NONEXPANSIVE MAPS IN NONLINEAR SMOOTH SPACES

PEDRO PINTO

Department of Mathematics, Technische Universität Darmstadt,
Schlossgartenstraße 7, 64289 Darmstadt, Germany,

E-mail: pinto@mathematik.tu-darmstadt.de

Abstract. We introduce the notion of a nonlinear smooth space generalizing both CAT(0) spaces as well as smooth Banach spaces. We show that this notion allows for a unified treatment of several results in functional analysis. Namely, we substantiate the usefulness of this setting by establishing a nonlinear generalization of an important result due to Reich in Banach spaces. On par with the linear context, this nonlinear version entails a convergence proof of several other methods. Here, we establish the convergence of a general form of the Halpern-type schema for resolvent-like families of functions. We furthermore prove the convergence of the viscosity generalization of Halpern's iteration (even for families of maps) generalizing a result due to Chang. This work is set in the context of the 'proof mining' program, and the results are complemented with quantitative information like rates of convergence and of metastability (in the sense of T. Tao).

Keywords: Proof mining; metastability; hyperbolic spaces; uniform smoothness; uniform convexity; sunny nonexpansive retractions
MSC2020 Classification: 47J25; 47H09; 47H10; 03F10

1. Introduction

Let X be a Banach space and $C \subseteq X$ a nonempty closed subset. We say that a map $T : C \to C$ is a strict contraction if there exists a real number $\alpha < 1$ satisfying

$$\|T(x) - T(y)\| \leq \alpha \cdot \|x - y\|, \text{ for all } x, y \in C.$$

Banach's contraction principle states that any such T has a unique fixed point, and moreover that such a point can be approximated from any initial first guess x_0 via the Picard iteration

$$x_{n+1} = T(x_n).$$

A central problem in fixed point theory is the study of the case when T is only required to be a nonexpansive map, i.e. when one allows $\alpha = 1$. In this case, Banach's contraction principle fails in all regards. There are nonexpansive maps with no fixed points (e.g. translations) and others with many (e.g. the identity). Moreover, even when T has indeed a unique fixed point, the Picard iteration may fail to approximate it. For example, in the real-valued function $T(x) := 1 - x$ the Picard iteration alternates indefinitely between x and $1-x$, unless initiated already at the fixed point.

One of the first crucial results regarding the study of fixed points of nonexpansive maps dates back to the sixties and is due to Browder [9], set in the context of Hilbert spaces.

Date: March 7, 2024.

Theorem 1.1 ([9]). *Let X be a Hilbert space, $C \subseteq X$ be a nonempty closed convex subset of X. Let $T : C \to C$ be a nonexpansive map on C, and $u \in C$. For each $t \in (0, 1)$, since the map defined by $T_t(x) := (1 - t)T(x) + tu$ is a strict contraction, consider $z_t \in C$ satisfying*

$$z_t = (1 - t)T(z_t) + tu. \tag{B}$$

If C is bounded (or $\mathrm{Fix}(T) \neq \emptyset$) and $t \to 0$, then $(z_t)_t$ converges strongly towards a fixed point of T, namely $P(u)$ where $P : C \to \mathrm{Fix}(T)$ is the metric projection onto $\mathrm{Fix}(T)$.

Browder's argument has been one of the most common techniques for establishing the convergence of several iterative methods. The approach crucially relies on both a projection argument and a sequential weak compactness principle. The compactness principle is troublesome in generalizations to nonlinear settings. In this case, Kirk [28] observed that a simpler proof due to Halpern in [25] adapts, essentially unchanged, to a geodesic setting (specifically, to CAT(0) spaces) – see also [23, Theorem 24.1]. On a different direction, the projection argument doesn't reach the desirable convergence statement, if one considers more general normed spaces. This was overcome by Reich [59] in a celebrated result which in particular extends Browder's theorem to uniformly smooth normed spaces (see also [13, 40]).

Theorem 1.2 ([59]). *Let X be a uniformly smooth Banach space, $C \subseteq X$ be a nonempty closed convex subset of X. Let $T : C \to C$ be a nonexpansive map on C, and $u \in C$. For each $t \in (0, 1)$, consider $z_t \in C$ satisfying (B). If C is bounded (or $\mathrm{Fix}(T) \neq \emptyset$) and $t \to 0$, then $(z_t)_t$ converges strongly towards a fixed point of T, namely $Q(u)$ where Q is the unique sunny nonexpansive retraction $Q : C \to \mathrm{Fix}(T)$.*

An explicit iterative version of Browder's approach was introduced by Halpern [25] (see also Lions [47]). Given $x_0 \in C$, one considers a sequence recursively defined by

$$x_{n+1} := (1 - \alpha_n)T(x_n) + \alpha_n u \tag{H}$$

where $(\alpha_n) \subseteq [0, 1]$ is a sequence of real numbers and $u \in C$.

In Hilbert spaces, Halpern's argument shows the strong convergence of (H) towards the metric projection of u onto the set of fixed points of T, provided the parameter sequence (α_n) is subject to three conditions. Moreover, he showed that the his first two conditions,

$$(i)\ \lim \alpha_n = 0 \ \text{ and } \ (ii)\ \sum \alpha_n = \infty,$$

are indeed necessary for strong convergence. The third condition, however, prevented the natural choice of $\alpha_n = 1/n+1$. This was later overcome by Wittmann [75] which establish the strong convergence considering instead as a third condition

$$\sum |\alpha_{k+1} - \alpha_k| < \infty.$$

For the natural choice of the parameters α_n, $u = x_0$ and linear T, the Halpern's iteration becomes the Cesàro's average,

$$x_{n+1} := \frac{1}{n+1} \sum_{k=0}^{n} T^{(k)}(x_0)$$

and thus Wittmann's result is considered an important generalization of the classic von Neumann Mean Ergodic Theorem to nonlinear maps.

Reich [60] and Shioji and Takahashi [65] generalized Wittmann's theorem to uniformly smooth Banach spaces. The strong convergence of the Halpern iteration (H) was also established in uniformly smooth Banach spaces by Xu [76, 77] under conditions incomparable to those of Wittmann but which still allow for the choice $\alpha_n = \frac{1}{n+1}$. The strong convergence of the Halpern iteration was also established in the nonlinear setting of CAT(0) spaces by Saejung in [64] (see also [62, 41]). Again in the context of Hilbert spaces, Moudafi [52] introduced the so-called viscosity algorithms which generalize (H) by replacing the anchor point u by the value of a strict contraction applied to the current term of the iteration,

$$x_{n+1} := (1-\alpha_n)T(x_n) + \alpha_n \phi(x_n). \tag{H_ϕ}$$

Moudafi's viscosity algorithms were subsequently extended to uniformly smooth Banach spaces in [78].

The schema (H) was also studied in the general form extended to families of nonexpansive maps $\{T_n\}$ instead of a single map T,

$$x_{n+1} := (1-\alpha_n)T_n(x_n) + \alpha_n u. \tag{H^{T_n}}$$

For example, Bauschke [6] generalized Wittmann's result by considering the Halpern iterative schema with a family of nonexpansive maps $\{T_n\}$ defined in a cyclic manner from an initial finite list of maps (not necessarily commutative but subject to a condition on the fixed point sets for ordered compositions). Bauschke's result has since been further generalized, e.g. in [26]. Another relevant iterative method also encompassed by the generalization of Halpern's schema to families of nonexpansive maps is the Halpern-type version of Rockafellar's well-known proximal point algorithm [49, 63],

$$x_{n+1} := (1-\alpha_n)J_{c_n}(x_n) + \alpha_n u,$$

where the J_{c_n} are resolvent functions associated with an accretive set-valued operator A subject to the range condition

$$\overline{\mathrm{D}(A)} \subseteq C \subseteq \mathrm{R}(\mathrm{Id} + \lambda A), \text{ for all } \lambda > 0$$

where $\overline{\mathrm{D}(A)}$ is the closure of the domain $\mathrm{D}(A)$ of A, and C is a nonempty closed convex subset of X. This method was studied for example in [8, 27, 77]. In uniformly convex Banach spaces whose norm is uniformly Gâteaux differentiable, Aoyama and Toyoda [3] showed that, when the zero set $A^{-1}(0)$ is nonempty, the Halpern-type proximal point algorithm strongly converges to a zero of A provided that $\inf c_n > 0$ and the necessary conditions (i) and (ii) hold.

Naturally, one may further combine these two generalizations of Halpern's schema,

$$x_{n+1} := (1-\alpha_n)T_n(x_n) + \alpha_n \phi(x_n), \tag{$\mathrm{H}_\phi^{T_n}$}$$

where $\{T_n : C \to C\}$ is a family of nonexpansive maps and $\phi : C \to C$ is a strict contraction. In the context of uniformly smooth Banach spaces, Chang [15] studied the schema ($\mathrm{H}_\phi^{T_n}$) with the family $\{T_n\}$ defined in a cyclic manner from a finite list $T_0, \cdots, T_{\ell-1}$ and under the condition introduced by Bauschke. In particular, he proved that the hypothesis (i) and (ii) on the sequence (α_n) are actually sufficient for establishing strong convergence of ($\mathrm{H}_\phi^{T_n}$) provided the iteration is asymptotically regular with respect to the composition $T = T_{\ell-1} \cdots T_0$. Concretely, he proved the following result.

Theorem 1.3 ([15]). *Let X be a uniformly smooth Banach space, C a nonempty closed convex subset of X, and $T_0, \cdots, T_{\ell-1}$, be $\ell \geq 1$ nonexpansive maps on C such that $\bigcap_{j=0}^{\ell-1} \mathrm{Fix}(T_j) \neq \emptyset$ and satisfies*

$$\bigcap_{j=0}^{\ell-1} \mathrm{Fix}(T_j) = \mathrm{Fix}(T_0 T_{\ell-1} \cdots T_1) = \cdots = \mathrm{Fix}(T),$$

where $T := T_{\ell-1} \cdots T_0$. Let $\phi : C \to C$ be a strict contraction, and $(\alpha_n) \subseteq [0,1]$ a sequence of real numbers satisfying

$$(i)\ \lim \alpha_n = 0, \quad (ii) \sum \alpha_n = \infty.$$

For $x_0 \in C$, let (x_n) be defined by $(\mathrm{H}_\phi^{T_n})$, with $\{T_n\}$ defined cyclically from $\{T_j\}_{j=0}^{\ell-1}$. We have:

1. *If $\lim \|x_n - T(x_n)\| = 0$, then (x_n) converges strongly to a common fixed point of $T_0, \cdots, T_{\ell-1}$,*
2. *for each ϕ strict contraction on C, if (x_n) converges strongly to a fixed point $z \in \bigcap_{j=0}^{\ell-1} \mathrm{Fix}(T_j)$ and if $Q(\phi) = \lim x_n$, then $Q(\phi)$ solves the variational inequality*

$$\forall p \in \bigcap_{j=0}^{\ell-1} \mathrm{Fix}(T_j) \left(\langle \phi(Q(\phi)) - Q(\phi), J(p - Q(\phi)) \rangle \leq 0 \right),$$

where J denotes the normalized duality map[1].

Chang's interesting argument reduces the convergence of the general Halpern-type iteration to the convergence result due to Reich, under the minimal conditions (i) and (ii) together with the obviously necessary condition of asymptotic regularity. By imposing further conditions on the scalars (α_n), one can then stablish the asymptotic regularity of the sequence, and subsequently recover several central results in the literature.

The work developed in this paper arises from the perspective of the proof mining research program [30, 33] in which techniques from Logic (namely, proof interpretations) are employed in the analysis of a priori noneffective mathematical proofs with the goal of obtaining additional information. This information is usually in the form of rates of convergence or rates of metastability (in the sense of T. Tao [73, 74]), providing computational information in the asymptotic behaviour of the iterations studied. We promptly remark that these proof-theoretical techniques operate backstage to the analysis, guiding it, and are usually absent from the presentation of the final results. This feature makes the results available to a general audience without any background in Logic, as is the case with this work. Another important aspect that emerges from the quantitative studies is the knowledge of which are the central elements of the mathematical proof analysed. This frequently allows for generalizations of the original result, and has been particularly successful in obtaining generalizations of convergence results to nonlinear settings. A quantitative analysis of Browder's theorem was obtained in [31] (see also [21]) as well as the quantitative analysis of the elementary proof by Halpern which extends essentially unchanged to CAT(0) spaces, the nonlinear counterpart of Hilbert spaces.

[1] The definition of the normalized duality map is given in section 2.

On the other hand, the proof-theoretical treatment of the more general result by Reich was obtained in [39] by Kohlenbach and Sipoş. It is undoubtedly the most complicated instance of proof mining to date, solving a problem that resisted a quantitative scrutiny for more than ten years. The complexity of the quantitative analysis in [39] is itself a reflection of the profound combinatorial and computational content of the proof and further points to the relevance of Reich's theorem. Any attempt to generalize Reich's theorem to a nonlinear setting either fails or stops at the level of CAT(0) spaces, in which case it simply becomes the nonlinear generalization of Browder's convergence theorem. The present work addresses this issue.

In this paper, we introduce the notion of a nonlinear smooth space generalizing both CAT(0) spaces as well as smooth Banach spaces. Throughout, we argue that this notion allows for a unified treatment of several results in functional analysis, and in particular we establish a nonlinear generalization of Reich's theorem in a suitable geodesic setting. Using this result, we establish the convergence of a Halpern-type iteration for a general form of resolvent-like families of functions. Lastly, in this novel nonlinear setting, we prove the convergence of the viscosity Halpern's iteration (even for families of maps) through a quantitative study of Chang's arguments which moreover provide a generalization (even in the linear setting) of Theorem 1.3.

The next section ('Preliminaries') is divided into collecting the necessary technical lemmata and quantitative notions (subsection 2.1), and in recalling several notions of linear and nonlinear spaces (subsection 2.2). The concept of nonlinear smooth space is introduced and discussed in section 3. Section 4 is dedicated to the study of basic properties regarding the path (z_t) in preparation to section 5, where we stablish our nonlinear generalization of Reich's result. Sections 6 and 7, respectively, deal with the convergence of the Halpern-type schema for resolvent-like families of functions, and the quantitative study and generalization of Chang's result. The last two sections are devoted to corollaries and final remarks.

2. Preliminaries

Throughout, we denote $\mathbb{N} := \{0, 1, 2, \cdots\}$ and $\mathbb{N}^* := \mathbb{N} \setminus \{0\}$. We also distinguished a notation for intervals of natural numbers, namely $[a; b] := [a, b] \cap \mathbb{N}$.

2.1. Quantitative notions and Lemmas.

We begin by recalling some useful technical lemmas. Let (x_n) be a sequence in a metric space (X, d) and consider $x \in X$.

Definition 2.1. *We say that*

(1) $\varphi : (0, \infty) \to \mathbb{N}$ *is a rate of convergence for* (x_n) *(towards* x*) if*
$$\forall \varepsilon > 0\ \forall i \geq \varphi(\varepsilon)\, (d(x_i, x) \leq \varepsilon)\,;$$

(2) $\Phi : (0, \infty) \times \mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ *is a quasi-rate of convergence for* (x_n) *(towards* x*) if*
$$\forall \varepsilon > 0\ \forall f \in \mathbb{N}^{\mathbb{N}}\ \exists n \leq \Phi(\varepsilon, f)\ \forall i \in [n; n + f(n)]\, (d(x_i, x) \leq \varepsilon)\,.$$

(3) $\varphi : (0, \infty) \to \mathbb{N}$ *is a Cauchy rate for* (x_n) *if*
$$\forall \varepsilon > 0\ \forall i, j \geq \varphi(\varepsilon)\, (d(x_i, x_j) \leq \varepsilon)\,;$$

(4) $\Phi : (0, \infty) \times \mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ *is a rate of metastability for* (x_n) *if*
$$\forall \varepsilon > 0\ \forall f \in \mathbb{N}^{\mathbb{N}}\ \exists n \leq \Phi(\varepsilon, f)\ \forall i, j \in [n; n + f(n)]\, (d(x_i, x_j) \leq \varepsilon)\,.$$

The seemingly weaker 'metastable' reformulation is actually equivalent to the original property. However, since in the non-trivial direction one argues by contradiction, the proof is noneffective and a rate of metastability (quasi-rate) is in general computationally weaker than a Cauchy rate (rate of convergence).

The next result explains how to convert a rate of metastability into a rate of metastability with lower bound (e.g. [38]).

Lemma 2.2. *Let (x_n) be a Cauchy sequence with rate of metastability Φ. Then, for any $\varepsilon > 0$, $f : \mathbb{N} \to \mathbb{N}$ and any (lower bound) $N \in \mathbb{N}$*

$$\exists n \in [N; \tilde{\Phi}(\varepsilon, f, N)]\ \forall i, j \in [n; n + f(n)]\, (d(x_i, x_j) \leq \varepsilon)\,,$$

where $\tilde{\Phi}(\varepsilon, f, N) := \max\{N, \Phi(\varepsilon, f_N)\}$ with f_N the function defined for all $r \in \mathbb{N}$ by $f_N(r) := \max\{N, r\} - r + f(\max\{N, r\})$.

Proof. Let ε, f, N be given. As Φ is a rate of metastability, we have

$$\exists n_0 \leq \Phi(\varepsilon, f_N)\ \forall i, j \in [n_0; n_0 + f_N(n_0)]\, (d(x_i, x_j) \leq \varepsilon)\,.$$

Consider $n := \max\{N, n_0\} \in [N; \tilde{\Phi}(\varepsilon, f, N)]$. As $n \geq n_0$ and $n + f(n) = n_0 + f_N(n_0)$, the result follows. ☐

Remark 2.3. *Obviously, the same construction $\Phi \mapsto \tilde{\Phi}$ also converts any quasi-rate of convergence into a corresponding quasi-rate of convergence with lower bound.*

We will also require the following notions of asymptotic regularity (cf. [5, 11]) as well as their corresponding quantitative perspectives.

Definition 2.4. *Consider a sequence (x_n) and a map $T : X \to X$. We say that:*

(1) *(x_n) is asymptotically regular if $\lim d(x_{n+1}, x_n) = 0$.*

A (quasi-)rate of asymptotic regularity for (x_n) is a (quasi-)rate of convergence for $(d(x_{n+1}, x_n))$ towards 0.

(2) *(x_n) is T-asymptotically regular if $\lim d(T(x_n), x_n) = 0$.*

A (quasi-)rate of T-asymptotic regularity for (x_n) is a (quasi-)rate of convergence for $(d(T(x_n), x_n))$ towards 0.[2]

We require quantitative versions of a well-known lemma by Xu (e.g. [76, Lemma 2.1]) which we recall next.

Lemma 2.5. *Consider sequences of real numbers $(a_n) \subseteq [0, \infty)$, (b_n), $(c_n) \subset [0, \infty)$, and $(\lambda_n) \subseteq (0, 1)$ such that $a_{n+1} \leq (1 - \lambda_n)a_n + \lambda_n b_n + c_n$, for all $n \in \mathbb{N}$. If*

$$(i)\ \sum \lambda_n = \infty, \quad (ii)\ \limsup b_n \leq 0, \quad (iii)\ \sum c_n < \infty,$$

then $\lim a_n = 0$.

We say that $\sum \lambda_n$ diverges with rate of divergence $\theta : \mathbb{N} \to \mathbb{N}$ if

$$(\mathrm{Q}\Sigma) \qquad \forall L \in \mathbb{N} \left(\sum_{k=0}^{\theta(L)} \lambda_k \geq L \right).$$

Whenever $\theta(L) \leq \theta(L + 1)$, for all $L \in \mathbb{N}$, we say that the function θ is monotone.

---

[2]Note that if (x_n) is the Picard iteration, then the notions of asymptotic regularity and T-asymptotic regularity coincide.

The following quantitative version of Lemma 2.5 is from [20] (and it is an easy variant of [36, Lemma 5.2]).

Lemma 2.6. *Consider sequences of real numbers $(a_n), (b_n) \subset \mathbb{R}$, and $(\lambda_n) \subseteq [0,1]$. Let $B \in \mathbb{N}^*$ be an upper bound on (a_n). Given $\varepsilon > 0$, $K, P \in \mathbb{N}$, assume that for all $n \in [K; P]$,*

$$(i)\ a_{n+1} \leq (1-\lambda_n)a_n + \lambda_n b_n + \frac{\varepsilon}{3(P+1)} \quad \text{and} \quad (ii)\ b_n \leq \frac{\varepsilon}{3}.$$

If θ is a function satisfying (QΣ), then $\forall n \in [\chi; P]\,(a_n \leq \varepsilon)$, where[3]

$$\chi := \chi[\theta, B](\varepsilon, K) := \theta\left(K + \left\lceil \ln\left(\frac{3B}{\varepsilon}\right)\right\rceil\right) + 1.$$

Remark 2.7. *Obviously, it suffices to have $(\lambda_n) \subseteq [0,1]$ for $n \in [K; P]$, as we are free to change the terms outside such an interval. Note that, since for all $L \in \mathbb{N}$*

$$L \leq \sum_{k=0}^{\theta(L)} \lambda_k \leq \theta(L) + 1,$$

we have $\chi[\theta, B](\varepsilon, K) \geq K$. Moreover, if θ is monotone, then χ is monotone in the variable K, i.e.

$$K \leq K' \to \chi[\theta, B](\varepsilon, K) \leq \chi[\theta, B](\varepsilon, K').$$

We will also use the following lemma.

Lemma 2.8 ([20, Lemma 2.9])**.** *Consider sequences of real numbers $(a_n) \subseteq \mathbb{R}^+$, $(b_n) \subseteq \mathbb{R}$, $(c_n) \subset \mathbb{R}^+$ and $(\lambda_n) \subseteq [0,1]$. Let $B \in \mathbb{N}^*$ be an upper bound on (a_n). Assume that for all $n \in \mathbb{N}$, $a_{n+1} \leq (1-\lambda_n)a_n + \lambda_n b_n + c_n$. Let θ_1 be a function satisfying (QΣ). Then,*

(1) if $c_n = 0$ for all $n \in \mathbb{N}$, and θ_2 satisfies $\forall \varepsilon > 0\ \forall n \geq \theta_2(\varepsilon)\,(b_n \leq \varepsilon)$, then $\lim a_n = 0$ *with rate of convergence*

$$\chi_1[\theta_1, \theta_2, B](\varepsilon) := \theta_1\left(\theta_2\left(\frac{\varepsilon}{2}\right) + \left\lceil \ln\left(\frac{2B}{\varepsilon}\right)\right\rceil\right) + 1;$$

(2) if $b_n = 0$ for all $n \in \mathbb{N}$, and θ_3 a Cauchy rate for $(\sum c_n)$, then $\lim a_n = 0$ *with rate of convergence*

$$\chi_2[\theta_1, \theta_3, B](\varepsilon) := \theta_1\left(\theta_3\left(\frac{\varepsilon}{2}\right) + 1 + \left\lceil \ln\left(\frac{2B}{\varepsilon}\right)\right\rceil\right) + 1.$$

Remark 2.9. *The condition $\sum \alpha_n < \infty$ is equivalent to the condition $\prod(1-\lambda_n) = 0$, when $(\lambda_n) \subseteq [0,1)$. Naturally, one may additionally consider the latter condition in a quantitative form, and provide computational information based on such assumption. In some instances, changing to that formulation may lower the complexity of the resulting bounds. It is not clear that to be the case in the results discussed in this paper and so, for simplicity, we don't do this here.*

We recall the finitary version of the monotone convergence principle (cf. [43, 74]).

[3] Our function $\lceil \cdot \rceil$ is capped below at zero, i.e. $\lceil r \rceil := \max\{\lceil r \rceil, 0\}$, with the one on the right-hand side with the usual definition.

Lemma 2.10. *Consider $b \in \mathbb{N}^*$ and $(a_n) \subseteq [0,b]$ a nonincreasing sequence of real numbers. Then, for any $\varepsilon > 0$, $f : \mathbb{N} \to \mathbb{N}$ and $N \in \mathbb{N}$*

$$\exists n \in [N; f^{+\left(\lceil \frac{b}{\varepsilon} \rceil\right)}(N)]\ \forall i,j \in [n; n+f(n)]\, (|a_i - a_j| \leq \varepsilon)\,,$$

with $f^+(n) := n + f(n)$, and $g^{(r)}$ stands for the r-fold composition.

Proof. If it were false, then in particular $a_{f^{+(i)}(N)} - a_{f^{+(i+1)}(N)} > \varepsilon$, for all $i \leq \lceil b/\varepsilon \rceil$, which entails the contradiction

$$b \geq a_N \geq a_N - a_{f^{+(\lceil b/\varepsilon \rceil + 1)}(N)} > \left\lceil \frac{b}{\varepsilon} \right\rceil \varepsilon \geq b. \qquad \square$$

2.2. **Classes of linear and nonlinear spaces.** We recall the notion of hyperbolic space introduced in [29] which we will consider throughout the paper. A triple (X, d, W) is called a hyperbolic space if $W : X \times X \times [0,1] \to X$ is a function satisfying for all $x, y, z, w \in X$ and $\lambda, \lambda' \in [0,1]$

(W1) $d(W(x,y,\lambda), z) \leq (1-\lambda) d(x,z) + \lambda d(y,z)$
(W2) $d(W(x,y,\lambda), W(x,y,\lambda')) = |\lambda - \lambda'| d(x,y)$
(W3) $W(x,y,\lambda) = W(y,x,1-\lambda)$
(W4) $d(W(x,y,\lambda), W(z,w,\lambda)) \leq (1-\lambda) d(x,z) + \lambda d(y,w)$.

Whenever it is convenient, we also write $(1-\lambda)x \oplus \lambda y$ for $W(x,y,\lambda)$. One easily sees that

$$d(x, (1-\lambda)x \oplus \lambda y) = \lambda d(x,y) \text{ and } d(y, (1-\lambda)x \oplus \lambda y) = (1-\lambda) d(x,y).$$

A set $C \subseteq X$ is said to be convex if

$$\forall x, y \in C\ \forall \lambda \in [0,1]\, ((1-\lambda)x \oplus \lambda y \in C)\,.$$

Hyperbolic spaces allow for discussions where convex combinations feature in an essential way. Several notions exist in the literature. The convexity function W was first considered by Takahashi in [71] with a triple (X, d, W) satisfying (W1) called a convex metric space. The notion used here was introduced by Kohlenbach in [29] motivated by proof-theoretical reasons, and it is frequently considered the nonlinear generalization of normed spaces. This notion is more general than that of hyperbolic spaces in the sense of Reich and Shafrir [61], and slightly more restrictive that the setting due to Goebel and Kirk [22] of spaces of hyperbolic type.

The class of hyperbolic spaces includes the normed spaces and their convex subsets (with $W(x,y,\lambda)$ being the usual convex combination, providing the motivation for the $\oplus$ notation), and the Hilbert ball [23].

We also recall the notion of uniformly convex hyperbolic spaces as introduced in [45] (inspired by [23, p.105], see also [61]), which generalizes to the nonlinear setting the notion of uniformly convex normed space. A uniformly convex hyperbolic space is a structure (X, η), where X is a hyperbolic space and $\eta : (0,\infty) \times (0,2] \to (0,1]$ is a function satisfying

$$\left.\begin{aligned} d(x,a) &\leq r \\ d(y,a) &\leq r \\ d(x,y) &\geq \varepsilon \cdot r \end{aligned}\right\} \to d\left(\frac{1}{2}x \oplus \frac{1}{2}y, a\right) \leq (1 - \eta(r,\varepsilon))\, r$$

for all $r > 0$, $\varepsilon \in (0,2]$ and $x, y, a \in X$. A function η satisfying the condition above is called a modulus of uniform convexity of the space X. As introduced in [46], we moreover say that X is a UCW hyperbolic space if the modulus of uniform convexity η is monotone in the following sense:

$$\forall \varepsilon \in (0,2]\ \forall r,s>0\,(r \leq s \to \eta(s,\varepsilon) \leq \eta(r,\varepsilon)).$$

The class of UCW-hyperbolic spaces was shown to be a suitable setting for discussing quantitative results of well-known iterative methods. See for example [35, 45, 46], where the asymptotic behaviour of the Mann iteration for (asymptotically) nonexpansive mappings was studied, and [4] for the behaviour of the Picard iteration for firmly nonexpansive mappings.

Another important subclass of hyperbolic spaces is that of CAT(0) spaces. These spaces, introduced by Aleksandrov in [1] and named as such by Gromov [24], are characterized as the hyperbolic spaces that satisfy the CN^- property (which, in the presence of the W-axioms, is equivalent to the Bruhat-Tits CN-inequality [14]):

$$(\mathrm{CN}^-) \qquad \forall x,y,z \in X \left(d^2\left(\frac{1}{2}x \oplus \frac{1}{2}y, z\right) \leq \frac{1}{2}d^2(x,z) + \frac{1}{2}d^2(y,z) - \frac{1}{4}d^2(x,y)\right),$$

which actually extends beyond midpoints (see e.g. [18, Lemma 2.5]), and we have for all $x,y,z \in X$ and $\lambda \in [0,1]$

$$(\mathrm{CN}^+) \qquad d^2\left((1-\lambda)x \oplus \lambda y, z\right) \leq (1-\lambda)d^2(x,z) + \lambda d^2(y,z) - \lambda(1-\lambda)d^2(x,y).$$

Recall that CAT(0) spaces are a particular example of UCW hyperbolic space, and in [45] Leuştean showed that it has a modulus of uniform convexity quadratic in $\varepsilon \in (0,2]$ and independent of $r>0$, namely $\eta(r,\varepsilon) := \varepsilon^2/8$.

In [7], Berg and Nikolaev showed that CAT(0) spaces are equivalently characterized as the hyperbolic spaces that satisfy the following four-point inequality,

$$(\mathrm{CS}) \qquad \frac{1}{2}\left(d^2(x,v) + d^2(y,u) - d^2(x,u) - d^2(y,v)\right) \leq d(x,y)d(u,v).$$

The expression on the left-hand side of the inequality above is of central relevance for arguments in a nonlinear setting as it defines the so-called quasi-linearization function, $\langle\cdot,\cdot\rangle : X^2 \times X^2 \to \mathbb{R}$

$$\langle \overrightarrow{xy}, \overrightarrow{uv} \rangle := \frac{1}{2}\left(d^2(x,v) + d^2(y,u) - d^2(x,u) - d^2(y,v)\right),$$

where $\overrightarrow{xy}$ denotes the pair (x,y). In any metric space, this function is the *unique* function satisfying the following properties (see [7, Proposition 14]):

(1) $\langle \overrightarrow{xy}, \overrightarrow{xy} \rangle = d^2(x,y)$,
(2) $\langle \overrightarrow{xy}, \overrightarrow{uv} \rangle = \langle \overrightarrow{uv}, \overrightarrow{xy} \rangle$,
(3) $\langle \overrightarrow{xy}, \overrightarrow{uv} \rangle = -\langle \overrightarrow{yx}, \overrightarrow{uv} \rangle$,
(4) $\langle \overrightarrow{xy}, \overrightarrow{uv} \rangle + \langle \overrightarrow{xy}, \overrightarrow{vw} \rangle = \langle \overrightarrow{xy}, \overrightarrow{uw} \rangle$.

Therefore, this function enjoys properties akin to an inner product, and the condition (CS) can be regarded as a metric version of the Cauchy-Schwarz inequality. Indeed, CAT(0) spaces are considered the canonical nonlinear counterpart of inner product spaces.

Proposition 2.11 ([2]). *Let $(X, \|\cdot\|)$ be a real normed space. Then (X,d) with the metric induced by the norm is a* CAT(0) *space if and only if the norm is induced by an inner product.*

Proof. Assume that X is a CAT(0) space. Hence, by the CN^- property, we derive the (equivalent formulation of the) parallelogram law

$$\|x+y\|^2 + \|x-y\|^2 \leq 2(\|x\|^2 + \|y\|^2)$$

and thus X is an inner product space with the inner product defined by the polarization identity

$$\langle x, y\rangle := \frac{\|x\|^2 + \|y\|^2 - \|x-y\|^2}{2},$$

which coincides with $\left\langle \overrightarrow{x0_X}, \overrightarrow{y0_X} \right\rangle$. For the reverse direction, assume that the norm is induced by an inner product. Then the quasi-linearization is defined by[4]

$$\langle \overrightarrow{xy}, \overrightarrow{uv}\rangle := \langle x-y, u-v\rangle = \langle y-x, v-u\rangle.$$

Hence the condition (CS) holds and X is a CAT(0) space. □

Several results that hold in Hilbert spaces, still hold in a more general setting of normed spaces provided 'inner product'-like arguments are still available. In this instance, one usually assumes some additional conditions on the norm which are more general than assuming it to arise from an inner-product.

Definition 2.12. *Let X be a (real) normed space. The normalized duality map, $J : X \to 2^{X^*}$ is defined for all $x \in X$ by*

$$J(x) := \{f \in X^* : \langle x, f\rangle = \|x\|^2 = \|f\|^2\},$$

where X^* is the dual space of X and $\langle y, f\rangle$ denotes the functional application $f(y)$.

A normed space X is said to be smooth (introduced in [17] under the name of *flattening*) if the norm is Gâteaux differentiable, i.e. for any $x, y \in X$ satisfying $\|x\| = \|y\| = 1$, the limit

$$(\star) \qquad \lim_{t\to 0} \frac{\|x+ty\| - \|x\|}{t}$$

exists. It is well-known that X is smooth if and only if the normalized duality map is single-valued. Moreover, X is uniformly smooth if the limit $(\star)$ is attained uniformly in x and y. In turn, X is uniformly smooth if and only if the normalized duality map is single-valued and is norm-to-norm uniformly continuous on bounded subsets. See e.g. [16, 23, 58, 72] for further discussions on these notions.

3. Smooth hyperbolic spaces

In the following, we introduce a nonlinear generalization of smooth normed spaces allowing for a uniform treatment of several results in functional analysis.

Let (X, d, W) be a hyperbolic space. Consider a function $\pi : X^2 \times X^2 \to \mathbb{R}$ satisfying for all $x, y, u, v, z \in X$

(P1) $\pi(\overrightarrow{xy}, \overrightarrow{xy}) = d^2(x, y)$
(P2) $\pi(\overrightarrow{xy}, \overrightarrow{uv}) = -\pi(\overrightarrow{yx}, \overrightarrow{uv}) = -\pi(\overrightarrow{xy}, \overrightarrow{vu})$
(P3) $\pi(\overrightarrow{xy}, \overrightarrow{uv}) + \pi(\overrightarrow{yz}, \overrightarrow{uv}) = \pi(\overrightarrow{xz}, \overrightarrow{uv})$
(P4) $\pi(\overrightarrow{xy}, \overrightarrow{uv}) \leq d(x, y)d(u, v)$

and for any $x, y, z \in X$ and $\lambda \in [0, 1]$

(P5) $d^2(W(x, y, \lambda), z) \leq (1-\lambda)^2 d^2(x, z) + 2\lambda\pi\left(\overrightarrow{yz}, \overrightarrow{W(x, y, \lambda)z}\right)$.

[4]This justifies the 'inner product' notation.

We call a space (X, d, W, π) under the conditions above a smooth hyperbolic spaces and, motivated by the results in this paper, we regarded them as a nonlinear counterpart to smooth normed vector spaces. The space is said to be a uniformly smooth hyperbolic space if it additionally satisfies

$$\text{(P6)} \begin{cases} \forall \varepsilon > 0 \; \forall r > 0 \; \exists \delta > 0 \; \forall a \in X \; \forall u, v \in \overline{B}_r(a) \\ \quad d(u, v) \leq \delta \to \forall x, y \in X \left(|\pi(\overrightarrow{xy}, \overrightarrow{ua}) - \pi(\overrightarrow{xy}, \overrightarrow{va})| \leq \varepsilon \cdot d(x, y)\right). \end{cases}$$

We call a function $\omega_X : (0, \infty)^2 \to (0, \infty)$ a modulus of uniform continuity for π if $\omega_X(r, \varepsilon)$ witnesses δ in (P6).

Further intermediate notions can be introduced by relaxing the uniformity of δ, e.g. one may allow $\delta > 0$ to additionally depend on $x, y \in X$, in which case the factor $d(x, y)$ becomes inconsequential.

Remark 3.1. *In this work, the underlying setting is that of hyperbolic spaces where the function W is required to satisfy (W1)–(W4). It is however clear that the conditions (P1)–(P6) can still be considered in broader contexts where the properties of W are weakened. That gives rise to further notions of a smooth geodesic space. For example, one may replace the condition (W4) with the weaker*

$(W5)$ $\forall x, y, z \in X \; \forall \lambda \in [0, 1] \left(d(W(x, y, \lambda), W(z, y, \lambda)) \leq d(x, z)\right)$.

This weakening of the function W allows to cover in particular the class of $CAT(\kappa)$ spaces with $\kappa > 0$ (cf. the proof-theoretical treatment in [37]). In particular, whenever our results make no use of (W4), they immediately extend to this weaker assumption on the function W. Such is the case in the reduction proofs in sections 6 and 7, but not in the proof of the nonlinear version of Reich's theorem (Theorem 5.2). For convenience we state our results in hyperbolic spaces in the sense of subsection 2.2.

Whenever the space is simultaneously a UCW hyperbolic space and (uniformly) smooth, then we say that it is a (uniformly) smooth UCW hyperbolic space. We have the following easy result.

Proposition 3.2. *Let (X, π) be a smooth hyperbolic space. If π is symmetric, i.e. $\pi(\overrightarrow{xy}, \overrightarrow{uv}) = \pi(\overrightarrow{uv}, \overrightarrow{xy})$, then X is a* CAT(0) *space. If for any $x, y, u, v, w \in X$, $\pi(\overrightarrow{xy}, \overrightarrow{uv}) + \pi(\overrightarrow{xy}, \overrightarrow{vw}) = \pi(\overrightarrow{xy}, \overrightarrow{uw})$, then X is a* CAT(0) *space.*

Proof. In the first case, the function π satisfies the conditions (1)–(4), and by uniqueness we get $\pi(\overrightarrow{xy}, \overrightarrow{uv}) = \langle \overrightarrow{xy}, \overrightarrow{uv} \rangle$. By (P4) the condition (CS) holds. Hence X is a CAT(0) space. For the second case, note that the conditions (P1)–(P4) are satisfied by $\pi'(\overrightarrow{xy}, \overrightarrow{uv}) := \dfrac{\pi(\overrightarrow{xy}, \overrightarrow{uv}) + \pi(\overrightarrow{uv}, \overrightarrow{xy})}{2}$. So $\pi'(\overrightarrow{xy}, \overrightarrow{uv}) = \langle \overrightarrow{xy}, \overrightarrow{uv} \rangle$ and (CS) holds. $\square$

We now discuss some connections with other spaces.

Proposition 3.3. *Any* CAT(0) *space is a uniformly smooth UCW hyperbolic space.*

Proof. Let X be a CAT(0) space. Consider the function $\pi(\overrightarrow{xy}, \overrightarrow{uv}) := \langle \overrightarrow{xy}, \overrightarrow{uv} \rangle$. The properties (P1)–(P3) are satisfied by the quasi-linearization function. As X is a CAT(0) space, the quasi-linearization function satisfies the metric Cauchy-Schwarz inequality and thus (P4) holds. To show that (P5) holds, for completeness, we reproduce the argument used in [67]. Multiplying (CN^+) by $(1 - \lambda)$ and using the fact that $(1 - \lambda) d(x, y) = d(y, W(x, y, \lambda))$, we obtain

$$\begin{aligned} (1 - \lambda) d^2(W(x, y, \lambda), z) &\leq (1 - \lambda)^2 d^2(x, z) + \lambda(1 - \lambda) d^2(y, z) - \lambda(1 - \lambda)^2 d^2(x, y) \\ &\leq (1 - \lambda)^2 d^2(x, z) + \lambda d^2(y, z) - \lambda d^2(y, W(x, y, \lambda)). \end{aligned}$$

From this inequality, we conclude that

$$\begin{aligned} d^2(W(x,y,\lambda),z) &\leq (1-\lambda)^2 d^2(x,z) + \lambda\left(d^2(y,z) + d^2(z,W) - d^2(y,W(x,y,\lambda))\right) \\ &\leq (1-\lambda)^2 d^2(x,z) + 2\lambda\left\langle \overrightarrow{yz}, \overrightarrow{W(x,y,\lambda)z}\right\rangle \end{aligned}$$

and (P5) holds. Therefore, X is a smooth hyperbolic space. By the 'additivity' property (4) of the quasi-linearization function and using (CS),

$$|\langle \overrightarrow{xy}, \overrightarrow{ua}\rangle - \langle \overrightarrow{xy}, \overrightarrow{va}\rangle| = |\langle \overrightarrow{xy}, \overrightarrow{ua}\rangle + \langle \overrightarrow{xy}, \overrightarrow{av}\rangle| = |\langle \overrightarrow{xy}, \overrightarrow{uv}\rangle| \leq d(x,y)d(u,v).$$

Therefore, condition (P6) holds with $\delta = \varepsilon$, and X is a uniformly smooth hyperbolic space. □

Proposition 3.4. *Any (uniformly) smooth normed space is a (uniformly) smooth hyperbolic space.*

Proof. Let X be a smooth normed space. Hence the duality map is single-valued and we write $j : X \to X^*$ for the unique selection of the element in the duality map. It follows from the definition of the duality map that the selection j is odd, i.e. $j(-x) = -j(x)$. The space X is a hyperbolic space with the canonical metric induced by the norm, $d(x,y) := \|x-y\|$, and the function W standing for the usual linear convex combination. For the function π, we define for all $x,y,u,v \in X$

$$\pi(\overrightarrow{xy}, \overrightarrow{uv}) := \langle x-y, j(u-v)\rangle.$$

It is easy to see that properties (P1)–(P4) hold. We know that (see e.g. [54])

$$\forall x,y \in X\left(\|x+y\|^2 \leq \|x\|^2 + 2\langle y, j(x+y)\rangle\right).$$

Taking $x = (1-\lambda)(x-z)$ and $y = \lambda(y-z)$, we obtain for $x,y,z \in X$ and $\lambda \in [0,1]$

$$\|W(x,y,\lambda) - z\|^2 \leq (1-\lambda)^2\|x-z\|^2 + 2\lambda\langle y-z, j(W(x,y,\lambda)-z)\rangle,$$

i.e.

$$d^2(W(x,y,\lambda),z) \leq (1-\lambda)^2 d(x,z)^2 + 2\lambda\pi\left(\overrightarrow{yz}, \overrightarrow{W(x,y,\lambda)z}\right).$$

Therefore (P5) holds and X is a smooth hyperbolic space.

If X is uniformly smooth, then the duality map is norm-to-norm uniformly continuous on bounded subsets, which entails that for all $\varepsilon > 0$ and $b > 0$ there exists some $\delta > 0$ such that for all $a \in X$

$$\forall u,v \in \overline{B}_b(a)\left(\|(u-a)-(v-a)\| \leq \delta \to \|j(u-a) - j(v-a)\|_{X^*} \leq \varepsilon\right).$$

Stating the operator norm in terms of the language of X, we get

$$\begin{cases} \forall u,v \in \overline{B}_b(a)\ \forall x,y \in X \\ (\|u-v\| \leq \delta \to |\langle x-y, j(u-a)\rangle - \langle x-y, j(v-a)\rangle| \leq \varepsilon \cdot \|x-y\|) \end{cases}$$

which is (P6). Hence X is a uniformly smooth hyperbolic space. □

As shown, the class of CAT(0) spaces is a proper subclass of uniformly smooth hyperbolic spaces. Indeed, any uniformly smooth normed space is also a uniformly smooth hyperbolic space and if it were a CAT(0) space, then it would be an inner product space (cf. Proposition 2.11). Since the class of uniformly smooth normed spaces is broader than that of inner product spaces, we conclude that the class of (uniformly) smooth hyperbolic spaces introduced here is more general than that of CAT(0) spaces.

4. Maps and Iterations in hyperbolic spaces

Consider (X, d) to be a metric space and C a nonempty subset of X. A metric setting still allows for the notions of strict contraction and of nonexpansive mapping.

Definition 4.1. *Consider a map $T : C \to C$. We say that*

(1) T is a strict contraction on C if there exists (a contracting factor) $\alpha < 1$ such that

$$\forall x, y \in C \left(d(T(x), T(y)) \leq \alpha \cdot d(x, y)\right),$$

(2) T is a nonexpansive map on C if

$$\forall x, y \in C \left(d(T(x), T(y)) \leq d(x, y)\right).$$

Trivially, any constant map is a strict contraction with a null factor. Also, clearly any strict contraction is a nonexpansive map, and the composition of a strict contraction with a nonexpansive map is again a strict contraction.

The following fact, valid in any metric spaces, will be useful in the sequel.

Lemma 4.2. *Let (X, d) be a metric space and $r > 0$. For any $x, y, z \in X$ with $d(x, y), d(x, z) \leq r$, we have $d^2(x, y) \leq d^2(x, z) + 2r \cdot d(y, z)$.*

Proof. By triangle inequality we get $d(x, y) - d(x, z) \leq d(y, z)$, and thus

$$d^2(x, y) - d^2(x, z) = \left(d(x, y) - d(x, z)\right)\left(d(x, y) + d(x, z)\right) \leq 2r \cdot d(y, z),$$

which entails the inequality. □

Let now (X, d, W) be a complete hyperbolic space and $C \subseteq X$ a nonempty closed convex subset. Consider $T : C \to C$ a nonexpansive map and $\phi : C \to C$ a strict contraction with a contracting factor $\alpha \in [0, 1)$. For $t \in (0, 1]$, consider the map defined for all $x \in C$ by

$$T_t(x) := (1 - t)T(x) \oplus t\phi(x).$$

As the map $T_t : C \to C$ is a strict contraction by Banach's contraction principle it has an unique fixed point. Therefore, we may consider a point $z_t \in C$ satisfying

$$z_t = (1 - t)T(z_t) \oplus t\phi(z_t). \tag{B$_\phi$}$$

We now discuss some easy properties of the path characterized by the equation (B$_\phi$).

Lemma 4.3. *Let $p \in \mathrm{Fix}(T)$. Then, for any $t \in (0, 1]$*

$$d(z_t, p) \leq \frac{d(\phi(p), p)}{1 - \alpha}.$$

Proof. Using (W1), we have

$$\begin{aligned} d(z_t, p) &\leq (1 - t)d(T(z_t), p) + td(\phi(z_t), p) \\ &\leq (1 - t)d(z_t, p) + td(\phi(z_t), p) \end{aligned}$$

which implies

$$\begin{aligned} d(z_t, p) \leq d(\phi(z_t), p) &\leq d(\phi(z_t), \phi(p)) + d(\phi(p), p) \\ &\leq \alpha \cdot d(z_t, p) + d(\phi(p), p) \end{aligned}$$

entailing the result. □

Consider $b > 0$ such that $b \geq \frac{d(\phi(p),p)}{1-\alpha}$, for some $p \in \mathrm{Fix}(T)$. The next result shows the T-asymptotic regularity of (z_t) in the sense that $d(z_t, T(z_t)) \to 0$, when $t \to 0$.

Lemma 4.4. *For all $\varepsilon > 0$ and $t \in (0,1]$,*

$$t \leq \frac{\varepsilon}{2b} \to d(z_t, T(z_t)) \leq \varepsilon.$$

Proof. Assume that $t \leq \dfrac{\varepsilon}{2b}$. We have,

$$\begin{aligned} d(z_t, T(z_t)) &= td(T(z_t), \phi(z_t)) \\ &\leq t\left(d(T(z_t), p) + d(p, \phi(p)) + d(\phi(p), \phi(z_t))\right) \\ &\leq t\left(d(z_t, p) + d(p, \phi(p)) + \alpha \cdot d(z_t, p)\right) \\ &\leq t(b + (1-\alpha)b + \alpha b) = 2b \cdot t \leq \varepsilon. \end{aligned}$$ □

When (X, π) is a smooth hyperbolic space, we have the following result.

Lemma 4.5. *Given $\varepsilon > 0$, $t \in (0,1]$, $r > 0$, $\eta > 0$, we have for all $q \in C$ such that $d(q,p) \leq r$,*

$$\left(\eta \leq t \leq \frac{\varepsilon}{(b+r)^2} \ \wedge \ d(T(q), q) \leq \frac{\varepsilon\eta}{2(b+r)}\right) \to \pi\left(\overrightarrow{z_t\phi(z_t)}, \overrightarrow{z_tq}\right) \leq \varepsilon.$$

Proof. Note that under the premises,

$$t\frac{(b+r)^2}{2} \leq \frac{\varepsilon}{2} \quad \text{and} \quad \frac{b+r}{t} d(T(q), q) \leq \frac{b+r}{\eta} d(T(q), q) \leq \frac{\varepsilon}{2}.$$

As $d(T(z_t), q), d(T(z_t), T(q)) \leq b + r$, using (P1),(P3) (P5) and Lemma 4.2, we get

$$\begin{aligned} d^2(z_t, q) &\leq (1-t)^2 d^2(T(z_t), q) + 2t\pi\left(\overrightarrow{\phi(z_t)q}, \overrightarrow{z_tq}\right) \\ &\leq (1-t)^2 d^2(T(z_t), T(q)) + 2(b+r)d(T(q), q) + 2t\pi\left(\overrightarrow{\phi(z_t)z_t}, \overrightarrow{z_tq}\right) \\ &\quad + 2t\pi(\overrightarrow{z_tq}, \overrightarrow{z_tq}) \\ &\leq (1-t)^2 d^2(z_t, q) + 2(b+r)d(T(q), q) + 2t\pi\left(\overrightarrow{\phi(z_t)z_t}, \overrightarrow{z_tq}\right) \\ &\quad + 2td^2(z_t, q) \\ &= (1+t^2) d^2(z_t, q) + 2(b+r)d(T(q), q) + 2t\pi\left(\overrightarrow{\phi(z_t)z_t}, \overrightarrow{z_tq}\right) \end{aligned}$$

which, using (P2) and $d(z_t, q) \leq b + r$, entails

$$\pi\left(\overrightarrow{z_t\phi(z_t)}, \overrightarrow{z_tq}\right) \leq t\frac{(b+r)^2}{2} + \frac{b+r}{t} \cdot d(T(q), q) \leq \varepsilon.$$ □

When t is instantiated by the sequence $\frac{1}{m+1}$, and $z_m = z_{t_m}$, we have the particular instance of the previous two lemmas.

Lemma 4.6. *For all $\varepsilon > 0$ and $m \in \mathbb{N}$*

$$m \geq \left\lfloor \frac{2b}{\varepsilon} \right\rfloor \to d(z_m, T(z_m)) \leq \varepsilon.$$

Proof. Immediate from Lemma 4.4, as $m \geq \lfloor \frac{2b}{\varepsilon} \rfloor$ implies $t_m \leq \frac{\varepsilon}{2b}$. □

Lemma 4.7. *Given $\varepsilon, r > 0$, and $m \in \mathbb{N}$, we have for all $q \in C$ such that $d(q,p) \leq r$,*

$$\left(m \geq \left\lfloor \frac{(b+r)^2}{\varepsilon} \right\rfloor \ \wedge\ d(T(q),q) \leq \frac{\varepsilon}{2(b+r)(m+1)}\right) \to \pi\left(\overrightarrow{z_m\phi(z_m)}, \overrightarrow{z_m q}\right) \leq \varepsilon.$$

Proof. Just note that the assumption on m entails the premise of Lemma 4.5 with $\eta = t_m > 0$. □

We conclude this section by introducing some notation for the relevant iterative schemas discussed in what follows. Consider that $\{T_n\}$ is an infinite family of nonexpansive maps on C. For given $x_0 \in C$, recursively define

$$x_{n+1} := (1-\alpha_n)T_n(x_n) \oplus \alpha_n\phi(x_n), \tag{$\mathrm{H}_\phi^{T_n}$}$$

where (α_n) is a sequence of real numbers in $[0,1]$. If $\{T_n\}$ consists of a single nonexpansive map T, we recover the viscosity Halpern iteration introduced by Moudafi [52] (in the linear setting)

$$x_{n+1} := (1-\alpha_n)T(x_n) \oplus \alpha_n\phi(x_n). \tag{H_ϕ^{T}}$$

If additionally ϕ is a constant function, this recursion becomes the original iterative schema defined by Halpern in [25] (in the linear setting),

$$x_{n+1} := (1-\alpha_n)T(x_n) \oplus \alpha_n u. \tag{H}$$

When the family $\{T_n\}$ is obtained cyclically from a finite family of maps, we get the nonlinear version of the iteration studied in Banach spaces by Xu [78] and Chang [15] (generalizing Bauschke [6] beyond Hilbert spaces and with the addition of a viscosity term).

5. A nonlinear version of Reich's theorem

We are interested in generalizing to a nonlinear setting the following pivotal result.

Theorem 5.1 (cf. [59])**.** *Let X be a Banach space which is uniformly smooth and uniformly convex, $C \subseteq X$ a closed nonempty bounded convex subset, and $u \in C$. Consider $T : C \to C$ a nonexpansive map on C. For any $t \in (0,1]$, let z_t denote the unique point in C satisfying $z_t = (1-t)T(z_t) + tu$. Then, for all $(t_n) \subseteq (0,1]$ such that $\lim t_n = 0$, we have that (z_{t_n}) strongly converges to a fixed point of T.*

This result was first established by Reich [59] without the uniform convexity hypothesis. Moreover, Reich's theorem actually goes beyond nonexpansive maps and deals with set-valued accretive operators satisfying the range condition, including in particular the important class of continuous pseudocontractions maps [10]. This class of functions is more general than that of nonexpansive mappings and plays a crucial role in the abstract formulation of Cauchy problems. We remark that one can still consider such class of maps in our setting (via a π-version of the variational inequality characterization) and believe that Theorem 5.2 below should also hold with that extension. For simplicity, we don't do this here and leave it for future research. We shall prove the following result.

Theorem 5.2. *Let X be a complete uniformly smooth UCW hyperbolic space, C a closed nonempty bounded convex subset, and $u \in C$. Consider $T : C \to C$ a nonexpansive map on C. For any $t \in (0,1]$, let z_t denote the unique point in C satisfying $z_t = (1-t)T(z_t) \oplus tu$. Then, for all $(t_n) \subseteq (0,1]$ such that $\lim t_n = 0$, we have that (z_{t_n}) converges to a fixed point of T.*

In the normed setting, Reich's argument makes use of the continuous convex function

$$F(z) := \limsup_{n\to\infty} \|z_{t_n} - z\|, \text{ for all } z \in C$$

for $(t_n) \subseteq (0,1]$ a sequence of real numbers converging to 0. Since X is uniformly smooth, it is reflexive and thus the function F attains its infimum on the closed convex bounded set C. In our nonlinear setting, this avenue is not available. As in [39], we instead rely on the additional assumption of uniform convexity. The proof-theoretical analysis given by Kohlenbach and Sipoş in [39], motivated by a proof due to Morales [51], starts by arguing that one may establish Theorem 5.1 (even with pseudocontractions) using only ε-infima instead of the (proof-theoretically trouble-some) exact infimum. It is this proof that we show in the sequel to extend to a nonlinear setting. We require the following two preliminary results.

Lemma 5.3 ([39, Lemma 3.2]). *Let (a_n), (b_n) be two bounded sequences of real numbers. Then,*

$$\liminf_{n\to\infty}(a_n - b_n) \leq \limsup_{n\to\infty} a_n - \limsup_{n\to\infty} b_n.$$

The following is a reformulation of [66, Proposition 2.5] and in our argument it replaces [39, Proposition 2.4] used in the linear setting.

Proposition 5.4. *Let X be a UCW hyperbolic space with monotone modulus of uniform convexity η. Then, for all $r > 0$, $\varepsilon \in (0,2]$ and $a, x, y \in X$, we have*

$$\left.\begin{aligned} d(x,a) &\leq r \\ d(y,a) &\leq r \\ d(x,y) &\geq \varepsilon r \end{aligned}\right\} \to d^2\left(\frac{1}{2}x \oplus \frac{1}{2}y, a\right) \leq \max\left\{d^2(x,a), d^2(y,a)\right\} - \psi_\eta(r,\varepsilon),$$

where $\psi_\eta(r,\varepsilon) := \frac{1}{4}\varepsilon^2 r^2 \eta(r,\varepsilon)$.

Proof. Let $\gamma := \max\{d(x,a), d(y,a)\} \leq r$. We have,

$$\varepsilon r \leq d(x,y) \leq d(x,a) + d(y,a) \leq 2\gamma,$$

and so $\gamma \geq \frac{\varepsilon r}{2}$. Then, by the fact that η is a monotone modulus of uniform convexity, we conclude

$$\begin{aligned} d^2\left(\frac{1}{2}x \oplus \frac{1}{2}y, a\right) &\leq ((1-\eta(\gamma,\varepsilon))\gamma)^2 \leq (1-\eta(\gamma,\varepsilon))\gamma^2 = \gamma^2 - \eta(\gamma,\varepsilon)\gamma^2 \\ &\leq \gamma^2 - \eta(r,\varepsilon)\frac{\varepsilon^2 r^2}{4} = \max\{d^2(x,a), d^2(y,a)\} - \psi_\eta(r,\varepsilon). \end{aligned}$$ □

For the rest of this section, we assume to be in the conditions of Theorem 5.2 which we want to prove. For (t_n) a sequence of real numbers $(0,1]$ with $t_n \to 0$, we simply write $z_n := z_{t_n}$. Since C is assumed to be bounded, we consider $b > 0$ to be a positive upper bound on its diameter, i.e. $d(x,y) \leq b$, for all $x, y \in C$.

Lemma 5.5. *For all $\varepsilon > 0$, there exists $y \in C$ such that for all $z \in C$*

$$\max\left\{\limsup_{n\to\infty} d^2(z_n, y), \limsup_{n\to\infty} d^2(z_n, T(y))\right\} \leq \limsup_{n\to\infty} d^2(z_n, z) + \varepsilon.$$

Proof. Assume towards a contradiction that for some $\varepsilon > 0$ we have

$$\forall y \in C\ \exists z \in C \left(\limsup_{n\to\infty} d^2(z_n, y) > \limsup_{n\to\infty} d^2(z_n, z) + \varepsilon \right). \tag{1}$$

With $K := \lceil \frac{b^2}{\varepsilon} \rceil$, recursively define a finite sequence $\{y_0, \ldots, y_K\} \subseteq C$ by: y_0 is some point in the nonempty set C and, for $i < K$, given $y_i \in C$ let $y_{i+1} \in C$ be such that

$$\limsup_{n\to\infty} d^2(z_n, y_i) > \limsup_{n\to\infty} d^2(z_n, y_{i+1}) + \varepsilon,$$

which is guaranteed to exists by the assumption (1). We then derive the contradiction

$$b^2 \geq \limsup_{n\to\infty} d^2(z_n, y_0) > \limsup_{n\to\infty} d^2(z_n, y_K) + K\varepsilon \geq \frac{b^2}{\varepsilon}\varepsilon = b^2.$$

For the second inequality, given $\varepsilon > 0$, take $y \in C$ such that

$$\forall z \in C \left(\limsup_{n\to\infty} d^2(z_n, y) \leq \limsup_{n\to\infty} d^2(z_n, z) + \varepsilon \right).$$

Since $t_n \to 0$, by Lemma 4.4 we have $\lim d(z_n, T(z_n)) = 0$ and so, for all $z \in C$

$$\begin{aligned}
\limsup_{n\to\infty} d^2(z_n, T(y)) &\leq \limsup_{n\to\infty} \left(d(z_n, T(z_n)) + d(T(z_n), T(y))\right)^2 \\
&\leq \limsup_{n\to\infty} d(z_n, T(z_n)) \left(d(z_n, T(z_n)) + 2d(z_n, y)\right) + d^2(z_n, y) \\
&\leq 3b \cdot \limsup_{n\to\infty} d(z_n, T(z_n)) + \limsup_{n\to\infty} d^2(z_n, y) \\
&= \limsup_{n\to\infty} d^2(z_n, y) \leq \limsup_{n\to\infty} d^2(z_n, z) + \varepsilon.
\end{aligned}$$

□

Lemma 5.6. *For all $\varepsilon > 0$, there exists $v \in C$ such that for all $z \in C$*

$$d(v, T(v)) \leq \varepsilon \ \wedge\ \limsup_{n\to\infty} d^2(z_n, v) \leq \limsup_{n\to\infty} d^2(z_n, z) + \varepsilon.$$

Proof. Assume w.l.g. that $\varepsilon \leq 2$. By Lemma 5.5 applied to $\varepsilon_0 := \min\{\varepsilon, \frac{1}{2}\psi_\eta(b, \varepsilon/b)\}$, we can consider $v \in C$ such that for all $z \in C$

$$\max\left\{ \limsup_{n\to\infty} d^2(z_n, v), \limsup_{n\to\infty} d^2(z_n, T(v)) \right\} \leq \limsup_{n\to\infty} d^2(z_n, z) + \varepsilon_0.$$

Assume towards a contradiction that $d(v, T(v)) > \varepsilon = \frac{\varepsilon}{b} b$. By the assumption on b, we have $d(v, z_n), d(T(v), z_n) \leq b$ and can apply Proposition 5.4 to conclude that for all $n \in \mathbb{N}$,

$$d^2\left(z_n, \frac{1}{2}v \oplus \frac{1}{2}T(v)\right) \leq \max\left\{d^2(z_n, v), d^2(z_n, T(v))\right\} - \psi_0,$$

with $\psi_0 := \psi_\eta(b, \varepsilon/b) > \varepsilon_0$. As by convexity $\frac{1}{2}v \oplus \frac{1}{2}T(v) \in C$, we derive

$$\begin{aligned}
\limsup_{n\to\infty} d^2\left(z_n, \frac{1}{2}v \oplus \frac{1}{2}T(v)\right) &\leq \limsup_{n\to\infty} \max\left\{d^2(z_n, v), d^2(z_n, T(v))\right\} - \psi_0 \\
&\leq \max\left\{ \limsup_{n\to\infty} d^2(z_n, v), \limsup_{n\to\infty} d^2(z_n, T(v)) \right\} - \psi_0 \\
&\leq \limsup_{n\to\infty} d^2\left(z_n, \frac{1}{2}v \oplus \frac{1}{2}T(v)\right) + \underbrace{\varepsilon_0 - \psi_0}_{<0},
\end{aligned}$$

which is a contradiction. $\square$

Proposition 5.7. *There exists $p \in \mathrm{Fix}(T)$ such that for all $z \in C$*

$$\limsup_{n\to\infty} d^2(z_n, p) \le \limsup_{n\to\infty} d^2(z_n, z).$$

Proof. For each $m \in \mathbb{N}$, consider $v_m \in C$ such that $d(v_m, T(v_m)) \le \frac{1}{m+1}$ and

$$\forall z \in C \left(\limsup_{n\to\infty} d^2(z_n, v_m) \le \limsup_{n\to\infty} d^2(z_n, z) + \frac{1}{m+1} \right).$$

Given $\varepsilon \in (0, 2]$, let $i, j \ge \lceil \frac{2}{\psi_0} \rceil$ with $\psi_0 := \psi_\eta(b, \varepsilon/b)$. If $d(v_i, v_j) > \varepsilon = \frac{\varepsilon}{b} b$, then similarly to before we can derive a contradiction by using Proposition 5.4,

$$\begin{aligned}
\limsup_{n\to\infty} d^2\left(z_n, \frac{1}{2}v_i \oplus \frac{1}{2}v_j\right) &\le \max\left\{\limsup_{n\to\infty} d^2(z_n, v_i), \limsup_{n\to\infty} d^2(z_n, v_j)\right\} - \psi_0 \\
&\le \limsup_{n\to\infty} d^2\left(z_n, \frac{1}{2}v_i \oplus \frac{1}{2}v_j\right) + \frac{1}{2}\psi_0 - \psi_0 \\
&< \limsup_{n\to\infty} d^2\left(z_n, \frac{1}{2}v_i \oplus \frac{1}{2}v_j\right)
\end{aligned}$$

Hence (v_m) is a Cauchy sequence. Since X is complete and C is closed, it converges to some point in C, and the limit point of (v_m) satisfies the desired properties. $\square$

We are now ready to give the proof of the theorem.

Proof of Theorem 5.2. Consider $p \in \mathrm{Fix}(T)$ as guaranteed to exist by Proposition 5.7. Consider $\varepsilon > 0$ arbitrary. Then, by (P6) there exists $\delta > 0$ such that

$$\forall n \in \mathbb{N}\ \forall v \in \overline{B}_b(z_n)\, (d(p, v) \le \delta \to |\pi(\overrightarrow{up}, \overrightarrow{z_n p}) - \pi(\overrightarrow{up}, \overrightarrow{z_n v})| \le \varepsilon).$$

Since $t_n \to 0$, we may consider $n_0 \in \mathbb{N}$ such that $t_n \le \min\{\delta/b, \varepsilon/2b\}$ for all $n \ge n_0$. Let $v := (1 - t_{n_0})p \oplus t_{n_0} u \in C$. Then, clearly for all $n \in \mathbb{N}$, $v \in \overline{B}_b(z_n)$ and

$$d(p, v) = d(p, W(p, u, t_{n_0})) = t_{n_0} d(p, u) \le \frac{\delta}{b} b = \delta.$$

Hence, for all $n \in \mathbb{N}$, we have $\pi(\overrightarrow{up}, \overrightarrow{z_n p}) \le \varepsilon + \pi(\overrightarrow{up}, \overrightarrow{z_n v})$. On the other hand, by (P5) and using (P3) and (P4), we have

$$\begin{aligned}
d^2(z_n, v) &\le (1 - t_{n_0})^2 d^2(z_n, p) + 2t_{n_0}\pi(\overrightarrow{uz_n}, \overrightarrow{vz_n}) \\
&\le (1 - t_{n_0})^2 d^2(z_n, p) + 2t_{n_0}\left(\pi(\overrightarrow{up}, \overrightarrow{vz_n}) + \pi(\overrightarrow{pz_n}, \overrightarrow{vz_n})\right) \\
&\le (1 - t_{n_0})^2 d^2(z_n, p) + 2t_{n_0}\pi(\overrightarrow{up}, \overrightarrow{vz_n}) + 2t_{n_0} d(z_n, p) d(z_n, v).
\end{aligned}$$

Note that, using (W2) and (W4), for all $n \ge n_0$

$$\begin{aligned}
d(z_n, v) &= d(W(T(z_n), u, t_n), W(p, u, t_{n_0})) \\
&\le d(W(T(z_n), u, t_n), W(T(z_n), u, t_{n_0})) + d(W(T(z_n), u, t_{n_0}), W(p, u, t_{n_0})) \\
&\le |t_n - t_{n_0}| d(T(z_n), u) + (1 - t_{n_0}) d(T(z_n), p) \\
&\le \varepsilon + (1 - t_{n_0}) d(z_n, p).
\end{aligned}$$

Therefore, for $n \geq n_0$

$$\begin{aligned} d^2(z_n, v) &\leq (1-t_{n_0})^2 d^2(z_n,p) + 2t_{n_0}\pi(\overrightarrow{up}, \overrightarrow{vz_n}) + 2t_{n_0} d(z_n,p)\left(\varepsilon + (1-t_{n_0})d(z_n,p)\right) \\ &\leq (1-t_{n_0})(1-t_{n_0}+2t_{n_0})d^2(z_n,p) + 2t_{n_0}\pi(\overrightarrow{up}, \overrightarrow{vz_n}) + 2t_{n_0}\varepsilon d(z_n,p) \\ &\leq (1-t_{n_0}^2)d^2(z_n,p) + 2t_{n_0}\pi(\overrightarrow{up}, \overrightarrow{vz_n}) + 2t_{n_0}\varepsilon \cdot b \\ &\leq d^2(z_n,p) + 2t_{n_0}\pi(\overrightarrow{up}, \overrightarrow{vz_n}) + 2t_{n_0}\varepsilon \cdot b \end{aligned}$$

Hence, using (P2), $\pi(\overrightarrow{up}, \overrightarrow{z_n v}) \leq \frac{1}{2t_{n_0}}\left(d^2(z_n,p) - d^2(z_n,v)\right) + \varepsilon \cdot b$. Thus, it follows that for all $n \geq n_0$

$$\pi(\overrightarrow{up}, \overrightarrow{z_n p}) \leq \varepsilon \cdot (b+1) + \frac{1}{2t_{n_0}}\left(d^2(z_n,p) - d^2(z_n,v)\right).$$

Hence, by Lemma 5.3,

$$\begin{aligned} \liminf_{n\to\infty} \pi(\overrightarrow{up}, \overrightarrow{z_n p}) &\leq \varepsilon \cdot (b+1) + \frac{1}{2t_{n_0}} \liminf_{n\to\infty}\left(d^2(z_n,p) - d^2(z_n,v)\right) \\ &\leq \varepsilon \cdot (b+1) + \frac{1}{2t_{n_0}} \underbrace{\left(\limsup_{n\to\infty} d^2(z_n,p) - \limsup_{n\to\infty} d^2(z_n,v)\right)}_{\leq 0} \\ &\leq \varepsilon \cdot (b+1). \end{aligned}$$

Since ε was arbitrary, we conclude that $\liminf_{n\to\infty} \pi(\overrightarrow{up}, \overrightarrow{z_n p}) \leq 0$. Hence, there is a subsequence (t_{α_n}) of (t_n) such that $\limsup_{n\to\infty} \pi(\overrightarrow{up}, \overrightarrow{z_{\alpha_n} p}) \leq 0$.
For a given $\varepsilon > 0$, consider $N \in \mathbb{N}$ such that $t_n \leq \frac{2}{b^2}\varepsilon$, for all $n \geq N$. Then, we have for $n \geq N$

$$\begin{aligned} d^2(z_n,p) &\overset{\text{(P5)}}{\leq} (1-t_n)^2 d^2(T(z_n),p) + 2t_n\pi(\overrightarrow{up}, \overrightarrow{z_n p}) \\ &\overset{\text{(P3)}}{\leq} (1-t_n)^2 d^2(z_n,p) + 2t_n\left(\pi(\overrightarrow{uz_n}, \overrightarrow{z_n p}) + \pi(\overrightarrow{z_n p}, \overrightarrow{z_n p})\right) \\ &\overset{\text{(P1)}}{=} (1-t_n)^2 d^2(z_n,p) + 2t_n\pi(\overrightarrow{uz_n}, \overrightarrow{z_n p}) + 2t_n d^2(z_n,p), \end{aligned}$$

Therefore, using (P2)

$$2t_n\pi(\overrightarrow{z_n u}, \overrightarrow{z_n p}) \leq \left(1 - 2t_n + t_n^2 + 2t_n - 1\right) d^2(z_n,p) \leq t_n^2 \cdot b^2,$$

and so, for all $n \geq N$

$$\pi(\overrightarrow{z_n u}, \overrightarrow{z_n p}) \leq t_n \frac{b^2}{2} \leq \varepsilon, \tag{2}$$

which shows that $\limsup_{n\to\infty} \pi(\overrightarrow{z_n u}, \overrightarrow{z_n p}) \leq 0$. Now, for arbitrary $\varepsilon > 0$, consider $N \in \mathbb{N}$ such that

$$\forall n \geq N \left(\pi(\overrightarrow{z_n u}, \overrightarrow{z_n p}) \leq \frac{\varepsilon^2}{2} \ \wedge\ \pi(\overrightarrow{up}, \overrightarrow{z_{\alpha_n} p}) \leq \frac{\varepsilon^2}{2}\right).$$

For $n \geq N$, as (α_n) is strictly increasing, $\alpha_n \geq N$ and we conclude

$$d^2(z_{\alpha_n}, p) = \pi(\overrightarrow{z_{\alpha_n} p}, \overrightarrow{z_{\alpha_n} p}) = \pi(\overrightarrow{z_{\alpha_n} u}, \overrightarrow{z_{\alpha_n} p}) + \pi(\overrightarrow{up}, \overrightarrow{z_{\alpha_n} p}) \leq \varepsilon^2,$$

which means that (z_{α_n}) converges to p. To conclude the proof, assume that (z_n) does not converge to p. Then there exist $\varepsilon > 0$ and a subsequence (t_{β_n}) of (t_n) such that

$$\forall n \in \mathbb{N}\left(d(z_{\beta_n}, p) > \varepsilon\right). \tag{3}$$

Let $t'_n := t_{\beta_n}$ and $z'_n := z_{t'_n}$. Since, $(t'_n) \subseteq (0,1]$ and $t'_n \to 0$, by the previous arguments, there is some subsequence (t'_{γ_n}) of (t'_n) and some $q \in \mathrm{Fix}(T)$ such that

$$z'_{\gamma_n} \to q \quad \text{and} \quad \limsup_{n\to\infty} \pi\left(\overrightarrow{z'_n u}, \overrightarrow{z'_n q}\right) \leq 0. \tag{4}$$

We show that $q = p$, which contradicts (3) and so (z_n) must converge towards p. By (P6), for any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\forall n \in \mathbb{N}\ \forall v \in \overline{B}_b(p) \left(d(q, v) \leq \delta \to \left| \pi\left(\overrightarrow{z'_{\gamma_n} u}, \overrightarrow{qp}\right) - \pi\left(\overrightarrow{z'_{\gamma_n} u}, \overrightarrow{vp}\right) \right| \leq \frac{\varepsilon^2}{6} \right).$$

By (4) we can take $N_0 \in \mathbb{N}$ large enough such that

$$d(q, z'_{\gamma_{N_0}}) \leq \min\left\{\delta, \frac{\varepsilon^2}{6b}\right\} \quad \text{and} \quad \pi\left(\overrightarrow{z'_{\gamma_{N_0}} u}, \overrightarrow{z'_{\gamma_{N_0}} q}\right) \leq \frac{\varepsilon^2}{6}.$$

Since $z'_{\gamma_{N_0}} \in \overline{B}_b(p)$, $\pi\left(\overrightarrow{z'_{\gamma_{N_0}} u}, \overrightarrow{qp}\right) \leq \pi\left(\overrightarrow{z'_{\gamma_{N_0}} u}, \overrightarrow{z'_{\gamma_{N_0}} p}\right) + \varepsilon^2/6$, and therefore

$$\begin{aligned}
\pi(\overrightarrow{qu}, \overrightarrow{qp}) &= \pi\left(\overrightarrow{q z'_{\gamma_{N_0}}}, \overrightarrow{qp}\right) + \pi\left(\overrightarrow{z'_{\gamma_{N_0}} u}, \overrightarrow{qp}\right) \\
&\leq d(q, z'_{\gamma_{N_0}}) \cdot b + \pi\left(\overrightarrow{z'_{\gamma_{N_0}} u}, \overrightarrow{z'_{\gamma_{N_0}} p}\right) + \frac{\varepsilon^2}{6} \\
&\leq \frac{\varepsilon^2}{6b} b + \frac{\varepsilon^2}{6} + \frac{\varepsilon^2}{6} = \frac{\varepsilon^2}{2}.
\end{aligned}$$

Similarly, one concludes that $\pi(\overrightarrow{pu}, \overrightarrow{pq}) \leq \varepsilon^2/2$, and so

$$d^2(q,p) = \pi(\overrightarrow{qp}, \overrightarrow{qp}) = \pi(\overrightarrow{qu}, \overrightarrow{qp}) + \pi(\overrightarrow{up}, \overrightarrow{qp}) = \pi(\overrightarrow{qu}, \overrightarrow{qp}) + \pi(\overrightarrow{pu}, \overrightarrow{pq}) \leq \varepsilon^2,$$

which shows that $q = p$ and concludes the proof of Theorem 5.2. ☐

The quantitative study in [39] further simplifies the argument by bypassing the use of exact $\limsup$'s and working instead with the relaxed notion of ε-$\limsup$. This is also possible in our nonlinear setting and, similar to [39], enables the extraction of the computational content associated with the asymptotic behaviour of (z_t). As such extraction is (necessarily) very technical it is not discussed here but instead in the forthcoming [55].

The relevance of Reich's theorem is tied to the last statement of Theorem 1.2, namely the fact that it provides an algorithmic approach to the construction of sunny nonexpansive retractions [57]. We also address this point. Recall that a map $Q : C \to E \subseteq C$ is called a retraction if $Q(x) = x$ whenever $x \in E$. In the smooth Banach spaces, sunny nonexpansive retractions are characterized as the functions that satisfy the following variational inequality

$$\forall x \in C\ \forall y \in E\, (\langle x - Q(x), j(y - Q(x))\rangle \leq 0)\,,$$

where j is the single-valued normalized duality map on X (cf. [23, Lemma 13.1]). This motivates the following definition.[5]

[5]We remark that the terminology might be slightly misleading. Indeed, in the nonlinear setting the connection with "sunny retractions" is not clear, as the usual arguments rely on the linearity of the underlying space.

Definition 5.8. *Let X be a smooth hyperbolic space and $E \subseteq C$ subsets of X. We say that a retraction $Q : C \to E$ is a (π-)sunny nonexpansive retraction if*

$$\forall x \in C \ \forall y \in E \left(\pi\left(\overrightarrow{xQ(x)}, \overrightarrow{yQ(x)}\right) \leq 0\right).$$

Lemma 5.9. *Let X be a smooth hyperbolic space and $E \subseteq C$ subsets of X.*

(1) Any sunny nonexpansive retraction is a (π-)firmly nonexpansive map[6] *i.e.*

$$\forall x, y \in C \left(d^2(Q(x), Q(y)) \leq \pi\left(\overrightarrow{xy}, \overrightarrow{Q(x)Q(y)}\right)\right),$$

and so, in particular, it is a nonexpansive map.

(2) There exists at most one sunny nonexpansive retraction from C onto E.

Proof. For item (1), consider $x, y \in C$. Since $Q(x), Q(y) \in E$ and Q is a sunny nonexpansive retraction

$$\pi\left(\overrightarrow{Q(x)x}, \overrightarrow{Q(x)Q(y)}\right) \overset{(P2)}{=} \pi\left(\overrightarrow{xQ(x)}, \overrightarrow{Q(y)Q(x)}\right) \leq 0 \text{ and } \pi\left(\overrightarrow{yQ(y)}, \overrightarrow{Q(x)Q(y)}\right) \leq 0$$

Hence,

$$\begin{aligned}
0 &\geq \pi\left(\overrightarrow{Q(x)x}, \overrightarrow{Q(x)Q(y)}\right) + \pi\left(\overrightarrow{yQ(y)}, \overrightarrow{Q(x)Q(y)}\right) \\
&\overset{(P3)}{=} \pi\left(\overrightarrow{Q(x)y}, \overrightarrow{Q(x)Q(y)}\right) + \pi\left(\overrightarrow{yx}, \overrightarrow{Q(x)Q(y)}\right) + \pi\left(\overrightarrow{yQ(y)}, \overrightarrow{Q(x)Q(y)}\right) \\
&\overset{(P3)}{=} \pi\left(\overrightarrow{Q(x)Q(y)}, \overrightarrow{Q(x)Q(y)}\right) + \pi\left(\overrightarrow{yx}, \overrightarrow{Q(x)Q(y)}\right) \\
&\overset{(P1)}{=} d^2(Q(x), Q(y)) + \pi\left(\overrightarrow{yx}, \overrightarrow{Q(x)Q(y)}\right).
\end{aligned}$$

Using (P2) we conclude that

$$d^2(Q(x), Q(y)) \leq -\pi\left(\overrightarrow{yx}, \overrightarrow{Q(x)Q(y)}\right) = \pi\left(\overrightarrow{xy}, \overrightarrow{Q(x)Q(y)}\right).$$

The claim that Q is nonexpansive now follows from (P4).
For (2), consider Q_1 and Q_2 two π-sunny nonexpansive retractions from C onto E. Let $x \in C$ be arbitrary. We have

$$\pi\left(\overrightarrow{Q_1(x)x}, \overrightarrow{Q_1(x)Q_2(x)}\right) \leq 0 \text{ and } \pi\left(\overrightarrow{xQ_2(x)}, \overrightarrow{Q_1(x)Q_2(x)}\right) \leq 0,$$

which together entail $d^2(Q_1(x), Q_2(x)) \leq 0$, and so $Q_1(x) = Q_2(x)$. □

In the conditions of Theorem 5.2, for each $u \in C$, define $Q(u) := \lim z_n$ with (z_n) satisfying $z_n = (1-t_n)T(z_n) \oplus t_n u$, for some sequence $(t_n) \subseteq (0, 1]$ with $t_n \to 0$. As the proof of Theorem 5.2 shows, the limit does not depend on the sequence (t_n) and so Q is a well-defined function from C onto $\mathrm{Fix}(T)$. Moreover, if $u \in \mathrm{Fix}(T)$, then for all $n \in \mathbb{N}$,

$$d(z_n, u) = (1-t_n)d(T(z_n), u) \leq (1-t_n)d(z_n, u),$$

which, since $t_n > 0$ entails $z_n \equiv u$. This shows that, $\forall u \in \mathrm{Fix}(T)\,(Q(u) = u)$, i.e. the function Q is a retraction from C onto $\mathrm{Fix}(T)$.

Proposition 5.10. *The map Q is the unique (π-)sunny nonexpansive retraction from C onto* $\mathrm{Fix}(T)$.

[6] In CAT(0) spaces, with $\pi = \langle \cdot, \cdot \rangle$ this corresponds to the so-called property (P_2).

Proof. Let $q \in \mathrm{Fix}(T)$ and $\varepsilon > 0$ be arbitrary. We can reuse the argument towards (2), noticing that it only required the point p to be a fixed point of T. Consider $N \in \mathbb{N}$ such that $t_n \leq \frac{2}{b^2}\varepsilon$, for all $n \geq N$. Then,

$$\begin{aligned} d^2(z_n, q) &\leq (1-t_n)^2 d^2(T(z_n), q) + 2t_n \pi(\overrightarrow{uq}, \overrightarrow{z_n q}) \\ &\leq (1+t_n^2) d^2(z_n, q) + 2t_n \pi(\overrightarrow{uz_n}, \overrightarrow{z_n q}), \end{aligned}$$

which implies $\pi(\overrightarrow{uz_n}, \overrightarrow{qz_n}) \leq \varepsilon$, and so $\limsup_{n\to\infty} \pi(\overrightarrow{uz_n}, \overrightarrow{qz_n}) \leq 0$. The result now follows by a continuity argument. By (P6), there exists $\delta > 0$ such that

$$\forall v \in \overline{B}_b(q) \left(d(Q(u), v) \leq \delta \to \left| \pi\left(\overrightarrow{uQ(u)}, \overrightarrow{qQ(u)}\right) - \pi\left(\overrightarrow{uQ(u)}, \overrightarrow{qv}\right) \right| \leq \frac{\varepsilon}{3} \right).$$

Take $N_0 \in \mathbb{N}$ such that $d(z_{n_0}, Q(u)) \leq \min\{\delta, \varepsilon/3b\}$ and $\pi(\overrightarrow{uz_{n_0}}, \overrightarrow{qz_{n_0}}) \leq \frac{\varepsilon}{3}$. Since clearly $z_{n_0} \in \overline{B}_b(q)$, we conclude

$$\begin{aligned} \pi\left(\overrightarrow{uQ(u)}, \overrightarrow{qQ(u)}\right) &\leq \pi\left(\overrightarrow{uQ(u)}, \overrightarrow{qz_{n_0}}\right) + \frac{\varepsilon}{3} \\ &= \pi(\overrightarrow{uz_{n_0}}, \overrightarrow{qz_{n_0}}) + \pi\left(\overrightarrow{z_{n_0}Q(u)}, \overrightarrow{qz_{n_0}}\right) + \frac{\varepsilon}{3} \\ &\leq \frac{\varepsilon}{3} + d(z_{n_0}, Q(u)) \cdot b + \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

As $\varepsilon > 0$ was arbitrary, the result follows. □

Remark 5.11. *The boundedness assumption on C can be replaced by the assumption that T has fixed points. Indeed, if $T : C \to C$ is a nonexpansive map on a nonempty closed convex subset of X and $\mathrm{Fix}(T) \neq \emptyset$, consider some $p \in \mathrm{Fix}(T)$ and $u \in C$. Let $b > 0$ be such that $b \geq d(u, p)$. It follows that T is a nonexpansive map on $C \cap \overline{B}_b(p)$, which clearly is a nonempty closed convex bounded subset of X.*

We conclude this section with some considerations on the viscosity variant of (z_t).

Definition 5.12. *Let (X, d) be a metric space and C a nonempty subset of X. Consider a mapping $\phi : C \to C$. We say that*

(1) ϕ is a Rakotch map on C, if there exists a function $\rho : (0,\infty) \to (0,1)$ satisfying

$$\forall \varepsilon > 0\, \forall x, y \in C\, (d(x,y) \geq \varepsilon \to d(\phi(x), \phi(y)) \leq \rho(\varepsilon) \cdot d(x,y)).$$

(2) ϕ is a Meir-Keeler contraction (MKC) on C, if

$$\forall \varepsilon > 0\, \exists \delta > 0\, \forall x, y \in C\, (d(x,y) < \varepsilon + \delta \to d(\phi(x), \phi(y)) < \varepsilon)$$

Clearly any strict contraction is a Rakotch map with a constant function ρ, and any Rakotch map is nonexpansive. Any Rakotch map is a MKC mapping (given $\varepsilon > 0$, take $\delta = \frac{1-\rho(\varepsilon)}{\rho(\varepsilon)}\varepsilon$) and any MKC mapping is a nonexpansive map. The function ρ is called a Rakotch modulus for the map ϕ. Meir-Keeler contraction were first introduced in [50] as a generalization of strict contractions. Actually, [50] defined this class of maps with the stronger premise $\varepsilon \leq d(x,y) < \varepsilon + \delta$, which turns out to be equivalent to the one above using the nonexpansivity of ϕ. In general hyperbolic spaces, when C is a convex set any MKC map is a Rakotch map (see [38, Lemma 2.2]; first shown for normed spaces in [69, Proposition 2]). Rakotch maps were introduced in [56] as the class of maps satisfying the condition

$$x \neq y \to d(T(x), T(y)) \leq \rho(d(x,y)) \cdot d(x,y)$$

for a decreasing function $\rho : (0,\infty) \to (0,1)$. However, the assumption that ρ is decreasing actually implies that the condition is equivalent to the one in the definition, and any Rakotch modulus can always be assumed to be decreasing, changing if necessary to

$$\rho'(\varepsilon) := \max\{1/2, \inf\{\rho(\varepsilon') : \varepsilon' \in (0,\varepsilon]\}\}.$$

In [56, 50], Banach's contraction principle was extended, respectively, to Rakotch and MKC maps. In complete hyperbolic spaces, for any nonexpansive map T and $t \in (0,1]$, if ϕ is a MKC, then the map $x \mapsto (1-t)T(x) \oplus t\phi(x)$ is again an MKC map. Hence, it has a unique fixed point, and Browder's implicit scheme (B_ϕ) is well-defined even for an MKC ϕ.

In [38, Theorem 3.5 and Remark 3.6], it was shown in particular that the convergence of the Browder-type iteration entails the convergence of the more general form (B_ϕ) (even for an MKC ϕ) in the general setting of hyperbolic spaces – the results in [38] were obtained from a proof-theoretical treatment of the main results in [69], in particular, generalizing them to hyperbolic spaces. We thus obtain the following corollary to Theorem 5.2.

Corollary 5.13. *Let X be a complete uniformly smooth UCW hyperbolic space, and C a closed nonempty bounded convex subset. Consider $T : C \to C$ a nonexpansive map on C, and ϕ an MKC map. For any $t \in (0,1]$, let z_t denote the unique point in C satisfying $z_t = (1-t)T(z_t) \oplus t\phi(z_t)$. Then, for all $(t_n) \subseteq (0,1]$ such that $\lim t_n = 0$, we have that (z_{t_n}) converges to the unique fixed point of $Q \circ \phi$, with Q as before.*

Remark 5.14. *The limit point z in Corollary 5.13 above satisfies*

$$\forall q \in \mathrm{Fix}(T) \left(\pi\left(\overrightarrow{\phi(z)z}, \overrightarrow{qz}\right) \leq 0\right).$$

Indeed, with $z = \lim z_n$, by Proposition 5.10, we have

$$\forall q \in \mathrm{Fix}(T) \left(\pi\left(\overrightarrow{\phi(z)Q(\phi(z))}, \overrightarrow{qQ(\phi(z))}\right) \leq 0\right),$$

which entails the result as $Q(\phi(z)) = z$.

6. Resolvent-like families of nonexpansive maps

Let X be a hyperbolic space, and C a nonempty subset of X.

Definition 6.1. *We say that a family $\{T_n : C \to C\}_{n\in\mathbb{N}}$ of nonexpansive maps is resolvent-like if for some (or for all [7]) $n \in \mathbb{N}$, $\mathrm{Fix}(T_n) \neq \emptyset$ and:*

(1) *There exists a (w.l.g.) monotone function $\mu : \mathbb{N} \to [1,\infty)$ satisfying*

$$\forall n, m \in \mathbb{N}\ \forall x \in C\ (d(x, T_n(x)) \leq \mu(n) \cdot d(x, T_m(x)));$$

(2) *There exists a function $\Delta : \mathbb{N} \times (0,\infty) \to (0,\infty)$ satisfying*

$$\forall \varepsilon > 0\ \forall r \in \mathbb{N}\ \forall n \in \mathbb{N}\ \forall p \in C\ \forall x \in C \cap \overline{B}_r(p)$$
$$\Big(\big(d(p, T_n(p) \leq \Delta(r,\varepsilon)\ \wedge\ d(x,p) - d(T_n(x),p) \leq \Delta(r,\varepsilon)\big) \to d(x, T_n(x)) \leq \varepsilon\Big).$$

[7] Indeed, by (1), the fixed point sets of all the maps T_n coincide.

Let us briefly give the motivation for these conditions, and clarify use of the terminology 'resolvent-like'. Let $(\gamma_n) \subseteq (0, \infty)$ be a sequence of real numbers such that $\inf \gamma_n \geq \widetilde{\gamma} > 0$. When X is a Banach space, consider T_n to be the resolvent functions $J_{\gamma_n A} = (\mathrm{Id} + \gamma_n A)^{-1}$ of an accretive operator A subject to the range condition

$$\overline{\mathrm{D}(A)} \subseteq C \subseteq \mathrm{R}(\mathrm{Id} + \gamma A), \quad \text{for all} \quad \gamma > 0,$$

where $\overline{\mathrm{D}(A)}$ is the closure of the domain of A, and C is a nonempty closed subset of X. In [3], the authors proved the strong convergence of the Halpern-type variant of the proximal point algorithm (HPPA) in uniformly convex Banach spaces with a uniformly Gâteaux differentiable norm under the weakest conditions known so far. From the well-known resolvent identity, it follows

$$\forall n, m \in \mathbb{N}\, \forall x \in C \left(\|x - J_{\gamma_n A}(x)\| \leq \left(2 + \frac{\gamma_n}{\gamma_m}\right) \|x - J_{\gamma_m A}(x)\| \right),$$

which means that condition (1) is satisfied with $\mu(n) := 2 + \frac{\gamma_n}{\widetilde{\gamma}}$. In [34], a proof-theoretical analysis of the central result in [3] was carried out in the slightly more restrictive setting of Banach spaces which are both uniformly smooth and uniformly convex. As pointed out by Kohlenbach [34], the central aspect in the convergence proof is the fact that the resolvents functions are strongly nonexpansive with a common SNE-modulus (introduced in [32]), and [34, Lemma 3.3] (essentially following from [32, Proposition 2.17]) shows that the condition (2) holds and that a function Δ can be easily computed from a modulus of uniform convexity on the space. Motivated by the proof mining work in [34], Sipoş in [67] proved the corresponding convergence result in the nonlinear setting of CAT(0) spaces, where the resolvent functions are replaced by a jointly (P_2) family of maps[8]. In [67] it is shown that, similarly to resolvent functions in the linear setting, a jointly (P_2) family of map $\{T_n : X \to X\}$ w.r.t. a sequence (γ_n) satisfies

$$\forall n, m \in \mathbb{N}\, \forall x \in X \left(d(x, T_n(x)) \leq \left(2 + \frac{\gamma_n}{\gamma_m}\right) d(x, T_m(x)) \right),$$

and so, again under the condition $\inf \gamma_n \geq \widetilde{\gamma} > 0$, we can take $\mu(n)$ as before. Moreover, Sipoş also observed that it is already sufficient to consider the notion of uniform strong quasi-nonexpansiveness (see [12, 32]), and obtained a property dubbed "quantitative quasiness" (cf. [67, Proposition 4.3]), which entails the condition (2) above (with the same function Δ that works for resolvent functions of monotone operators in Hilbert spaces). Overall, these properties are essential in establishing the convergence of both the HPPA in Banach spaces which are simultaneously uniformly smooth and uniformly convex [3] (in light of the proof mining study in [34]), as well as of the Halpern-type schema with jointly (P_2) families of maps in CAT(0) spaces by the work in [67].

This section gives a uniform treatment to the main results in [34] and [67]. Namely, we will prove the following conditional convergence result.

Theorem 6.2. *Let X be a complete uniformly smooth hyperbolic space and C be a nonempty closed convex subset. Consider $\{T_n\}$ to be a resolvent-like family of*

[8]The notion of jointly (P_2) family of maps has been considered for the study of an abstract form of the proximal point algorithm in a geodesic setting [44, 68].

nonexpansive maps on C. Let $(\alpha_n) \subseteq (0,1]$ be a sequence of positive real numbers satisfying $\lim \alpha_n = 0$ and $\sum \alpha_n = \infty$, and for $u \in C$ let (z_m) be the sequence characterized by the equation

$$z_m = \frac{m}{m+1} T_0(z_m) \oplus \frac{1}{m+1} u, \text{ for all } m \in \mathbb{N}$$

and assume it be a Cauchy sequence. If for a given initial $x_0 \in C$, (x_n) is recursively generated by the schema

$$(\mathrm{H_{ppa}}) \qquad x_{n+1} := (1-\alpha_n) T_n(x_n) \oplus \alpha_n u,$$

then (x_n) converges to a common fixed point of $\{T_n\}$.

We recall the following lemma by Maingé which is a frequent tool in these convergence proofs with very weak assumptions on the parameters.

Lemma 6.3 ([48, Lemma 3.1])**.** *Let (a_n) be a sequence of real numbers and (n_j) be a strictly increasing sequence of natural numbers. Assume that for all $j \in \mathbb{N}$, $a_{n_j} < a_{n_j+1}$. Define $\tau : \mathbb{N} \to \mathbb{N}$ for all $n \in \mathbb{N}$ by*

$$\tau(n) := \max\{k \le \max\{n_0, n\} : a_k < a_{k+1}\}.$$

Then:

(1) τ is unboundedly increasing;
(2) for all $n \in \mathbb{N}$, $a_{\tau(n)} \le a_{\tau(n)+1}$ and, for all $n \ge n_0$, $a_n \le a_{\tau(n)+1}$.

Quantitative formulations of Lemma 6.3 have featured before in proof mining e.g. in [19]. Here we will use the following simple statement from [34] corresponding to a quantitative version of [3, Lemma 2.7] (itself a simple variation of Lemma 6.3).

Lemma 6.4 ([34, Lemma 3.4.1])**.** *Let $b \in \mathbb{N}^*$ and (a_n) be a sequence of nonnegative real numbers in $[0, b]$.*

1. *Let $\tau : \mathbb{N} \to \mathbb{N}$ be such that*

$$(\star) \quad \forall n, k \in \mathbb{N}(k \le n \wedge a_k < a_{k+1} \to k \le \tau(n)).$$

Define for each $N \in \mathbb{N}$, $g \in \mathbb{N}^{\mathbb{N}}$, and $\varepsilon > 0$

$$\psi(\varepsilon, f, N, b) := f^{+(\lceil b/\varepsilon \rceil)}(N),$$

with $f^+(n) := n + f(n)$. If $\tau(\psi(\varepsilon, f, N, b)) < N$, then

$$\exists n \in [N; \psi(\varepsilon, f, N, b)]\ \forall i, j \in [n; n + f(n)]\ (|a_i - a_j| \le \varepsilon).$$

2. *Let $n_0 \in \mathbb{N}$ be such that $\exists n \le n_0\, (a_n < a_{n+1})$. Define*

$$\tau(n) := \max\{k \le \max\{n_0, n\} : a_k < a_{k+1}\}.$$

Then τ is well-defined, satisfies $(\star)$ and moreover
(i) $\forall n \in \mathbb{N}\left(a_{\tau(n)} \le a_{\tau(n)+1}\right)$,
(ii) $\forall n \in \mathbb{N}\,(\tau(n) \le \tau(n+1))$,
(iii) $\forall n \ge n_0\left(a_n \le a_{\tau(n)+1}\right)$.

We prove Theorem 6.2 by first establishing its finitary counterpart. For some given points $x_0, u \in C$, we assume the following quantitative conditions:

(Q_1) $\lim \alpha_n = 0$ with rate of convergence $\sigma_1 : (0, \infty) \to \mathbb{N}$.
(Q_2) $\sum \alpha_n = \infty$ with rate of divergence $\sigma_2 : \mathbb{N} \to \mathbb{N}$, which we additionally assume w.l.g. to be monotone.

(Q_3) $\widetilde{\alpha}:\mathbb{N}\to(0,1]$ is a function witnessing that (α_n) is strictly positive satisfying

$$0<\widetilde{\alpha}(k)\leq\min\{\alpha_{k'}:k'\leq k\}.$$

(Q_4) $p\in\mathrm{Fix}(T_0)$ and $b\in\mathbb{N}^*$ is such that $b\geq 2\cdot\max\{d(x_0,p),d(u,p)\}$.[9]

From condition (Q_4), we have that the sequences (x_n) and (z_m) are bounded.

Proposition 6.5. *For all $n,m\in\mathbb{N}$, $d(x_n,p),d(z_m,p)\leq b/2$.*

Proof. The first inequality is proven by an easy induction. The second inequality follows from Lemma 4.3 (with $\phi\equiv u$ and $\alpha=0$). □

Theorem 6.6. *Let X be a uniformly smooth hyperbolic space with ω_X a modulus of uniform continuity for π. Assume that (z_m) is a Cauchy sequence with rate of metastability ξ, and let (x_n) be generated by (H_{ppa}). Then, for all $\varepsilon>0$ and $f\in\mathbb{N}^{\mathbb{N}}$,*

$$\exists n\leq\Omega\ \exists w\in C\ \forall i\in[n;n+f(n)]\left(d(w,T_i(w))\leq\frac{\varepsilon}{2}\wedge d(x_i,w)\leq\frac{\varepsilon}{2}\right),$$

with $\Omega:=\Omega(\varepsilon,f,\sigma_1,\sigma_2,\widetilde{\alpha},b,\mu,\Delta,\xi,\omega_X):=\hat{\chi}(\theta^M(\hat{\xi}))$, where $\hat{\xi}:=\tilde{\xi}(\varepsilon_0,f_0,N_0)$ as per the construction in Lemma 2.2,

$$\varepsilon_0:=\min\left\{\frac{\tilde{\varepsilon}}{2},\omega_X\left(b,\frac{\tilde{\varepsilon}}{2}\right)\right\},\ f_0(k):=\left\lceil\frac{b}{\delta_1(k)}\right\rceil,\ N_0:=\left\lfloor\frac{3b}{2\tilde{\varepsilon}}\right\rfloor,\ \tilde{\varepsilon}:=\frac{\varepsilon^2}{48b},$$

and, taking χ from Lemma 2.6 and writing g^M for $g^M(k):=\max\{g(k'):k'\leq k\}$,

$$\delta_1(k):=\frac{\min\left\{\frac{\varepsilon}{2},\Delta\left(b,\frac{\eta_k}{\mu(0)}\right),\Delta\left(b,\frac{\omega_X(b,\tilde{\varepsilon})}{2}\right),3\widetilde{\alpha}(K(k))\cdot\tilde{\varepsilon},\frac{2\tilde{\varepsilon}}{K(k)}\right\}}{\mu(K(k))},$$

$$\eta_k:=\frac{\tilde{\varepsilon}}{3(k+1)},\ K(k):=\max\{k'+f_\chi(k'):k'\leq\theta(k)\},$$

$$f_\chi(k)\underset{10}{:=}\hat{\chi}(k)-k+f^M(\hat{\chi}(k))+2,\ \hat{\chi}(k):=\chi\left[\sigma_2,b^2\right]\left(\frac{\varepsilon^2}{4},k\right),$$

and also, with ψ as in Lemma 6.4,

$$\theta(k):=\psi(\varepsilon_1(k),f_\chi,N_1(k),b):=f_\chi^{+\left(\left\lceil\frac{b}{\varepsilon_1(k)}\right\rceil\right)}(N_1(k))$$

$$\varepsilon_1(k):=\frac{1}{2}\Delta\left(b,\frac{\eta_k}{\mu(0)}\right),\ N_1(k):=\sigma_1\left(\frac{\delta_2(k)}{b}\right),$$

$$\delta_2(k):=\min\left\{\varepsilon_1(k),\Delta\left(b,\frac{\omega_X(b,\tilde{\varepsilon})}{2}\right),\frac{\omega_X(b,\tilde{\varepsilon})}{2}\right\}.$$

In particular, (x_n) is a Cauchy sequence with rate of metastability Ω.

Proof. Let $\varepsilon>0$ and $f:\mathbb{N}\to\mathbb{N}$ be given. By Lemma 2.2, we consider $m_0\in[N_0;\hat{\xi}]$ such that

$$\forall i\in[m_0;m_0+f_0(m_0)]\,(d(z_i,z_{m_0})\leq\varepsilon_0)$$

and, with $m_1:=m_0+f_0(m_0)$, in particular $d(z_{m_1},z_{m_0})\leq\varepsilon_0$. Note that[11]

$$d(z_{m_1},T_0(z_{m_1}))=\frac{1}{m_1+1}d(u,T_0(z_{m_1}))\leq\frac{b}{m_1+1}\leq\delta_1(m_0).\tag{5}$$

[9]The factor 2 is just to simplify the computations.

[10]By Remark 2.7, $\hat{\chi}(k)\geq k$, for all $k\in\mathbb{N}$, and so the function $f_\chi:\mathbb{N}\to\mathbb{N}$ is well-defined.

[11]This already holds with f_0 defined by $f_0(k):=\max\left\{0,\lceil\frac{b}{\delta_1(k)}\rceil-k-1\right\}$, for all $k\in\mathbb{N}$.

The proof in now argued by cases. We write,

$$\varepsilon_1 := \varepsilon_1(m_0),\ N_1 := N_1(m_0),\ \eta := \eta_{m_0},\ \delta_1 := \delta_1(m_0),\ \delta_2 := \delta_2(m_0),$$
$$\text{and}\quad \theta := \theta(m_0) := \psi\left(\varepsilon_1, f_\chi, N_1, b\right).$$

Set for each $j \in \mathbb{N}$, $a_j := d(x_j, z_{m_1}) \leq b$.
Case I. $\forall j \leq \theta\, (a_{j+1} \leq a_j)$:
From the definition of θ and Lemma 2.10, we get

$$\exists n \in [N_1; \theta]\ \forall i, j \in [n; n + f_\chi(n)]\, (|a_i - a_j| \leq \varepsilon_1)\,. \tag{6}$$

We put this case on hold for the moment.

Case II. $\exists j \leq \theta\, (a_j < a_{j+1})$:
As in Lemma 6.4, define for all $m \in \mathbb{N}$

$$\tau(m) := \max\left\{k \leq \max\{\theta, m\} : a_k < a_{k+1}\right\}.$$

We now discuss two further sub-cases. The first entails the desired conclusion of the theorem and the second coalesces into the conclusion (6) of case I.

Case II.a. $\forall m \in [\theta; \theta + f_\chi(\theta)]\, (\tau(m) \geq N_1)$:
Consider $m \in [\theta; \theta + f_\chi(\theta)]$ arbitrary. By (W1),

$$d(x_{\tau(m)+1}, z_{m_1}) \leq (1 - \alpha_{\tau(m)})d(T_{\tau(m)}(x_{\tau(m)}), z_{m_1}) + \alpha_{\tau(m)}d(u, z_{m_1}),$$

which entails, using Lemma 6.4.2.(i)

$$d(x_{\tau(m)}, z_{m_1}) - d(T_{\tau(m)}(x_{\tau(m)}), z_{m_1}) \leq \alpha_{\tau(m)}d(u, z_{m_1}) \leq b \cdot \alpha_{\tau(m)}. \tag{7}$$

Since by assumption $\tau(m) \geq N_1 = \sigma_1\left(\frac{\delta_2}{b}\right)$, condition (Q_1) entails

$$\begin{aligned} d(x_{\tau(m)}, z_{m_1}) - d(T_{\tau(m)}(x_{\tau(m)}), z_{m_1}) &\leq \delta_2 \\ &\leq \min\left\{\Delta\left(b, \frac{\eta}{\mu(0)}\right), \Delta\left(b, \frac{\omega_X(b, \tilde{\varepsilon})}{2}\right)\right\}. \end{aligned} \tag{8}$$

As $\tau(m) \leq m \leq \theta + f_\chi(\theta) \leq K(m_0)$, by the assumptions on μ and inequality (5),

$$\begin{aligned} d(z_{m_1}, T_{\tau(m)}(z_{m_1})) &\leq \mu(\tau(m))d(z_{m_1}, T_0(z_{m_1})) \leq \mu(K(m_0)) \cdot \delta_1 \\ &\leq \min\left\{\Delta\left(b, \frac{\eta}{\mu(0)}\right), \Delta\left(b, \frac{\omega_X(b, \tilde{\varepsilon})}{2}\right), 3\widetilde{\alpha}(K(m_0)) \cdot \tilde{\varepsilon}\right\}. \end{aligned} \tag{9}$$

The assumption on the function Δ therefore entails that

$$d(x_{\tau(m)}, T_{\tau(m)}(x_{\tau(m)})) \leq \min\left\{\frac{\eta}{\mu(0)}, \frac{\omega_X(b, \tilde{\varepsilon})}{2}\right\}.$$

Using again the assumption that $\tau(m) \geq N_1 = \sigma_1\left(\frac{\delta_2}{b}\right)$, we get

$$\begin{aligned} d(x_{\tau(m)}, x_{\tau(m)+1}) &\leq d(x_{\tau(m)}, T_{\tau(m)}(x_{\tau(m)})) + d(T_{\tau(m)}(x_{\tau(m)}), x_{\tau(m)+1}) \\ &\leq \frac{\omega_X(b, \tilde{\varepsilon})}{2} + \alpha_{\tau(m)}d(u, T_{\tau(m)}(x_{\tau(m)})) \\ &\leq \frac{\omega_X(b, \tilde{\varepsilon})}{2} + b \cdot \alpha_{\tau(m)} \\ &\leq \frac{\omega_X(b, \tilde{\varepsilon})}{2} + \delta_2 \leq \omega_X(b, \tilde{\varepsilon}) \end{aligned}$$

At the same time we have

$$d(x_{\tau(m)}, T_0(x_{\tau(m)})) \leq \mu(0) d(x_{\tau(m)}, T_{\tau(m)}(x_{\tau(m)})) \leq \eta < \frac{\tilde{\varepsilon}}{2(m_0+1)} = \frac{\varepsilon^2/48}{2b(m_0+1)}.$$

Since $m_0 \geq N_0 \geq \left\lfloor \frac{b}{\tilde{\varepsilon}} \right\rfloor = \left\lfloor \frac{b^2}{\varepsilon^2/48} \right\rfloor$, by Lemma 4.7 we conclude

$$\pi\left(\overrightarrow{uz_{m_0}}, \overrightarrow{x_{\tau(m)}z_{m_0}}\right) \leq \frac{\varepsilon^2}{48}. \tag{10}$$

Since $d(x_{\tau(m)}, x_{\tau(m)+1}) \leq \omega_X(b, \tilde{\varepsilon}) = \omega_X\left(b, \frac{\varepsilon^2}{48b}\right)$, by (P6) we get

$$\pi\left(\overrightarrow{uz_{m_0}}, \overrightarrow{x_{\tau(m)+1}z_{m_0}}\right) \leq \pi\left(\overrightarrow{uz_{m_0}}, \overrightarrow{x_{\tau(m)}z_{m_0}}\right) + \frac{\varepsilon^2}{48b} d(u, z_{m_0}) \leq \frac{\varepsilon^2}{48} + \frac{\varepsilon^2}{48} = \frac{\varepsilon^2}{24}. \tag{11}$$

Since $d(z_{m_1}, z_{m_0}) \leq \varepsilon_0 = \min\left\{\frac{\tilde{\varepsilon}}{2}, \omega_X\left(b, \frac{\tilde{\varepsilon}}{2}\right)\right\}$, by (P6) again we obtain

$$\begin{aligned}
\pi\left(\overrightarrow{uz_{m_1}}, \overrightarrow{x_{\tau(m)+1}z_{m_1}}\right) &\leq \pi\left(\overrightarrow{uz_{m_1}}, \overrightarrow{x_{\tau(m)+1}z_{m_0}}\right) + \frac{\tilde{\varepsilon}}{2} d(u, z_{m_1}) \\
&\leq \pi\left(\overrightarrow{uz_{m_0}}, \overrightarrow{x_{\tau(m)+1}z_{m_0}}\right) + \pi\left(\overrightarrow{z_{m_0}z_{m_1}}, \overrightarrow{x_{\tau(m)+1}z_{m_0}}\right) + \frac{\varepsilon^2}{96} \\
&\leq \frac{\varepsilon^2}{24} + b \cdot d(z_{m_1}, z_{m_0}) + \frac{\varepsilon^2}{96} \\
&\leq \frac{\varepsilon^2}{24} + \frac{\varepsilon^2}{96} + \frac{\varepsilon^2}{96} = \frac{\varepsilon^2}{16}. \qquad (12)
\end{aligned}$$

Now, using (P5), Lemma 4.2 and Lemma 6.4.2.(i), we obtain

$$\begin{aligned}
d^2(x_{\tau(m)+1}, z_{m_1}) &\leq (1-\alpha_{\tau(m)})^2 d^2(T_{\tau(m)}(x_{\tau(m)}), z_{m_1}) + 2\alpha_{\tau(m)} \pi\left(\overrightarrow{uz_{m_1}}, \overrightarrow{x_{\tau(m)+1}z_{m_1}}\right) \\
&\leq (1-\alpha_{\tau(m)})^2 d^2(T_{\tau(m)}(x_{\tau(m)}), T_{\tau(m)}(z_{m_1})) + 2b \cdot d(z_{m_1}, T_{\tau(m)}(z_{m_1})) \\
&\quad + 2\alpha_{\tau(m)} \pi\left(\overrightarrow{uz_{m_1}}, \overrightarrow{x_{\tau(m)+1}z_{m_1}}\right) \\
&\leq (1-\alpha_{\tau(m)}) d^2(x_{\tau(m)}, z_{m_1}) + 2b \cdot d(z_{m_1}, T_{\tau(m)}(z_{m_1})) \\
&\quad + 2\alpha_{\tau(m)} \pi\left(\overrightarrow{uz_{m_1}}, \overrightarrow{x_{\tau(m)+1}z_{m_1}}\right) \\
&\leq (1-\alpha_{\tau(m)}) d^2(x_{\tau(m)+1}, z_{m_1}) + 2b \cdot d(z_{m_1}, T_{\tau(m)}(z_{m_1})) \\
&\quad + 2\alpha_{\tau(m)} \pi\left(\overrightarrow{uz_{m_1}}, \overrightarrow{x_{\tau(m)+1}z_{m_1}}\right),
\end{aligned}$$

which since $\tau(m) \leq K(m_0)$, using (9), (12) and (Q_3), we conclude

$$\begin{aligned}
d^2(x_{\tau(m)+1}, z_{m_1}) &\leq \frac{2b \cdot d(z_{m_1}, T_{\tau(m)}(z_{m_1}))}{\alpha_{\tau(m)}} + 2\pi\left(\overrightarrow{uz_{m_1}}, \overrightarrow{x_{\tau(m)+1}z_{m_1}}\right) \\
&\leq \frac{6b \cdot \widetilde{\alpha}(K(m_0)) \cdot \tilde{\varepsilon}}{\alpha_{\tau(m)}} + \frac{\varepsilon^2}{8} \leq \frac{\varepsilon^2}{4}.
\end{aligned}$$

Therefore, as $m \geq \theta$ by Lemma 6.4.2(iii),

$$d(x_m, z_{m_1}) \leq \frac{\varepsilon}{2}.$$

Moreover, as $m \leq \theta + f_\chi(\theta) \leq K(m_0)$, we get using the monotonicity of μ and (5)

$$d(z_{m_1}, T_m(z_{m_1})) \leq \mu(m) d(z_{m_1}, T_0(z_{m_1})) \leq \mu(K(m_0)) \cdot \delta_1 \leq \frac{\varepsilon}{2}$$

Overall, we conclude that

$$\forall m \in [\theta; \theta + f_\chi(\theta)] \left(d(z_{m_1}, T_m(z_{m_1})) \leq \frac{\varepsilon}{2} \wedge d(x_m, z_{m_1}) \leq \frac{\varepsilon}{2} \right).$$

If now suffices to verify that $\theta \leq \Omega$, to conclude that in this case the result holds with $n = \theta$ and $w = z_{m_1}$. Indeed, since $\hat{\chi}(\theta) \geq \theta$, we have

$$\theta + f_\chi(\theta) = \hat{\chi}(\theta) + f^M(\hat{\chi}(\theta)) + 2 \geq \theta + f(\theta),$$

and so $[\theta; \theta + f_\chi(\theta)] \supseteq [\theta; \theta + f(\theta)]$. On the other hand, we have

$$\theta \leq \theta^M(m_0) \leq \theta^M(\hat{\xi}) \leq \hat{\chi}(\theta^M(\hat{\xi})) = \Omega,$$

which concludes the proof of this case.

Case II.b. $\exists m \in [\theta; \theta + f_\chi(\theta)]\, (\tau(m) < N_1)$:
Take $m \in [\theta; \theta + f_\chi(\theta)]$ such that $\tau(m) < N_1$. Since $\theta \leq m$, we get by the monotonicity of τ (Lemma 6.4.2(ii)) that $\tau(\theta) \leq \tau(m) < N_1$. Hence, by Lemma 6.4.1 we conclude once again

$$\exists n \in [N_1; \theta]\, \forall i, j \in [n; n + f_\chi(n)]\, (|a_i - a_j| \leq \varepsilon_1). \tag{6}$$

We now show that (6) entails the conclusion of the theorem.

Conclusion of Cases I and II.b: For all $j \in \mathbb{N}$, by (P5) and Lemma 4.2, we get

$$\begin{aligned} d^2(x_{j+1}, z_{m_1}) &\leq (1-\alpha_j)^2 d^2(T_j(x_j), z_{m_1}) + 2\alpha_j \pi(\overrightarrow{uz_{m_1}}, \overrightarrow{x_{j+1}z_{m_1}}) \\ &\leq (1-\alpha_j)^2 d^2(T_j(x_j), T_j(z_{m_1})) + 2b \cdot d(z_{m_1}, T_j(z_{m_1})) \\ &\quad + 2\alpha_j \pi(\overrightarrow{uz_{m_1}}, \overrightarrow{x_{j+1}z_{m_1}}) \\ &\leq (1-\alpha_j) d^2(x_j, z_{m_1}) + \alpha_j b_j + c_j, \end{aligned}$$

with $b_j := 2\pi(\overrightarrow{uz_{m_1}}, \overrightarrow{x_{j+1}z_{m_1}})$ and $c_j := 2b \cdot d(z_{m_1}, T_j(z_{m_1}))$. Our goal is to apply Lemma 2.6 with $K := n$ witnessing (6) and $P := n + f_\chi(n) - 2 = \hat{\chi}(n) + f^M(\hat{\chi}(n))$. Consider $n \in [N_1; \theta]$ as in (6). Since $n \leq \theta$, we have $n + f_\chi(n) \leq K(m_0)$. On the other hand, since $n \geq N_1 = \sigma_1\left(\frac{\delta_2}{b}\right)$ by condition (Q$_1$)

$$\forall j \geq n \left(b \cdot \alpha_j \leq \delta_2 \leq \frac{1}{2}\Delta\left(b, \frac{\eta}{\mu(0)}\right)\right).$$

Consider now an arbitrary number $j \in [n; n + f_\chi(n) - 1] = [n; \hat{\chi}(n) + f^M(\hat{\chi}(n)) + 1]$. Then,

$$\begin{aligned} d(x_j, z_{m_1}) - d(T_j(x_j), z_{m_1}) &= (d(x_{j+1}, z_{m_1}) - d(T_j(x_j), z_{m_1})) + (a_j - a_{j+1}) \\ &\leq \alpha_j \left(d(u, z_{m_1}) - d(T_j(x_j), z_{m_1})\right) + \varepsilon_1 \\ &\leq b \cdot \alpha_j + \frac{1}{2}\Delta\left(b, \frac{\eta}{\mu(0)}\right) \leq \Delta\left(b, \frac{\eta}{\mu(0)}\right). \end{aligned}$$

On the other hand, since $j \leq n + f_\chi(n) \leq K(m_0)$, the monotonicity of μ and the inquality (5) entail

$$d(z_{m_1}, T_j(z_{m_1})) \leq \mu(j) d(z_{m_1}, T_0(z_{m_1})) \leq \mu(K(m_0))\delta_1 \leq \Delta\left(b, \frac{\eta}{\mu(0)}\right)$$

We can now use the assumption on Δ to conclude that $d(x_j, T_j(x_j)) \leq \frac{\eta}{\mu(0)}$, and therefore

$$d(x_j, T_0(x_j)) \leq \mu(0)d(x_j, T_j(x_j)) \leq \eta = \frac{\tilde{\varepsilon}}{3(m_0+1)} = \frac{\varepsilon^2/72}{2b(m_0+1)}.$$

Since $m_0 \geq N_0 = \left\lfloor \frac{3b}{2\tilde{\varepsilon}} \right\rfloor = \left\lfloor \frac{b^2}{\varepsilon^2/72} \right\rfloor$, by Lemma 4.7 we have

$$\pi(\overrightarrow{uz_{m_0}}, \overrightarrow{x_j z_{m_0}}) \leq \frac{\varepsilon^2}{72}.$$

We now use (P6) to switch from z_{m_0} to z_{m_1}. Indeed, the definition of ε_0 entails,

$$\begin{aligned}
\pi(\overrightarrow{uz_{m_1}}, \overrightarrow{x_j z_{m_1}}) &\leq \pi(\overrightarrow{uz_{m_1}}, \overrightarrow{x_j z_{m_0}}) + \frac{\tilde{\varepsilon}}{2} d(u, z_{m_1})\\
&\leq \pi(\overrightarrow{uz_{m_0}}, \overrightarrow{x_j z_{m_0}}) + \pi(\overrightarrow{z_{m_0} z_{m_1}}, \overrightarrow{x_j z_{m_0}}) + \frac{\varepsilon^2}{72}\\
&\leq \frac{2\varepsilon^2}{72} + b \cdot \varepsilon_0 \leq \frac{\varepsilon^2}{24}.
\end{aligned}$$

We therefore conclude that

$$\forall j \in [n; \hat{\chi}(n) + f^M(\hat{\chi}(n))] \left(b_j = 2\pi(\overrightarrow{uz_{m_1}}, \overrightarrow{x_{j+1} z_{m_1}}) \leq \frac{\varepsilon^2}{12} = \frac{\varepsilon^2/4}{3} \right).$$

Now for $j \in [n; \hat{\chi}(n) + f^M(\hat{\chi}(n))]$, since $j \leq K(m_0)$ and $n \leq \theta$ (which entails that $K(m_0) \geq \hat{\chi}(n) + f^M(\hat{\chi}(n)) + 1$), we get using (5) and the definition of δ_1,

$$\begin{aligned}
c_j &= 2b \cdot d(z_{m_1}, T_j(z_{m_1})) \leq 2b\mu(j) d(z_{m_1}, T_0(z_{m_1}))\\
&\leq 2b\mu(K(m_0))\delta_1 \leq \frac{4b\tilde{\varepsilon}}{K(m_0)} = \frac{\varepsilon^2/4}{3K(m_0)} \leq \frac{\varepsilon^2/4}{3(\hat{\chi}(n) + f^M(\hat{\chi}(n)) + 1)} \qquad (13)
\end{aligned}$$

Hence, by Lemma 2.6, we obtain

$$\forall j \in [\hat{\chi}(n), \hat{\chi}(n) + f^M(\hat{\chi}(n))] \left(d(x_j, z_{m_1}) \leq \frac{\varepsilon}{2} \right).$$

Since $\hat{\chi}(n) \geq n$, the inequality (13) clearly entails $d(z_{m_1}, T_j(z_{m_1})) \leq \varepsilon/2$, for any $j \in [\hat{\chi}(n), \hat{\chi}(n) + f^M(\hat{\chi}(n))]$. We have, $n \leq \theta \leq \theta^M(m_0) \leq \theta^M(\hat{\xi})$. Finally by the assumption that σ_2 is monotone and Remark 2.7 the function $\hat{\chi}$ is also monotone, which gives $\hat{\chi}(n) \leq \hat{\chi}(\theta^M(\hat{\xi})) = \Omega$. As $f^M(\hat{\chi}(n)) \geq f(\hat{\chi}(n))$, this shows the conclusion of the theorem with $n = \hat{\chi}(n)$ and $w = z_{m_1}$. □

We can now easily argue the full infinitary version.

Proof of Theorem 6.2. Since X is complete and (x_n) is a Cauchy sequence, it converges to some point z and, as C is closed, $z \in C$. We now argue that $z \in \mathrm{Fix}(T_0)$, which by (1) entails that $z \in \mathrm{Fix}(\{T_n\})$. By triangle inequality, from the conclusion of Theorem 6.6 it follows

$$\forall \varepsilon > 0\, \forall f \in \mathbb{N}^{\mathbb{N}}\, \exists n \in \mathbb{N}\, \forall i \in [n; n + f(n)]\, (d(x_i, T_i(x_i)) \leq \varepsilon),$$

which, arguing by contradiction, entails

$$\forall \varepsilon > 0\, \exists n \in \mathbb{N}\, \forall i > n\, (d(x_i, T_i(x_i)) \leq \varepsilon).$$

Using the condition (1) of $\{T_n\}$, we derive that (x_n) is asymptotically regular with respect to the map T_0, i.e. $\lim d(x_n, T_0(x_n)) = 0$. Finally, by the continuity of T_0, we conclude that $z \in \mathrm{Fix}(T_0)$. □

In light of the nonlinear generalization of Reich's theorem, we have the following.

Corollary 6.7. *Let X be a complete uniformly smooth UCW hyperbolic space, C a nonempty closed convex subset of X, and $\{T_n\}$ a resolvent-like family of nonexpansive maps on C. For given $x_0, u \in C$, if (x_n) is generated by $(\mathrm{H}_{\mathrm{ppa}})$ with $\{T_n\}$ and a sequence $(\alpha_n) \subseteq (0,1]$ satisfying $\lim \alpha_n = 0$ and $\sum \alpha_n = \infty$, then (x_n) converges towards $Q(u)$, where Q is the sunny nonexpansive retraction of C onto $\mathrm{Fix}(\{T_n\})$.*

Proof. The proof of Theorem 6.6 further shows that

$$\forall \varepsilon > 0\, \forall f \in \mathbb{N}^{\mathbb{N}}\, \forall N \in \mathbb{N}\, \exists n, m \in \mathbb{N} \left(m \geq N \wedge \forall i \in [n; n + f(n)] \left(d(x_i, z_m) \leq \frac{\varepsilon}{2} \right) \right).$$

Since by Theorem 5.2 and Proposition 5.10, $\lim z_m = Q(u)$ and Q is the unique π-sunny nonexpansive retraction from C onto $\mathrm{Fix}(T_0) = \mathrm{Fix}(\{T_n\})$, it follows that

$$\forall \varepsilon > 0\, \forall f \in \mathbb{N}^{\mathbb{N}}\, \exists n \in \mathbb{N}\, \forall i \in [n; n + f(n)] \left(d(x_i, Q(u)) \leq \varepsilon \right).$$

Arguing by contradiction, we conclude

$$\forall \varepsilon > 0\, \exists n \in \mathbb{N}\, \forall i \geq n \left(d(x_i, Q(u)) \leq \varepsilon \right),$$

i.e. (x_n) converges towards $Q(u)$. □

Remark 6.8. *Moreover, an application of Theorem 6.6 with the rate of metastability ξ for (z_m) from the forthcoming [55], outputs a rate of metastability for (x_n) generated by $(\mathrm{H}_{\mathrm{ppa}})$ which additionally only depends on quantitative information regarding the assumptions on (α_n), on the functions μ and Δ, on the bounding constant b and on a modulus of continuity for π, but otherwise is uniform on the specifics of the problem.*

The results in this section provide a uniform treatment of the main theorems in [3] (and [34]) as well as in [67]. On one hand, in [3] (and [34]) while one needs to content merely with the properties of the normalized duality map in uniformly convex Banach spaces with a uniformly Gâteaux differentiable norm, the underlying setting is nevertheless linear. On the other hand, [67] in CAT(0) spaces, requires an adaptation of the arguments to the nonlinear setting but relies on the use of the quasilinearization function which essentially mimics an inner-product function. Benefiting from the previous proof mining studies [34, 67], the work in this section tackles these problematic issues simultaneously: the nonlinearity and the use of the normalized duality map.

Finally, we remark that in light of the results in [38] (a proof mining generalization of [69] to general hyperbolic spaces), we can extend both the finitary and infinitary results in this section to the general viscosity schema

$$x_{n+1} := (1 - \alpha_n) T_n(x_n) \oplus \alpha_n \phi(x_n),$$

where $\{T_n\}$ is a resolvent-like family of nonexpansive maps on C (cf. Definition 6.1) and ϕ is a MKC contraction on C (cf. Definition 5.12).

7. A reduction proof in smooth hyperbolic spaces

Let X be a hyperbolic space and $C \subseteq X$ a nonempty convex set. In this section, we consider (x_n) to be the iteration generated from an initial point $x_0 \in C$ by the general viscosity schema $(\mathrm{H}_{\phi}^{T_n})$ associated with some arbitrary family $\{T_n\}$ of nonexpansive maps on C, a strict contraction $\phi : C \to C$ with a contracting factor $\alpha < 1$, and a sequence $(\alpha_n) \subseteq [0,1]$ of real numbers,

$$x_{n+1} := (1-\alpha_n)T_n(x_n) \oplus \alpha_n \phi(x_n).$$

Given T some nonexpansive map on C, we denote by (z_m) the unique iteration characterized by the equation (B_ϕ) with t instantiate by the sequence $t_m = \frac{1}{m+1}$, i.e.

$$z_m = \frac{m}{m+1}T(z_m) \oplus \frac{1}{m+1}\phi(z_m).$$

We consider the following conditions:

(Q_1') $\lim \alpha_n = 0$ with rate of convergence $\sigma_1 : (0,\infty) \to \mathbb{N}$.

(Q_2') $\sum \alpha_n = \infty$ with rate of divergence $\sigma_2 : \mathbb{N} \to \mathbb{N}$, which we additionally assume w.l.g. to be monotone.

(Q_3') $p \in \mathrm{Fix}(T) \cap \bigcap_{n=0}^{\infty} \mathrm{Fix}(T_n)$ and $b \in \mathbb{N}^*$ is such that

$$b \geq \max\left\{d(x_0,p), \frac{d(\phi(p),p)}{1-\alpha}\right\};$$

(Q_4') $\mathrm{Fix}(T) \subseteq \bigcap_{n=0}^{\infty} \mathrm{Fix}(T_n)$ and $\tau : \mathbb{N} \times \mathbb{N} \times (0,\infty) \to (0,\infty)$ is a function satisfying for all $\varepsilon > 0$, $r, N \in \mathbb{N}$ and $x \in C$

$$\left.\begin{array}{l} d(x,p) \leq r \\ d(x,T(x)) \leq \tau(r,N,\varepsilon) \end{array}\right\} \to \forall n \leq N \left(d(x,T_n(x)) \leq \varepsilon\right).$$

We additionally assume w.l.g. τ to be monotone in the following way:

$$(N \leq N' \ \wedge\ \varepsilon' \leq \varepsilon) \ \to\ \tau(r,N',\varepsilon') \leq \tau(r,N,\varepsilon).$$

Using the condition (Q_3') we have the following.

Lemma 7.1. *We have for all $n, m \in \mathbb{N}$ and for $y_n \in \{z_n, x_n\}$*

$$\max\{d(y_n,p), d(T(y_n),p), d(T_m(y_n),p), d(\phi(y_n),p)\} \leq b.$$

Proof. From Lemma 4.3 and the hypothesis on b, we have $d(z_n,p) \leq b$. By induction, we argue that for all $n \in \mathbb{N}$

$$d(x_n,p) \leq \max\left\{d(x_0,p), \frac{d(\phi(p),p)}{1-\alpha}\right\}.$$

The base case is trivial and for the inductive step we have

$$\begin{aligned} d(x_{n+1},p) &\overset{\text{(W1)}}{\leq} (1-\alpha_n)d(T_n(x_n),p) + \alpha_n d(\phi(x_n),p) \\ &\leq (1-\alpha_n)d(x_n,p) + \alpha_n d(\phi(x_n),\phi(p)) + \alpha_n d(\phi(p),p) \\ &\leq (1-(1-\alpha)\alpha_n)d(x_n,p) + (1-\alpha)\alpha_n \frac{d(\phi(p),p)}{1-\alpha} \\ &\overset{\text{(IH)}}{\leq} \max\left\{d(x_0,p), \frac{d(\phi(p),p)}{1-\alpha}\right\}. \end{aligned}$$

Hence, the hypothesis on b gives $d(x_n,p) \leq b$. Since p is a fixed point of both T and T_m, and the maps are nonexpansive, we have

$$d(T(y_n),p) = d(T(y_n),T(p)) \leq d(y_n,p) \leq b,$$

and similarly for T_m. Lastly, noting that $(1-\alpha)b \geq d(\phi(p), p)$ and using $d(y_n, p) \leq b$, the fact that ϕ is a α-contraction entails

$$\begin{aligned} d(\phi(y_n), p) &\leq d(\phi(y_n), \phi(p)) + d(\phi(p), p) \\ &\leq \alpha d(y_n, p) + (1-\alpha)b \leq b. \end{aligned}$$

□

From this point on, we assume X to be a smooth hyperbolic space associated with some function π satisfying (P1)–(P5).

Proposition 7.2. *Assume that* $\lim d(x_n, T(x_n)) = 0$ *and that* Φ *is a quasi-rate of* T*-asymptotic regularity. Then, for all* $\varepsilon > 0$, $f : \mathbb{N} \to \mathbb{N}$ *and* $N, m \in \mathbb{N}$ *there is* $K \in \left[N; \tilde{\Phi}\left(\frac{\varepsilon}{4b(m+1)}, f, N\right)\right]$ *such that*

$$m \geq \left\lfloor \frac{4b^2}{\varepsilon} \right\rfloor \to \forall n \in [K; K + f(K)] \left(\pi\left(\overrightarrow{z_m\phi(z_m)}, \overrightarrow{z_m x_n}\right) \leq \varepsilon \right),$$

where $\tilde{\Phi}$ *is the quasi-rate of* T*-asymptotic regularity with lower bound for* (x_n) *defined from* Φ *(cf. Remark 2.3).*

Proof. Assume that $m \geq \lfloor 4b^2/\varepsilon \rfloor$. Hence, $t_m \leq \varepsilon/4b^2$. Since, $\tilde{\Phi}$ is a quasi-rate of asymptotic regularity with lower bound, we have

$$\exists K \in \left[N; \tilde{\Phi}\left(\frac{\varepsilon}{4b(m+1)}, f, N\right)\right] \; \forall n \in [K; K+f(K)] \left(d(x_n, T(x_n)) \leq \frac{\varepsilon}{4b(m+1)} \right).$$

The result now follows from Lemma 4.5 with $r = b$ and $\eta = \frac{1}{m+1}$. □

If we instead have a rate of T-asymptotic regularity, we can use the same argument to conclude the following.

Corollary 7.3. *Assume that* $\lim d(x_n, T(x_n)) = 0$ *and let* φ *be a rate of* T*-asymptotic regularity. Then, for all* $\varepsilon > 0$ *and* $m \in \mathbb{N}$

$$m \geq \left\lfloor \frac{4b^2}{\varepsilon} \right\rfloor \to \forall n \geq \varphi\left(\frac{\varepsilon}{6b(m+1)}\right) \left(\pi\left(\overrightarrow{z_m\phi(z_m)}, \overrightarrow{z_m x_n}\right) \leq \varepsilon \right).$$

The next result plays a central role in allowing for a discussion with Cauchy points of the iteration instead of the actual limit point. We now assume that X is a uniformly smooth hyperbolic space and that $\omega_X : (0, \infty)^2 \to (0, \infty)$ is a modulus of uniform continuity for π in the sense of (P6), which recall means it satisfies

$$\begin{cases} \forall \varepsilon > 0 \; \forall r > 0 \; \forall a \in X \; \forall u, v \in \overline{B}_r(a) \\ \quad \left(d(u,v) \leq \omega_X(r, \varepsilon) \to \forall x, y \in X \left(|\pi(\overrightarrow{xy}, \overrightarrow{ua}) - \pi(\overrightarrow{xy}, \overrightarrow{va})| \leq \varepsilon \cdot d(x,y) \right) \right) \end{cases}$$

Proposition 7.4. *For all* $\varepsilon > 0$, *and* $x, z, z' \in \overline{B}_b(p)$,

$$\left. \begin{aligned} & d(z, z') \leq \min\left\{ \frac{\varepsilon}{12b}, \omega_X\left(2b, \frac{\varepsilon}{9b}\right) \right\} \\ & \pi\left(\overrightarrow{z\phi(z)}, \overrightarrow{zx}\right) \leq \frac{\varepsilon}{3} \end{aligned} \right\} \to \pi\left(\overrightarrow{z'\phi(z')}, \overrightarrow{z'x}\right) \leq \varepsilon.$$

Proof. Since $x, z, z' \in \overline{B}_b(p)$, we have $z, z' \in \overline{B}_{2b}(x)$. From the assumption on ω_X, as $d(z, z') \leq \omega_X\left(2b, \frac{\varepsilon}{9b}\right)$, we get

$$\left| \pi\left(\overrightarrow{z'\phi(z')}, \overrightarrow{z'x}\right) - \pi\left(\overrightarrow{z'\phi(z')}, \overrightarrow{zx}\right) \right| \leq \frac{\varepsilon}{9b} \cdot d(z', \phi(z')) \leq \frac{\varepsilon}{3},$$

using also the fact that the assumption on b entails $3b \geq d(z', \phi(z'))$. Therefore,

$$\begin{aligned}\pi\left(\overrightarrow{z'\phi(z')}, \overrightarrow{z'x}\right) &\leq \frac{\varepsilon}{3} + \pi\left(\overrightarrow{z'z}, \overrightarrow{zx}\right) + \pi\left(\overrightarrow{z\phi(z)}, \overrightarrow{zx}\right) + \pi\left(\overrightarrow{\phi(z)\phi(z')}, \overrightarrow{zx}\right)\\ &\leq \frac{2\varepsilon}{3} + (d(z,z') + d(\phi(z), \phi(z')))d(z,x)\\ &\leq \frac{2\varepsilon}{3} + 4b \cdot d(z,z') \leq \frac{2\varepsilon}{3} + 4b\frac{\varepsilon}{12b} = \varepsilon. \qquad \square\end{aligned}$$

We now discuss a quantitative version of (a generalized version of) Theorem 1.3. Chang's proof reduces the convergence of the schema ($\mathrm{H}_\phi^{T_n}$) to that of (B_ϕ) and, as such, we additionally assume a rate of metastability for (z_n). Therefore, we obtain a transformation of such rate of metastability (and the parameters corresponding to the other assumptions) into a rate of metastability for the general form of the Halpern iteration ($\mathrm{H}_\phi^{T_n}$). Moreover, our argument extends Chang's result to a non-linear setting and to an arbitrary family of nonexpansive maps on C, $\{T_n\}$, and with T any nonexpansive map on C subject to the assumption that its fixed points are common fixed points of the maps T_n.

Theorem 7.5. *Let X be a uniformly smooth hyperbolic space, and (x_n) be given by ($\mathrm{H}_\phi^{T_n}$). Assume that the conditions (Q_1')–(Q_4') hold and that*

(i) Φ is a quasi-rate of T-asymptotic regularity for (x_n),

(ii) (z_n) is a Cauchy sequence with a rate of metastability ξ.

Then, for all $\varepsilon > 0$, $f : \mathbb{N} \to \mathbb{N}$ and $N \in \mathbb{N}$,

$$\exists n \in [N; \Omega(\varepsilon, f, N)]\ \exists m \in \left[N; \hat{\xi} + f_0(\hat{\xi})\right]\ \forall i \in [n; n + f(n)] \left(d(x_i, z_m) \leq \frac{\varepsilon}{2}\right),$$

where $\Omega(\varepsilon, f, N) := \Omega(\varepsilon, f, N, b, \alpha, \sigma_1, \sigma_2, \tau, \Phi, \xi) := \hat{\chi}(\hat{\Phi}(\hat{\xi}))$, with

$$\tilde{\varepsilon} = \frac{\varepsilon^2(1-\alpha)}{48},\ \varepsilon_0 = \min\left\{\frac{\tilde{\varepsilon}}{12b}, \omega_X\left(2b, \frac{\tilde{\varepsilon}}{9b}\right)\right\},$$

$$N_0 := \max\left\{N, \left\lfloor\frac{12b^2}{\tilde{\varepsilon}}\right\rfloor\right\},\ N_1 := \max\left\{1, N, \sigma_1\left(\frac{1}{2}\right), \sigma_1\left(\frac{\tilde{\varepsilon}}{2b^2}\right)\right\},$$

$$\hat{\xi} := \tilde{\xi}(\varepsilon_0, f_0, N_0),$$

together with the functions defined for all $r \in \mathbb{N}$ and $L \in \mathbb{N}$ by

$$\hat{\sigma}(L) := \sigma_2\left(\left\lceil\frac{L}{2(1-\alpha)}\right\rceil\right),$$

$$\hat{\chi}(r) := \chi\left[\hat{\sigma}, 4b^2\right]\left(\frac{\varepsilon^2}{4}, r\right),\ f_1(r) := \max\left\{\hat{\chi}(r') + f(\hat{\chi}(r')) : r' \leq r\right\}$$

$$\hat{\Phi}(r) := \max\left\{\tilde{\Phi}\left(\frac{\tilde{\varepsilon}}{12b(r'+1)}, f_1, N_1\right) : r' \leq r\right\},\ \hat{\varepsilon}(r) := \frac{\varepsilon^2}{96b(f_1(r)+1)}$$

$$f_0(r) := \left\lfloor\frac{2b}{\tau\left(b, f_1(\hat{\Phi}(r)), \hat{\varepsilon}(\hat{\Phi}(r))\right)}\right\rfloor,$$

where $\tilde{\xi}$, $\tilde{\Phi}$ are defined from ξ, Φ as in Lemma 2.2 (and Remark 2.3), and χ is as in Lemma 2.6. In particular, (x_n) is a Cauchy sequence with rate of metastability Ω (with lower bound N).

Proof. Let $\varepsilon > 0$, $f : \mathbb{N} \to \mathbb{N}$ and $N \in \mathbb{N}$ be given. Since $\tilde{\xi}$ is a rate of metastability with lower bound for (z_n), there exists $m_0 \in [N_0, \tilde{\xi}(\varepsilon_0, f_0, N_0)]$ such that

$$\forall m \in [m_0; m_0 + f_0(m_0)] \left(d(z_m, z_{m_0}) \leq \varepsilon_0\right).$$

Since $m_0 \geq N_0 \geq \left\lfloor \frac{12b^2}{\tilde{\varepsilon}} \right\rfloor$ and $N_1 \geq 1$, by Proposition 7.2 and the definition of $\hat{\Phi}$, we get

$$\exists K \in [N_1; \hat{\Phi}(m_0)]\ \forall i \in [K; f_1(K)] \left(\pi\left(\overrightarrow{z_{m_0}\phi(z_{m_0})}, \overrightarrow{z_{m_0}x_{i+1}}\right) \leq \frac{\tilde{\varepsilon}}{3}\right). \tag{14}$$

Define $m_1 := \max\left\{m_0, \left\lfloor \frac{2b}{\tau(b, f_1(K), \hat{\varepsilon}(K))} \right\rfloor\right\}$. Observe that, since $K \leq \hat{\Phi}(m_0)$, we get $f_1(K) \leq f_1(\hat{\Phi}(m_0))$, which then also entails $\hat{\varepsilon}(\hat{\Phi}(m_0)) \leq \hat{\varepsilon}(K)$. By the monotonicity assumption on the function τ (cf. (Q_4')), we get

$$\tau(b, f_1(\hat{\Phi}(m_0)), \hat{\varepsilon}(\hat{\Phi}(m_0))) \leq \tau(b, f_1(K), \hat{\varepsilon}(K)),$$

and thus $m_1 \in [m_0; m_0 + f_0(m_0)]$. Hence $d(z_{m_1}, z_{m_0}) \leq \varepsilon_0$. From the definition of ε_0 and (14), using Proposition 7.4 we conclude

$$\forall i \in [K; f_1(K)] \left(\pi\left(\overrightarrow{z_{m_1}\phi(z_{m_1})}, \overrightarrow{z_{m_1}x_{i+1}}\right) \leq \tilde{\varepsilon}\right). \tag{15}$$

At the same time, since $m_1 \geq \left\lfloor \frac{2b}{\tau(b, f_1(K), \hat{\varepsilon}(K))} \right\rfloor$, by Lemma 4.6 we get

$$d(z_{m_1}, T(z_{m_1})) \leq \tau(b, f_1(K), \hat{\varepsilon}(K)),$$

which by (Q_4') entails

$$\forall i \leq f_1(K) \left(d(z_{m_1}, T_i(z_{m_1})) \leq \hat{\varepsilon}(K) = \frac{\varepsilon^2}{96b(f_1(K) + 1)}\right). \tag{16}$$

For $i \in [K; f_1(K)]$, using (P2)–(P5) and Lemma 4.2, we have

$$\begin{aligned}
d^2(x_{i+1}, z_{m_1}) &\leq (1 - \alpha_i)^2 d^2(T_i(x_i), z_{m_1}) + 2\alpha_i \pi\left(\overrightarrow{\phi(x_i)z_{m_1}}, \overrightarrow{x_{i+1}z_{m_1}}\right)\\
&\leq (1 - \alpha_i)^2 d^2(T_i(x_i), T_i(z_{m_1})) + 4b \cdot d(T_i(z_{m_1}), z_{m_1})\\
&\quad + 2\alpha_i \left(\pi\left(\overrightarrow{\phi(x_i)\phi(z_{m_1})}, \overrightarrow{x_{i+1}z_{m_1}}\right) + \pi\left(\overrightarrow{\phi(z_{m_1})z_{m_1}}, \overrightarrow{x_{i+1}z_{m_1}}\right)\right)\\
&\leq (1 - \alpha_i)^2 d^2(x_i, z_{m_1}) + 4b \cdot d(T_i(z_{m_1}), z_{m_1})\\
&\quad + 2\alpha_i\alpha \cdot d(x_i, z_{m_1}) d(x_{i+1}, z_{m_1}) + 2\alpha_i \pi\left(\overrightarrow{\phi(z_{m_1})z_{m_1}}, \overrightarrow{x_{i+1}z_{m_1}}\right)\\
&\leq (1 - \alpha_i)^2 d^2(x_i, z_{m_1}) + 4b \cdot d(T_i(z_{m_1}), z_{m_1})\\
&\quad + \alpha_i\alpha \left[d^2(x_i, z_{m_1}) + d^2(x_{i+1}, z_{m_1})\right] + 2\alpha_i \pi\left(\overrightarrow{z_{m_1}\phi(z_{m_1})}, \overrightarrow{z_{m_1}x_{i+1}}\right),
\end{aligned}$$

using (for Lemma 4.2) that $d(T_i(x_i), z_{m_1}), d(T_i(x_i), T_i(z_{m_1})) \leq 2b$ and the general inequality $2ab \leq a^2 + b^2$. Hence

$$\begin{aligned}
&(1 - \alpha_i\alpha) d^2(x_{i+1}, z_{m_1}) \leq (1 - \alpha_i(2 - \alpha)) d^2(x_i, z_{m_1})\\
&\qquad + \alpha_i \left[\alpha_i d^2(x_i, z_{m_1}) + 2\pi\left(\overrightarrow{z_{m_1}\phi(z_{m_1})}, \overrightarrow{z_{m_1}x_{i+1}}\right)\right] + 4b \cdot d(T_i(z_{m_1}), z_{m_1}).
\end{aligned}$$

Since $1 - \alpha_i\alpha \in (0, 1]$, we have

$$\frac{1 - \alpha_i(2 - \alpha)}{1 - \alpha_i\alpha} = 1 - \frac{2(1 - \alpha)\alpha_i}{1 - \alpha_i\alpha} \leq 1 - 2(1 - \alpha)\alpha_i.$$

When $i \geq K \geq N_1 \geq \sigma_1(1/2)$ we have $1-\alpha_i\alpha \geq 1/2$, and thus for $i \in [K; f_1(K)]$

$$d^2(x_{i+1}, z_{m_1}) \leq (1-\lambda_i)d^2(x_i, z_{m_1}) + \lambda_i b_i + c_i,$$

with $\lambda_i := 2(1-\alpha)\alpha_i \in [0,1]$ (when $i \geq K$), $c_i := 8b \cdot d(T_i(z_{m_1}), z_{m_1})$ and

$$b_i := \frac{1}{1-\alpha}\left(\alpha_i d^2(x_i, z_{m_1}) + 2\pi\left(\overrightarrow{z_{m_1}\phi(z_{m_1})}, \overrightarrow{z_{m_1}x_{i+1}}\right)\right).$$

Hence, by (16), we have for all $i \in [K; f_1(K)]$,

$$d^2(x_{i+1}, z_{m_1}) \leq (1-\lambda_i)d^2(x_i, z_{m_1}) + \lambda_i b_i + \frac{\varepsilon^2}{12(f_1(K)+1)}.$$

Clearly the function $\hat{\sigma}$ is a monotone rate of divergence for $\sum \lambda_n$. Indeed, for $L \in \mathbb{N}$

$$\sum_{k=0}^{\hat{\sigma}(L)} \lambda_k = 2(1-\alpha) \cdot \sum_{k=0}^{\sigma_2(\lceil L/2(1-\alpha)\rceil)} \alpha_k \geq 2(1-\alpha)\left\lceil \frac{L}{2(1-\alpha)} \right\rceil \geq L.$$

Since $K \geq N_1 \geq \sigma_1\left(\tilde{\varepsilon}/2b^2\right)$, it follows that for $i \geq K$

$$\alpha_i d^2(x_i, z_{m_1}) \leq \frac{\tilde{\varepsilon}}{2b^2}4b^2 = \frac{\varepsilon^2(1-\alpha)}{24}.$$

Therefore, by (15) we conclude that for all $i \in [K; f_1(K)]$

$$b_i \leq \frac{1}{1-\alpha}\left(\frac{\varepsilon^2(1-\alpha)}{24} + 2\tilde{\varepsilon}\right) = \frac{1}{1-\alpha}\left(\frac{\varepsilon^2(1-\alpha)}{24} + 2\frac{\varepsilon^2(1-\alpha)}{48}\right) = \frac{\varepsilon^2}{12}.$$

Hence, we are in the conditions of Lemma 2.6 (with $a_n = d^2(x_n, z_{m_1})$, $B = 4b^2$, $\varepsilon = \varepsilon^2/4$ and $P = f_1(K)$). By the definition of $\hat{\chi}$, we conclude that

$$\forall i \in [\hat{\chi}(K); f_1(K)] \left(d^2(x_i, z_{m_1}) \leq \frac{\varepsilon^2}{4}\right),$$

which entails for $n := \hat{\chi}(K)$

$$\forall i \in [n; n+f(n)] \left(d(x_i, z_{m_1}) \leq \frac{\varepsilon}{2}\right).$$

It just remains to verify that the bounding information is correct. Since we have $m_0 \leq \tilde{\xi}(\varepsilon_0, f_0, N_0)$, the monotone definition of $\hat{\Phi}$ entails

$$K \leq \hat{\Phi}(m_0) \leq \hat{\Phi}(\tilde{\xi}(\varepsilon_0, f_0, N_0)).$$

By the hypothesis (Q_2'), the function σ_2 is monotone, which entails that $\hat{\sigma}$ is also monotone. By Remark 2.7, we get that the function $\hat{\chi}$ is monotone, and thus

$$n = \hat{\chi}(K) \leq \hat{\chi}(\hat{\Phi}(\tilde{\xi}(\varepsilon_0, f_0, N_0))) = \hat{\chi}(\hat{\Phi}(\hat{\xi})) = \Omega(\varepsilon, f, N, b, \alpha, \sigma_1, \sigma_2, \tau, \xi).$$

Remark 2.7, also allows for the conclusion that

$$n = \hat{\chi}(K) \geq K \geq N_1 \geq N.$$

On the other hand, that the monotone definition of f_1 and the monotonicity hypothesis on τ, entails that f_0 is a monotone function. Thus,

$$m_1 \in [m_0; m_0 + f_0(m_0)] \subseteq [N; \hat{\xi} + f_0(\hat{\xi})],$$

which concludes the main statement of the theorem with $m = m_1 \in [m_0; m_0 + f_0(m_0)]$ since $m_0 \geq N$ and $m_0 + f_0(m_0) \leq \hat{\xi} + f_0(\hat{\xi})$. Lastly, the fact that Ω is a rate of metastability follows trivially by triangle inequality. $\square$

The above finitary result entails a corresponding infinitary counterpart which generalizes Theorem 1.3.

Theorem 7.6. *Let X be a complete uniformly smooth hyperbolic space, and $C \subseteq X$ a nonempty closed convex set. Consider T a nonexpansive map on C, $\{T_n\}$ an infinite family of nonexpansive maps on C, and ϕ a strict contraction on C. Let (z_m) be the sequence characterized by the identity (B_ϕ) when $t = \frac{1}{m+1}$. With $(\alpha_n) \subseteq [0,1]$, assume the following conditions*

(1) $\lim \alpha_n = 0$, *and* $\sum \alpha_n = \infty$,
(2) $\emptyset \neq \mathrm{Fix}(T) \subseteq \bigcap_{n\in\mathbb{N}} \mathrm{Fix}(T_n)$, *and there exists τ as in (Q_4'),*[12]
(3) (z_m) *is a Cauchy sequence.*

Given $x_0 \in C$, let (x_n) be the sequence defined by ($\mathrm{H}_\phi^{T_n}$). If $\lim d(x_n, T(x_n)) = 0$, then (x_n) is a Cauchy sequence. Moreover, its limit is equal to the limit of (z_m) which is a common fixed point z of T and the maps T_n satisfying the inequality

$$\forall q \in \mathrm{Fix}(T) \left(\pi\left(\overrightarrow{z\phi(z)}, \overrightarrow{zq}\right) \leq 0\right).$$

Proof. By the assumption that X is complete and (z_m) is a Cauchy sequence, we get that it converges to some point $z \in X$. By the T-asymptotic regularity of (z_m) (cf. Lemma 4.6) and the fact that C is closed, we get that $z \in \mathrm{Fix}(T)$. By the assumption (2), z is also a common fixed point of the maps T_n. Moreover, by Lemma 4.5 and the uniform continuity of π (in the sense of (P6)), z satisfies the desired variational inequality. Now, under the assumption that $\lim d(x_n, T(x_n)) = 0$, by Theorem 7.5, it follows that (x_n) is a Cauchy sequence, and thus converges to some point $x \in X$. Again by T-asymptotic regularity, we conclude that $x \in \mathrm{Fix}(T)$. Let $N \in \mathbb{N}$ be such that $d(x_n, x) \leq \varepsilon/4$ and $d(z_m, z) \leq \varepsilon/4$, for all $n, m \geq N$. By Theorem 7.5, there exist $n, m \geq N$ such that $d(x_n, z_m) \leq \varepsilon/2$. Therefore,

$$d(x, z) \leq d(x, x_n) + d(x_n, z_m) + d(z_m, z) \leq \varepsilon,$$

and so we must have $x = z$, which concludes the proof. □

8. Corollaries

Here we discuss some instances of application of the main theorems in section 7. We first discuss the viscosity schema with a single nonexpansive map. Afterwards, we focus on families of nonexpansive maps as considered in Bauschke's schema (still allowing viscosity terms).

Corollary 8.1. *Under the conditions (1) and (3) of Theorem 7.6, with $T_n = T$ for all $n \in \mathbb{N}$ and assuming that $\mathrm{Fix}(T) \neq \emptyset$, if $d(x_n, T(x_n)) = 0$, then (x_n) generated by (H_ϕ^T) converges to a fixed point of T satisfying the inequality*

$$\forall q \in \mathrm{Fix}(T) \left(\pi\left(\overrightarrow{z\phi(z)}, \overrightarrow{zq}\right) \leq 0\right).$$

Moreover, $\Omega(\varepsilon, f, 0, b, \alpha, \sigma_1, \sigma_2, \tau_0, \Phi, \xi)$ is a rate of metastability for (x_n), with Ω as in Theorem 7.5 and $\tau_0(r, N, \varepsilon) = \varepsilon$, for all $(r, N, \varepsilon) \in \mathbb{N} \times \mathbb{N} \times (0, \infty)$.

Proof. Just see that if $T_n \equiv T$, then the inclusion of condition (2) in Theorem 7.6 trivially holds, and moreover one can apply Theorem 7.5 with τ_0 as it satisfies the condition (Q_4'). □

---

[12]Note that, in a compact setting, the property $\mathrm{Fix}(T) \subseteq \bigcap_{n\in\mathbb{N}} \mathrm{Fix}(T_n)$ entails the existence of such a function τ.

Under Wittmann's third condition on the sequence (α_n), we can establish the condition $d(x_n, T(x_n)) \to 0$ and therefore conclude the convergence of (x_n).

Corollary 8.2. *Under the conditions (1) and (3) of Theorem 7.6, with $T_n = T$ for all $n \in \mathbb{N}$, assume that $\mathrm{Fix}(T) \neq \emptyset$ and*

$$\lim \frac{\alpha_n}{\alpha_{n+1}} = 1 \quad or \quad \sum |\alpha_{n+1} - \alpha_n| < \infty.$$

Then (x_n) generated by (H^T_ϕ) converges to a fixed point of T satisfying the inequality

$$\forall q \in \mathrm{Fix}(T) \left(\pi\left(\overrightarrow{z\phi(z)}, \overrightarrow{zq}\right) \leq 0\right).$$

Moreover, $\Omega(\varepsilon, f, 0, b, \alpha, \sigma_1, \sigma_2, \tau_0, \Phi_0, \xi)$ is a rate of metastability for (x_n), for Ω as in Theorem 7.5, τ_0 as before,

$$\Phi_0(\varepsilon) := \max\left\{\sigma_1\left(\frac{\varepsilon}{4b}\right),\ \theta_1\left(\sigma_3\left(\frac{(1-\alpha)\varepsilon}{8b}\right) + 1 + \left\lceil \ln\left(\frac{8b}{\varepsilon}\right)\right\rceil\right) + 1\right\},$$

with $\theta_1(L) := \sigma_2\left(\left\lceil \frac{L}{1-\alpha} + 1\right\rceil\right) - 1$ and where σ_3 is either a rate of convergence for $\lim \frac{\alpha_n}{\alpha_{n+1}} = 1$ or a Cauchy rate for $(\sum |\alpha_{n+1} - \alpha_n|)$.

Proof. It suffices to check that Φ_0 is a rate of T-asymptotic regularity for (x_n). Recall that by Lemma 7.1, we have for all $n \in \mathbb{N}$

$$d(T(x_n), \phi(x_n)) \leq d(T(x_n), p) + d(\phi(x_n), p) \leq 2b.$$

and

$$d(x_{n+1}, T(x_n)) = \alpha_n d(T(x_n), \phi(x_n)) \leq 2b\alpha_n.$$

Hence, $d(x_{n+1}, T(x_n)) \to 0$ with rate of convergence $\varepsilon \mapsto \sigma_1(\varepsilon/2b)$. For all $n \in \mathbb{N}$, using (W4) and (W2), we obtain

$$\begin{aligned}
d(x_{n+2}, x_{n+1}) &= d(W(T(x_{n+1}), \phi(x_{n+1}), \alpha_{n+1}), W(T(x_n), \phi(x_n), \alpha_n)) \\
&\leq d(W(T(x_{n+1}), \phi(x_{n+1}), \alpha_{n+1}), W(T(x_n), \phi(x_n), \alpha_{n+1})) \\
&\quad + d(W(T(x_n), \phi(x_n), \alpha_{n+1}), W(T(x_n), \phi(x_n), \alpha_n)) \\
&\leq (1 - \alpha_{n+1}) d(T(x_{n+1}), T(x_n)) + \alpha_{n+1} d(\phi(x_{n+1}), \phi(x_n)) \\
&\quad + |\alpha_{n+1} - \alpha_n| d(T(x_n), \phi(x_n)) \\
&\leq (1 - (1-\alpha)\alpha_{n+1}) d(x_{n+1}, x_n) + 2b \cdot |\alpha_{n+1} - \alpha_n|
\end{aligned}$$

Thus, we have

$$d(x_{n+2}, x_{n+1}) \leq (1 - \lambda_n) d(x_{n+1}, x_n) + \lambda_n b_n + c_n,$$

with $\lambda_n := (1-\alpha)\alpha_{n+1}$, for which $\sum \lambda_n = \infty$ with rate of divergence θ, and

(i) if $\lim \frac{\alpha_n}{\alpha_{n+1}} = 1$ with rate of convergence σ_3,

$$b_n := \frac{2b|\alpha_{n+1} - \alpha_n|}{(1-\alpha)\alpha_{n+1}} = \frac{2b}{1-\alpha}\left|1 - \frac{\alpha_n}{\alpha_{n+1}}\right|, \quad \text{and} \quad c_n := 0.$$

(ii) if $\sum |\alpha_{n+1} - \alpha_n| < \infty$ with Cauchy rate σ_3,

$$b_n := 0, \quad \text{and} \quad c_n := 2b \cdot |\alpha_{n+1} - \alpha_n|.$$

The result now follows from Lemma 2.8 (with the functions $\theta_3(\varepsilon) := \sigma_3((1-\alpha)\varepsilon/2b)$ and $\theta_2(\varepsilon) := \theta_3(\varepsilon) + 1$). $\square$

Recalling Corollary 5.13 and Remark 5.14, we know that in complete uniformly smooth UCW hyperbolic spaces, the implicit Browder-schema (B_ϕ) (even with ϕ an MKC map on C, and so in particular for strict contractions) converges to a fixed point of T and so condition (3) holds. We can therefore apply Corollary 8.2 to conclude the convergence of (x_n) generated via (H^T_ϕ), which generalizes to a nonlinear setting both Moudafi's [52] and Wittmann's [75] convergence results. Moreover, instantiating ξ with a rate of metastability for (z_m) generated by (B_ϕ) (which can be obtained from a rate of metastability for the original schema without viscosity terms [55] via the construction in [38]), the construction in Corollary 8.2 gives a rate of metastability for (x_n).

8.1. **Cyclic families of nonexpansive maps.** We now discuss the situation of infinite families of nonexpansive maps as considered by Bauschke. Let $T_0, \cdots, T_{\ell-1}$ be $\ell \geq 1$ nonexpansive maps on C, and cyclically define $T_n := T_{n \bmod \ell}$. Setting $T := T_{\ell-1} \cdots T_0$ which is a nonexpansive map on C, we assume Bauschke's condition

$$(+) \qquad \forall k < \ell \left(\emptyset \neq \mathrm{Fix}(T_{\ell-k-1}T_{\ell-k-2} \cdots T_0 T_{\ell-1} \cdots T_{\ell-k}) \subset \bigcap_{j=0}^{\ell-1} \mathrm{Fix}(T_j) \right)$$

In [70], Suzuki remarked that Bauschke's condition (+) already follows from the seemingly weaker statement (+) restricted to $k = 0$,

$$(++) \qquad \emptyset \neq \mathrm{Fix}(\underbrace{T_{\ell-1} \cdots T_0}_{=:T}) \subset \bigcap_{j=0}^{\ell-1} \mathrm{Fix}(T_j),$$

and so we consider the corresponding quantitative information: Let p be some point in $\mathrm{Fix}(T)$, and consider a function $\tau : \mathbb{N} \times (0, \infty) \to (0, \infty)$ satisfying for all $\varepsilon > 0$, $r \in \mathbb{N}$ and $x \in C$

$$(\mathrm{Q}_\tau) \qquad \Big(d(x, p) \leq r \wedge d(x, T(x)) \leq \tau(r, \varepsilon) \Big) \to \forall j < \ell \left(d(x, T_j(x)) \leq \varepsilon \right),$$

and w.l.g. we additionally assume τ to be monotone in ε, and that $\tau(r, \varepsilon) \leq \varepsilon$ for all $r \in \mathbb{N}$ and $\varepsilon > 0$. We require a quantitative version of Suzuki's remark, which was first given by Körnlein in [42, Theorem 8.7.1] in the context of Hilbert spaces. We argue that it actually immediately extends to hyperbolic spaces.

Lemma 8.3. *For any $\varepsilon > 0$, $r \in \mathbb{N}$ and $x \in C$ such that $d(x, p) \leq r$, assume that for some $k < \ell$*

$$d(x, T_{\ell-k-1}T_{\ell-k-2} \cdots T_0 T_{\ell-1} \cdots T_{\ell-k}(x)) \leq \tau \left(r, \frac{\varepsilon}{2\ell + 1} \right).$$

Then, for all $j < \ell$, $d(x, T_j(x)) \leq \varepsilon$.

Proof. Let $\varepsilon > 0$ and $r \in \mathbb{N}$ be given, and write $\delta = \dfrac{\varepsilon}{2\ell + 1}$ and $\tau := \tau(r, \delta)$. Assume that the premise holds for some $k \in [1; \ell - 1]$, since for $k = 0$ it is immediate by the assumption on the function τ. Using the nonexpansivity of the functions, we have for $y = T_{\ell-1} \cdots T_{\ell-k}(x)$,

$$\begin{aligned} d(y, T(y)) &= d(T_{\ell-1} \cdots T_{\ell-k}(x), \underbrace{T_{\ell-1} \cdots T_{\ell-k} T_{\ell-k-1} \cdots T_0}_{=:T} \underbrace{T_{\ell-1} \cdots T_{\ell-k}(x)}_{=y}) \\ &\leq d(x, T_{\ell-k-1} \cdots T_0 T_{\ell-1} \cdots T_{\ell-k}(x)) \leq \tau. \end{aligned}$$

Since $d(y,p) \leq d(x,p) \leq r$, by the assumption on the function τ, it follows

$$\forall j < \ell \left(d\left(T_{\ell-1}\cdots T_{\ell-k}(x), T_j T_{\ell-1}\cdots T_{\ell-k}(x)\right) = d(y, T_j(y)) \leq \delta \right).$$

For any $j < \ell - k$, see that, $d(T_{\ell-1}\cdots T_{\ell-k}(x), T_{\ell-k-1}\cdots T_j T_{\ell-1}\cdots T_{\ell-k}(x))$ is bounded above by

$$\begin{aligned}
& d(T_{\ell-1}\cdots T_{\ell-k}(x), T_{\ell-k-1}\cdots T_{j+1}T_{\ell-1}\cdots T_{\ell-k}(x)) \\
&\qquad\qquad + d(T_{\ell-k-1}\cdots T_{j+1}T_{\ell-1}\cdots T_{\ell-k}(x), T_{\ell-k-1}\cdots T_{j+1}T_j T_{\ell-1}\cdots T_{\ell-k}(x)) \\
&\leq d(T_{\ell-1}\cdots T_{\ell-k}(x), T_{\ell-k-1}\cdots T_{j+1}T_{\ell-1}\cdots T_{\ell-k}(x)) \\
&\qquad\qquad\qquad\qquad + d(T_{\ell-1}\cdots T_{\ell-k}(x), T_j T_{\ell-1}\cdots T_{\ell-k}(x)) \\
&\leq d(T_{\ell-1}\cdots T_{\ell-k}(x), T_{\ell-k-1}\cdots T_{j+1}T_{\ell-1}\cdots T_{\ell-k}(x)) + \delta
\end{aligned}$$

Therefore, inductively we conclude

$$d(T_{\ell-1}\cdots T_{\ell-k}(x), T_{\ell-k-1}\cdots T_0 T_{\ell-1}\cdots T_{\ell-k}(x)) \leq (\ell-k)\delta \leq (\ell-1)\delta.$$

As $\tau = \tau(r,\delta) \leq \delta$, it follows

$$\begin{aligned}
d(x, T_{\ell-1}\cdots T_{\ell-k}(x)) &\leq d(x, T_{\ell-k-1}\cdots T_0 T_{\ell-1}\cdots T_{\ell-k}(x)) \\
&\qquad + d(T_{\ell-k-1}\cdots T_0 T_{\ell-1}\cdots T_{\ell-k}(x), T_{\ell-1}\cdots T_{\ell-k}(x)) \\
&\leq \tau + (\ell-1)\delta \leq \ell\cdot\delta.
\end{aligned}$$

Finally, for any $j < \ell$,

$$\begin{aligned}
d(x, T_j(x)) &\leq d(x, T_{\ell-1}\cdots T_{\ell-k}(x)) + d(T_{\ell-1}\cdots T_{\ell-k}(x), T_j T_{\ell-1}\cdots T_{\ell-k}(x)) \\
&\quad + d(T_j T_{\ell-1}\cdots T_{\ell-k}(x), T_j(x)) \\
&\leq 2d(x, T_{\ell-1}\cdots T_{\ell-k}(x)) + \delta \leq (2\ell+1)\,\delta = \varepsilon.
\end{aligned}$$

□

We have the following nonlinear generalization of [15, Theorem 5], which in particularly removes an unnecessary commutativity condition.

Theorem 8.4. *Let X be a uniformly smooth hyperbolic space and C a nonempty convex subset. Let ϕ be a strict contraction on C with factor $\alpha \in [0,1)$, and $T_0, \cdots, T_{\ell-1}$ be $\ell \geq 1$ nonexpansive maps on C. Define $T := T_{\ell-1}\cdots T_0$ and consider an infinite family of maps $\{T_n\}$ cyclically defined from $T_0, \cdots, T_{\ell-1}$. For $x_0 \in C$, consider $p \in \mathrm{Fix}(T)$ and $b \in \mathbb{N}^*$ such that condition (Q′3) holds, and let $\tau : \mathbb{N}\times(0,\infty) \to (0,\infty)$ be a monotone function satisfying (Q_τ). Let $(\alpha_n) \subseteq [0,1]$ be a sequence of real numbers, and assume that conditions (Q′1), (Q′2) hold with functions σ_1 and σ_2 respectively. Furthermore, assume that $\sum |\alpha_n - \alpha_{n+\ell}| < \infty$, and that there exists a function $\sigma_3 : (0,\infty) \to \mathbb{N}$ satisfying*

$$(\mathrm{Q}_5') \qquad \forall \varepsilon > 0\ \forall n \in \mathbb{N} \left(\sum_{j=\sigma_3(\varepsilon)+1}^{n} |\alpha_j - \alpha_{j+\ell}| \leq \varepsilon \right).$$

Let (z_m) be the sequence defined by the identity (B_ϕ) when $t = \frac{1}{m+1}$, and assume it to be a Cauchy sequence with rate of metastability ξ. Then, (x_n) generated by ($\mathrm{H}_\phi^{T_n}$) is a Cauchy sequence with rate of metastability

$$\Omega(\varepsilon, f, 0, b, \alpha, \sigma_1, \sigma_2, \tau_0, \Phi_0, \xi),$$

with Ω as in Theorem 7.5, $\tau_0(r,N,\varepsilon)=\tau(r,\varepsilon)$ and

$$\Phi_0(\varepsilon) := \max\left\{\sigma_1\left(\frac{\tilde{\varepsilon}}{4b\ell}\right), \theta_1\left(\sigma_3\left(\frac{\tilde{\varepsilon}}{8b}\right)+1+\left\lceil \ln\left(\frac{8b}{\tilde{\varepsilon}}\right)\right\rceil\right)+1\right\},$$

$$\theta_1(L) := \sigma_2\left(\left\lceil \frac{L}{1-\alpha}+\ell\right\rceil\right)-\ell, \quad \tilde{\varepsilon} := \tau\left(b, \frac{\varepsilon}{(2\ell+1)\ell}\right).$$

Proof. By Lemma 7.1, for all $n\in\mathbb{N}$

$$d(x_{n+1},T_n(x_n)) = \alpha_n d(T_n(x_n),\phi(x_n)) \leq 2b\cdot\alpha_n,$$

and so, by (Q$'_1$), $d(x_{n+1},T_n(x_n))\to 0$ with rate of convergence $\varepsilon\mapsto\sigma_1(\varepsilon/2b)$. Now, using (W4), (W2) and the fact that $T_{n+\ell}=T_n$,

$$\begin{aligned}
d(x_{n+\ell+1},x_{n+1}) &= d(W(T_{n+\ell}(x_{n+\ell}),\phi(x_{n+\ell}),\alpha_{n+\ell}),W(T_n(x_n),\phi(x_n),\alpha_n))\\
&\leq d(W(T_{n+\ell}(x_{n+\ell}),\phi(x_{n+\ell}),\alpha_{n+\ell}),W(T_n(x_n),\phi(x_n),\alpha_{n+\ell}))\\
&\quad + d(W(T_n(x_n),\phi(x_n),\alpha_{n+\ell}),W(T_n(x_n),\phi(x_n),\alpha_n))\\
&\leq (1-\alpha_{n+\ell})d(T_{n+\ell}(x_{n+\ell}),T_n(x_n)) + \alpha_{n+\ell}d(\phi(x_{n+\ell}),\phi(x_n))\\
&\quad + |\alpha_n-\alpha_{n+\ell}|d(T_n(x_n),\phi(x_n))\\
&\leq (1-\lambda_n)d(x_{n+\ell},x_n)+c_n.
\end{aligned}$$

with $\lambda_n := (1-\alpha)\alpha_{n+\ell}$ and $c_n := 2b|\alpha_n-\alpha_{n+\ell}|$. From the condition (Q$'_2$) it follows that $\sum\lambda_n=\infty$ with rate of divergence θ_1. Indeed, for any $L\in\mathbb{N}$,

$$\begin{aligned}
\sum_{k=0}^{\theta_1(L)}\lambda_k &= (1-\alpha)\sum_{k=\ell}^{\theta_1(L)+\ell}\alpha_k = (1-\alpha)\sum_{k=0}^{\theta_1(L)+\ell}\alpha_k - (1-\alpha)\sum_{k=0}^{\ell-1}\alpha_k\\
&\geq (1-\alpha)\left(\frac{L}{1-\alpha}+\ell\right) - (1-\alpha)\sum_{k=0}^{\ell-1}\alpha_k \geq L.
\end{aligned}$$

On the other hand, from (Q$'_5$), we get that $\sum c_n<\infty$ and immediately see that $\theta_3(\varepsilon) := \sigma_3(\varepsilon/2b)$ is a Cauchy rate for $(\sum c_n)$. We are therefore in the conditions of Lemma 2.8(2), and conclude that $d(x_{n+\ell},x_n)\to 0$ with rate of convergence

$$\chi_2(\varepsilon) := \chi_2[\theta_1,\theta_3,2b](\varepsilon) := \theta_1\left(\sigma_3\left(\frac{\varepsilon}{4b}\right)+1+\left\lceil\ln\left(\frac{4b}{\varepsilon}\right)\right\rceil\right)+1.$$

Noting that for every $n\in\mathbb{N}$,

$$\begin{aligned}
d(x_{n+\ell},T_{n+\ell-1}\cdots T_n(x_n)) &\leq d(x_{n+\ell},T_{n+\ell-1}(x_{n+\ell-1}))\\
&\quad + d(T_{n+\ell-1}(x_{n+\ell-1}),T_{n+\ell-1}\cdots T_n(x_n))\\
&\leq d(x_{n+\ell},T_{n+\ell-1}(x_{n+\ell-1})) + d(x_{n+\ell-1},T_{n+\ell-2}\cdots T_n(x_n)),
\end{aligned}$$

we derive that for any $\varepsilon>0$ and $n\geq\max\left\{\sigma_1\left(\frac{\varepsilon}{4b\ell}\right),\chi_2\left(\frac{\varepsilon}{2}\right)\right\}$

$$\begin{aligned}
d(x_n,T_{n+\ell-1}\cdots T_n(x_n)) &\leq d(x_n,x_{n+\ell}) + d(x_{n+\ell},T_{n+\ell-1}\cdots T_n(x_n))\\
&\leq d(x_n,x_{n+\ell}) + \sum_{k=0}^{\ell-1} d(x_{n+k+1},T_{n+k}(x_{n+k})) \leq \frac{\varepsilon}{2}+\frac{\varepsilon}{2\ell}\ell=\varepsilon.
\end{aligned}$$

Given $\varepsilon>0$, consider $n\geq\Phi_0(\varepsilon)$ and $k<\ell$ such that $\ell-k=n \mod \ell$. Then,

$$d(x_n,T_{\ell-k-1}\cdots T_0T_{\ell-1}\cdots T_{\ell-k}(x_n)) = d(x_n,T_{n+\ell-1}\cdots T_n(x_n)) \leq \tau\left(b,\frac{\varepsilon}{(2\ell+1)\ell}\right),$$

which by Lemma 8.3 entails

$$\forall j < \ell \left(d(x_n, T_j(x_n)) \leq \frac{\varepsilon}{\ell} \right).$$

Therefore, using the nonexpansivity of the maps $T_0, \ldots, T_{\ell-1}$, we get

$$\begin{aligned} d(x_n, T(x_n)) &\leq d(x_n, T_{\ell-1}(x_n)) + d(T_{\ell-1}(x_n), T_{\ell-1} \cdots T_0(x_n)) \\ &\leq d(x_n, T_{\ell-1}(x_n)) + d(x_n, T_{\ell-2} \cdots T_0(x_n)) \\ &\leq \cdots \leq \sum_{j=0}^{\ell-1} d(x_n, T_j(x_n)) \leq \ell \frac{\varepsilon}{\ell} = \varepsilon. \end{aligned}$$

Hence, $d(x_n, T(x_n)) \to 0$, and Φ_0 is a rate of T-asymptotic regularity. We are then in the conditions of Theorem 7.5, and the result follows. □

The previous theorem also holds if we replace the assumption Q_5' with the condition $\lim \frac{\alpha_n}{\alpha_{n+\ell}} = 1$ and a corresponding rate of convergence σ_3. In this case, one uses instead Lemma 2.8(1), and the argument showing that (a slight adaptation of) Φ_0 is a rate of T-asymptotic regularity for (x_n) is similar to that of Corollary 8.2. When $\ell = 1$ one moreover recovers Corollary 8.2.

Like before, by the results in section 5, we know that the sequence (z_m) converges if the space is additionally a complete UCW hyperbolic space, and so the previously theorem entails that (x_n) also converges. Furthermore, note that from the results in [38] the convergence result above can be extended to ϕ being a MKC map on C.

9. Final Remarks

In Banach spaces, one frequently relies on the notion of duality map. Providing a generalization to inner products, the use of the normalized duality map allows for arguments similar to those employed within the context of Hilbert spaces, specifically when the underlying norm is Gâteaux differentiable (i.e. the space is smooth). In nonlinear spaces, there is a proxy to duality maps, namely the quasi-linearization function when in the context of CAT(0) spaces. However, such spaces are substantially similar to the Hilbert context and the quasi-linearization function is in fact a nonlinear counterpart to inner products, not to the more general notion of normalized duality map. In this paper, we address this issue by describing a notion of smooth hyperbolic space (section 3). This class of spaces is, in principle, more general than CAT(0) spaces as well as smooth Banach spaces. The key feature is the existence of a function $\pi : X^2 \times X^2 \to \mathbb{R}$ asked to satisfy relaxed versions of the conditions characterizing the quasi-linearization function in CAT(0) spaces, providing a nonlinear perspective to the normalized duality map of smooth Banach spaces.

This work is set in the context of the proof mining research program. Although frequently associated solely with the extraction of computational information, like rates of convergence or of metastability, another primary feature of proof mining is the generalization of proofs. In this case, its techniques provided the crucial insight to know which are the qualities of the duality map essential in its frequent use in functional analysis. In this regard, the paper follows the successful endeavour of proof mining in generalizing results from the linear to the nonlinear setting.

A first question raised upon the introduction of any new system is what can one prove within it. We answer this by establishing a nonlinear generalization of the pivotal result due to Reich [59] regarding sunny nonexpansive retractions in Banach spaces (section 5). Any previous attempt to generalize Reich's theorem would either fail (due to lack of structure in the underlying space) or would (in the case of CAT(0) spaces) collapse into a nonlinear generalization of the simpler result due to Browder regarding metric projections. In this regard, the notion of smooth hyperbolic spaces introduced allowed for a proper discussion of Reich's theorem in a nonlinear context. Moreover, using this result, we show that several variants of Halpern's schema still converge. In section 6, we show the convergence of Halpern's schema applied to a family of nonexpansive maps which have, as explained, properties akin to those of families of resolvent maps when in a linear setting, extending [3] – and unifying [34, 67]. Section 7 extends a result due to Chang [15] addressing that the necessary conditions originally studied by Halpern [25], are indeed sufficient to establish convergence, provided one knows that the iteration is asymptotically regular. Section 8 then considers particular instances of the results in the previous section in order to discuss central results in the literature, in particular, generalizing results due to Halpern [25], Lions [47], Wittmann [75], Reich [60], Xu [78], Bauschke [6], and O'Hara, Pillay and Xu [53].

Together with the new infinitary results, we also provided a quantitative account of the convergence results in the form of convergence and metastability rates. These finitary results arise from a quantitative study and subsequent generalization of their linear counterparts, which then give rise to the full infinitary results. The only exception, where no quantitative information is provided, is regarding the nonlinear generalization of Reich's theorem. Already in the linear setting, its proof mining study in [39] is very technical and, an extension to a nonlinear setting is necessarily at least as complex. For this reason, we decouple its quantitative analysis to the forthcoming [55].

The existence of a nonlinear variant of smooth Banach spaces allows for the study of iterations and results separate from the linearity of the underlying space. This is clear in the proof of the nonlinear version of Reich's theorem, showing that the result is actually independent of any linearity argument. Moreover, these spaces allow for a proper extension to a nonlinear context of the main result in [3], while in this regard [67] in the setting of CAT(0) spaces must necessarily fall short. Still, a pressing question remains: is there a concrete example of an intrinsically nonlinear space which is a smooth hyperbolic space without it being a CAT(0) space? At the moment, we don't have an answer to this question and must leave it for future research. Further questions are left open. For example, is there a relation between uniform smooth hyperbolic spaces and uniform convexity in (some notion of) a dual-type space, similar to the linear setting? Regarding classes of functions, it is known that characterizations which are equivalent in normed spaces may become disconnected in a nonlinear setting. It will be interesting to understand if known relations in CAT(0) spaces remain true: for example, are firmly nonexpansive maps (metrically characterized) always π-firmly nonexpansive?; are π-sunny nonexpansive retractions actually 'sunny'?; and so on.

Acknowledgment: The author was supported by the DFG Project KO 1737/6-2. The proof mining study of Chang's theorem was one of the objectives in Ulrich Kohlenbach's DFG Project KO 1737/6-2. This work benefited from discussions with Ulrich Kohlenbach, Adriana Nicolae and Nicholas Pischke.

References

[1] A. D. Aleksandrov. A theorem on triangles in a metric space and some of its applications. *Trudy Matematicheskogo Instituta imeni VA Steklova*, 38:5–23, 1951.

[2] A. D. Aleksandrov. Ruled surfaces in metric spaces. *Vestnik Leningrad Univ.*, 12(1):5–26, 1957.

[3] K. Aoyama and M. Toyoda. Approximation of zeros of accretive operators in a Banach space. *Israel Journal of Mathematics*, 220(2):803–816, 2017.

[4] D. Ariza-Ruiz, L. Leuştean, and G. López-Acedo. Firmly nonexpansive mappings in classes of geodesic spaces. *Transactions of the American Mathematical Society*, 366(8):4299–4322, 2014.

[5] J. B. Baillon, R. E. Bruck, and S. Reich. On the asymptotic behavior of nonexpansive mappings and semigroups in Banach spaces. *Houston Journal of Mathematics*, 4(1):1–9, 1978.

[6] H. H. Bauschke. The approximation of fixed points of compositions of nonexpansive mappings in Hilbert space. *Journal of Mathematical Analysis and Applications*, 202(1):150–159, 1996.

[7] I. D. Berg and I. G. Nikolaev. Quasilinearization and curvature of Aleksandrov spaces. *Geometriae Dedicata*, 133:195–218, 2008.

[8] O. A. Boikanyo and G. Moroşanu. Inexact Halpern-type proximal point algorithm. *Journal of Global Optimization*, 51(1):11–26, 2011.

[9] F. E. Browder. Convergence of approximants to fixed points of nonexpansive nonlinear mappings in Banach spaces. *Archive for Rational Mechanics and Analysis*, 24(1):82–90, 1967.

[10] F. E. Browder. Nonlinear mappings of nonexpansive and accretive type in Banach spaces. *Bulletin of the American Mathematical Society*, 73:875–882, 1967.

[11] F. E. Browder and W. V. Petryshyn. The solution by iteration of nonlinear functional equations in Banach spaces. *Bulletin of the American Mathematical Society*, 72:571–575, 1966.

[12] R. E. Bruck. Random products of contractions in metric and Banach spaces. *Journal of Mathematical Analysis and Applications*, 88(2):319–332, 1982.

[13] R. E. Bruck and S. Reich. Accretive operators, Banach limits, and dual ergodic theorems. *L'Académie Polonaise des Sciences. Bulletin. Série des Sciences Mathématiques*, 29(11-12):585–589, 1981.

[14] F. Bruhat and J. Tits. Groupes réductifs sur un corps local. *Institut des Hautes Études Scientifiques. Publications Mathématiques*, 41:5–251, 1972.

[15] S.-S. Chang. Viscosity approximation methods for a finite family of nonexpansive mappings in Banach spaces. *Journal of Mathematical Analysis and Applications*, 323(2):1402–1416, 2006.

[16] C. Chidume. *Geometric properties of Banach spaces and nonlinear iterations*, volume 1965 of *Lecture Notes in Mathematics*. Springer-Verlag London, Ltd., London, 2009.

[17] M. M. Day. Uniform convexity in factor and conjugate spaces. *Annals of Mathematics. Second Series*, 45:375–385, 1944.

[18] S. Dhompongsa and B. Panyanak. On Δ-convergence theorems in CAT(0) spaces. *Computers & Mathematics with Applications. An International Journal*, 56(10):2572–2579, 2008.

[19] B. Dinis and P. Pinto. Quantitative results on the multi-parameters proximal point algorithm. *Journal of Convex Analysis*, 28(3):729–750, 2021.

[20] B. Dinis and P. Pinto. Strong convergence for the alternating Halpern-Mann iteration in CAT(0) spaces. *SIAM Journal on Optimization*, 33(2):785–815, 2023.

[21] F. Ferreira, L. Leuştean, and P. Pinto. On the removal of weak compactness arguments in proof mining. *Advances in Mathematics*, 354:106728, 55, 2019.

[22] K. Goebel and W. A. Kirk. Iteration processes for nonexpansive mappings. In *Topological methods in nonlinear functional analysis (Toronto, Ont., 1982)*, volume 21 of *Contemp. Math.*, pages 115–123. Amer. Math. Soc., Providence, RI, 1983.

[23] K. Goebel and S. Reich. *Uniform convexity, hyperbolic geometry, and nonexpansive mappings*, volume 83 of *Monographs and Textbooks in Pure and Applied Mathematics*. Marcel Dekker, Inc., New York, 1984.

[24] M. Gromov. Hyperbolic groups. In *Essays in group theory*, volume 8 of *Math. Sci. Res. Inst. Publ.*, pages 75–263. Springer, New York, 1987.

[25] B. Halpern. Fixed points of nonexpanding maps. *Bulletin of the American Mathematical Society*, 73(6):957–961, 1967.

[26] J. S. Jung, Y. J. Cho, and R. P. Agarwal. Iterative schemes with some control conditions for a family of finite nonexpansive mappings in Banach spaces. *Fixed Point Theory and Applications*, 2005(2):125–135, 2005.
[27] S. Kamimura and W. Takahashi. Approximating solutions of maximal monotone operators in Hilbert spaces. *Journal of Approximation Theory*, 106(2):226–240, 2000.
[28] W. A. Kirk. Geodesic geometry and fixed point theory. In *Seminar of Mathematical Analysis (Malaga/Seville, 2002/2003)*, volume 64 of *Colecc. Abierta*, pages 195–225. Univ. Sevilla Secr. Publ., Seville, 2003.
[29] U. Kohlenbach. Some logical metatheorems with applications in functional analysis. *Transactions of the American Mathematical Society*, 357(1):89–128, 2005.
[30] U. Kohlenbach. *Applied Proof Theory: Proof Interpretations and their Use in Mathematics.* Springer Science & Business Media, 2008.
[31] U. Kohlenbach. On quantitative versions of theorems due to F.E. Browder and R. Wittmann. *Advances in Mathematics*, 226(3):2764–2795, 2011.
[32] U. Kohlenbach. On the quantitative asymptotic behavior of strongly nonexpansive mappings in Banach and geodesic spaces. *Israel Journal of Mathematics*, 216(1):215–246, 2016.
[33] U. Kohlenbach. Proof-theoretic Methods in Nonlinear Analysis. In Boyan Sirakov, Paulo Ney de Souza, and Marcelo Viana, editors, *Proceedings of the International Congress of Mathematicians 2018*, volume 2, pages 62–82. World Scientific, 2019.
[34] U. Kohlenbach. Quantitative analysis of a Halpern-type proximal point algorithm for accretive operators in Banach spaces. *Journal of Nonlinear and Convex Analysis. An International Journal*, 21(9):2125–2138, 2020.
[35] U. Kohlenbach and L. Leuştean. Asymptotically nonexpansive mappings in uniformly convex hyperbolic spaces. *Journal of the European Mathematical Society*, 12(1):71–92, 2010.
[36] U. Kohlenbach and L. Leuştean. Effective metastability of Halpern iterates in CAT(0) spaces. *Advances in Mathematics*, 231(5):2526–2556, 2012. Addendum in: Adv. Math., 250:650–651, 2014.
[37] U. Kohlenbach and A. Nicolae. A proof-theoretic bound extraction theorem for CAT(κ)-spaces. *Studia Logica*, 105(3):611–624, 2017.
[38] U. Kohlenbach and P. Pinto. Quantitative translations for viscosity approximation methods in hyperbolic spaces. *Journal of Mathematical Analysis and Applications*, 507(2):Paper No. 125823, 33pp, 2022.
[39] U. Kohlenbach and A. Sipoş. The finitary content of sunny nonexpansive retractions. *Communications in Contemporary Mathematics*, 23(1):Paper No. 1950093, 63pp, 2021.
[40] E. Kopecká and S. Reich. Nonexpansive retracts in Banach spaces. In *Fixed point theory and its applications*, volume 77 of *Banach Center Publ.*, pages 161–174. Polish Acad. Sci. Inst. Math., Warsaw, 2007.
[41] E. Kopecká and S. Reich. Approximating fixed points in the Hilbert ball. *Journal of Nonlinear and Convex Analysis*, 15(4):819–829, 2014.
[42] D. Körnlein. *Quantitative Analysis of Iterative Algorithms in Fixed Point Theory and Convex Optimization.* PhD thesis, Technische Universität Darmstadt, 2016.
[43] G. Kreisel. On the interpretation of non-finitist proofs–Part II. *The Journal of Symbolic Logic*, 17(1):43–58, 1952.
[44] L. Leuştean, A. Nicolae, and A. Sipoş. An abstract proximal point algorithm. *Journal of Global Optimization. An International Journal Dealing with Theoretical and Computational Aspects of Seeking Global Optima and Their Applications in Science, Management and Engineering*, 72(3):553–577, 2018.
[45] L. Leuştean. A quadratic rate of asymptotic regularity for CAT(0)-spaces. *Journal of Mathematical Analysis and Applications*, 325(1):386–399, 2007.
[46] L. Leuştean. Nonexpansive iterations in uniformly convex W-hyperbolic spaces. In *Nonlinear analysis and optimization I. Nonlinear analysis*, volume 513 of *Contemp. Math.*, pages 193–210. Amer. Math. Soc., Providence, RI, 2010.
[47] P.-L. Lions. Approximation de points fixes de contractions. *Comptes Rendus Hebdomadaires des Séances de l'Académie des Sciences. Séries A et B*, 284(21):A1357–A1359, 1977.

[48] P.-E. Maingé. Strong convergence of projected subgradient methods for nonsmooth and non-strictly convex minimization. *Set-Valued Analysis. An International Journal Devoted to the Theory of Multifunctions and its Applications*, 16(7-8):899–912, 2008.

[49] B. Martinet. Régularisation d'inéquations variationnelles par approximations successives. *Rev. Française Informat. Recherche Opérationnelle*, 4(Sér. R-3,):154–158, 1970.

[50] A. Meir and E. Keeler. A theorem on contraction mappings. *Journal of Mathematical Analysis and Applications*, 28:326–329, 1969.

[51] C. H. Morales. Strong convergence theorems for pseudo-contractive mappings in Banach space. *Houston Journal of Mathematics*, 16(4):549–557, 1990.

[52] A. Moudafi. Viscosity approximation methods for fixed-points problems. *Journal of Mathematical Analysis and Applications*, 241(1):46–55, 2000.

[53] J. G. O'Hara, P. Pillay, and H.-K. Xu. Iterative approaches to convex feasibility problems in Banach spaces. *Nonlinear Analysis. Theory, Methods & Applications. An International Multidisciplinary Journal*, 64(9):2022–2042, 2006.

[54] W. V. Petryshyn. A characterization of strict convexity of Banach spaces and other uses of duality mappings. *Journal of Functional Analysis*, pages 282–291, 1970.

[55] P. Pinto. The finitary content of sunny nonexpansive retractions in nonlinear spaces. *Manuscript in preparation*, 2023.

[56] E. Rakotch. A note on contractive mappings. *Proceedings of the American Mathematical Society*, 13:459–465, 1962.

[57] S. Reich. Asymptotic behavior of contractions in Banach spaces. *Journal of Mathematical Analysis and Applications*, 44(1):57–70, 1973.

[58] S. Reich. Product formulas, nonlinear semigroups, and accretive operators. *Journal of Functional Analysis*, 36(2):147–168, 1980.

[59] S. Reich. Strong convergence theorems for resolvents of accretive operators in Banach spaces. *Journal of Mathematical Analysis and Applications*, 75(1):287–292, 1980.

[60] S. Reich. Approximating fixed points of nonexpansive mappings. *Panamer. Math. J*, 4(2):23–28, 1994.

[61] S. Reich and I. Shafrir. Nonexpansive iterations in hyperbolic spaces. *Nonlinear Analysis. Theory, Methods & Applications. An International Multidisciplinary Journal*, 15(6):537–558, 1990.

[62] S. Reich and L. Shemen. A note on halpern's algorithm in the Hilbert ball. *Journal of Nonlinear and Convex Analysis*, 14(4):853–862, 2013.

[63] R. T. Rockafellar. Monotone operators and the proximal point algorithm. *SIAM Journal on Control and Optimization*, 14(5):877–898, 1976.

[64] S. Saejung. Halpern's iteration in CAT(0) spaces. *Fixed Point Theory and Applications*, 2010. Art. ID 471781, 13pp.

[65] N. Shioji and W. Takahashi. Strong convergence of approximated sequences for nonexpansive mappings in Banach spaces. *Proceedings of the American Mathematical Society*, 125(12):3641–3645, 1997.

[66] A. Sipoş. Construction of fixed points of asymptotically nonexpansive mappings in uniformly convex hyperbolic spaces. *Numerical Functional Analysis and Optimization. An International Journal*, 42(6):696–711, 2021.

[67] A. Sipoş. Abstract strongly convergent variants of the proximal point algorithm. *Computational Optimization and Applications. An International Journal*, 83(1):349–380, 2022.

[68] A. Sipoş. Revisiting jointly firmly nonexpansive families of mappings. *Optimization. A Journal of Mathematical Programming and Operations Research*, 71(13):3819–3834, 2022.

[69] T. Suzuki. Moudafi's viscosity approximations with Meir-Keeler contractions. *Journal of Mathematical Analysis and Applications*, 325(1):342–352, 2007.

[70] T. Suzuki. Some notes on Bauschke's condition. *Nonlinear Analysis. Theory, Methods & Applications. An International Multidisciplinary Journal*, 67(7):2224–2231, 2007.

[71] W. Takahashi. A convexity in metric space and nonexpansive mappings. I. *Kodai Mathematical Seminar Reports*, 22:142–149, 1970.

[72] W. Takahashi. *Nonlinear functional analysis: Fixed point theory and its applications.* Yokohama Publishers, Yokohama, 2000.

[73] T. Tao. Norm convergence of multiple ergodic averages for commuting transformations. *Ergodic Theory and Dynamical Systems*, 28(2):657–688, 2008.

[74] T. Tao. *Structure and Randomness: Pages from Year One of a Mathematical Blog*. American Mathematical Society Providence, RI, 2008.

[75] R. Wittmann. Approximation of fixed points of nonexpansive mappings. *Archiv der Mathematik*, 58(5):486–491, 1992.

[76] H.-K. Xu. Another control condition in an iterative method for nonexpansive mappings. *Bulletin of the Australian Mathematical Society*, 65(1):109–113, 2002.

[77] H.-K. Xu. Iterative algorithms for nonlinear operators. *Journal of the London Mathematical Society*, 66(1):240–256, 2002.

[78] H.-K. Xu. Viscosity approximation methods for nonexpansive mappings. *Journal of Mathematical Analysis and Applications*, 298(1):279–291, 2004.