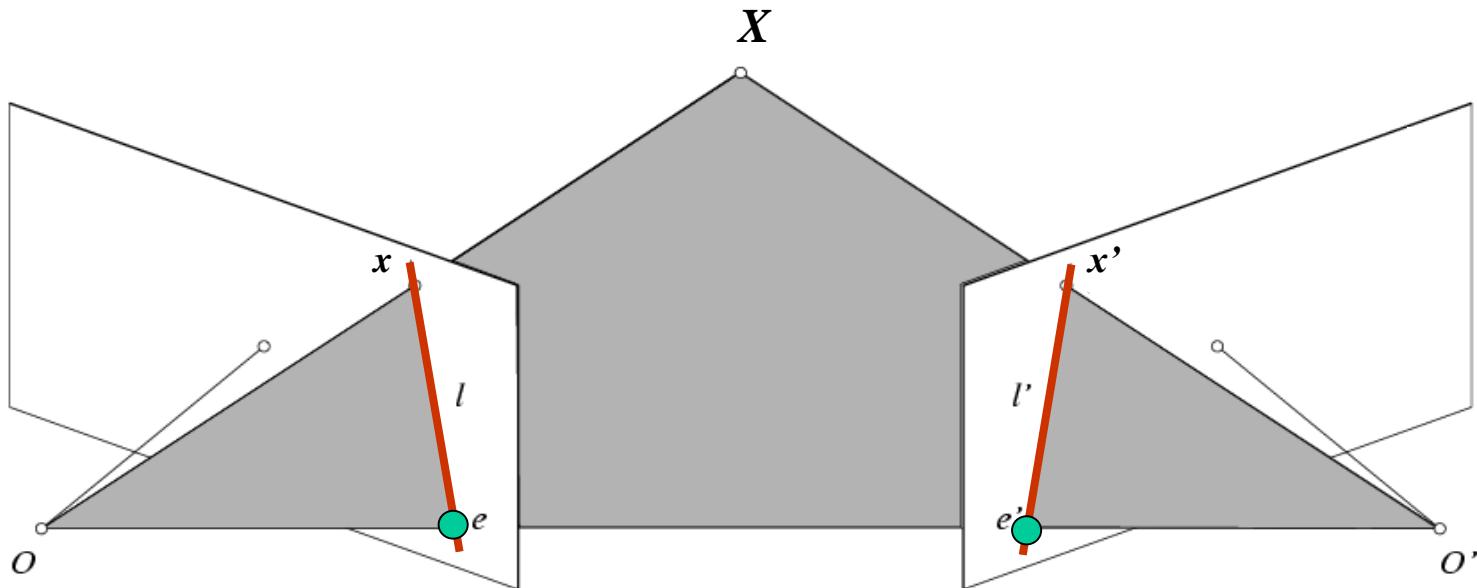


$$\mathbf{x}^T \mathbf{E} \mathbf{x} = 0$$

- $\mathbf{E} \mathbf{x}$ is the epipolar line associated with \mathbf{x} ($\mathbf{l}' = \mathbf{E} \mathbf{x}$)
 - Recall: a line is given by $ax + by + c = 0$ or

$$\mathbf{l}^T \mathbf{x} = 0 \quad \text{where} \quad \mathbf{l} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Epipolar constraint: Calibrated case



$$\mathbf{x}^T \mathbf{E} \mathbf{x} = 0$$

- $\mathbf{E} \mathbf{x}$ is the epipolar line associated with \mathbf{x} ($\mathbf{l}' = \mathbf{E} \mathbf{x}$)
- $\mathbf{E}^T \mathbf{x}'$ is the epipolar line associated with \mathbf{x}' ($\mathbf{l} = \mathbf{E}^T \mathbf{x}'$)
- $\mathbf{E} \mathbf{e} = 0$ and $\mathbf{E}^T \mathbf{e}' = 0$
- \mathbf{E} is singular (rank two)
- \mathbf{E} has five degrees of freedom

Why is the epipolar matrix rank 2?

Question: With known camera (i.e. essential matrix),
can you map a point on the left image to the right?

A: No! You don't know where exactly the point is,
just that it will be on a line.

2D point → 1D line, lost a dimension

Why? Because camera does not give you depth.
You need depth for precise point correspondance

$R, T \sim$ Essential Matrix

Recall, knowing the camera gives you the essential matrix (i.e. the plane per point)

So the DoF has to match up

Essential matrix: 3×3 , 9 numbers, but rank 2 means 2 columns fully define = 6 parameters
-1 for scale = 5 DoF

Extrinsic Camera (R, T): 3 for rotation, 3 for translation, but -1 for scale = 5 DoF!

Epipolar constraint: Uncalibrated case

- Recall that we normalized the coordinates

$$x = K^{-1}\hat{x} \quad x' = K'^{-1}\hat{x}'$$

$$\hat{x} = \begin{bmatrix} u \\ v \\ 1 \end{bmatrix}$$

where \hat{x} is the image coordinates

- But in the *uncalibrated* case, K and K' are unknown!
- We can write the epipolar constraint in terms of *unknown* normalized coordinates:

$$x'^T E x = 0$$

$$(K'^{-1}\hat{x}')^T E (K^{-1}\hat{x}) = 0$$

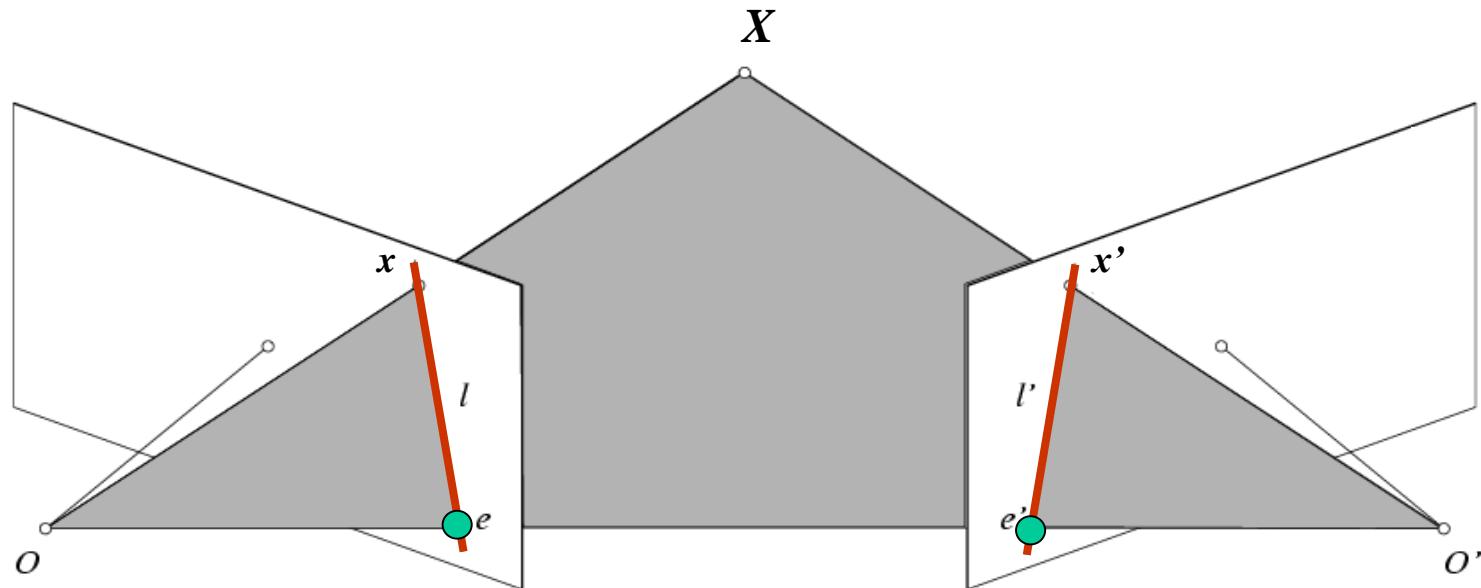
$$F = K'^{-T} E K^{-1}$$

$$\hat{x}'^T \underbrace{K'^{-T} E (K^{-1}\hat{x})}_{\hat{x}'^T F \hat{x}} = 0$$

$$\hat{x}'^T F \hat{x} = 0$$

Fundamental Matrix
(Faugeras and Luong, 1992)

Epipolar constraint: Uncalibrated case



$$x'^T E x = 0 \quad \rightarrow \quad \hat{x}'^T F \hat{x} = 0 \quad \text{with} \quad F = K'^{-T} E K^{-1}$$

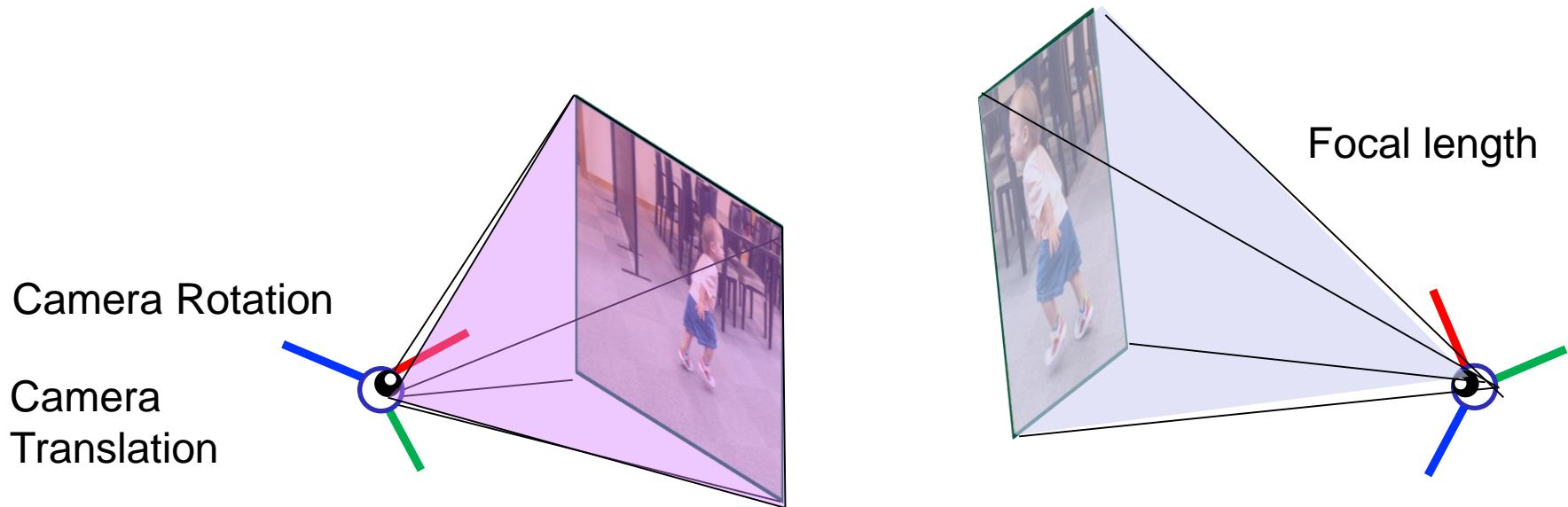
- $F \hat{x}$ is the epipolar line associated with \hat{x} ($I' = F \hat{x}$)
- $F^T \hat{x}'$ is the epipolar line associated with \hat{x}' ($I = F^T \hat{x}'$)
- $F \mathbf{e} = 0$ and $F^T \mathbf{e}' = 0$
- F is singular (rank two)
- F has seven degrees of freedom

Why 7 DoF?

Think about the frustum, this is the unknown!

It's a scaling factor on the frustum, thus 1 unknown for each camera

5 of Essential matrix + 2 = 7.



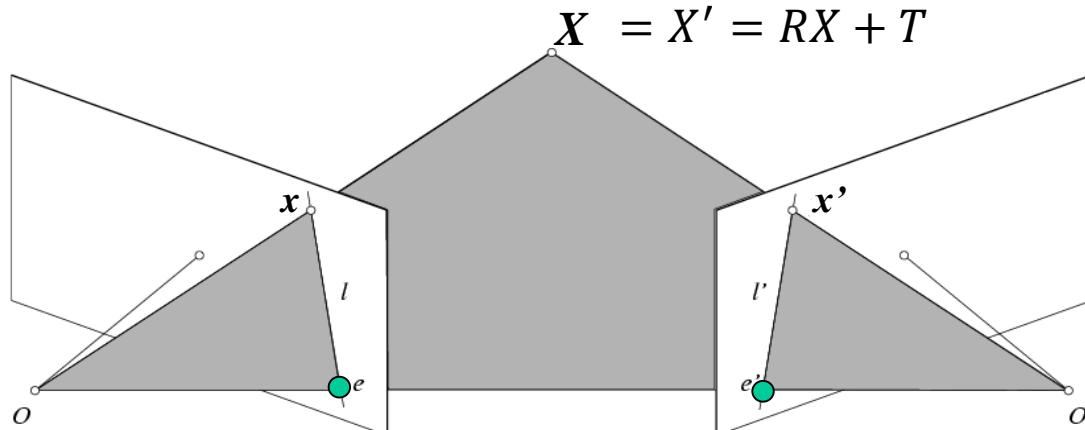
Where are we? (in the original setup)

We have two images with calibrated cameras, want the 3D points!

1. Solve for correspondences using epipolar constraints from known camera (1D search)
 - **Now we know the exact equation of this line**
2. Triangulate to get depth!

Finally: computing depth by triangulation

We know about the camera, K_1 , K_2 and $[R \ t]$:



and found the corresponding points: $x \leftrightarrow x'$

$$x = K \boxed{X}$$

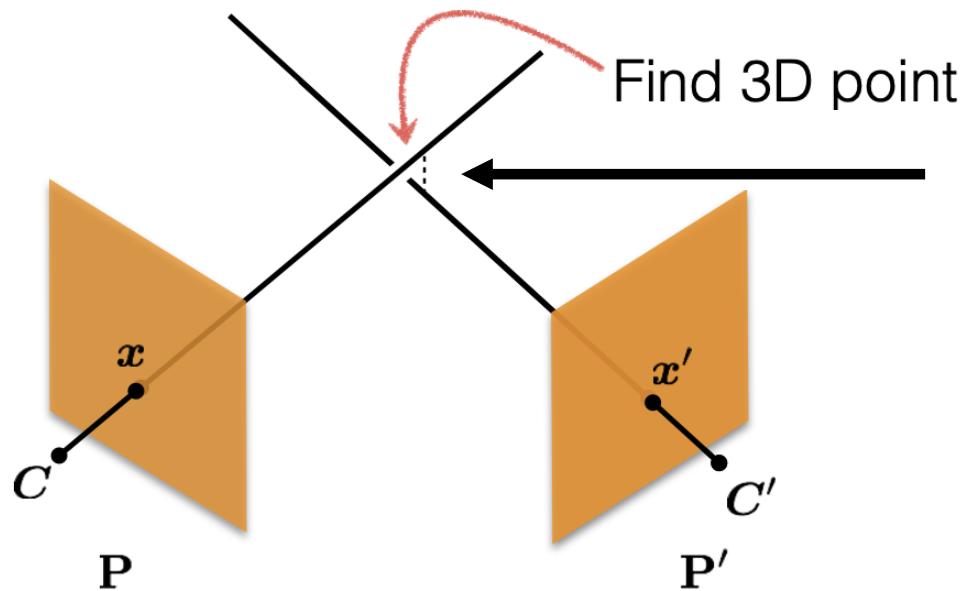
$$\begin{aligned} x' &= K' X' \\ &= K' (R \boxed{X} + T) \end{aligned}$$

How many unknowns
+ how many equations
do we have?

only unknowns!

Solve by formulating
 $Ax=0$, see H&Z ch.12

In practice the rays may not intersect!



Ray's don't always intersect
because of noise!!!

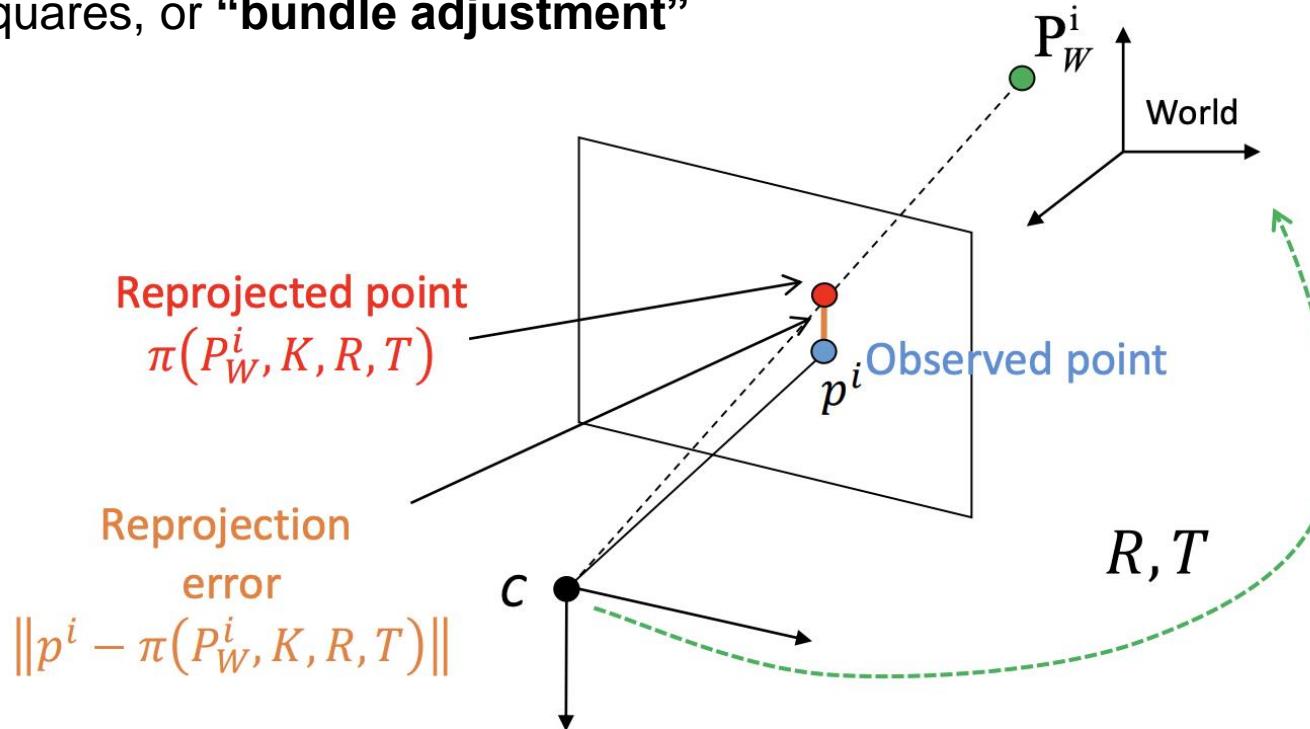
\mathbf{X} s.t.

$$\mathbf{x} = \mathbf{P}\mathbf{X}, \quad \mathbf{x}' = \mathbf{P}'\mathbf{X}$$

Reprojection error

Even if you do everything right, you will still be off because of noise, this is called the Reprojection Error

In practice with noise, want to directly minimize this with non-linear least squares, or “**bundle adjustment**”

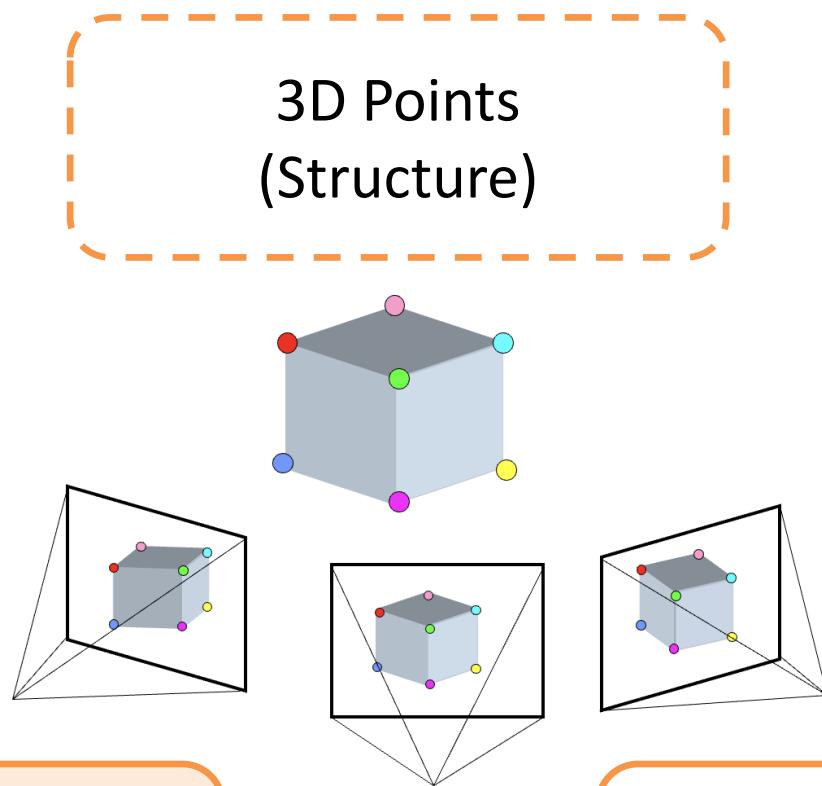


Solve with non-linear least squares, iteratively

Summary: Two-view, known camera

0. Assuming known camera intrinsics + extrinsics
1. Find correspondences:
 - Reduce this to 1D search with Epipolar Geometry!
2. Get depth:
 - If simple stereo, disparity (difference of corresponding points) is inversely proportional to depth
 - In the general case, triangulate.

What if we don't know the camera?



Correspondences



Camera
(Motion)

What if we don't know the camera?

Assume we know the correspondences (\hat{x}', \hat{x}) :
Solve for F s.t.

$$\hat{x}'^T F \hat{x} = 0 \quad \hat{x} = \begin{bmatrix} u \\ v \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} u' & v' & 1 \end{bmatrix} \begin{bmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{bmatrix} \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = 0$$

How many correspondences do we need?

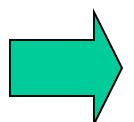
Estimating the fundamental matrix



The eight-point algorithm

$$\mathbf{x} = (u, v, 1)^T, \quad \mathbf{x}' = (u', v', 1)$$

$$[u' \quad v' \quad 1] \begin{bmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{bmatrix} \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = 0$$



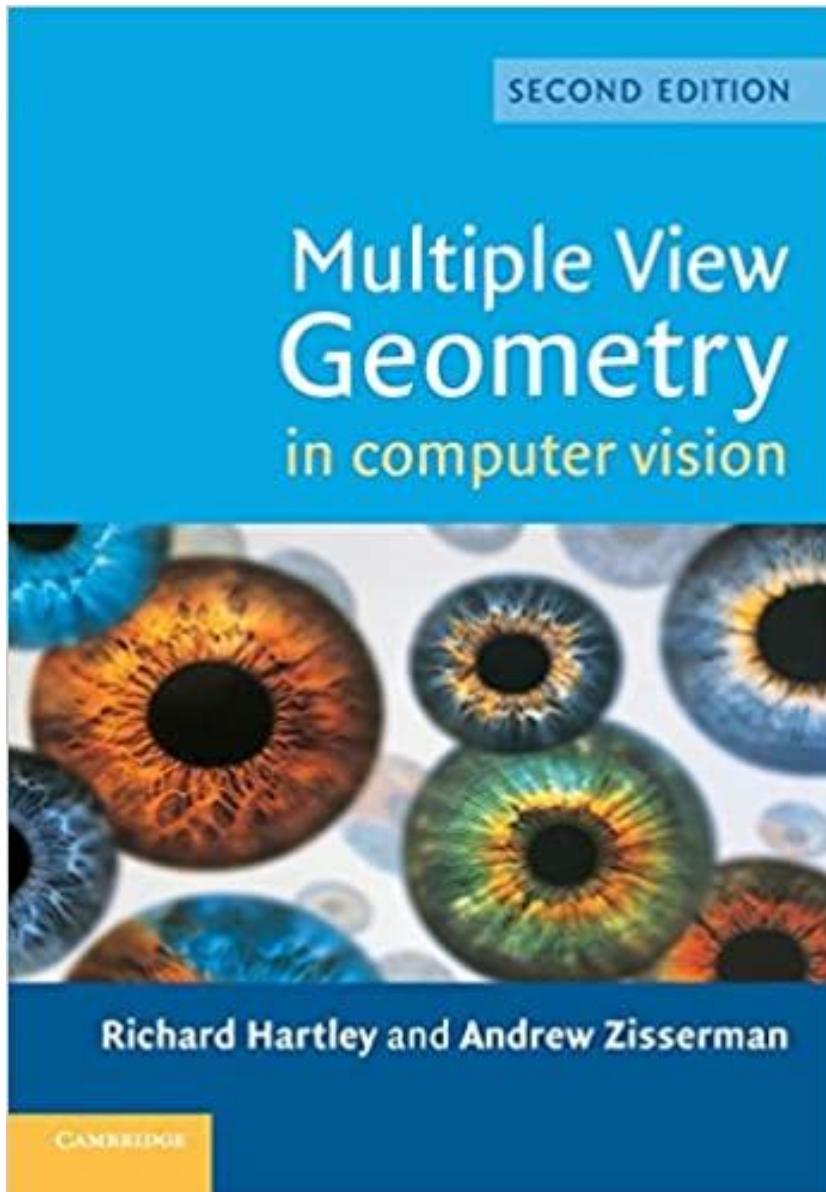
$$[u'u \quad u'v \quad u' \quad v'u \quad v'v \quad v' \quad u \quad v \quad 1] \begin{bmatrix} f_{11} \\ f_{12} \\ f_{13} \\ f_{21} \\ f_{22} \\ f_{23} \\ f_{31} \\ f_{32} \\ f_{33} \end{bmatrix} = 0$$

Solve homogeneous
linear system using
eight or more matches



Enforce rank-2
constraint (take SVD
of \mathbf{F} and throw out the
smallest singular value)

The Bible by Hartley & Zisserman



The Fundamental Matrix Song

In the other view passing through x-prime



<http://danielwedge.com/fmatrix/>

https://www.youtube.com/watch?time_continue=8&v=DgGV3I82NTk&feature=emb_title

Going from F -> E -> Camera

Get the essential matrix with K (or some estimates of K) ...

in practice you calibrate your cameras so you know K or have a very good estimate

$$E = K'^T F K.$$

Essential matrix can be decomposed

$$E = T_x R$$

If we know E, we can recover t and R

$$\begin{bmatrix} e_{11} & e_{12} & e_{13} \\ e_{21} & e_{22} & e_{23} \\ e_{31} & e_{32} & e_{33} \end{bmatrix} = \begin{bmatrix} 0 & -t_z & t_y \\ t_z & 0 & -t_x \\ -t_y & t_x & 0 \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

Given that T_x is a **Skew-Symmetric** matrix ($a_{ij} = -a_{ji}$) and R is an **Orthonormal** matrix, it is possible to “**decouple**” T_x and R from their product using “**Singular Value Decomposition**”.

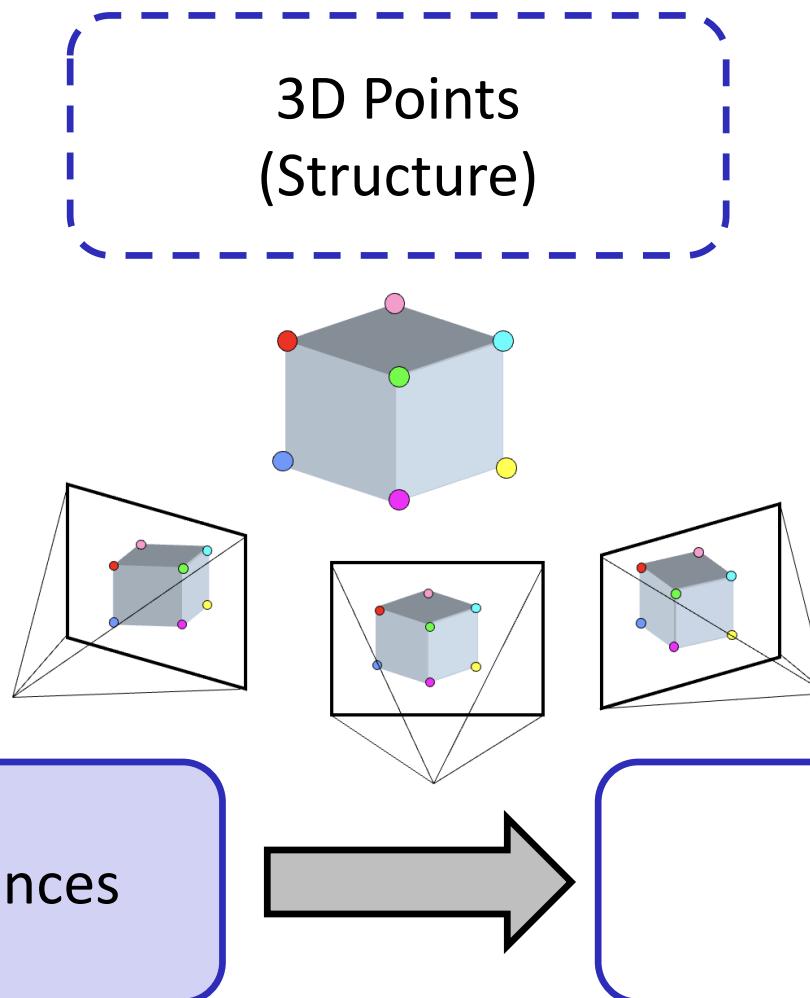
What about more than two views?

The geometry of three views is described by a $3 \times 3 \times 3$ tensor called the *trifocal tensor*

The geometry of four views is described by a $3 \times 3 \times 3 \times 3$ tensor called the *quadrifocal tensor*

After this it starts to get complicated...

This completes: Corresp to Camera



How to estimate the camera?

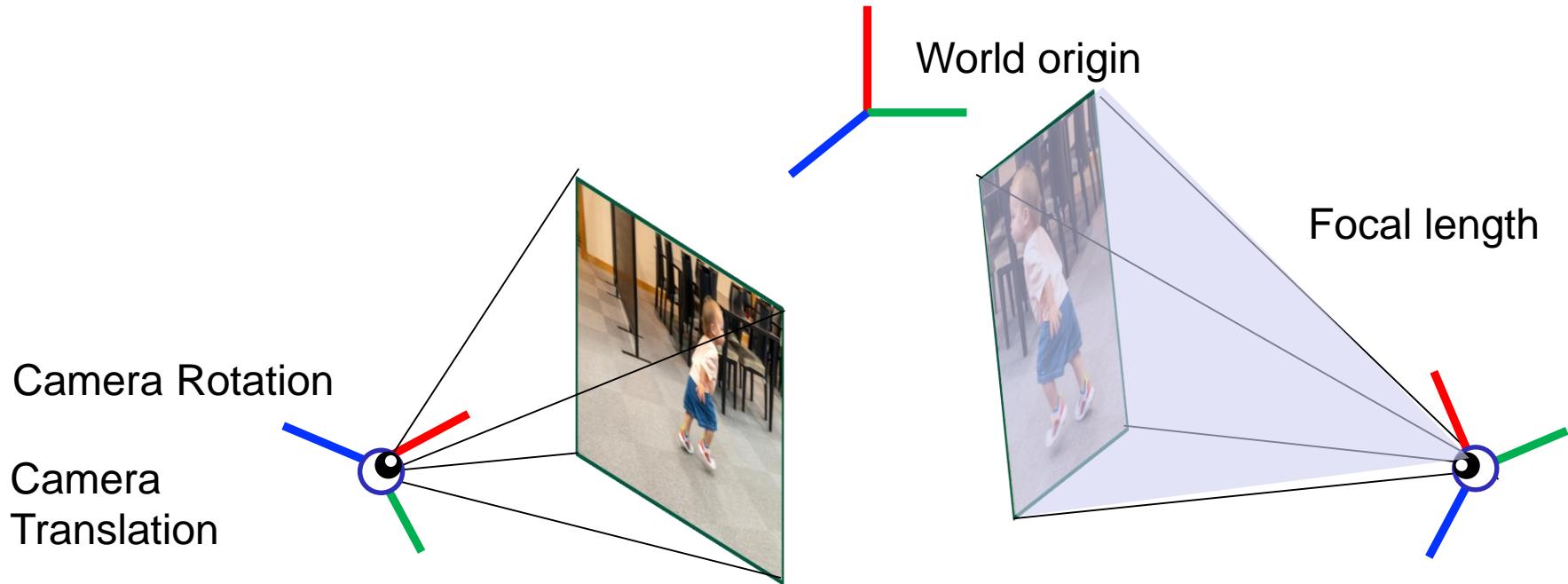
1. Estimate the fundamental/essential matrix!
2. Another method: Calibration

Problem: Solve for the camera

What are the camera parameters?

- Extrinsic (R, T)
- Intrinsic (K)

How am I situated in the world + what is the shape of the ray

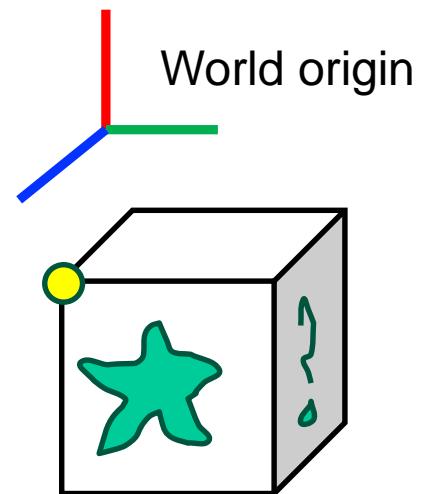
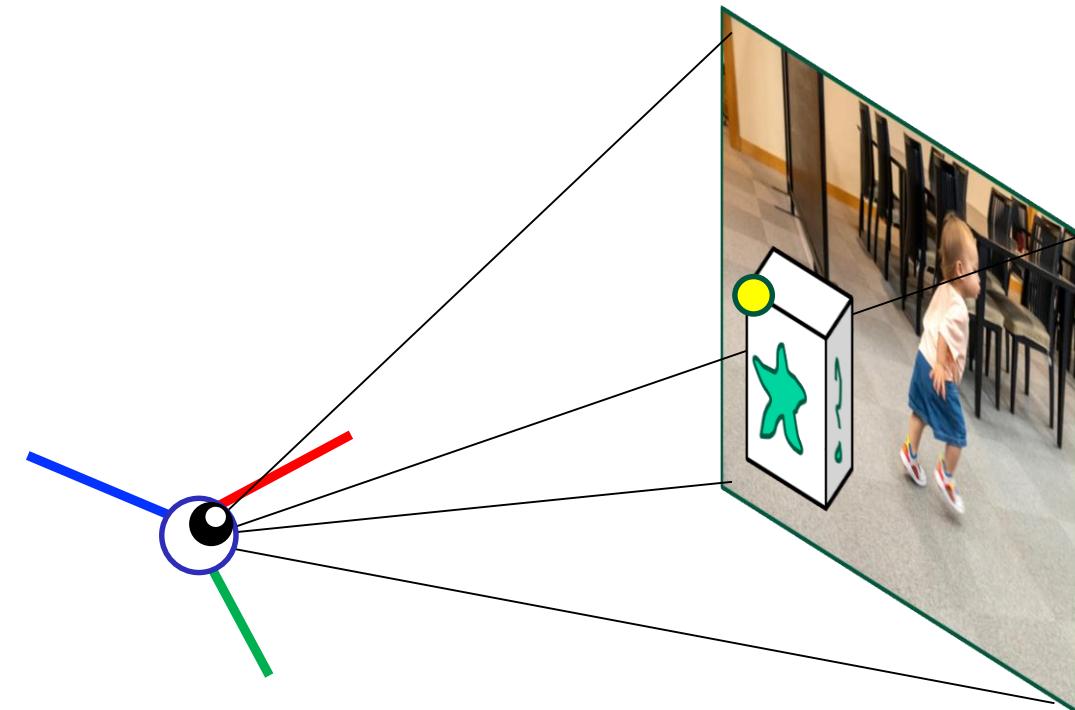


Calibration

Definition: Solve for camera using a known 3D structure + where it is in the image

Invasive / active

Can't be done on existing pictures



Only have to do it once
if the cameras are static

How to calibrate the camera?

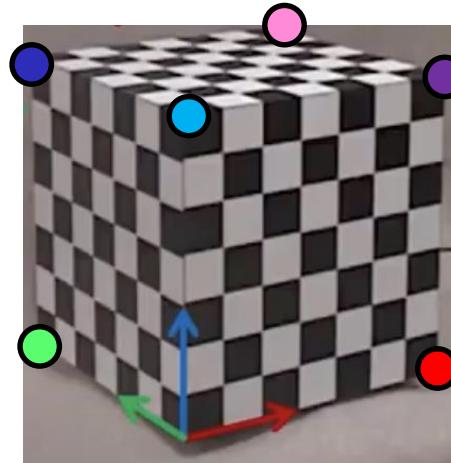
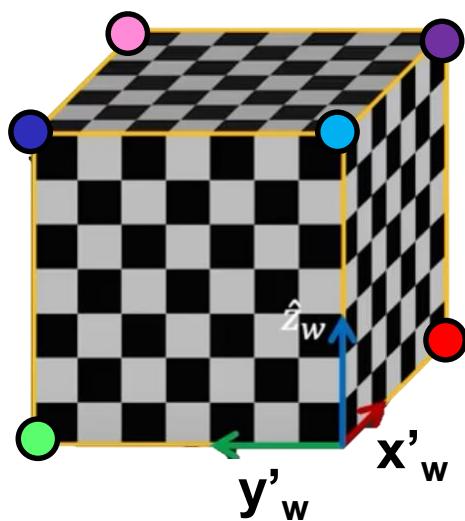
$$\mathbf{x} = \mathbf{K}[\mathbf{R} \quad \mathbf{t}] \mathbf{X}$$

$$\begin{bmatrix} su \\ sv \\ s \end{bmatrix} = \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

If we know the points in 3D we can estimate the camera!!

Step 1: With a known 3D object

1. Take a picture of an object with known 3D geometry



2. Identify correspondences

How do we calibrate a camera?

880	214
43	203
270	197
886	347
745	302
943	128
476	590
419	214
317	335
783	521
235	427
665	429
655	362
427	333
412	415
746	351
434	415
525	234
716	308
602	187

$$\begin{bmatrix} su \\ sv \\ s \end{bmatrix} = \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

312.747	309.140	30.086
305.796	311.649	30.356
307.694	312.358	30.418
310.149	307.186	29.298
311.937	310.105	29.216
311.202	307.572	30.682
307.106	306.876	28.660
309.317	312.490	30.230
307.435	310.151	29.318
308.253	306.300	28.881
306.650	309.301	28.905
308.069	306.831	29.189
309.671	308.834	29.029
308.255	309.955	29.267
307.546	308.613	28.963
311.036	309.206	28.913
307.518	308.175	29.069
309.950	311.262	29.990
312.160	310.772	29.080
311.988	312.709	30.514

Method: Set up a linear system

$$\begin{bmatrix} su \\ sv \\ s \end{bmatrix} = \begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

Solve for m's entries using linear least squares

Ax=0 form

$$\begin{bmatrix} X_1 & Y_1 & Z_1 & 1 & 0 & 0 & 0 & 0 & -u_1X_1 & -u_1Y_1 & -u_1Z_1 & -u_1 \\ 0 & 0 & 0 & 0 & X_1 & Y_1 & Z_1 & 1 & -v_1X_1 & -v_1Y_1 & -v_1Z_1 & -v_1 \\ X_n & Y_n & Z_n & 1 & 0 & 0 & 0 & 0 & -u_nX_n & -u_nY_n & -u_nZ_n & -u_n \\ 0 & 0 & 0 & 0 & X_n & Y_n & Z_n & 1 & -v_nX_n & -v_nY_n & -v_nZ_n & -v_n \end{bmatrix} \begin{bmatrix} m_{11} \\ m_{12} \\ m_{13} \\ m_{14} \\ m_{21} \\ m_{22} \\ m_{23} \\ m_{24} \\ m_{31} \\ m_{32} \\ m_{33} \\ m_{34} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Similar to how you solved for homography!

Can we factorize M back to K [R | T]?

Yes.

Why? because K and R have a very special form:

$$\begin{bmatrix} f_x & s & o_x \\ 0 & f_y & o_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

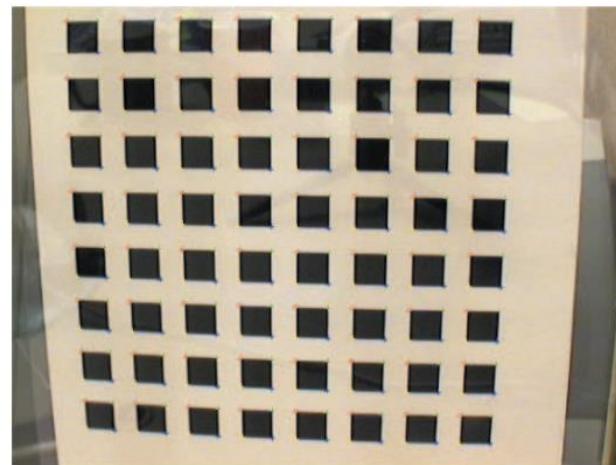
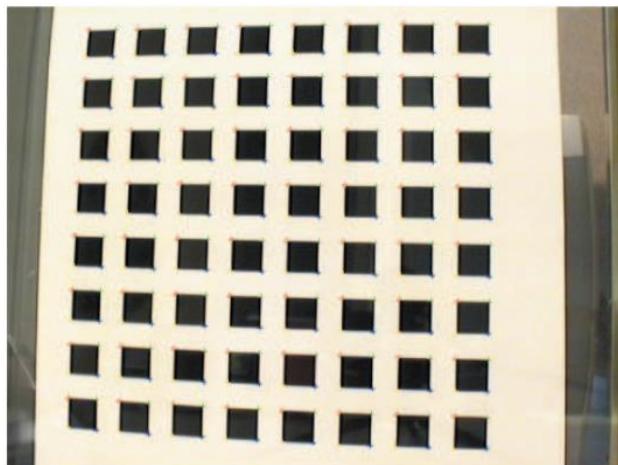
QR decomposition

Practically, use camera calibration packages
(there is a good one in OpenCV)

Inserting a 3D known object...

Also called “Tsai’scalibration” requires non-coplanar 3D points, is not very practical...

Modern day calibration uses a planar calibration target



Developed in 2000 by Zhang at Microsoft research

Zhang, A flexible new technique for camera calibration, IEEE Transactions on Pattern Analysis and Machine Intelligence, 2000

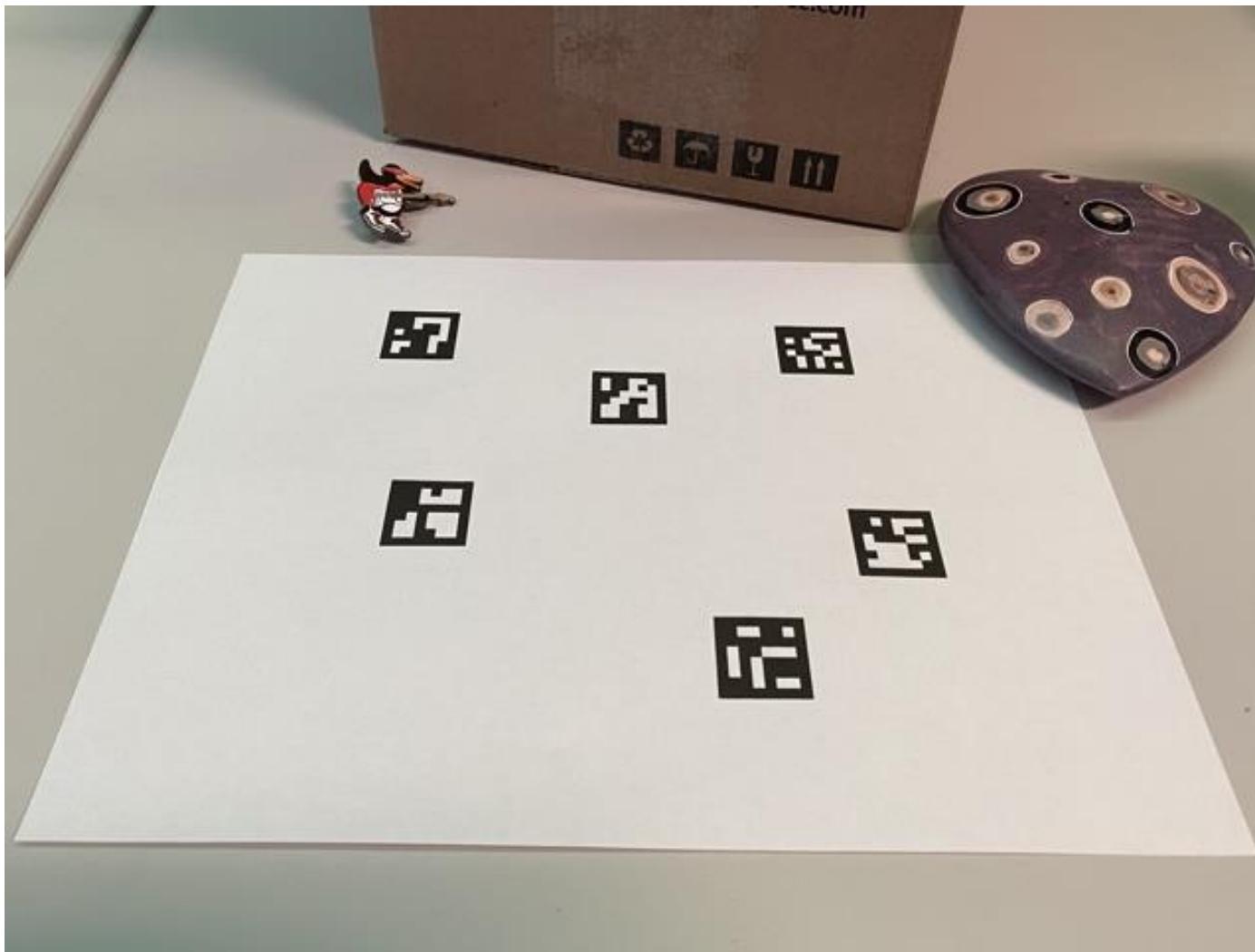
Doesn't plane give you homography?

Yes! If it's a plane, it's only a homography, so instead of recovering 3x4 matrix, you will recover 3x3 in Zhang's method

The 3x3 gives first two columns of R and T

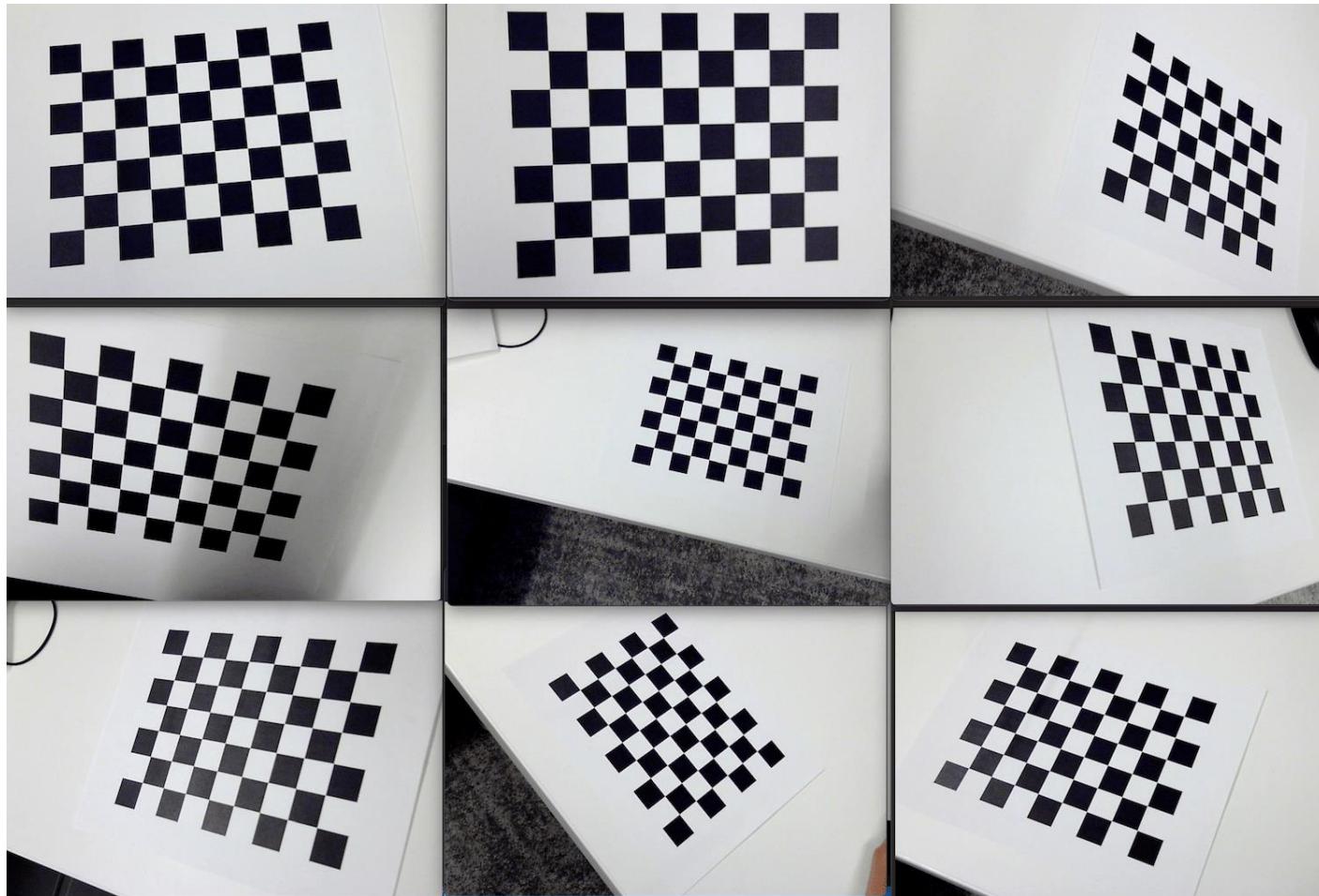
$$\begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} = \begin{bmatrix} \alpha_u & 0 & u_0 \\ 0 & \alpha_v & v_0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} r_{11} & r_{12} & t_1 \\ r_{21} & r_{22} & t_2 \\ r_{31} & r_{32} & t_3 \end{bmatrix}$$

You will use Aruco tags



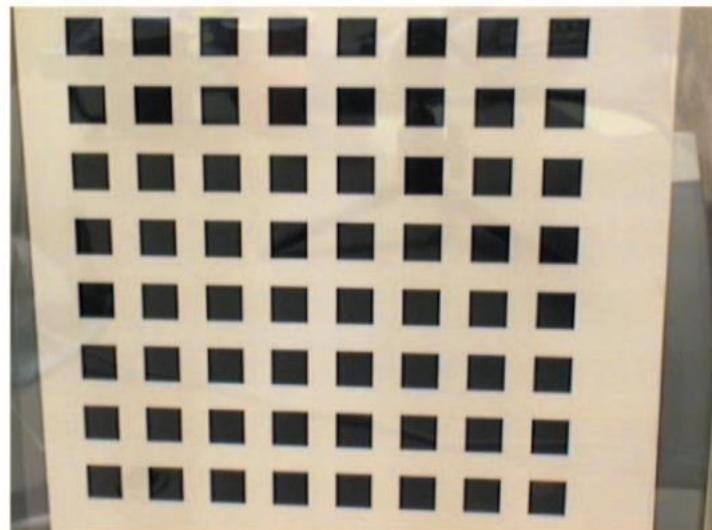
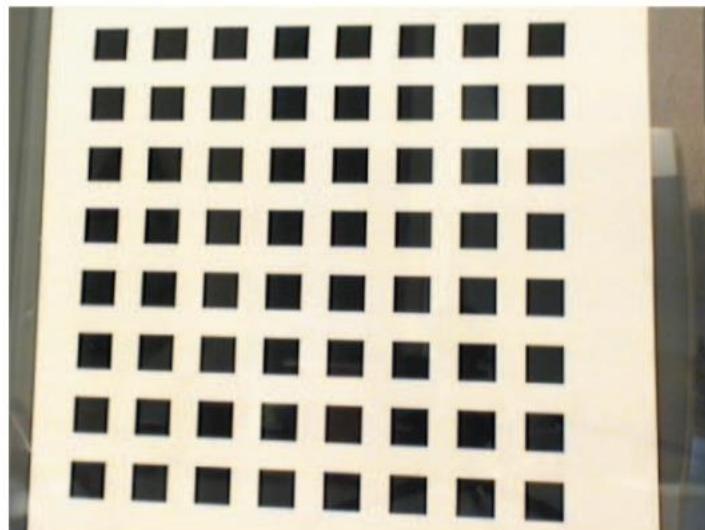
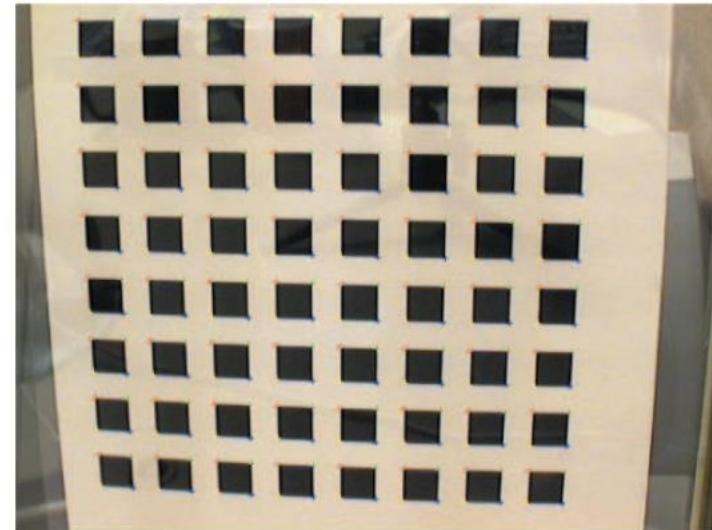
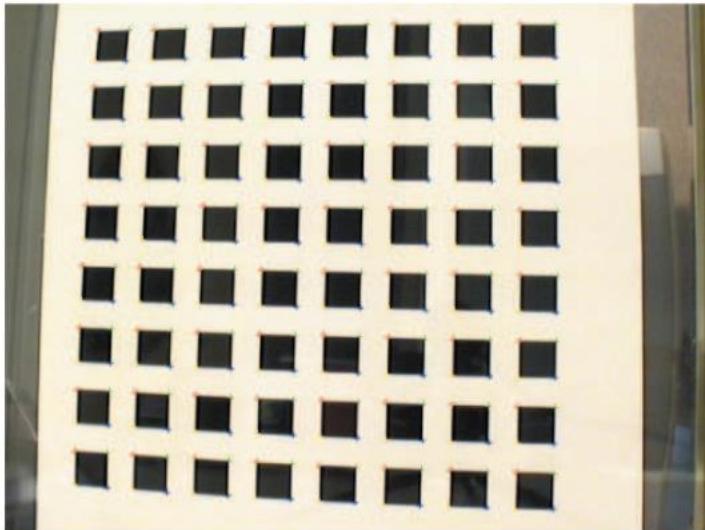
In practice: Step 0

Calibrate your intrinsics first, also estimates lens distortion (cv2.calibrateCamera)



Cv2.undistort()

Step 1: Undistort your image



Step 2: Estimate camera with PnP

PnP – “Perspective-n-Point” problem:
Estimate extrinsic parameters given n
correspondences

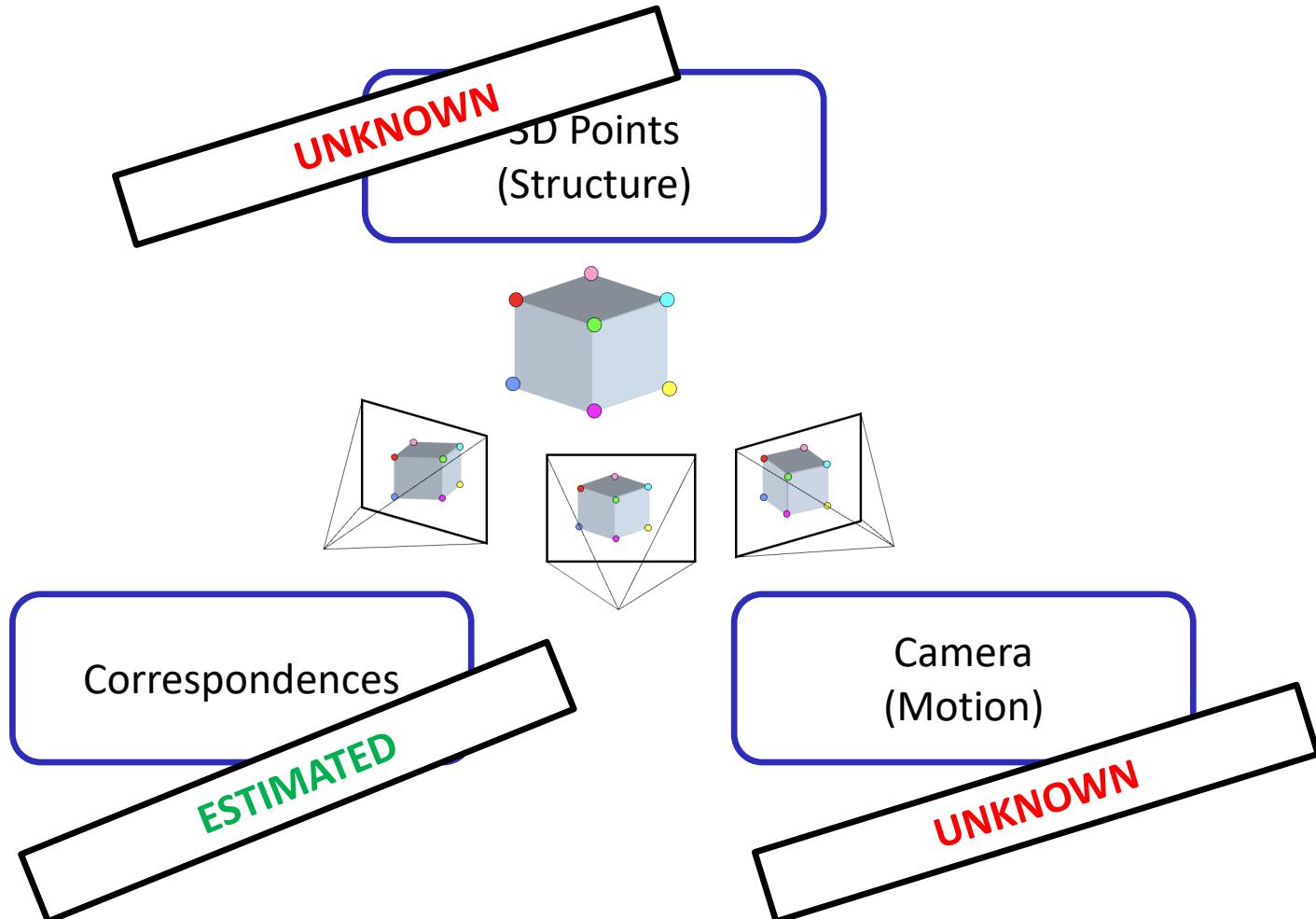
Minimize reprojection loss with non-linear least squares

$$\min_{R,T} \sum \|x_i - K[R \ T]X\|^2$$

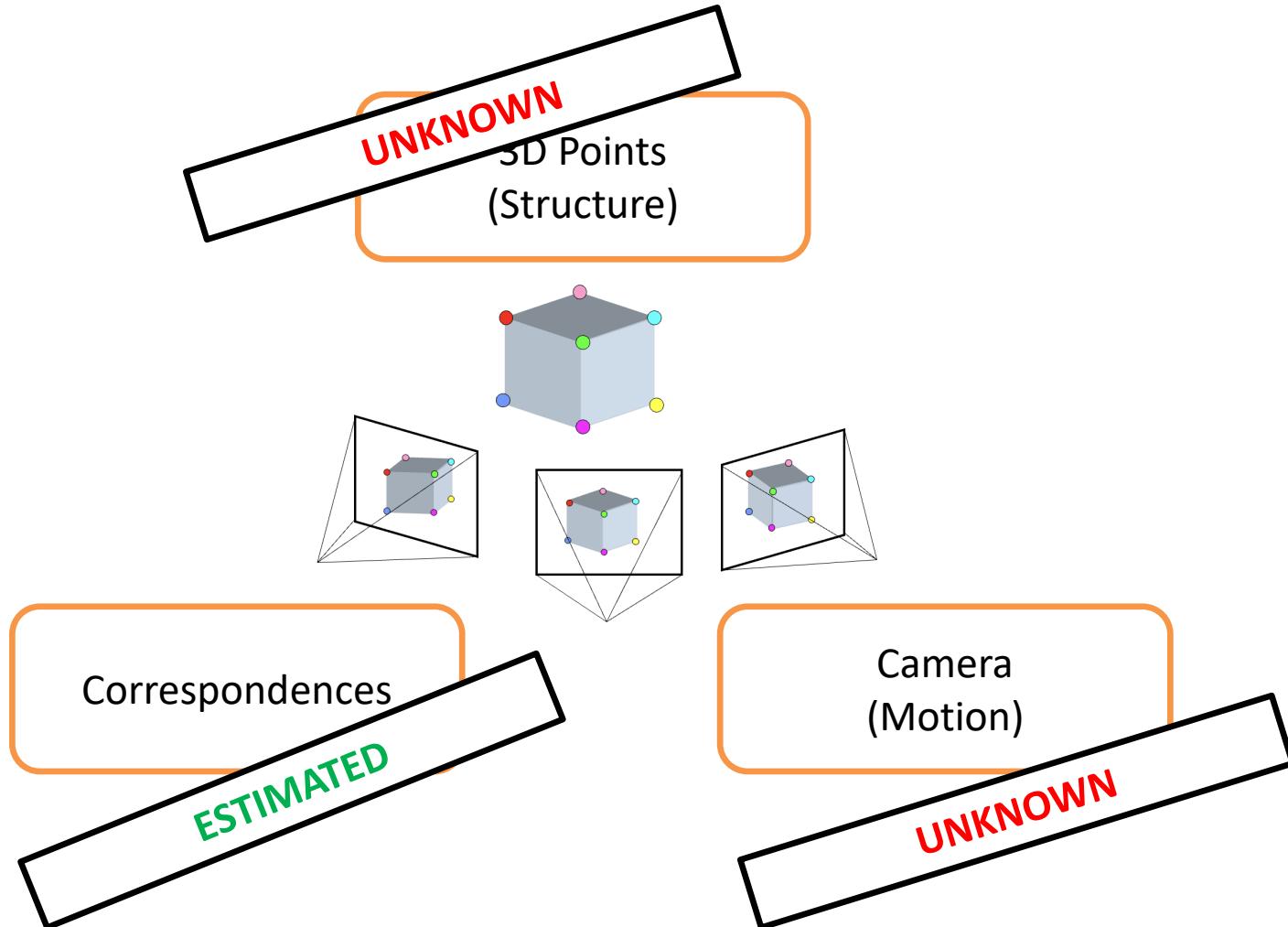
In general you do DLT first ($Ax=0$), then use that as initialization, or do other algorithms like Efficient PnP

Putting it all together

Structure-from-Motion: You know nothing!
(except ok maybe intrinsics)

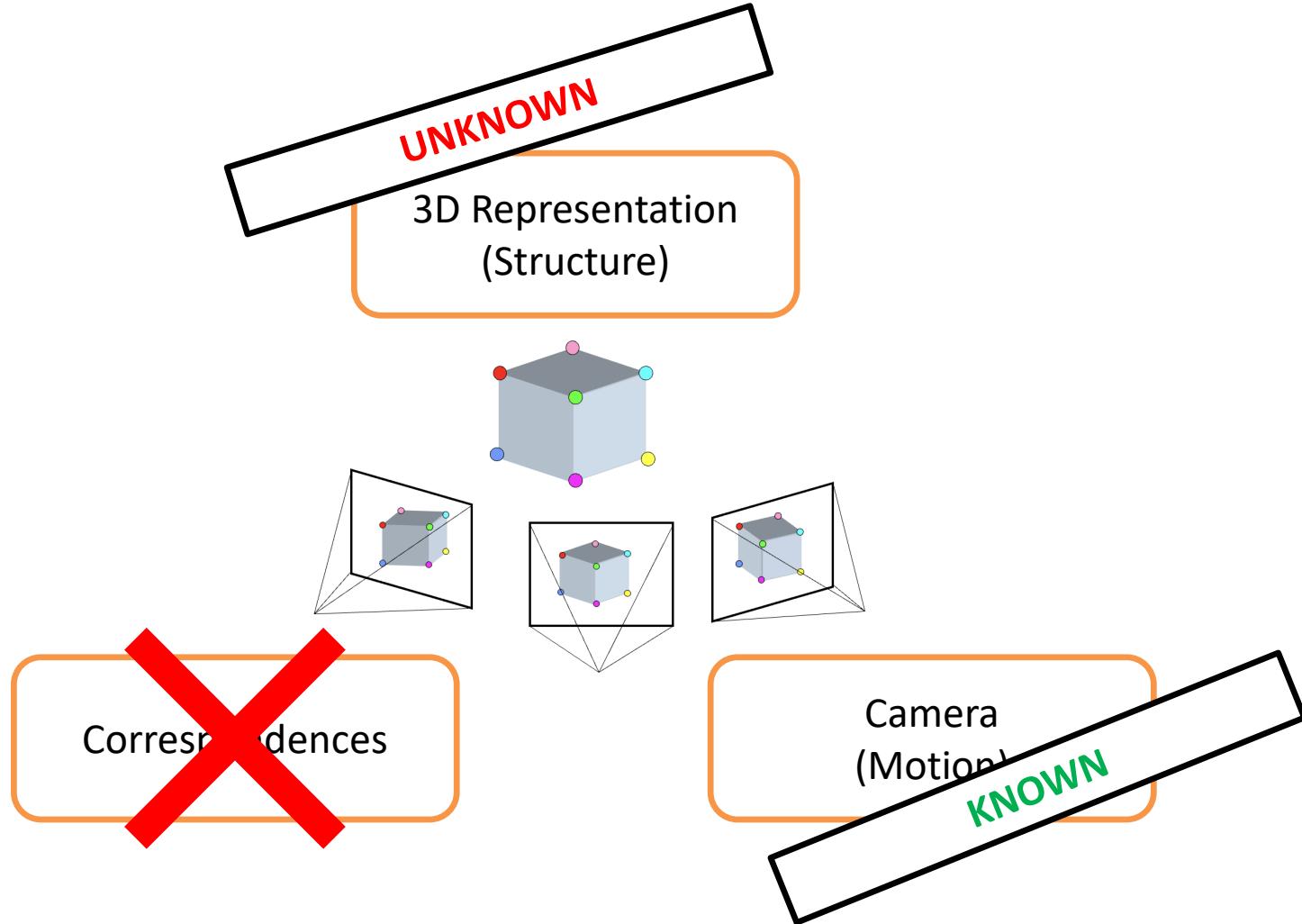


(Next lecture) Ultimate: Structure-from-Motion/SLAM



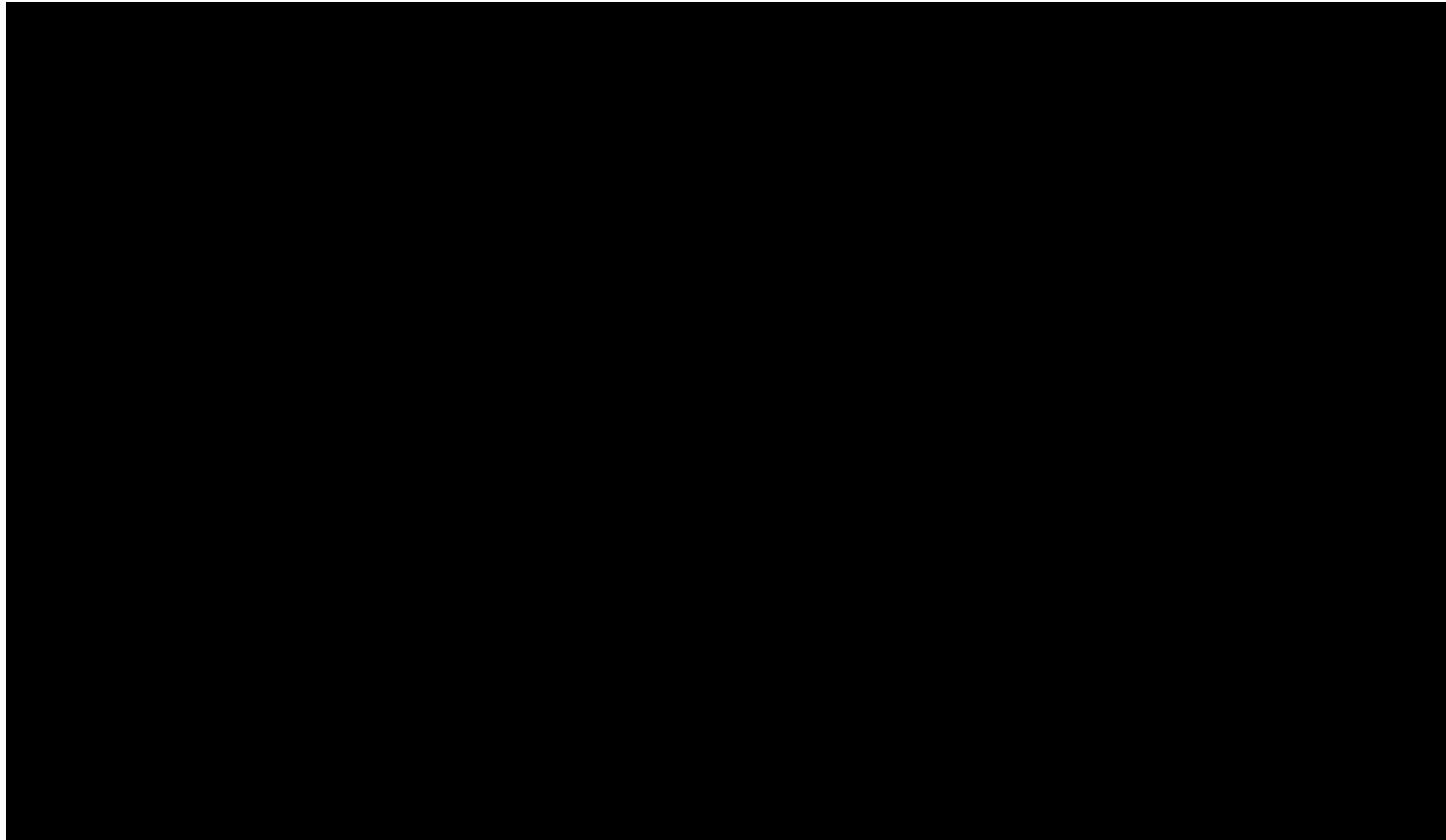
The starting point for all problems where you can't calibrate actively

(after that): Neural Rendering



A form of multi-view stereo, more on this in the NeRF lecture.

Next: Large-scale structure from motion



Dubrovnik, Croatia. 4,619 images (out of an initial 57,845).

Total reconstruction time: 23 hours

Number of cores: 352

Building Rome in a Day, Agarwal et al. ICCV 2009

Slide courtesy of Noah Snavely

Large-scale structure from motion



Result using COLMAP: Schönberger et al. CVPR '16