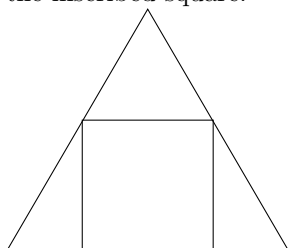


Maths Problems Set 1 (13 December 2019)

1. Each side length of the equilateral triangle below is 1. Find the area of the inscribed square.



2. Initially there are m balls in one bag, and n in the other, where $m, n > 0$. Two different operations are allowed:

- (a) Remove an equal number of balls from each bag;
- (b) Double the number of balls in one bag.

Is it always possible to empty both bags after a finite sequence of operations?

Operation (b) is now replaced with

- (b') Triple the number of balls in one bag.

Is it now always possible to empty both bags after a finite sequence of operations?

3. Prove that there are infinitely many non-trivial Pythagorean triples, i.e. not a multiple of a different Pythagorean triple.
4. Straight lines are drawn on an infinite 2D plane. Show that the resulting regions can be coloured with two colors such that no adjacent regions have the same color.
5. For a four digit number n (using at least two different digits with leading zeros allowed), the digits of n are rearranged to form the largest possible four digit number a and the smallest possible four digit numbers b (with leading zeros if necessary). The difference between the a and b is found, and this process is repeated with n being this difference.

Prove that the process eventually hits 6174 and remains there in at most 7 iterations for any four digit number n .

6. Gnoms have friendships. Friendship is commutative. A gnome is odd if it has an odd number of friends. Show that there is always an even number of odd gnomes.

7. Prove that if A and B are coprime where $A, B \in \mathbb{Z}$, then $\exists s, t \in \mathbb{Z}$ such that $As + Bt = 1$.
8. Evaluate $\int_0^1 \frac{1}{x + \sqrt{1-x^2}} dx$.
9. Find the smallest $a \geq 1$ such that $e^{y-x} \geq \frac{a + \sin x}{a + \sin y}$ for all $y \geq x$.
10. Evaluate $\lim_{n \rightarrow \infty} \left(\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} \right)$.
11. Find an expression for $\int_1^n (-1)^{\lfloor x \rfloor} \lfloor x \rfloor^{-1} dx$ where $n \in \mathbb{N}$.
12. Prove that every prime has infinitely many multiples in the Fibonacci sequence. Prove that they have a constant frequency. Deduce that there are infinitely many Fibonacci numbers that are the product of only primes that had not previously occurred.
13. Find all solutions to the Diophantine equation $4x^2 = y^3 + 1$.
14. Let $A = \{0, 1, \dots, 2^n - 1, 2^n\}$ and $B = \{0, \dots, n\}$. How many functions g can be defined from A to B such that both of the following conditions hold:
 - for all $x \in B$ we have $g(2^x) = x$
 - for all $y, z \in A$ with $y \leq z$ we have $g(y) \leq g(z)$
15. Find all functions $f : \mathbb{Z} \rightarrow \mathbb{Z}$ such that for all $a, b \in \mathbb{Z}$:
 - (a) $f(a) + f(b) = f(f(a+b))$, or
 - (b) $f(a) + f(2b) = f(f(a+b))$, or
 - (c) $f(a) + 2f(b) = f(f(a+b))$
16. There are n points on 2D plane. Show that it is always possible to select at least \sqrt{n} of this points so that the points selected do not form any equilateral triangles.