Exercicios Algebra Linear

Prof Helder Matos

Eduardo Furtado Sa Correa

Matricula: 09/0111575

Chapter 1

Nocoes basicas

1.1 Grupos

1) Seja **G** grupo e único

a^{-1} único

Suponha, e_1, e_2 identidade

$e_1 = e_1 \quad e_2 = e_2$

Suponha a_1, a_2 inversos de a

$a_1 a = a a_1 = e, \quad a_2 a = a a_2 = e$

$a_1 = e\ a_1 = (a_2\ a)\ a_1 = a_2\ (a\ a_1) = a_2\ e = a_2$

1.2 Corpos

1)

($\mathbb{Q}$ [$\sqrt{5}$],+, .)

($\mathbb{Q}$ [$\sqrt{5}$],+) grupo abeliano

i) ((a+b$\sqrt{5}$) + (c+d$\sqrt{5}$)) + (e+f$\sqrt{5}$) = (a+b$\sqrt{5}$)+((c+d$\sqrt{5}$) + (e+f$\sqrt{5}$))

ii) (a+b$\sqrt{5}$) + 0 = (a+b$\sqrt{5}$)

iii) (a+b$\sqrt{5}$) (- (a+$\sqrt{5}$))=0

iv) (a+b$\sqrt{5}$) + (c+d$\sqrt{5}$) = (c+d$\sqrt{5}$) + (a+b$\sqrt{5}$)

($\mathbb{Q}$ [$\sqrt{5}$]*, .)

i)$((a+b\sqrt{5})\ .\ (c+d\sqrt{5}))\ .\ (e+f\sqrt{5}) = (a+b\sqrt{5})\ .\ ((c+d\sqrt{5})\ .\ (e+f\sqrt{5}))$

ii) $(a+b\sqrt{5})\ .\ 1 = (a+b\sqrt{5})$

iii) $(a+b\sqrt{5})\ .\ (\frac{1}{a+b\sqrt{5}}) = 1$

Chapter 2

Espacos Vetoriais

2.1 Espacos e Subespacos Vetoriais

1) Def : I) $(V, +)$ grupo abeliano
II) $(\lambda_1\ \lambda_2)$ v = λ_1 (λ_2 v)

III) $(\lambda_1 + \lambda_2)$v = λ_1 v + λ_2v

IV) λ(v+w) = λv+λw

V) 1. v = v

v) 1(x_1 ,y_1) = (1x_1,0) $\neq$ (x_1,y_1)

iv) $\lambda((x_1, y_1) + (x_2, y_2)) = (\lambda(x_1 + x_2), 0) = (\lambda x_1 + \lambda x_2, 0)$

ii) $(\lambda_1\lambda_2)(x_1, y_1) = ((\lambda_1\lambda_2)x_1, 0)\ \lambda_1(\lambda_2 x_1, 0)$

i) $((x_1, y_1) + (x_2, y_2)) + (x_3, y_3) = (x_1 + x_2, 0) + (x_3, y_3) = ((x_1 + x_2) + x_3, 0)$
$(x_1, y_1) + ((x_2, y_2) + (x_3, y_3)) = (x_1, y_1) + (x_2 + x_3, 0) = ((x_2 + x_3) + x_1, 0)$

comutativo: $(x_1, x_2) + (y_1, y_2) = (x_1 + x_2, 0)(x_2, y_2)(x_1, y_1) = (x_2 + x_1, 0)$

2.2 Dimensao e Bases

2)
$\beta = \{(\bar{1}, \bar{2}, \bar{0}, \bar{1}), (\bar{1}, \bar{1}, \bar{1}, \bar{1})\}$

$a(1, 2, 0, 1) + b(1, 1, 1, 1) = (0, 0, 0, 0) \rightarrow \{a + b = 0, 2a + b = 0, b = 0, a + b = 0\}$ $\rightarrow$b = 0 e a = 0 $\rightarrow \beta$ é L.I

$(1, 0, 0, 0) = a(1, 2, 0, 1) + b(1, 1, 1, 1) \rightarrow \{a + b = 1, 2a + b = 0, b = 0, a + b = 0\} \rightarrow$ $a = 0, a = 1$ Absurdo $\rightarrow \beta_1 = \{(1, 2, 0, 1), (1, 1, 1, 1), (1, 0, 0, 0)\}$ $L.I$

$(0,1,0,0) = a(1,2,0,1)+b(1,1,1,1)+c(1,0,0,0) \to \{a+b+c=0, 2a+b=1, b=0, a+b=0\}$

6) $w_1, w_2 \leq$ V dim V finita $dimw_1 + dimw_2 = dim(w_1, w_2) + dim(w_1 \cap w_2)$

Def:$w_1 \cap w_2 \leq w_1$ e $w_1 \cap w_2 \leq w_2$

Seja B $=\{v_1, ..., v_n\}$ base de $(w_1 \cap w_2)$ $dimw_1 \cap w_2 = n \Longrightarrow \exists w_i$ tq.$B \cup \{w_1, ..., w_2\} ebasedew_1$ $(dimw_1 = n + l) e/existsu_i$ tq $B \cup \{u_1, ..., u_k\} ebasedew_2$ $comdimw_2 = n + u$

i)$\Longrightarrow w = \sum_1^n a_i v_i + \sum_1^l b_i u_i$
$n = \sum_1^n a_i v_i + \sum_1^k b_i u_i$

16)
$au + bv + cw = 0$

$a(\alpha + \beta) + b(\beta + \gamma) + c(\gamma + \alpha) = 0$

$(a+c)\alpha + (a+b)\beta + (b+c)\gamma = 0$

$a(\bar{1}, \bar{1}, 0) + b(0, \bar{1}, \bar{1}) + c(0, 0, \bar{1}) = 0$
$a = 0, b = 0, c = 0$

$a + b = 0$

$b + c = 0$ é L.I

$\beta' = \{(1,0,1); (0,1,0); (1,1,1)\}$ é L.D

Chapter 3

Transformacoes Lieneares

3.1 Transformacoes Lineares

8)

Dem :$a_1T(v_1)+...+a_nT(v_n) = 0 \rightarrow T(a_1v_1+...+a_nv_n) = 0 = T(0) \rightarrow a_1v_1+...+a_nv_n = 0 \rightarrow a_i = 0\ \forall i$

Logo$\{T(v_1),...,T(v_n)\}$é L.I

3.2 Associacoes de Matrizes as Transformacoes Lineares

2)

i)$T(x_1 + x_2, y_1 + y_2, z_1 + z_2) = ((x_1 + x_2) + (y_1 + y_2), 2(z_1 + z_2) - (x_1 + x_2)) = ((x_1 + y_1) + (x_2 + y_2), (2z_1 - x_1) + (2z_2 - x_2)) = T(v_1) + T(v_2)$

$T(\lambda v) = \lambda T(v)$

T(λ x,λ y,λz) = (λ x + λ y, 2λ z - λ x) = λ(x+y,2z-x) = λ T(v) 'e linear

ii)$y_1 = \{(1,0);(0,1)\}$

$y_2 = \{(1,0);(0,1)\}$

$y_3 = \{(1,0,0);(0,1,0);(0,0,1)\}$

$T^{\alpha}_{\gamma} : T(1,0,-1) = (1,-3) = a(1,0) + b(0,1) \quad (1,-3)$

$T^{\alpha}_{\gamma} : T(1,1,1) = (2,1) = a(1,0) + b(0,1) \quad (2,1)$

$T^{\alpha}_{\gamma} : T(1,0,0) = (1,-1) = a(1,0) + b(0,1) \quad (1,-1)$

$T^{\alpha}_{\gamma} = \{(1,-3);(2,1);(1,-1)\}$

$T^{\gamma}_{\gamma} = T(1,0,0) = (1,-1) = 1e_1 - 1e_2$

$T(0,1,0) = (1,0) = 1e_1 + 0e_2$

$T(0,0,1) = (0,2) = 0e_1 + 2e_2$

$T^{\gamma}_{\gamma} : \{(1,-1);(1,0);(0,2)\}$

Chapter 4

Diagonalicacao de operadores

4.1 Autovalores e Autovetores

1)

$$T_c^c = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 1 & 4 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 0 & 0 \\ -1 & 1 & 4 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \lambda I \begin{bmatrix} x \\ y \\ z \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 0 & 0 \\ -1 & 1 & 4 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} - \lambda I \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \rightarrow$$

$$\begin{bmatrix} 2-\lambda & 0 & 0 \\ -1 & 1-\lambda & 4 \\ 0 & 0 & 2-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$det = (2-\lambda)(1-\lambda)(2-\lambda) = 0$

$\lambda = (2, 1)$

$$\lambda = 2 \quad \begin{bmatrix} 0 & 0 & 0 \\ -1 & -1 & 4 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} - x - y - 4z = 0$$

$v = (1, -1, 0)$

$\lambda = 1$

$$\begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & 4 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} - x + 4z = 0$$

$v = (0, 0, 0)$

$\beta = \{(1, 3, 1), (0, 0, 0), (1, 3, 1)\}$

$$T_\beta^\beta = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & \\ 0 & 0 & 2 \end{bmatrix} \text{ T é diagonalizavel}$$

12) $V \in Im(E) \cap Im(I-E) \to v = E(w), v = (I-E)(u) \to E(v) = E(E(w)) = E^2(w) = v \to V = (I-E)(u) = u - E(u) \to V = E(v) = E(u-E(u)) = E(u) - E(E(u) = E(u) - E(u) = 0$

$V = E(v) + V - E(v) = E(v) + (I-E)(v) \to V \in Im(E) + Im(I - E_{\forall} v \to V = Im(E) \diamond Im(I-E)$

III)$Sejav \in Im(E) \to v = E(w) \to E(v) = E(E(w)) = E^2(w) = E(w) = v \to v - E(v)m = 0 \to (I-E)(v) = 0 \to v \in Nuc(I-E) \to Im(E) \sqsubseteq Nuc(I-E)$

Seja $v \in Nuc(I-E) \to (I-E)(v) = 0 \to V - E(v) = 0 \to V = E(v) \to v \in Im(E) \to Nuc(I-E) \sqsubseteq Im(E)$

Logo Im(E) = Nuc (I-E)

IV) V $\in Im(I-E) \to v = (I-E)(w) \to v = w - E(w) \to E(v) = E(w - E(w)) = E(w) - E(E(w)) = E(w) - E(w) = 0 \to V \in Nuc(E) \to Im(I-E) \sqsubseteq Nuc(E)$

v$\in Nuc(E) \to E(v) = 0 \to v = E(v) + v - E(v) = v - E(v) = (I-E)(v) \to V \in Im(I-E) \to Nuc(E) \sqsubseteq Im(I-E) \to Nuc(E) = Im(I-E)$

4.2 Polinomios

5)

R(x) = A

I = $\frac{(x^2+x-2)}{a} R(x) + \frac{(x^2-4x+3)}{b} R(x) \leq R(x)$

I =$d(x)d = mdc(a,b)ia \in Iai \in I$

a =$bq_1 + r_1$

b= $r_1 q_2 + r_2$

$r_1 = r_2 q_3 + r_3$

$r_n = r_n + 1q_n + 2 + r_n + 2$

$mdc(x^2 + x - 2, x^2 - 4x + 3) = 5x - 5$

monico $\to x - 1 = mdc$

$x - 1 = (x^2 + x - 2)r(x) + (x^2 - 4x + 3)5(x)$

$x^2+x-2=(x^2-4x+3)+(5x-5) \to 5x-5=(x^2+x-2)-(x^2-4x+3)$

$(x-1)=(x^2+x-2)\frac{1}{3}+(x^2-4x+3)(-\frac{-1}{5} \in L+J=I \leq R(x) \to (x-1) \subseteq I$

$\frac{r(x)=1}{5}$

$\frac{s(x)=-1}{5}$

9)
Dem: $f(x)=a_0+a_1x+...+a_nx^n$

g(x) $=b_0+b_1x+...+a_kx^k$

$\partial f < \partial g \to q=0 \quad r=f$

$\partial f=\partial g \to k=n$

$a_nx^n+a_n-1x^n-1+...+a_0$

$-a_nx^n-\frac{a_nb_n-1x^n-1}{b_n}+...+\frac{a_n}{b_n}$

$q=\frac{a_n}{b_n}(b_n \neq 0)\ e\ r(x)$

10) Suponha f(x) $=q_1(x)g(x)+r_1(x), 0 \leq \partial r < \partial g$

f(x) $=q_2(x)g(x)+r_2(x), 0 \leq \partial r_2 < \partial g$

$0=q_2(x)g(x)+r_2(x)-(q_1(x)g(x)+r_1(x))=(q_2(x)^2-q_1(x))g(x)+r_2(x)-r_1(x)=0 \to (q_2(x)-q_1(x))g(x)=r_1(x)-r_2(x) \to \partial((q_2(x)-q_19x)g(x)=\partial(r_1(x)-r_2(x)) \leq max\,\{\partial r_1, \partial r_2\} \leq n-1 \to \partial(q_2-q_1)+\partial g(x) \leq n-1 \to \partial(q_2-q_1)+n \leq n-1 \to Se\ \partial(q_2-q_1) \in \mathbb{N}\ entao\ \partial(q_2-q_1)+n \leq n-1\ Absurdo$

Logo $\partial(q_2-q_1)$ = - $\infty \to q_2-q_1$= $0 \to q_2=q_1 \to r_1-r_2=0 \to r_1=r_2$

4.3 Subespacos Invariantes

4) $m(x)=x^2 \quad c(x)=x^3 \to \lambda=0$

$$A^2=\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}=0$$

8)
I) $\begin{bmatrix} 1-\lambda & 2 \\ 0 & 2-\lambda \end{bmatrix} det=(1-\lambda)(2-\lambda)=0$

$\lambda=1,2$

$\lambda = 1 \begin{bmatrix} 0 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad y = 0$

$v_1(1,0)$

$\lambda = 2 \begin{bmatrix} -1 & 2 \\ 0 & 0 \end{bmatrix}$
$\begin{bmatrix} x \\ y \end{bmatrix} =$
$\begin{bmatrix} 0 \\ 0 \end{bmatrix} \{-x + 2y = 0, x = 2y\}$

$v_2(2,1)$

B $= [\begin{matrix} 3 & -8 \\ 0 & -1 \end{matrix}] [\begin{matrix} 1 \\ 0 \end{matrix}] = [\begin{matrix} 3 \\ 0 \end{matrix}] = 3 [\begin{matrix} 1 \\ 0 \end{matrix}]$

$\begin{bmatrix} 3 & -8 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \end{bmatrix} = - \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$\rightarrow \beta_b^a = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}$

21) T = λ I

$P^-1TP = P^-1\lambda IP = P^-1\lambda P = P^-1P\lambda = I\lambda = \lambda I =$ T

I) m(x) = $x - 5 \rightarrow$ T = 5I

II) m(x) = $(x-5)^2 \rightarrow T \begin{bmatrix} (5 & 0) & 0 \\ (1 & 5) & 0 \\ 0 & 1 & 5 \end{bmatrix}$

III) m(x) = $(x-5)^3 \rightarrow T \begin{bmatrix} 5 & 0 & 0 \\ 1 & 5 & 0 \\ 0 & 1 & 5 \end{bmatrix}$

22)
c(x) = $(x-1)^2(x-2)$

$\lambda = 2 \rightarrow \begin{bmatrix} -1 & 0 & 0 \\ 2 & -1 & 0 \\ 2 & 3 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} 4 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \rightarrow x = 0, y = 0$

$v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} 4 \text{ , } v_2 \begin{bmatrix} 0,5 \\ -1 \\ -1 \end{bmatrix} \text{ , } v_1 \begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix},$

$\beta \{V_1, V_2, V_3\}$

$\lambda = 1 \rightarrow \begin{bmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \rightarrow \{z = -3y, v = [\begin{matrix} 0 \\ y \\ -3y \end{matrix} \}$

$(T - I) = \begin{bmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \lambda \begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix} \rightarrow \{y = z = -1, 2x = 1, x = 0,5, 2x + 3y +$

$z = -3\}$

(T- I)v $= \lambda V_1$

T(v) $= v + \lambda v_1$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & 3 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix} =$$

$$\begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & 3 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 0,5 \\ -1 \\ -1 \end{bmatrix} =$$

$$\begin{bmatrix} 0,5 \\ 1 \\ -3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix} + \begin{bmatrix} 0,5 \\ -1 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & 3 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} = 2 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$T_\beta^\beta = \begin{bmatrix} (1 & 1) & 0 \\ (0 & 1) & 0 \\ 0 & 0 & 2 \end{bmatrix} \quad \text{Jordan}$$

Chapter 5

Produto Interno

5.1 Produto Interno

4) I) $\langle ax+b, ax+b\rangle = \frac{a^2}{3} + ab + b^2 = b^2 + 2b\frac{a}{2} + (\frac{a}{2})^2 - (\frac{a}{2})^2 + \frac{a^2}{3} = (b+\frac{a}{2})^2 + \frac{a^2}{12} \geq 0$

$\langle ax+b, ax+b\rangle = 0 \to \left\{b + \frac{a}{2} = 0 <=> b = 0, a = 0\right\}$

II)$\langle \lambda(ax+b), cx+d\rangle = \frac{\lambda ac}{3} + \frac{\lambda bc}{2} + \frac{\lambda ad}{2} + \lambda bd = \lambda(\frac{ac}{3} + \frac{bc}{} + \frac{ad}{} + bd) = \lambda \langle v, w\rangle$

III) $\langle v+w, u\rangle = \langle (a+c)x + b + d, ex + f\rangle = \frac{(a+c)e}{} + \frac{(a+c)f}{2} + \frac{e(b+d)}{2} + (b+d)f = \frac{ae}{3} + \frac{ce}{3} + \frac{af}{2} + \frac{cf}{2} + \frac{eb}{2} + \frac{ed}{2} + bf + df = \langle v, u\rangle + \langle w, u\rangle$

IV)$\langle v, w\rangle = \frac{ac}{3} + \frac{bc}{2} + \frac{ad}{2} + bd = \frac{ca}{3} + \frac{cb}{2} + \frac{da}{2} + db = \langle w, v\rangle$

$cos\theta = \frac{\langle v,w\rangle}{|v||w|}$

$\langle ax+b, cx+d\rangle = \int_0^1 (ax+b)(cx+d)dx = \int_0^1 acx^2 + bcx + adx + bddx = (\frac{acx^3}{3} + \frac{bcx^2}{2} + \frac{adx^2}{2} + bdx)_a^b = \frac{ac}{3} + \frac{bc}{2} + \frac{ad}{2} + bd$

5.2 Ortogonalizacao