

Quantitative translations for viscosity approximation methods in hyperbolic spaces

Ulrich Kohlenbach[†] and Pedro Pinto[‡]

Department of Mathematics, Technische Universität Darmstadt,
Schlossgartenstraße 7, 64289 Darmstadt, Germany

[†]kohlenbach@mathematik.tu-darmstadt.de

[‡]pinto@mathematik.tu-darmstadt.de

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Abstract

In the setting of hyperbolic spaces, we show that the convergence of Browder-type sequences and Halpern iterations respectively entail the convergence of their viscosity version with a Rakotch map. We also show that the convergence of a hybrid viscosity version of the Krasnoselskii-Mann iteration follows from the convergence of the Browder type sequence. Our results follow from proof-theoretic techniques (proof mining). From an analysis of theorems due to T. Suzuki, we extract a transformation of rates for the original Browder type and Halpern iterations into rates for the corresponding viscosity versions. We show that these transformations can be applied to earlier quantitative studies of these iterations. From an analysis of a theorem due to H.-K. Xu, N. Altwaijry and S. Chebbi, we obtain similar results. Finally, in uniformly convex Banach spaces we study a strong notion of accretive operator due to Brezis and Sibony and extract an uniform modulus of uniqueness for the property of being a zero point. In this context, we show that it is possible to obtain Cauchy rates for the Browder type and the Halpern iterations (and hence also for their viscosity versions).

Keywords: Viscosity method, Rates of convergence, Rates of metastability, Proof mining. MSC2020: 47H09 47J25 03F10 53C23

1 Introduction

In this paper we study strongly convergent iteration schemes for nonexpansive self mappings $T : C \rightarrow C$ of bounded closed convex subsets of Banach spaces and – more generally – complete (W -)hyperbolic spaces X in the sense of [27].

In the framework of Hilbert spaces, Browder [8] and Halpern [19] proved strong convergence theorems for implicit and explicit iterations, respectively.

Browder's implicit scheme is

$$y_n(u) = (1 - \alpha_n)T(y_n(u)) \oplus \alpha_n u, \quad u \in C, \quad (1)$$

while Halpern introduced the explicit iteration scheme

$$w_{n+1}(u) = (1 - \alpha_n)T(w_n(u)) \oplus \alpha_n u, \quad w_0(u) = u \in C. \quad (2)$$

Here $(\alpha_n) \subseteq (0, 1]$ and ' \oplus ' is the result of applying the convexity operator W from our hyperbolic space (see the next section for details) which is the usual linear convex combination in the case of normed spaces.

The implicit schema $(y_n(u))$ was introduced by Browder in [8] and shown to be strongly convergent in Hilbert spaces to the metric projection of u onto the set $Fix(T)$ of fixed points of T (see also [19] for a more elementary proof). In [47], Reich extended the convergence of the Browder sequence to uniformly smooth Banach spaces where the sequence converges to the unique sunny nonexpansive retraction of C onto $Fix(T)$ applied to u .

For the explicit scheme $(w_n(u))$, Halpern showed the strong convergence to the metric projection of the anchor point u onto the set $Fix(T)$. The conditions considered by Halpern prevented the natural choice $\alpha_n = \frac{1}{n+1}$, which was later overcome by Wittmann [55]. [50] generalized Wittmann's theorem in particular to uniformly smooth Banach spaces. In [57, 56], Xu proved the strong convergence of Halpern type iterations in uniformly smooth Banach spaces under conditions which are incomparable to those used by Wittmann but which also allow for the choice $\alpha_n = \frac{1}{n+1}$.

Both schemes have also been considered for families of nonexpansive mappings (S_n) instead of the single map T : the sequence implicitly defined by

$$y_n(u) = (1 - \alpha_n)S_n(y_n(u)) \oplus \alpha_n u, \quad (3)$$

is called the (S_n) -Browder sequence with anchor point u . The original Browder sequence is (3) for a constant sequence (S_n) .

Likewise, the sequence explicitly defined by

$$w_0(u) = u \in C \text{ and } w_{n+1}(u) = (1 - \alpha_n)S_n(w_n(u)) \oplus \alpha_n u, \quad (4)$$

is called the (S_n) -Halpern iteration (with anchor and starting point u). The Halpern iteration [19] is the particular case of (4) for constant (S_n) . In [4], Bauschke considered a finite sequence of nonexpansive maps (not necessarily commutative but under a condition on the fixed point sets for ordered compositions) to define in a cyclic manner an infinite sequence of nonexpansive maps (S_n) from an initial finite list of maps. In this way, Bauschke's result is a particular instance of convergence for an iteration in the style of (4). Bauschke's result was further generalized in [21].

A different instance of convergence for iterations (4) can be found in the methods for finding zeros of accretive operators, the so-called proximal point algorithms.

The Halpern-type proximal point algorithm (HPPA), introduced by Kamimura and Takahashi [22] and independently by Xu [57], is the particular case of (4) where the sequence of nonexpansive maps is given by resolvent functions of an accretive operator. The HPPA has been extensively studied in the literature and several results give conditions that guarantee the strong convergence of the algorithm both in Hilbert spaces as well as e.g. in Banach spaces which are both uniformly smooth and uniformly convex (e.g. [22, 57, 5, 23, 1]).

In [43], Moudafi introduced in the context of Hilbert spaces the so-called viscosity algorithms in which the fixed anchor u is replaced by the value of a strict contraction ϕ applied to the current iteration. Moudafi's viscosity algorithms were extended to uniformly smooth Banach spaces by Xu in [58]. While Moudafi only considered a single mapping T this has also subsequently been generalized to families of such mappings (S_n) : the sequence implicitly defined by

$$x_n = (1 - \alpha_n)S_n(x_n) \oplus \alpha_n\phi(x_n), \quad (5)$$

is called the (S_n) -viscosity-Browder sequence (for (α_n)).

The (S_n) -viscosity-Halpern iteration is an explicit counterpart of (5), namely, given $x_0 \in C$, the iteration is defined inductively by

$$x_{n+1} = (1 - \alpha_n)S_n(x_n) \oplus \alpha_n\phi(x_n). \quad (6)$$

Viscosity generalizations of Bauschke's result for finite families of mappings and of the results in [21] are given in [20] and - under very general conditions - in [10] which is closely related to the prior Hybrid Steepest Descent Method, introduced by Yamada in [60] for finite sets of nonexpansive functions. In fact, the proof in [60] can be adapted to provide a strong convergence result for the viscosity version of Bauschke's theorem (see [38]).

In the important paper [51], Suzuki showed in the context of normed spaces that under very general conditions the convergence of the Browder and Halpern schemes (for families of mappings) implies the convergence of the corresponding viscosity schemes. Moreover, he showed that the limit of (x_n) is the unique fixed point of $P \circ \phi$, where $P(u) := \lim y_n(u)$ in the Browder case and $P(u) := \lim w_n(u)$ in the Halpern case respectively. Suzuki's result actually applies to a larger class of mappings ϕ than only strict contractions, namely to so-called Meir-Keeler contractions (MKC) introduced in [42] which in the case of convex sets (even of hyperbolic spaces) coincide with the uniformly contractive mappings introduced earlier by Rakotch in [46]. With ϕ , also the map $x \mapsto (1 - \alpha_n)S_n(x) \oplus \alpha_n\phi(x)$ is an MKC mapping and so - by [42, 46] - has a unique fixed point. Thus the implicit Browder scheme (5) (even for ϕ a MKC) is well-defined.

In section 3 of this paper, we give a complete quantitative analysis of Suzuki's reduction technique both in terms of rates of convergence as well as in terms of rates of metastability in the sense of Tao. Since in many important cases, explicit rates of metastability have been constructed in recent years for both Browder-type sequences as well as for Halpern iterations, we now get in all

these situations also rates of metastability for the corresponding viscosity generalizations. Usually, effective rates of convergence can be ruled out for Browder and Halpern sequences (see e.g. [44]) and so it is important that our quantitative analysis of Suzuki's theorem allows us to operate on the level of rates of metastability, by which we mean for a sequence (x_n) in a metric space (X, d) any bound $\varphi : (0, \infty) \times \mathbb{N} \rightarrow \mathbb{N}$ such that

$$\forall \varepsilon > 0 \forall f : \mathbb{N} \rightarrow \mathbb{N} \exists n \leq \varphi(\varepsilon, f) \forall i, j \in [n, f(n)] (d(x_i, x_j) \leq \varepsilon)$$

(see [26, 54]). Note that noneffectively, the property bounded by φ is equivalent to the usual Cauchy property. So a rate of metastability provides a finitary quantitative statement which (noneffectively) implies back the convergence statement. Moreover, we augment the rate of metastability with further bounding information which entails that the limits of (x_n) is the unique fixed point of $P \circ \phi$ with P as above (see Theorems 3.5 and 3.10).

Our analysis uses ideas from the logic-based approach of ‘proof mining’ by which the extractability of such quantitative data follows from certain logical transformations of the given proof (see [28] for more on this). However, while ideas from logic played a key role in arriving at our results, the final proofs make no references to logic. As a common by-product of such a logical analysis of proofs one often obtains in addition to quantitative information also new qualitative generalizations of the original theorems. E.g. the generalization to the setting of hyperbolic spaces is such an offspring of the logic-based approach.

If one is in a special situation where an actual rate of convergence for the Browder or the Halpern sequences is available, then our quantitative transformation also yields a rate of convergence for the corresponding viscosity version.

A different method for finding fixed points of nonexpansive mappings is the Krasnoselskii-Mann iteration

$$x_{n+1} = (1 - \beta_n)x_n \oplus \beta_n T(x_n). \quad (7)$$

This iteration has nice properties, notably Fejér monotonicity w.r.t. the fixed point set of T , but in general only approximates a fixed point weakly. This motivated several modifications to (7) to ensure strong convergence. Here we consider a hybrid version of the Krasnoselskii-Mann iteration and the viscosity approximation method. Namely, for $(\beta_n) \subseteq (0, 1]$ and $x_0 \in C$, the vKM iteration is defined by

$$x_{n+1} = (1 - \beta_n)x_n \oplus \beta_n ((1 - \alpha_n)T(x_n) \oplus \alpha_n \phi(x_n)), \quad (8)$$

introduced in [18] and shown there to be strongly convergent in Banach spaces under very strong conditions. In [59] this is studied further and much improved. Note that for $\alpha_n \equiv 0$, (8) reduces to the original schema (7), and for $\beta_n \equiv 1$, (8) is the viscosity generalization of the Halpern iteration.

Again we provide a quantitative analysis of the results from [18, 59].

In the final section 4 we study a situation in which explicit rates of convergence

(rather than only rates of metastability) for Browder-type sequences, Halpern iterations as well as Krasnoselskii-Mann iterations can be obtained. Here the nonexpansive mapping $T : C \rightarrow C$ used in these sequences is of the form $I - A$, where A is an accretive operator which satisfies a condition of being uniformly accretive which was introduced in [6] (see also [18] and [59]) and which is more general than other notions of uniform Ψ or Φ -accretivity studied in the literature. In uniformly convex Banach spaces X , this liberal notion suffices to conclude that A has at most one zero. From this uniqueness proof due to [6] we extract a so-called modulus of uniqueness (see [28] and the references given there) which only depends on a general modulus function witnessing the uniform accretivity of A and a modulus of uniform convexity of X . From this we obtain explicit and low complexity rates of convergence for the iterations listed above involving the mapping $T = I - A$.

2 Preliminaries

In the following, we consider the set of natural numbers as starting at zero, i.e. $\mathbb{N} := \{0, 1, 2, \dots\}$, and denote $\mathbb{N}^* := \mathbb{N} \setminus \{0\}$. We consider sequences (x_n) as indexed by $n \in \mathbb{N}$.

2.1 Classes of maps and hyperbolic spaces

Let (X, d) be a metric space and C a nonempty subset of X .

Definition 2.1. *Consider a mapping $T : C \rightarrow C$. We say that*

- *T is a nonexpansive map on C if for all $x, y \in C$*

$$d(T(x), T(y)) \leq d(x, y);$$

- *T is a strict contraction on C if there is $r \in [0, 1)$ such that for all $x, y \in C$*

$$d(T(x), T(y)) \leq rd(x, y),$$

in which case we say that T is an r -contraction on C ;

- *T is a Meir-Keeler contraction (MKC) on C if for any $\varepsilon > 0$ there is $\delta > 0$ such that*

$$d(x, y) < \varepsilon + \delta \rightarrow d(T(x), T(y)) < \varepsilon, \text{ for all } x, y \in C.$$

Actually, [42] considers a condition with the stronger premise $\varepsilon \leq d(x, y) < \varepsilon + \delta$, which ends up being equivalent to the one above using the nonexpansivity of T . Clearly, any contraction is a MKC mapping and any MKC mapping is a nonexpansive map. Meir-Keeler contraction were introduced in [42] as a generalization of metric contractions.

We now recall the notion of hyperbolic space. There are several distinct notions of hyperbolic space in the literature [24, 15, 16, 48]. Our results will be in the setting of W -hyperbolic spaces as introduced by the first author [27], here labeled simply by *hyperbolic spaces*. This setting is slightly more restrictive than the notion of space of hyperbolic type by Goebel/Kirk [15], but more general than the hyperbolic spaces in the sense of Reich/Shafir [48].

The triple (X, d, W) is called a hyperbolic space if (X, d) is a metric space and $W : X \times X \times [0, 1] \rightarrow X$ is a function satisfying

$$(W1) \quad \forall x, y \in X \forall \lambda \in [0, 1] \quad (d(z, W(x, y, \lambda)) \leq (1 - \lambda)d(z, x) + \lambda d(z, y)),$$

$$(W2) \quad \forall x, y \in X \forall \lambda_1, \lambda_2 \in [0, 1] \quad (d(W(x, y, \lambda_1), W(x, y, \lambda_2)) = |\lambda_1 - \lambda_2|d(x, y)),$$

$$(W3) \quad \forall x, y \in X \forall \lambda \in [0, 1] \quad (W(x, y, \lambda) = W(y, x, 1 - \lambda)),$$

$$(W4) \quad \begin{cases} \forall x, y, z, w \in X \forall \lambda \in [0, 1] \\ (d(W(x, z, \lambda), W(y, w, \lambda)) \leq (1 - \lambda)d(x, y) + \lambda d(z, w)). \end{cases}$$

The convexity function W was first considered by Takahashi [53], where a triple (X, d, W) with W satisfying (W1) is called a convex metric space. The classe of hyperbolic spaces includes normed spaces and their convex subsets, the Hilbert ball [16] and CAT(0)-spaces in the sense of Gromov. (see e.g. [7] for a detailed treatment). In turn, hyperbolic spaces are CAT(0)-spaces if they satisfy the property CN^- (which in the presence of other axioms is equivalent to the Bruhat-Tits CN -inequality [9] but in contrast to the latter purely universal):

$$CN^- : \quad \forall x, y, z \in X \quad \left(d^2(z, W(x, y, \frac{1}{2})) \leq \frac{1}{2}d^2(z, x) + \frac{1}{2}d^2(z, y) - \frac{1}{4}d^2(x, y) \right).$$

If $x, y \in X$ and $\lambda \in [0, 1]$, we shall denote $W(x, y, \lambda)$ by $(1 - \lambda)x \oplus \lambda y$. Using (W1), it is easy to see that

$$d(x, (1 - \lambda)x \oplus \lambda y) = \lambda d(x, y) \quad \text{and} \quad d(y, (1 - \lambda)x \oplus \lambda y) = (1 - \lambda)d(x, y).$$

For $x, y \in X$, the set $\{(1 - \lambda)x \oplus \lambda y : \lambda \in [0, 1]\}$ is called the metric segment with endpoints x, y and is denote by $[x, y]$. A nonempty subset $C \subseteq X$ is called *convex* if $[x, y] \subseteq C$, for all $x, y \in C$.

For convex C , MKC have the useful property that for each $\varepsilon > 0$ there is a contraction factor for points ε -apart. This was shown for convex subsets of Banach spaces in [51, Proposition 2]. We generalize this result to the geodesic setting:

Lemma 2.2. *Let C be a convex subset of a hyperbolic space X and ϕ a MKC on C . Then there is a function $\delta : (0, \infty) \rightarrow (0, 1)$ such that for all $\varepsilon > 0$ and $x, y \in C$*

$$d(x, y) \geq \varepsilon \rightarrow d(\phi(x), \phi(y)) \leq (1 - \delta(\varepsilon)) \cdot d(x, y).$$

Proof. Let $\varepsilon > 0$ be given. Since ϕ is a MKC mapping, there is $\sigma > 0$ satisfying for all $x, y \in C$

$$d(x, y) < \frac{\varepsilon}{4} + \sigma \rightarrow d(\phi(x), \phi(y)) \leq \frac{\varepsilon}{4}.$$

Since this property holds a fortiori for $0 < \sigma' \leq \sigma$ we may assume that $\sigma < \varepsilon$. We claim that for all $x, y \in C$,

$$d(x, y) \geq \varepsilon \rightarrow d(\phi(x), \phi(y)) \leq \left(1 - \frac{\sigma}{4\varepsilon}\right) \cdot d(x, y), \quad (9)$$

and thus for each $\varepsilon > 0$ we can define $\delta(\varepsilon)$ by

$$\frac{1}{4\varepsilon} \sup \left\{ \sigma \in (0, \varepsilon) : \forall x, y \in C \left(d(x, y) \geq \varepsilon \rightarrow d(\phi(x), \phi(y)) \leq \left(1 - \frac{\sigma}{4\varepsilon}\right) \cdot d(x, y) \right) \right\}.$$

It remains to prove (9). Take $x, y \in C$ such that $d(x, y) \geq \varepsilon$ and write $a := d(x, y)$. For each $j \in \{0, \dots, \lfloor \frac{a}{\varepsilon} \rfloor + 1\}$, consider $\lambda_j \in [0, 1]$ defined by $(\frac{\varepsilon + \sigma}{4a})j$. For $j \in \{0, \dots, \lfloor \frac{a}{\varepsilon} \rfloor\}$, by (W2)

$$d(W(x, y, \lambda_j), W(x, y, \lambda_{j+1})) = (\lambda_{j+1} - \lambda_j)d(x, y) < \frac{\varepsilon}{4} + \sigma,$$

and hence, by the hypothesis on σ ,

$$d(\phi(W(x, y, \lambda_j)), \phi(W(x, y, \lambda_{j+1}))) \leq \frac{\varepsilon}{4}.$$

Since ϕ is nonexpansive, we have

$$\begin{aligned} d(\phi(x), \phi(y)) &\leq \sum_{j=0}^{\lfloor \frac{a}{\varepsilon} \rfloor} d(\phi(W(x, y, \lambda_j)), \phi(W(x, y, \lambda_{j+1}))) \\ &\quad + d(\phi(W(x, y, \lambda_{\lfloor \frac{a}{\varepsilon} \rfloor + 1})), \phi(y)) \\ &\leq \frac{\varepsilon}{4} \left(\left\lfloor \frac{a}{\varepsilon} \right\rfloor + 1 \right) + (1 - \lambda_{\lfloor \frac{a}{\varepsilon} \rfloor + 1})d(x, y) \\ &= a - \frac{\sigma}{4} \left(\left\lfloor \frac{a}{\varepsilon} \right\rfloor + 1 \right) \leq a \left(1 - \frac{\sigma}{4\varepsilon} \right) = \left(1 - \frac{\sigma}{4\varepsilon} \right) d(x, y), \end{aligned}$$

showing (9) and concluding the proof. \square

In [46], Rakotch generalized Banach's contraction principle to maps satisfying the condition

$$x \neq y \rightarrow d(\phi(x), \phi(y)) \leq \alpha(d(x, y)) \cdot d(x, y), \quad (10)$$

where $\alpha : (0, \infty) \rightarrow (0, 1)$ is a decreasing function. However, the assumption that α is decreasing implies that condition (10) is equivalent to

$$d(x, y) \geq \varepsilon \rightarrow d(\phi(x), \phi(y)) \leq \alpha(\varepsilon) \cdot d(x, y). \quad (11)$$

We may always assume that α in (11) is decreasing, switching if necessary to

$$\alpha'(\varepsilon) := \max\{1/2, \inf\{\alpha(\delta) : \delta \in (0, \varepsilon]\}\}.$$

This argument shows that [51, Proposition 2] actually entails that in normed spaces MKC restricted to convex sets are Rakotch maps. The converse is also true: take $\delta := \frac{1-\alpha(\varepsilon)}{\alpha(\varepsilon)} \cdot \varepsilon$ to satisfy the MKC-condition.

Our Lemma 2.2 generalizes this result to the setting of hyperbolic spaces. In our results, C is always convex and so MKC are only seemingly more general than Rakotch maps. In what follows, we call a function $\delta : (0, \infty) \rightarrow (0, 1)$ a Rakotch-modulus (see [35]) if it satisfies

$$\forall \varepsilon > 0 \forall x, y \in C \ (d(x, y) \geq \varepsilon \rightarrow d(\phi(x), \phi(y)) \leq (1 - \delta(\varepsilon))d(x, y))$$

The particular case where ϕ is an r -contraction coincides with the case where there is a constant Rakotch-modulus (i.e. for all $\varepsilon > 0$, $\delta(\varepsilon) = \delta$, where $\delta \in (0, 1)$ is such that $r \leq 1 - \delta$).

Notational convention: Throughout this paper $\lceil x \rceil$ is defined as $\max\{0, \lceil x \rceil\}$ with the usual definition of $\lceil \cdot \rceil$.

2.2 Quantitative notions

Let (x_n) be a Cauchy sequence in a metric space (X, d) .

Definition 2.3. A function $\rho : (0, \infty) \rightarrow \mathbb{N}$ is a *Cauchy rate* of (x_n) if

$$\forall \varepsilon > 0 \forall i, j \geq \rho(\varepsilon) \ (d(x_i, x_j) \leq \varepsilon).$$

If (X, d) is complete so that (x_n) converges, then clearly ρ is a rate of convergence of (x_n) towards its limit x , i.e.

$$\forall \varepsilon > 0 \forall n \geq \rho(\varepsilon) \ (d(x_n, x) \leq \varepsilon).$$

Definition 2.4 ([26, 54]). A function $\varphi : (0, \infty) \times \mathbb{N}^{\mathbb{N}} \rightarrow \mathbb{N}$ is a *rate of metastability* for (x_n) if

$$\forall \varepsilon > 0 \forall f : \mathbb{N} \rightarrow \mathbb{N} \exists n \leq \varphi(\varepsilon, f) \forall i, j \in [n, f(n)] \ (d(x_i, x_j) \leq \varepsilon).$$

Remark 2.5. In the official definition of ‘rate of stability’, one states that $\forall i, j \in [n, n + f(n)] \ (d(x_i, x_j) \leq \varepsilon)$ which clearly implies the formulation above and which in turn is implied by the latter taking $\tilde{f}(n) := n + f(n)$ instead of f .

Proposition 2.6. $\rho : (0, \infty) \rightarrow (0, \infty)$ is a *Cauchy rate* iff $\varphi(\varepsilon, f) := \rho(\varepsilon)$ is a *rate of metastability*.

Proof. If ρ is a Cauchy rate, then clearly $\varphi(\varepsilon, f) := \rho(\varepsilon)$ is a rate of metastability. Let $\varphi(\varepsilon, f)$ be a rate of metastability which is a function $\rho(\varepsilon)$ which does not depend on f . Let $\tilde{m} \geq m \geq \rho(\varepsilon)$. Take $f(k) := \tilde{m}$ for all k . Then there exists an $n \leq \rho(\varepsilon)$ such that

$$\forall i, j \in [n, f(n)] \supseteq [\rho(\varepsilon), \tilde{m}] \ (d(x_i, x_j) \leq \varepsilon)$$

and so, in particular, $d(x_{\tilde{m}}, x_m) \leq \varepsilon$. □

2.3 Versions of a Lemma by Xu

A well-known lemma by Xu (e.g. [56, Lemma 2.1]) is used in the original proofs.

Lemma 2.7. *Consider sequences of nonnegative real numbers $(\lambda_n) \subseteq (0, 1)$, (a_n) and a sequence $(b_n) \subseteq \mathbb{R}$ such that $a_{n+1} \leq (1 - \lambda_n)a_n + \lambda_n b_n$, for all $n \in \mathbb{N}$. If*

$$(i) \sum_{n=0}^{\infty} \lambda_n = \infty \text{ (or equivalently } \prod_{n=0}^{\infty} (1 - \lambda_n) = 0), \text{ (ii) } \limsup b_n \leq 0,$$

then $\lim a_n = 0$.

We say that $A : \mathbb{N} \rightarrow \mathbb{N}$ is a rate of divergence for $\sum \lambda_n = \infty$ if

$$\forall k \in \mathbb{N} \left(\sum_{i=0}^{A(k)} \lambda_i \geq k \right). \quad (\text{Q}\Sigma)$$

A quantitative treatment is also possible using instead the equivalent condition $\prod (1 - \lambda_n) = 0$, if we have a function $A' : \mathbb{N} \times (0, 1] \rightarrow \mathbb{N}$ such that for all $m \in \mathbb{N}$, $A'(m, \cdot)$ is a rate of convergence for $\prod_{i=m}^{\infty} (1 - \lambda_i) = 0$, i.e.

$$\forall m \in \mathbb{N} \forall \varepsilon \in (0, 1] \left(\prod_{i=m}^{A'(m, \varepsilon)} (1 - \lambda_i) \leq \varepsilon \right). \quad (\text{Q}\Pi)$$

For our analysis, we will need a quantitative version of this lemma, which we give below. The proof is identical to several similar previous results (see e.g. [33, Lemmas 5.2, 5.3] and [45, Lemmas 12, 13]).

Notation 2.8. *Consider a function φ on tuples of variables \bar{x}, \bar{y} . If we wish to consider the variables \bar{x} as parameters we write $\varphi[\bar{x}](\bar{y})$. For simplicity of notation we may then even omit the parameters and simply write $\varphi(\bar{y})$.*

Lemma 2.9. *Consider sequences of real numbers $(\lambda_n) \subseteq [0, 1]$, $(a_n), (b_n) \subseteq \mathbb{R}$ and let $B \in \mathbb{N}^*$ be an upper bound on (a_n) and assume that for all $n \in \mathbb{N}$*

$$a_{n+1} \leq (1 - \lambda_n)a_n + \lambda_n b_n.$$

Consider $\varepsilon > 0, p, N \in \mathbb{N}$ such that $\forall i \in [N, p] \ (b_n \leq \frac{\varepsilon}{2})$.

(i) If $A : \mathbb{N} \rightarrow \mathbb{N}$ satisfies (QΣ), then

$$\forall i \in [\sigma_1(\varepsilon, N), p] \ (a_i \leq \varepsilon),$$

$$\text{where } \sigma_1(\varepsilon, N) = \sigma_1[A, B](\varepsilon, N) := A(N + \lceil \ln(\frac{2B}{\varepsilon}) \rceil) + 1;$$

(ii) If $A' : \mathbb{N} \times (0, 1] \rightarrow \mathbb{N}$ satisfies (QΠ), then

$$\forall i \in [\sigma_2(\varepsilon, N), p] \ (a_i \leq \varepsilon),$$

$$\text{where } \sigma_2(\varepsilon, N) = \sigma_2[A', B](\varepsilon, N) := \max\{A'(N, \frac{\varepsilon}{2B}), N\} + 1.$$

The conclusion of Lemma 2.9 still holds true if the main inequality is only valid in the interval $[N, p]$.

Lemma 2.10. *Consider sequences of real numbers $(\lambda_n) \subseteq [0, 1]$, $(a_n), (b_n) \subseteq \mathbb{R}$. Let $B \in \mathbb{N}^*$ be an upper bound on (a_n) and $A : \mathbb{N} \rightarrow \mathbb{N}$ a function satisfying (Q Σ). Let $\varepsilon \in (0, 1]$ and $p, N \in \mathbb{N}$ be given. If*

$$\forall i \in [N, p] \left(a_{i+1} \leq (1 - \lambda_i)a_i + \lambda_i b_i \text{ and } b_i \leq \frac{\varepsilon}{2} \right),$$

then

$$\forall i \in [\sigma_1(\varepsilon, N), p] \left(a_i \leq \varepsilon \right),$$

where $\sigma_1(\varepsilon, N) = \sigma_1[A, B](\varepsilon, N) := A(N + \lceil \ln(\frac{2B}{\varepsilon}) \rceil) + 1$.

3 Viscosity approximation methods

3.1 Stability conditions

In [51], Suzuki starts by giving perspicuous proofs of Xu's strong convergence results for the viscosity method (with strict contractions) of Browder and Halpern type iterations in uniformly smooth Banach spaces.

Theorem 3.1 ([58, Thm.4.1]). *Let X be an uniformly smooth Banach space and C a nonempty, closed, convex subset of X . Let T and ϕ be a nonexpansive and a strict contraction on C , respectively. Assume that $\text{Fix}(T) \neq \emptyset$. Consider the net (x_α) satisfying for $\alpha \in (0, 1)$,*

$$x_\alpha = (1 - \alpha)T(x_\alpha) + \alpha\phi(x_\alpha).$$

Then (x_α) strongly converges, as $\alpha \rightarrow 0$, to the unique z satisfying $P\phi(z) = z$, where P is the unique sunny nonexpansive retraction of C onto $\text{Fix}(T)$.

Theorem 3.2 ([58, Thm.4.2]). *Under the same conditions as before, let $x_0 \in C$, and $(\alpha_n) \subseteq (0, 1)$ satisfy*

$$(C1) \lim \alpha_n = 0, (C2) \sum \alpha_n = \infty, (C3) \lim \frac{\alpha_{n+1}}{\alpha_n} = 1.$$

Consider the iteration (x_n) defined by

$$x_{n+1} = (1 - \alpha_n)T(x_n) + \alpha_n\phi(x_n).$$

Then (x_n) strongly converges to the unique z satisfying $P\phi(z) = z$.

The new proofs compare the viscosity iteration with the convergent original one. Suzuki, then goes on to employ similar arguments to obtain extended convergence results to MKC-mappings ϕ and with T replaced by a sequence of nonexpansive maps (S_n) together with a stability condition that the original iteration defined on that sequence of maps is convergent. From a logical point

of view, the only complicated aspect in Suzuki's proofs is the use of the sunny nonexpansive retraction (metric projection in Hilbert spaces) required to characterize the limit point z . We avoid this step by considering Cauchy points of the relevant iterations instead of the actual limit. This entails that a quantitative analysis of these results will extract a function that given a metastability rate for the assumed convergence (and some other parameters that illustrate the uniformity of the bound), outputs a rate of metastability for the viscosity iteration in question. In this section, we elaborate on the quantitative conditions of our main results given in the next two sections. We then apply our main theorems in particular cases where metastability rates have been extracted and witness the assumption, to obtain several rates of metastability for viscosity-type iterations.

The next lemma states that we can always transform a given rate of metastability such that the interval of metastability is to the right of a given $N \in \mathbb{N}$.

Lemma 3.3. *If (x_n) is a Cauchy sequence with rate of metastability φ , then*

$$\forall \varepsilon > 0 \forall f : \mathbb{N} \rightarrow \mathbb{N} \forall N \in \mathbb{N} \exists n \in [N, \theta(\varepsilon, f, N)] \forall i, j \in [n, f(n)] (d(x_i, x_j) \leq \varepsilon),$$

where $\theta(\varepsilon, f, N) := \theta[\varphi](\varepsilon, f, N) := \max\{N, \varphi(\varepsilon, f_N)\}$, where for all $m \in \mathbb{N}$ $f_N(m) := f(\max\{N, m\})$.

Proof. Let ε, f and N be given. Since φ is a rate of metastability for (x_n) , we have that for some $n' \leq \varphi(\varepsilon, f_N)$,

$$\forall i, j \in [n', f_N(n')] (d(x_i, x_j) \leq \varepsilon).$$

Let $n = \max\{N, n'\}$. Then $n \in [N, \max\{N, \varphi(\varepsilon, f_N)\}] = [N, \theta(\varepsilon, f, N)]$. Since $n' \leq n$ and $f_N(n') = f(n)$, the result follows. \square

Let $C \subseteq X$ be a nonempty, bounded, convex subset of a hyperbolic space X and $b \in \mathbb{N}^*$ such that $b \geq \text{diam}(C)$. For $u \in C$ let $(y_n(u))$ be the (S_n) -Browder sequence with anchor point u satisfying (3). We say that a monotone function θ_b satisfies the condition $(B[S_n])$ if for all $u \in C$,

$$\begin{aligned} \forall \varepsilon > 0 \forall f : \mathbb{N} \rightarrow \mathbb{N} \forall N \in \mathbb{N} \exists n \in [N, \theta_b(\varepsilon, f, N)] \\ \forall i, j \in [n, f(n)] (d(y_i(u), y_j(u)) \leq \varepsilon). \end{aligned} \quad (B[S_n])$$

Remark 3.4. *Note that the function θ_b is uniform on $u \in C$, depending only on the bound $b \in \mathbb{N}^*$ for the diameter of C . Furthermore, the monotonicity condition here is to be understood in the sense that the function θ_b is nondecreasing in N . Such assumption is by convenience, as one can always consider $\theta_b^{\text{maj}}(\varepsilon, f, N) := \max_{N' \leq N} \{\theta_b(\varepsilon, f, N')\}$. Finally, we remark that by Lemma 3.3, such a function θ_b exists whenever we have a uniform metastability rate φ_b for $(y_n(u))$.*

Similarly, for $u \in C$ let $(w_n(u))$ be the (S_n) -Halpern iteration with anchor and starting point u defined by (4). We say that a monotone function θ'_b satisfies

the condition $(\mathbf{H}[S_n])$ if for all $u \in C$,

$$\begin{aligned} \forall \varepsilon > 0 \forall f : \mathbb{N} \rightarrow \mathbb{N} \forall N \in \mathbb{N} \exists n \in [N, \theta'_b(\varepsilon, f, N)] \\ \forall i, j \in [n, f(n)] \quad (d(w_i(u), w_j(u)) \leq \varepsilon). \end{aligned} \quad (\mathbf{H}[S_n])$$

We also write the previous conditions for constant sequences (S_n) to discuss particular instances of the main theorems. Let T be a nonexpansive map.

A monotone function θ_b satisfies the condition $(\mathbf{B}[T])$ if for all $u \in C$,

$$\begin{aligned} \forall \varepsilon > 0 \forall f : \mathbb{N} \rightarrow \mathbb{N} \forall N \in \mathbb{N} \exists n \in [N, \theta_b(\varepsilon, f, N)] \\ \forall i, j \in [n, f(n)] \quad (d(y_i(u), y_j(u)) \leq \varepsilon), \end{aligned} \quad (\mathbf{B}[T])$$

where $(y_n(u))$ is the (S_n) -Browder sequence for $S_n \equiv T$, i.e. the original Browder type iteration.

A monotone function θ'_b satisfies the condition $(\mathbf{H}[T])$ if for all $u \in C$,

$$\begin{aligned} \forall \varepsilon > 0 \forall f : \mathbb{N} \rightarrow \mathbb{N} \forall N \in \mathbb{N} \exists n \in [N, \theta'_b(\varepsilon, f, N)] \\ \forall i, j \in [n, f(n)] \quad (d(w_i(u), w_j(u)) \leq \varepsilon), \end{aligned} \quad (\mathbf{H}[T])$$

where $(w_n(u))$ is the (S_n) -Halpern iteration for $S_n \equiv T$, i.e. the original Halpern type iteration.

3.2 Browder-viscosity

In this section we prove a quantitative version of Theorem 7 in [51] generalized to the geodesic setting.

Theorem 3.5. *Let C be a nonempty bounded convex subset of a hyperbolic space and let $b \in \mathbb{N}^*$ be a bound on the diameter of C . Consider (S_n) a family of nonexpansive maps on C , an MKC mapping $\phi : C \rightarrow C$ with Rakotch-modulus δ , and a sequence $(\alpha_n) \subseteq (0, 1]$. Let θ_b be a monotone function satisfying $(\mathbf{B}[S_n])$. Define*

$$\Psi(\varepsilon, f, N) = \Psi[b, \delta, \theta_b](\varepsilon, f, N) = \theta_b(\varepsilon_0, f, \Psi_M),$$

where

$$\begin{aligned} \Psi_0 = N \text{ and } \Psi_{m+1} = \theta_b(\varepsilon_0, f_{M-m}, \Psi_m), \text{ with} \\ \tilde{\varepsilon} = \frac{\varepsilon \delta(\varepsilon/2)}{4}, \varepsilon_0 = \frac{\tilde{\varepsilon} \delta(\tilde{\varepsilon})}{2}, M = \left\lceil \log_{1-\delta(\tilde{\varepsilon})} \left(\frac{\tilde{\varepsilon}}{2b} \right) \right\rceil \text{ and} \\ \text{for all } p \in \mathbb{N}, \begin{cases} f_0(p) = f(p), \\ f_{m+1}(p) = \max \{f(p), \theta_b(\varepsilon_0, f_m, p)\} \end{cases} \text{ for } m < M. \end{aligned}$$

Then for (x_n) satisfying (5) we have

$$\begin{aligned} \forall N \in \mathbb{N} \forall \varepsilon > 0 \forall f : \mathbb{N} \rightarrow \mathbb{N} \exists n \in [N, \Psi(\varepsilon, f, N)] \exists z \in C \\ (d(z, y_n(\phi(z))) \leq \varepsilon \wedge \forall i \in [n, f(n)] (d(x_i, z) \leq \frac{\varepsilon}{2})). \end{aligned}$$

In particular, $\Psi(\varepsilon, f, 0)$ is a rate of metastability for (x_n) .

Proof. Let $N \in \mathbb{N}$, $\varepsilon > 0$ and $f : \mathbb{N} \rightarrow \mathbb{N}$ be given. Note that the definition of the natural number M implies $(1 - \delta(\tilde{\varepsilon}))^M \leq \frac{\tilde{\varepsilon}}{2b}$. Consider the functions f_0, \dots, f_M as in the theorem. Define $z_0 = x_0$ and $n_0 = N$. For $m \leq M$, assume that z_m and n_m are already defined and let $(y_n^m)_n$ be the (S_n) -Browder sequence with anchor point $\phi(z_m)$, i.e. $y_n^m = y_n(\phi(z_m))$. By (B[S_n]), we can take $n_{m+1} \in [n_m, \theta_b(\varepsilon_0, f_{M-m}, n_m)]$ such that

$$\forall i \in [n_{m+1}, f_{M-m}(n_{m+1})] \left(d(y_i^m, y_{n_{m+1}}^m) \leq \varepsilon_0 \right).$$

Define $z_{m+1} = y_{n_{m+1}}^m$.

Notice that for $m \leq M-1$, we have $n_{m+2} \in [n_{m+1}, \theta_b(\varepsilon_0, f_{M-m-1}, n_{m+1})] \subseteq [n_{m+1}, f_{M-m}(n_{m+1})]$, and so

$$d(y_{n_{m+2}}^m, y_{n_{m+1}}^m) \leq \varepsilon_0.$$

For $m \in [1, M]$, by (W4) and the fact that $S_{n_{m+1}}$ is nonexpansive, we have

$$d(y_{n_{m+1}}^m, y_{n_{m+1}}^{m-1}) \leq (1 - \alpha_{n_{m+1}})d(y_{n_{m+1}}^m, y_{n_{m+1}}^{m-1}) + \alpha_{n_{m+1}}d(\phi(z_m), \phi(z_{m-1})),$$

which gives $d(y_{n_{m+1}}^m, y_{n_{m+1}}^{m-1}) \leq d(\phi(z_m), \phi(z_{m-1}))$. Thus, for $m \in [1, M]$,

$$d(z_{m+1}, z_m) \leq d(y_{n_{m+1}}^m, y_{n_{m+1}}^{m-1}) + d(y_{n_{m+1}}^{m-1}, y_{n_m}^{m-1}) \leq d(\phi(z_m), \phi(z_{m-1})) + \varepsilon_0.$$

Now we argue that there is $m_0 \leq M$ such that $d(z_{m_0+1}, z_{m_0}) \leq \tilde{\varepsilon}$. Assume that for all $m \leq M-1$, $d(z_{m+1}, z_m) > \tilde{\varepsilon}$, which gives $d(\phi(z_{m+1}), \phi(z_m)) \leq (1 - \delta(\tilde{\varepsilon}))d(z_{m+1}, z_m)$, since δ is a Rakotch-modulus for ϕ . Hence, for $m \in [1, M]$

$$d(z_{m+1}, z_m) \leq (1 - \delta(\tilde{\varepsilon}))d(z_m, z_{m-1}) + \varepsilon_0.$$

By induction

$$d(z_{M+1}, z_M) \leq (1 - \delta(\tilde{\varepsilon}))^M d(z_1, z_0) + \varepsilon_0 \sum_{i=0}^{M-1} (1 - \delta(\tilde{\varepsilon}))^i \leq \frac{\tilde{\varepsilon}}{2b} b + \frac{\varepsilon_0}{\delta(\tilde{\varepsilon})} = \tilde{\varepsilon}.$$

Let $m_0 \leq M$ be such that $d(z_{m_0+1}, z_{m_0}) \leq \tilde{\varepsilon}$. We now write $y_n \equiv y_n^{m_0}$ and $z = z_{m_0+1}$. We have $[n_{m_0+1}, f(n_{m_0+1})] \subseteq [n_{m_0+1}, f_{M-m_0}(n_{m_0+1})]$ and so

$$\forall i \in [n_{m_0+1}, f(n_{m_0+1})] \left(d(y_i, z) \leq \varepsilon_0 \leq \tilde{\varepsilon} \right). \quad (12)$$

By (W4) we have for $i \in [n_{m_0+1}, f(n_{m_0+1})]$ (using again that S_i is nonexpansive),

$$d(x_i, y_i) \leq (1 - \alpha_i)d(x_i, y_i) + \alpha_i d(\phi(x_i), \phi(z_{m_0})),$$

which implies, using the fact that ϕ is nonexpansive,

$$\begin{aligned} d(x_i, y_i) &\leq d(\phi(x_i), \phi(z_{m_0})) \leq d(\phi(x_i), \phi(z)) + d(\phi(z), \phi(z_{m_0})) \\ &\leq d(\phi(x_i), \phi(z)) + d(z_{m_0+1}, z_{m_0}) \leq d(\phi(x_i), \phi(z)) + \tilde{\varepsilon}. \end{aligned} \quad (13)$$

An inductive argument and the monotonicity of θ_b in the last variable entail $N \leq n_{m_0+1} \leq n_{M+1} \leq \Psi(\varepsilon, f, N)$. We now argue that for $i \in [n_{m_0+1}, f(n_{m_0+1})]$, $d(x_i, z) \leq \frac{\varepsilon}{2}$. Assume that for some $i \in [n_{m_0+1}, f(n_{m_0+1})]$, $d(x_i, z) > \frac{\varepsilon}{2}$. Since δ is a Rakotch-modulus for ϕ ,

$$d(\phi(x_i), \phi(z)) \leq (1 - \delta(\varepsilon/2))d(x_i, z),$$

and then, using (12) and (13),

$$\begin{aligned} d(x_i, z) &\leq d(x_i, y_i) + d(y_i, z) \leq d(\phi(x_i), \phi(z)) + \tilde{\varepsilon} + d(y_i, z) \\ &\leq (1 - \delta(\varepsilon/2))d(x_i, z) + 2\tilde{\varepsilon}. \end{aligned}$$

This implies the contradiction $d(x_i, z) \leq \frac{2\tilde{\varepsilon}}{\delta(\varepsilon/2)} = \frac{\varepsilon}{2}$.

Using $d(z, z_{m_0}) \leq \tilde{\varepsilon} \leq \varepsilon$, (W4) and the nonexpansivity of $S_{n_{m_0+1}}$ and ϕ , we get

$$d(z, y_{n_{m_0+1}}(\phi(z))) = d(y_{n_{m_0+1}}(\phi(z_{m_0})), y_{n_{m_0+1}}(\phi(z))) \leq \varepsilon,$$

which entails that the theorem is satisfied with z and $n := n_{m_0+1}$. \square

Remark 3.6. Theorem 3.5 implies that (already for $N = 0$) $\Psi(\varepsilon, f, N)$ is a rate of metastability for (x_n) . We wrote the statement in a more informative form so that it implies in an elementary way that (if X is complete and C is closed) the limit x of (x_n) actually is the (unique) fixed point of $P \circ \phi$, where $P(u) := \lim y_n(u)$, which is an important further information in Suzuki's theorem: let $\varepsilon > 0$ and $N \in \mathbb{N}$ be so large that

$$\forall k \geq N \quad (d(x_k, x), d(y_k(\phi(x)), P(\phi(x))) \leq \varepsilon). \quad (14)$$

Then by the theorem, we get $n \geq N$ and $z \in C$ such that

$$d(z, y_n(\phi(z))), d(x_n, z) \leq \varepsilon. \quad (15)$$

Using (14) and (15), $d(x, z) \leq 2\varepsilon$ and so by (W4) and the nonexpansivity of S_n and ϕ

$$d(y_n(\phi(x)), y_n(\phi(z))) \leq 2\varepsilon.$$

Hence using (15),

$$d(x, y_n(\phi(x))) \leq d(z, y_n(\phi(z))) + 4\varepsilon \leq 5\varepsilon$$

and so by (14)

$$d(x, P(\phi(x))) \leq 6\varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, we have that $x = P(\phi(x))$.

Remark 3.7. Theorem 3.5 can be adapted to dispense with the boundedness assumption on the set C , and assume the existence of a natural number $b \in \mathbb{N}^*$ that only satisfies $b \geq \max\{d(x_0, \phi(x_0)), d(x_0, y_n(\phi(x_0))) : n \in \mathbb{N}\}$. In the original proof by Suzuki, one considers that all the (S_n) -Browder type sequences converge and thus, the particular fact that $(y_n(\phi(x_0)))$ is bounded trivially follows.

In this situation, one assumes that for each $r \in \mathbb{N}^*$, the function θ_r satisfies the formula (B[S_n]) only for anchor points $u \in C \cap \overline{B}_r(x_0)$, where $\overline{B}_r(x_0)$ denotes the closed ball centred at x_0 with radius r . Then, Theorem 3.5 can be adapted to hold with the bound $\Psi(\varepsilon, f, N) = \theta_{(M+1)b}(\varepsilon_0, f, \Psi_M)$, where $\tilde{\varepsilon}$, ε_0 , M are as before and

$$\Psi_0 = N \text{ and } \Psi_{m+1} = \theta_{(m+1)b}(\varepsilon_0, f_{M-m}, \Psi_m),$$

with, for all $p \in \mathbb{N}$

$$\begin{cases} f_0(p) = f(p), \\ f_{m+1}(p) = \max \{f(p), \theta_{(M-m+1)b}(\varepsilon_0, f_m, p)\} \quad \text{for } m < M. \end{cases}$$

Indeed, the proof can be carried out with similar arguments. The definition of the natural numbers n_m and the points z_m is similar to before. Assume n_m and z_m are already defined, with $d(x_0, z_m) \leq m \cdot b$. From the hypothesis on $\theta_{(\cdot)}$ and the fact that

$$d(x_0, \phi(z_m)) = d(x_0, \phi(x_0)) + d(\phi(x_0), \phi(z_m)) \leq (m+1)b,$$

we can define $n_{m+1} \in [n_m, \theta_{(m+1)b}(\varepsilon_0, f_{M-m}, n_m)]$ satisfying

$$\forall i \in [n_{m+1}, f_{M-m}(n_{m+1})] \quad (d(y_i^m, y_{n_{m+1}}^m) \leq \varepsilon_0).$$

Moreover, with $z_{m+1} = y_{n_{m+1}}^m$, we have

$$\begin{aligned} d(x_0, z_{m+1}) &\leq d(x_0, y_{n_{m+1}}(\phi(x_0))) + d(y_{n_{m+1}}(\phi(x_0)), y_{n_{m+1}}(z_m)) \\ &\leq b + d(x_0, z_m) \leq (m+1)b, \end{aligned}$$

which ensures that this construction holds for all $m \leq M$. Since it still holds for $m \leq M-1$ that $[n_{m+1}, \theta_{(m+2)b}(\varepsilon_0, f_{M-m-1}, n_{m+1})] \subseteq [n_{m+1}, f_{M-m}(n_{m+1})]$, we again have $d(y_{n_{m+2}}^m, y_{n_{m+1}}^m) \leq \varepsilon_0$. The remainder of the proof follows the same arguments as before, noticing that the new condition on the natural number b suffices to show the existence of z_{m_0} satisfying $d(z_{m_0+1}, z_{m_0}) \leq \tilde{\varepsilon}$, as it still entails $(1 - \delta(\tilde{\varepsilon}))^M d(z_1, z_0) \leq \tilde{\varepsilon}/2$.

An application of Theorem 3.5 gives a rate of metastability for the Browder-viscosity iteration from a function θ_b satisfying (B[T]).

Corollary 3.8. *Let C be a nonempty bounded convex subset of a hyperbolic space and let $b \in \mathbb{N}^*$ be a bound on the diameter of C . Consider T a nonexpansive map on C , an r -contraction $\phi : C \rightarrow C$ with $\delta \in (0, 1)$ such that $r \leq 1 - \delta$, and a sequence $(\alpha_n) \subseteq (0, 1]$. Let θ_b be a monotone function satisfying (B[T]). Then (x_n) satisfying*

$$x_n = (1 - \alpha_n)T(x_n) \oplus \alpha_n\phi(x_n), \quad n \in \mathbb{N}$$

is a Cauchy sequence with metastability rate

$$\Psi(\varepsilon, f) = \Psi[b, \delta, \theta_b](\varepsilon, f) = \theta_b(\varepsilon_0, f, \Psi_M),$$

where

$$\begin{aligned} \Psi_0 &= 0 \text{ and } \Psi_{m+1} = \theta_b(\varepsilon_0, f_{M-m}, \Psi_m), \text{ with} \\ \varepsilon_0 &= \frac{\varepsilon\delta^2}{8}, M = \left\lceil \log_{1-\delta} \left(\frac{\varepsilon\delta}{8b} \right) \right\rceil \text{ and} \\ \text{for all } p \in \mathbb{N}, \quad &\begin{cases} f_0(p) = f(p), \\ f_{m+1}(p) = \max\{f(p), \theta_b(\varepsilon_0, f_m, p)\} \end{cases} \text{ for } m < M. \end{aligned}$$

Corollary 3.9. *Let $X, C, T, \phi, b, r, \delta, (\alpha_n), (x_n)$ be as in Corollary 3.8. Let ρ be a common Cauchy rate for the T -Browder iteration $(y_n(u))$ for all $u \in C$ used as anchor points. Then*

$$\Psi[b, \delta, \rho](\varepsilon) := \rho\left(\frac{\varepsilon\delta^2}{8}\right)$$

is a Cauchy rate for (x_n) .

Proof. By Proposition 2.6 and Lemma 3.3,

$$\theta_b[\rho](\varepsilon, f, N) := \rho'(\varepsilon, N) := \max\{N, \rho(\varepsilon)\}$$

is a monotone function satisfying (B[T]). Hence by Corollary 3.8

$$\Psi[b, \delta, \rho](\varepsilon) := \rho'(\varepsilon_0, \Psi_M),$$

where

$$\Psi_0 = 0 \text{ and } \Psi_{m+1} = \rho'(\varepsilon_0, \Psi_m), \text{ with } \varepsilon_0 = \frac{\varepsilon\delta^2}{8}, M = \left\lceil \log_{1-\delta} \left(\frac{\varepsilon\delta}{8b} \right) \right\rceil,$$

is a rate of metastability for (x_n) which does not depend on f and hence – again by Proposition 2.6 – is a Cauchy rate. An easy induction shows that $\Psi_m \leq \rho(\varepsilon_0)$ for all m and so $\rho'(\varepsilon_0, \Psi_M) = \rho(\varepsilon_0)$. \square

The Browder type sequence has been the focus of many quantitative studies using techniques from proof mining. In [29], the first author analysed Browder’s strong convergence theorem in Hilbert spaces showing that it was possible to avoid sequential weak compactness used in the original proof by Browder. A simpler proof of Browder’s result due to Halpern in [19] that already avoids weak compactness, was also analyzed and a metastability bound was obtained. Kirk [25] showed that Halpern’s proof can be extended to CAT(0) spaces (essentially unchanged), and thus the same holds true for its quantitative version [33, Prop.9.3]. By Theorem 3.5, using the rates of metastability extracted in [33], we obtain rates of metastability for the viscosity-Browder type sequences in the setting of CAT(0) spaces. In [38] and [39, Theorem 6.7], for the special case where X is a Hilbert space, similar rates of metastability have already been extracted – again using methods of proof mining – from a direct proof of the

strong convergence of (x_n) (i.e. without a reduction to Browder's theorem) from [60].

In [36], a quantitative treatment of Reich's extension of Browder fixed point theorem to uniformly smooth Banach spaces was carried out under the additional hypothesis that the space is uniformly convex. Using the rate of metastability extracted in [36], once again by an application of Theorem 3.5, we obtain a rate of metastability for the viscosity-Browder type sequence in Banach spaces that are both uniformly convex and uniformly smooth.

3.3 Halpern-viscosity

In this section we establish a quantitative version of Theorem 8 in [51], generalized here again to the geodesic setting.

Theorem 3.10. *Let C be a nonempty bounded convex subset of a hyperbolic space and let $b \in \mathbb{N}^*$ be a bound on the diameter of C . Consider (S_n) a family of nonexpansive maps on C , a MKC $\phi : C \rightarrow C$ with Rakotch-modulus δ , and a sequence $(\alpha_n) \subseteq [0, 1]$ satisfying $\sum \alpha_n = \infty$ with a monotone rate of divergence A . Let θ'_b be a monotone function satisfying (H[S_n]) and σ_1 be as in Lemma 2.9. Define*

$$\Psi(\varepsilon, f, N) = \Psi[b, \delta, A, \theta'_b](\varepsilon, f, N) = \sigma_1[\tilde{A}_\varepsilon, b] \left(\frac{\varepsilon}{3}, \Psi_{M+1} \right),$$

where

$$\begin{aligned} \tilde{A}_\varepsilon(k) &:= A \left(\left\lceil \frac{k}{\delta(\varepsilon/3)} \right\rceil \right) \text{ for all } k \in \mathbb{N} \\ \Psi_0 &= N \text{ and } \Psi_{m+1} = \theta'_b(\varepsilon_0, f_{M-m}, \Psi_m), \text{ with} \\ \tilde{\varepsilon} &= \frac{\varepsilon \delta(\varepsilon/3)}{9}, \varepsilon_0 = \frac{\tilde{\varepsilon} \delta(\tilde{\varepsilon})}{2}, M = \left\lceil \log_{1-\delta(\tilde{\varepsilon})} \left(\frac{\tilde{\varepsilon}}{2b} \right) \right\rceil \text{ and} \\ \text{for all } p \in \mathbb{N}, \quad &\begin{cases} f_0(p) = \max \left\{ f \left(\sigma_1[\tilde{A}_\varepsilon, b] \left(\frac{\varepsilon}{3}, p \right) \right), \sigma_1[\tilde{A}_\varepsilon, b] \left(\frac{\varepsilon}{3}, p \right) \right\}, \\ f_{m+1}(p) = \max \{ f_0(p), \theta'_b(\varepsilon_0, f_m, p) \} \end{cases} \text{ for } m < M. \end{aligned}$$

Then for (x_n) satisfying (6) we have

$$\begin{aligned} \forall N \in \mathbb{N} \forall \varepsilon > 0 \forall f : \mathbb{N} \rightarrow \mathbb{N} \exists n \in [N, \Psi(\varepsilon, f, N)] \exists k \in [N, n], z \in C \\ (d(z, w_k(\phi(z))) \leq \varepsilon \wedge \forall i \in [n, f(n)] (d(x_i, z) \leq \frac{\varepsilon}{2})). \end{aligned}$$

In particular, $\Psi(\varepsilon, f, 0)$ is a rate of metastability for (x_n) .

Proof. Let $N \in \mathbb{N}$, $\varepsilon > 0$ and $f : \mathbb{N} \rightarrow \mathbb{N}$ be given. From the definition of the natural number M , we have $(1 - \delta(\tilde{\varepsilon}))^M \leq \frac{\tilde{\varepsilon}}{2b}$. Consider the functions f_0, \dots, f_M as in the theorem. Define $z_0 = x_0$ and $n_0 = N$. For $m \leq M$, assume that z_m and n_m are already defined and let $(w_n^m)_n$ denote the (S_n) -Halpern

sequence with anchor and starting point $\phi(z_m)$, i.e. $w_n^m = w_n(\phi(z_m))$. By $(H[S_n])$, we consider $n_{m+1} \in [n_m, \theta'_b(\varepsilon_0, f_{M-m}, n_m)]$ such that

$$\forall i \in [n_{m+1}, f_{M-m}(n_{m+1})] \left(d(w_i^m, w_{n_{m+1}}^m) \leq \varepsilon_0 \right),$$

and define $z_{m+1} = w_{n_{m+1}}^m$. Notice that for $m \leq M-1$,

$$n_{m+2} \in [n_{m+1}, \theta'_b(\varepsilon_0, f_{M-m-1}, n_{m+1})] \subseteq [n_{m+1}, f_{M-m}(n_{m+1})]$$

and so $d(w_{n_{m+2}}^m, z_{m+1}) = d(w_{n_{m+2}}^m, w_{n_{m+1}}^m) \leq \varepsilon_0$.

We start by showing that there is $m_0 \leq M$ such that $d(z_{m_0+1}, z_{m_0}) \leq \tilde{\varepsilon}$. Assume that for all $m \leq M-1$, $d(z_{m+1}, z_m) > \tilde{\varepsilon}$. For $m \in [1, M]$,

$$d(z_{m+1}, z_m) \leq d(w_{n_{m+1}}^m, w_{n_{m+1}}^{m-1}) + d(w_{n_{m+1}}^{m-1}, w_{n_m}^{m-1}) \leq d(w_{n_{m+1}}^m, w_{n_{m+1}}^{m-1}) + \varepsilon_0,$$

and by induction on n , we have (using (W4) and the nonexpansivity of S_n)

$$d(w_n^m, w_n^{m-1}) \leq (1 - \delta(\tilde{\varepsilon})) d(z_m, z_{m-1}).$$

Hence,

$$\begin{aligned} d(z_{m+1}, z_m) &\leq (1 - \delta(\tilde{\varepsilon})) d(z_m, z_{m-1}) + \varepsilon_0 \\ &\leq (1 - \delta(\tilde{\varepsilon}))^m d(z_1, z_0) + \varepsilon_0 \sum_{i=0}^{m-1} (1 - \delta(\tilde{\varepsilon}))^i, \end{aligned}$$

and we conclude,

$$d(z_{M+1}, z_M) \leq (1 - \delta(\tilde{\varepsilon}))^M b + \frac{\varepsilon_0}{\delta(\tilde{\varepsilon})} \leq \frac{\tilde{\varepsilon}}{2b} b + \frac{\tilde{\varepsilon}}{2} = \tilde{\varepsilon}.$$

Thus, we can take $m_0 \leq M$ such that $d(z_{m_0+1}, z_{m_0}) \leq \tilde{\varepsilon}$. Write $w_n \equiv w_n^{m_0}$, $z = z_{m_0+1} = w_{n_{m_0+1}}^{m_0}$ and $\sigma_1 = \sigma_1[\tilde{A}_\varepsilon, b](\frac{\varepsilon}{3}, n_{m_0+1}) \geq n_{m_0+1}$. Assume towards a contradiction that

$$\forall i \in [n_{m_0+1}, \sigma_1] \left(d(x_i, w_i) > \frac{\varepsilon}{3} \right).$$

For all $i \in [n_{m_0+1}, \sigma_1]$, by (W4)

$$\begin{aligned} d(x_{i+1}, w_{i+1}) &\leq (1 - \alpha_i) d(S_i(x_i), S_i(w_i)) + \alpha_i d(\phi(x_i), \phi(z_{m_0})) \\ &\leq (1 - \alpha_i) d(x_i, w_i) + \alpha_i (d(\phi(x_i), \phi(w_i)) + d(\phi(w_i), \phi(z)) + d(\phi(z), \phi(z_{m_0}))) \\ &\leq (1 - \delta(\varepsilon/3) \alpha_i) d(x_i, w_i) + \delta(\varepsilon/3) \alpha_i \frac{d(w_i, z) + d(z, z_{m_0})}{\delta(\varepsilon/3)} \\ &\leq (1 - \tilde{\alpha}_i) d(x_i, w_i) + \tilde{\alpha}_i b_i, \end{aligned}$$

where $\tilde{\alpha}_i = \delta(\varepsilon/3) \alpha_i$ and $b_i = \frac{d(w_i, z) + d(z, z_{m_0})}{\delta(\varepsilon/3)}$.

Consider the following easy remarks:

- (1) Since A is a rate of divergence for $\sum \alpha_i = \infty$, we have that \tilde{A}_ε is a rate of divergence for $\sum \tilde{\alpha}_i = \sum \delta(\varepsilon/3)\alpha_i = \infty$.
- (2) From the monotonicity of the function A it follows that the function $\sigma_1[\tilde{A}_\varepsilon, b](\frac{\varepsilon}{3}, \cdot)$ is also monotone.
- (3) The monotonicity of the function θ'_b entails that $\forall m \leq M+1$ ($n_m \leq \Psi_m$).
- (4) By construction, we have $n_{m_0+1} \leq n_{M+1}$.

From the previous points, we derive $\sigma_1 \leq \Psi(\varepsilon, f, N)$. By the definition of the functions f_m , it follows $\sigma_1 \leq f_{M-m_0}(n_{m_0+1})$. Hence

$$\forall i \in [n_{m_0+1}, \sigma_1] \left(d(w_i, z) \leq \varepsilon_0 \leq \frac{\tilde{\varepsilon}}{2} \right),$$

and so,

$$b_i \leq \frac{\tilde{\varepsilon}/2 + \tilde{\varepsilon}}{\delta(\varepsilon/3)} = \frac{\varepsilon}{6} = \frac{(\varepsilon/3)}{2}.$$

By Lemma 2.10, we have that $d(x_{\sigma_1}, w_{\sigma_1}) \leq \frac{\varepsilon}{3}$, which contradicts the assumption since $\sigma_1 \geq n_{m_0+1}$. Consider i_0 such that

$$i_0 \in [n_{m_0+1}, \sigma_1] \text{ and } d(x_{i_0}, w_{i_0}) \leq \frac{\varepsilon}{3}.$$

Note that for all $u, v \in C$, it holds $d(\phi(u), \phi(v)) \leq \max\{(1 - \delta(\varepsilon/3))d(u, v), \varepsilon/3\}$. For any $i \in [i_0, f_{M-m_0}(n_{m_0+1}) - 1]$ such that $d(x_i, w_i) \leq \frac{\varepsilon}{3}$, by (W4) and the nonexpansivity of S_i , we have

$$\begin{aligned} d(x_{i+1}, w_{i+1}) &\leq (1 - \alpha_i)d(x_i, w_i) + \alpha_i d(\phi(x_i), \phi(z_{m_0})) \\ &\leq (1 - \alpha_i)d(x_i, w_i) + \alpha_i \max\left\{\left(1 - \delta\left(\frac{\varepsilon}{3}\right)\right)d(x_i, z_{m_0}), \frac{\varepsilon}{3}\right\} \\ &= \max\left\{(1 - \alpha_i)d(x_i, w_i) + \alpha_i\left(1 - \delta\left(\frac{\varepsilon}{3}\right)\right)d(x_i, z_{m_0}), (1 - \alpha_i)d(x_i, w_i) + \alpha_i\frac{\varepsilon}{3}\right\} \\ &\leq \max\left\{\left(1 - \delta\left(\frac{\varepsilon}{3}\right)\alpha_i\right)d(x_i, w_i) + \alpha_i\delta\left(\frac{\varepsilon}{3}\right)\left(1 - \delta\left(\frac{\varepsilon}{3}\right)\right)\frac{d(w_i, z) + d(z, z_{m_0})}{\delta(\frac{\varepsilon}{3})}, \right. \\ &\quad \left.(1 - \alpha_i)d(x_i, w_i) + \alpha_i\frac{\varepsilon}{3}\right\} \\ &\leq \max\left\{(1 - \tilde{\alpha}_i)\frac{\varepsilon}{3} + \tilde{\alpha}_i\frac{\varepsilon}{3}, (1 - \alpha_i)\frac{\varepsilon}{3} + \alpha_i\frac{\varepsilon}{3}\right\} = \frac{\varepsilon}{3}, \end{aligned}$$

using the inequalities $d(x_i, w_i) \leq \frac{\varepsilon}{3}$, $1 - \delta(\varepsilon/3) \leq 1$ and

$$\frac{d(w_i, z) + d(z, z_{m_0})}{\delta(\varepsilon/3)} \leq \frac{\varepsilon_0 + \tilde{\varepsilon}}{\delta(\varepsilon/3)} \leq \frac{\varepsilon}{3}.$$

Hence, $d(x_i, w_i) \leq \frac{\varepsilon}{3}$ for all $i \in [i_0, f_{M-m_0}(n_{m_0+1})]$ and in particular, the inequality holds for $i \in [\sigma_1, f(\sigma_1)]$ since by the definition of f_0 one has that

$f(\sigma_1) \leq f_{M-m_0}(n_{m_0} + 1)$. The theorem is now satisfied with $n := \sigma_1$, $k := n_{m_0+1}$ and z as defined above. Indeed, since $d(z, z_{m_0}) = d(z_{m_0+1}, z_{m_0}) \leq \varepsilon$ and $[n, f(n)] \subseteq [n_{m_0+1}, f_{M-m_0}(n_{m_0+1})]$, using (W4) and the nonexpansivity of $S_{n_{m_0+1}}$ and ϕ , we have

$$d(z, w_k(\phi(z))) = d(w_{n_{m_0+1}}(\phi(z_{m_0})), w_{n_{m_0+1}}(\phi(z_{m_0+1}))) \leq \varepsilon$$

and for all $i \in [n, f(n)]$

$$d(x_i, z) \leq d(x_i, w_i) + d(w_i, z) \leq \frac{\varepsilon}{3} + \varepsilon_0 \leq \frac{\varepsilon}{2}. \quad \square$$

Remark 3.11. As in Remark 3.6, it follows from Theorem 3.10 in an elementary way that (for X being complete and C being closed) the limit $x := \lim x_n$ is the unique fixed point of $P \circ \phi$, where $P(u) := \lim w_n(u)$.

Remark 3.12. As in Remark 3.7, instead of assuming that C is bounded and working with a bound on its diameter, we can adapt Theorem 3.10 to the situation where we only have $b \in \mathbb{N}^*$ satisfying $b \geq \max\{d(x_0, w_n(\phi(x_0))) : n \in \mathbb{N}\}$. Now consider a function θ'_r satisfying the formula (H[S_n]) only for anchor points $u \in C \cap \overline{B}_r(x_0)$. We also require bounding information on the initial displacement of the relevant mappings, i.e. for each $n \in \mathbb{N}$ let $c_n \in \mathbb{N}$ be such that

$$c_n \geq \max\{d(x_0, \phi(x_0)), d(x_0, S_n(x_0))\}.$$

In this case, the bound takes the form $\Psi(\varepsilon, f, N) = \sigma_1[\tilde{A}_\varepsilon, b'](\frac{\varepsilon}{3}, \Psi_{M+1})$, where $\tilde{\varepsilon}, \varepsilon_0, \tilde{A}_\varepsilon$ and M are as before,

$$\Psi_0 = N \text{ and } \Psi_{m+1} = \theta'_{(m+1)b}(\varepsilon_0, f_{M-m}, \Psi_m),$$

$$b' = (M+1) \cdot b + \sum_{k=0}^{\Psi_{M+1}-1} c_k,$$

and for all $p \in \mathbb{N}$,

$$\begin{cases} f_0(p) = \max\{f(\sigma_1[\tilde{A}_\varepsilon, b'](\frac{\varepsilon}{3}, p)), \sigma_1[\tilde{A}_\varepsilon, b'](\frac{\varepsilon}{3}, p)\}, \\ f_{m+1}(p) = \max\{f_0(p), \theta'_{(M-m+1)b}(\varepsilon_0, f_m, p)\} \end{cases} \quad \text{for } m < M.$$

We can define n_m and z_m in a way similar to Remark 3.7. Since it still holds that $(1 - \delta(\tilde{\varepsilon}))^M d(z_1, z_0) \leq \tilde{\varepsilon}/2$, we can conclude the existence of z_{m_0} satisfying $d(z_{m_0+1}, z_{m_0}) \leq \tilde{\varepsilon}$. The next argument requires an application of Lemma 2.10. Let $w_n \equiv w_n^{m_0}$ be as before, write $\sigma_1 = \sigma_1[\tilde{A}_\varepsilon, b'](\frac{\varepsilon}{3}, n_{m_0+1})$ and assume that

$$\forall i \in [n_{m_0+1}, \sigma_1] \quad (d(x_i, w_i) > \frac{\varepsilon}{3}).$$

To apply the proof of Lemma 2.10, it is enough to see that $d(x_i, w_i) \leq b'$ for all $i \in [n_{m_0+1}, \sigma_1 + 1]$. First, inductively it holds $d(x_i, w_i) \leq d(x_{n_{m_0+1}}, w_{n_{m_0+1}})$,

for $i \in [n_{m_0+1}, \sigma_1 + 1]$. The base case is trivial. In the induction step we argue as before: For $i \in [n_{m_0+1}, \sigma_1]$, we have

$$\begin{aligned} d(x_{i+1}, w_{i+1}) &\leq (1 - \tilde{\alpha}_i)d(x_i, w_i) + \tilde{\alpha}_i b_i \\ &\leq (1 - \tilde{\alpha}_i)d(x_i, w_i) + \tilde{\alpha}_i d(x_{n_{m_0+1}}, w_{n_{m_0+1}}) \\ &\leq d(x_{n_{m_0+1}}, w_{n_{m_0+1}}), \end{aligned}$$

using the induction hypothesis and $b_i \leq \varepsilon/6 < \varepsilon/3 < d(x_{n_{m_0+1}}, w_{n_{m_0+1}})$. Now by induction one shows, using (W1) and the nonexpansivity of S_n and ϕ , that $d(x_0, x_n) \leq \sum_{k=0}^{n-1} c_k$, for all $n \in \mathbb{N}$. Then, from $n_{m_0+1} \leq n_{M+1} \leq \Psi_{M+1}$ and $d(x_0, z_{m_0+1}) \leq (m_0 + 1)b \leq (M + 1)b$, we conclude

$$d(x_{n_{m_0+1}}, w_{n_{m_0+1}}) \leq d(x_0, x_{n_{m_0+1}}) + d(x_0, z_{m_0+1}) \leq b'.$$

The remainder of the proof follows unchanged.

An application of Theorem 3.10 gives a rate of metastability for the Halpern-viscosity iteration from a function θ'_b satisfying (H[T]).

Corollary 3.13. *Let C be a nonempty bounded convex subset of a hyperbolic space and let $b \in \mathbb{N}^*$ be a bound on the diameter of C . Consider T a nonexpansive map on C , an r -contraction $\phi : C \rightarrow C$ with $\delta \in (0, 1)$ such that $r \leq 1 - \delta$, and a sequence $(\alpha_n) \subseteq [0, 1]$ satisfying $\sum \alpha_n = \infty$ with a monotone rate of divergence A . Let θ'_b be a monotone function satisfying (H[T]) and σ_1 be as in Lemma 2.9. Then (x_n) defined by*

$$x_0 \in C, \quad x_{n+1} = (1 - \alpha_n)T(x_n) \oplus \alpha_n \phi(x_n), \quad n \in \mathbb{N},$$

is a Cauchy sequence with metastability rate

$$\Psi(\varepsilon, f) = \Psi[b, \delta, A, \theta'_b](\varepsilon, f) = \sigma_1[\tilde{A}, b] \left(\frac{\varepsilon}{3}, \Psi_{M+1} \right)$$

where

$$\begin{aligned} \tilde{A}(k) &:= A \left(\left\lceil \frac{k}{\delta} \right\rceil \right) \text{ for all } k \in \mathbb{N} \\ \Psi_0 &= 0 \text{ and } \Psi_{m+1} = \theta'_b(\varepsilon_0, f_{M-m}, \Psi_m), \text{ with} \\ \varepsilon_0 &= \frac{\varepsilon \delta^2}{18}, \quad M = \left\lceil \log_{1-\delta} \left(\frac{\varepsilon \delta}{18b} \right) \right\rceil \text{ and} \\ \text{for all } p \in \mathbb{N}, \quad &\begin{cases} f_0(p) = f \left(\sigma_1[\tilde{A}, b] \left(\frac{\varepsilon}{3}, p \right) \right), \\ f_{m+1}(p) = \max\{f_0(p), \theta'_b(\varepsilon_0, f_m, p)\} \end{cases} \text{ for } m < M. \end{aligned}$$

Just as in the case of Corollary 3.9 we obtain the following consequence of Corollary 3.13:

Corollary 3.14. *Let $X, C, b, T, \phi, r, \delta, (\alpha_n), A, (x_n), \sigma_1$ be as in Corollary 3.13. Let $\rho(\varepsilon)$ be a common (for all anchor points $u \in C$) Cauchy rate for the sequence $(w_n(u))$ of Halpern iterations of T . Then*

$$\Psi(\varepsilon, f) = \Psi[b, \delta, A, \rho](\varepsilon, f) = \sigma_1[\tilde{A}, b] \left(\frac{\varepsilon}{3}, \rho \left(\frac{\varepsilon \delta^2}{18} \right) \right)$$

with $\tilde{A}(k) := A \left(\lceil \frac{k}{\delta} \rceil \right)$, is a Cauchy rate for (x_n) .

Rates of metastability for the Halpern iterations have been extracted before. In the setting of CAT(0) spaces, under appropriate conditions, rates of metastability were obtained in [33]. The convergence of Halpern iterations was also studied by proof mining methods in the setting of Banach spaces. The results [34] in and [37] (under different sets of conditions, both including the natural choice $\alpha_n = \frac{1}{n+1}$) extracted a transformation of a rate of metastability for a certain Browder type sequence into a rate of metastability for the Halpern iteration. Thus, the recent rate obtained in [36] entails a rates of metastability for Halpern iterations in the setting of Banach spaces that are simultaneously uniformly convex and uniformly smooth (see [36, Theorem 7.1]). Subsequently, by an application of Theorem 3.10, instantiated with the corresponding rates of metastability for Halpern iterations, we obtain rates of metastability for the viscosity-Halpern iterations in the setting of CAT(0) spaces and in the setting of Banach spaces that are both uniformly convex and uniformly smooth.

In Hilbert spaces, a different generalization of Wittmann's theorem was that of Bauschke [4] to a finite family of nonexpansive maps T_1, \dots, T_ℓ satisfying:

$$\bigcap_{i=1}^{\ell} \text{Fix}(T_i) = \text{Fix}(T_\ell \cdots T_1) = \cdots = \text{Fix}(T_{\ell-1} \cdots T_1 T_\ell) \quad (+)$$

Bauschke's schema is the particular case of (4) when one considers the sequence of nonexpansive maps (S_n) defined cyclically by $S_n := T_{[n]+1}$, for $n \in \mathbb{N}$, where $[n] := n \bmod \ell$. Similarly to Wittmann's proof, Bauschke's generalization also follows from a sequential weak compactness argument. In [13], a theoretical approach to eliminate this principle was developed, and subsequently a quantitative version of Bauschke's theorem was obtained. From an application of Theorem 3.10, we then obtain a rate of metastability for a viscosity version of Bauschke's schema. In [38, 39], Körnlein also obtained rates of metastability for the viscosity version of Bauschke's schema. The rates obtained are different, since Körnlein's extraction follows from a direct proof (i.e. without a reduction to Bauschke's theorem). In the end, however, both rates look to be of a similar complexity: Körnlein still needed to carry out a convoluted quantitative treatment for the existence of (a proxy to) $z = P_F \phi z$ (see [38, Problem 8.4.4.]). Körnlein also considered Suzuki's comment on Bauschke's condition stated in [52]: without any commutativity assumption, the condition (+) already follows from having $\bigcap_{i=1}^{\ell} \text{Fix}(T_i) = \text{Fix}(T_\ell \cdots T_1)$. In [38, Theorem 8.7.1], Körnlein proved a finitary version of this remark, and thus [13, Theorem 6.10] can easily

be adapted to the situation where the function τ instead satisfies

$$\|x - T_\ell \cdots T_1(x)\| \leq \frac{1}{\tau(k) + 1} \rightarrow \forall i < \ell \|x - S_i(x)\| \leq \frac{1}{k + 1}.$$

Our result entails the strong convergence of the viscosity version of Bauschke's schema with a Rakotch map towards a common fixed point. In [20], Jung extended the convergence of the viscosity approximation method of finite families to the setting of (suitable) Banach spaces under a slightly more general condition than (C3). Thus, we obtained a rate of metastability for Jung's schema in Hilbert spaces under the slightly stronger condition but for more general contractions ϕ .

As a last application we look at proximal point algorithms, a setting in which sequences of nonexpansive maps are also naturally considered. Let X be a Hilbert space, and $A : X \rightarrow 2^X$ a monotone operator, i.e. $\langle x - x', y - y' \rangle \geq 0$ for any $y \in A(x), y' \in A(x')$. Furthermore, assume that A is maximal in the sense that its graph is not properly contained in the graph of any other monotone operator. For a real number $\lambda > 0$, the resolvent function $J_\lambda : X \rightarrow X$ is the single-valued function $(\text{Id} + \lambda A)^{-1}$. It is well known that J_λ is nonexpansive and its fixed point set coincides with the set of zeros of A . First introduced by Martinet in [41], the so-called proximal point algorithm (PPA) is defined by $x_{n+1} = J_{\lambda_n}(x_n)$, where (λ_n) is a sequence of positive real numbers. It is known that the PPA is only weakly convergent ([49] and [17]), and so modifications to the algorithm were proposed. Introduced in [22], and independently in [57], we recall one such generalization: For $u, w_0 \in X$, sequences of real numbers $(\alpha_n) \subseteq [0, 1]$ and $(\lambda_n) \subseteq \mathbb{R}^+$, the iteration (w_n) defined by

$$w_{n+1} = \alpha_n u + (1 - \alpha_n) J_{\lambda_n}(w_n), \quad (\text{HPPA})$$

is the Halpern type proximal point algorithm. Note that when $w_0 = u$ this iteration is the particular case of (4) with $S_n = J_{\lambda_n}$ for all $n \in \mathbb{N}$.

Quantitative versions of the strong convergence of this iteration were carried out in [45], [40] and [12]. Recently a generalization of (HPPA) to Banach spaces and accretive operators was studied in [1] and a rate of metastability was subsequently obtained in [30]. Pertaining to the Halpern type proximal point algorithm, this result is the most general so far, as strong convergence is established only asking that $(\alpha_n) \subseteq (0, 1]$ be a sequence slowly converging to zero and that $\inf \lambda_n > 0$. (The rate extracted in [30], receives as an input a rate of metastability for a Browder type sequence, which can be instantiated with that of [36].) Hence, Theorem 3.10 entails a rate of metastability for a viscosity version of the Halpern type proximal point algorithm under appropriate conditions (see also [2]).

3.4 Hybrid viscosity KM-iteration

In this subsection, we give a rate of metastability for the vKM iteration (8), provided one has a rate of metastability for Browder-type sequences. Our approach makes use of a quantitative analysis of Theorem 5 in [59] (for the case

with no error terms). The following result computes a rate of metastability for the vKM iteration from a rate of metastability for viscosity-Browder sequences.

Theorem 3.15. *Let C be a nonempty bounded convex subset of a hyperbolic space and let $b \in \mathbb{N}^*$ be a bound on the diameter of C . Consider T a nonexpansive map on C , an r -contraction $\phi : C \rightarrow C$ with $\delta \in (0, 1)$ such that $r \leq 1 - \delta$, and sequences $(\alpha_n), (\beta_n) \subseteq (0, 1]$ satisfying $\sum_{n=1}^{\infty} \alpha_n \beta_n = \infty$ with a monotone rate of divergence μ_1 , and $\lim \frac{|\alpha_n - \alpha_{n-1}|}{\alpha_n^2 \beta_n} = 0$ with a monotone rate of convergence μ_2 . Let (\tilde{x}_n) be the viscosity-Browder sequence defined with T, ϕ and the sequence (α_n) and (x_n) be defined by (8). Then $\lim d(x_{n+1}, \tilde{x}_n) = 0$ with rate of convergence*

$$\Xi(\varepsilon) := \sigma_1[\widetilde{\mu_1}, b] \left(\varepsilon, \mu_2 \left(\frac{\delta^2 \varepsilon}{2b} \right) \right), \text{ where}$$

$$\sigma_1 \text{ is as in Lemma 2.9, and } \widetilde{\mu_1}(k) := \mu_1 \left(\left\lceil \frac{k}{\delta} \right\rceil \right).$$

In particular, if (\tilde{x}_n) is a Cauchy sequence with rate of metastability Ψ then (x_n) is a Cauchy sequence with metastability rate

$$\Omega(\varepsilon, f) = \Omega[b, \delta, \Psi, \mu_1, \mu_2](\varepsilon, f) = \max\{\Xi(\varepsilon/3), \Psi(\varepsilon/3, \hat{f})\} + 1, \text{ where}$$

$$\hat{f}(n) := f(\max\{\Xi(\varepsilon/3), n\} + 1).$$

Proof. Let (\tilde{x}_n) denote for the viscosity-Browder sequence, i.e.

$$\tilde{x}_n = (1 - \alpha_n)T(\tilde{x}_n) \oplus \alpha_n \phi(\tilde{x}_n), \quad n \in \mathbb{N}.$$

Using (W2) and (W4),

$$\begin{aligned} d(\tilde{x}_{n+1}, \tilde{x}_n) &\leq d(\tilde{x}_{n+1}, (1 - \alpha_{n+1})T(\tilde{x}_n) \oplus \alpha_{n+1}\phi(\tilde{x}_n)) \\ &\quad + d((1 - \alpha_{n+1})T(\tilde{x}_n) \oplus \alpha_{n+1}\phi(\tilde{x}_n), \tilde{x}_n) \\ &\stackrel{(W2)}{=} d(\tilde{x}_{n+1}, (1 - \alpha_{n+1})T(\tilde{x}_n) \oplus \alpha_{n+1}\phi(\tilde{x}_n)) \\ &\quad + |\alpha_{n+1} - \alpha_n| d(T(\tilde{x}_n), \phi(\tilde{x}_n)) \\ &\stackrel{(W4)}{\leq} (1 - \alpha_{n+1} + \alpha_{n+1}r) d(\tilde{x}_{n+1}, \tilde{x}_n) + |\alpha_{n+1} - \alpha_n| d(T(\tilde{x}_n), \phi(\tilde{x}_n)) \\ &\leq (1 - \delta \alpha_{n+1}) d(\tilde{x}_{n+1}, \tilde{x}_n) + |\alpha_{n+1} - \alpha_n| \cdot b, \end{aligned}$$

which implies

$$d(\tilde{x}_{n+1}, \tilde{x}_n) \leq \frac{b|\alpha_{n+1} - \alpha_n|}{\delta \alpha_{n+1}}.$$

Then, by (8) and using (W1) and (W4), we have for $n \geq 1$

$$\begin{aligned} d(x_{n+1}, \tilde{x}_n) &\stackrel{W1}{\leq} (1 - \beta_n) d(x_n, \tilde{x}_n) + \beta_n d((1 - \alpha_n)T(x_n) \oplus \alpha_n \phi(x_n), \tilde{x}_n) \\ &\stackrel{W4}{\leq} (1 - \beta_n) d(x_n, \tilde{x}_n) + \beta_n ((1 - \alpha_n) d(x_n, \tilde{x}_n) + \alpha_n r d(x_n, \tilde{x}_n)) \\ &\leq (1 - \delta \alpha_n \beta_n) d(x_n, \tilde{x}_n) \leq (1 - \lambda_n) d(x_n, \tilde{x}_{n-1}) + \lambda_n \gamma_n, \end{aligned}$$

where $\lambda_n = \delta\alpha_n\beta_n$ and $\gamma_n = \frac{b|\alpha_n - \alpha_{n-1}|}{\delta^2\alpha_n^2\beta_n}$.

Since μ_1 is a rate of divergence for $(\sum \alpha_n\beta_n)$, we have that $\widetilde{\mu}_1$ is a rate of divergence for $(\sum \lambda_n)$. Hence by Lemma 2.9 (for an arbitrary p), we conclude that $d(x_{n+1}, \tilde{x}_n)$ converges to zero with rate of convergence Ξ .

Let $\varepsilon > 0$ and $f : \mathbb{N} \rightarrow \mathbb{N}$ be given. Since Ψ is a rate of metastability for (\tilde{x}_n) , there exists $n' \leq \Psi(\varepsilon/3, \hat{f})$ be such that $d(\tilde{x}_i, \tilde{x}_j) \leq \frac{\varepsilon}{3}$, for $i, j \in [n', \hat{f}(n')]$. Take $n := \max\{\Xi(\varepsilon/3), n'\}$. We have $n \leq \Omega(\varepsilon, f) - 1$ and for $i, j \in [n, f(n+1)]$

$$d(x_{i+1}, x_{j+1}) \leq d(x_{i+1}, \tilde{x}_i) + d(\tilde{x}_i, \tilde{x}_j) + d(x_{j+1}, \tilde{x}_j) \leq \varepsilon,$$

since $[n, f(n+1)] \subseteq [n', \hat{f}(n')]$. This entails the result. \square

Due to Corollary 3.8, we can just use a rate of metastability for the Browder sequence in the previous result (see also Lemma 3.3).

Corollary 3.16. *Let $X, C, b, T, \phi, r, \delta, (\alpha_n), (\beta_n), \mu_1, \mu_2, (x_n), \sigma_1$ be as in Theorem 3.15. Let θ_b be a monotone function satisfying (B[T]) for (y_n) the Browder sequence defined with T and the sequence (α_n) . Then (x_n) defined by (8) is a Cauchy sequence with metastability rate*

$$\Omega(\varepsilon, f) = \Omega[b, \delta, \Psi, \mu_1, \mu_2](\varepsilon, f)$$

where $\Psi = \Psi[b, \delta, \theta_b]$ is as in Corollary 3.8 and Ω as in Theorem 3.15.

As before we also get the following corollary

Corollary 3.17. *Let $X, C, b, T, \phi, r, \delta, (\alpha_n), (\beta_n), \mu_1, \mu_2, (x_n), \sigma_1$ be as in Theorem 3.15. Let $\rho(\varepsilon)$ be a common (for all anchor points $u \in C$) rate of convergence for the Browder sequence $(y_n(u))$ (w.r.t. (α_n)) of T . Then (x_n) defined by (8) is a Cauchy sequence with Cauchy rate*

$$\Omega(\varepsilon) = \Omega[b, \delta, \theta_b, \mu_1, \mu_2](\varepsilon) = \max\{\Xi(\varepsilon/3), \Psi(\varepsilon/3)\} + 1,$$

where

$$\Xi(\varepsilon) := \sigma_1[\widetilde{\mu}_1, b] \left(\varepsilon, \mu_2 \left(\frac{\delta^2 \varepsilon}{2b} \right) \right),$$

with $\widetilde{\mu}_1(k) := \mu_1(\lceil \frac{k}{\delta} \rceil)$, σ_1 as in Lemma 2.9 and $\Psi = \Psi[b, \delta, \theta_b]$ as in Corollary 3.9.

3.5 Inexact algorithms

In many applications of the algorithms studied above it is useful to allow for the iteration to contain error terms. In this subsection we provide rates of metastability of such relaxed versions under appropriate assumptions on the sequence of error terms. In the following, consider (S_n) a family of nonexpansive maps, ϕ an r -contraction, $(\alpha_n) \subseteq (0, 1]$ and $(\varepsilon_n) \subseteq \mathbb{R}_0^+$ a sequence of allowed errors. Assume that $\delta \in (0, 1)$ is such that $r \leq 1 - \delta$, and $b \in \mathbb{N}^*$ is a bound on the diameter of C .

Proposition 3.18. *Let (x_n) be a (S_n) -viscosity Browder sequence (5) and (x'_n) be a sequence in C satisfying for all $n \in \mathbb{N}$,*

$$d(x'_n, (1 - \alpha_n)S_n(x'_n) \oplus \alpha_n\phi(x'_n)) \leq \varepsilon_n.$$

- (i) *If $\lim \frac{\varepsilon_n}{\alpha_n} = 0$ with a rate of convergence ρ , then $\lim d(x_n, x'_n) = 0$ with rate of convergence $\rho_\delta(\varepsilon) := \rho(\varepsilon\delta)$.*
- (ii) *If (x_n) is a Cauchy sequence with rate of metastability Ψ , then (x'_n) is also a Cauchy sequence with metastability rate*

$$\Psi'(\varepsilon, f) = \theta[\Psi](\varepsilon/3, f, \rho_\delta(\varepsilon/3)),$$

where $\theta[\Psi]$ is as in Lemma 3.3.

Proof. For (i), using (W4) we have

$$\begin{aligned} d(x_n, x'_n) &\leq (1 - \alpha_n)d(S_n(x_n), S_n(x'_n)) + \alpha_n d(\phi(x_n), \phi(x'_n)) + \varepsilon_n \\ &\leq (1 - \alpha_n)d(x_n, x'_n) + \alpha_n r d(x_n, x'_n) + \varepsilon_n. \end{aligned}$$

So $d(x_n, x'_n) \leq \frac{\varepsilon_n}{\delta\alpha_n}$ and ρ_δ is a rate of convergence for $\lim d(x_n, x'_n) = 0$.

For (ii), let $\varepsilon > 0$ and $f : \mathbb{N} \rightarrow \mathbb{N}$ be given. By Lemma 3.3, there is $n \in [\rho_\delta(\varepsilon/3), \Psi'(\varepsilon, f)]$ such that $d(x_i, x_j) \leq \varepsilon/3$, for all $i, j \in [n, f(n)]$. Since $n \geq \rho_\delta(\varepsilon/3)$, by part (i) we get for all $i, j \in [n, f(n)]$

$$d(x'_i, x'_j) \leq d(x'_i, x_i) + d(x_i, x_j) + d(x_j, x'_j) \leq \varepsilon,$$

which concludes (ii). \square

Similarly one can study the metastability of a relaxed Halpern type iteration.

Proposition 3.19. *Let (x_n) be a (S_n) -viscosity Halpern iteration (6) and (x'_n) be a sequence in C satisfying $x'_0 = x_0$ and for all $n \in \mathbb{N}$,*

$$d(x'_{n+1}, (1 - \alpha_n)S_n(x'_n) \oplus \alpha_n\phi(x'_n)) \leq \varepsilon_n.$$

- (i) *If $\sum \alpha_n = \infty$ with rate of divergence A and $\lim \frac{\varepsilon_n}{\alpha_n} = 0$ with rate of convergence ρ , then $\lim d(x_n, x'_n) = 0$ with rate of convergence*

$$\Gamma(\varepsilon) := \sigma_1[\tilde{A}, b] \left(\varepsilon, \rho \left(\frac{\delta\varepsilon}{2} \right) \right),$$

where σ_1 is as in Lemma 2.9 and $\tilde{A}(k) := A(\lceil \frac{k}{\delta} \rceil)$.

- (ii) *If (x_n) is a Cauchy sequence with rate of metastability Ψ , then (x'_n) is also a Cauchy sequence with metastability rate*

$$\Psi'(\varepsilon, f) = \theta[\Psi](\varepsilon/3, f, \Gamma(\varepsilon/3)),$$

where $\theta[\Psi]$ is as in Lemma 3.3.

Proof. For part (i), we have for all $n \in \mathbb{N}$

$$\begin{aligned} d(x_{n+1}, x'_{n+1}) &\leq (1 - \alpha_n)d(S_n(x_n), S_n(x'_n)) + \alpha_n d(\phi(x_n), \phi(x'_n)) + \varepsilon_n \\ &\leq (1 - \alpha_n)d(x_n, x'_n) + \alpha_n r d(x_n, x'_n) + \varepsilon_n \\ &\leq (1 - \delta\alpha_n)d(x_n, x'_n) + \delta\alpha_n \frac{\varepsilon_n}{\delta\alpha_n}. \end{aligned}$$

Since $\sum \delta\alpha_n = \infty$ and $\lim \frac{\varepsilon_n}{\delta\alpha_n} = 0$, by Lemma 2.7 we conclude that $d(x_n, x'_n)$ converges to zero. From the fact that A is a rate of divergence for $(\sum \alpha_n)$ it follows that \tilde{A} is a rate of divergence for $(\sum \delta\alpha_n)$. From an application of Lemma 2.9 (for an arbitrary p) it follows that Γ is a rate of convergence towards zero for $d(x_n, x'_n)$. For (ii), one argues as in the proof of Proposition 3.18. \square

Remark 3.20. One can also conclude $\lim d(x_n, x'_n) = 0$ under the assumption $\sum \varepsilon_n < \infty$, using a different version of Xu's lemma – e.g. [57, Lemma 2.5]. In that case, a metastability rate can easily be obtained using a Cauchy rate for $(\sum \varepsilon_n)$ – see [40, Lemma 3.4] – instead of the convergence rate ρ .

We now consider additionally a sequence $(\beta_n) \subseteq (0, 1]$ and discuss the metastability of the relaxed version of vKM.

Proposition 3.21. Let (x_n) be a vKM iteration given by (8) and (x'_n) be a sequence in C satisfying $x'_0 = x_0$ and for all $n \in \mathbb{N}$,

$$d(x'_{n+1}, (1 - \beta_n)x'_n \oplus \beta_n((1 - \alpha_n)T(x'_n) \oplus \alpha_n\phi(x'_n))) \leq \varepsilon_n.$$

- (i) If $\sum \alpha_n \beta_n = \infty$ with rate of divergence A and $\lim \frac{\varepsilon_n}{\alpha_n \beta_n} = 0$ with rate of convergence ρ , then $\lim d(x_n, x'_n) = 0$ with rate of convergence

$$\Gamma(\varepsilon) := \sigma_1[\tilde{A}, b] \left(\varepsilon, \rho \left(\frac{\delta\varepsilon}{2} \right) \right),$$

where σ_1 is as in Lemma 2.9 and $\tilde{A}(k) := A(\lceil \frac{k}{\delta} \rceil)$.

- (ii) If (x_n) is a Cauchy sequence with rate of metastability Ψ , then (x'_n) is also a Cauchy sequence with metastability rate

$$\Psi'(\varepsilon, f) = \theta[\Psi](\varepsilon/3, f, \Gamma(\varepsilon/3)),$$

where $\theta[\Psi]$ is as in Lemma 3.3.

Proof. The proof of this result is similar to that of Proposition 3.19. \square

Remark 3.22. Similar to Remark 3.20, it is also possible to conclude that $\lim d(x_n, x'_n) = 0$ under the assumption $\sum \varepsilon_n < \infty$, and to obtain a rate of metastability rate for (x'_n) using a Cauchy rate for $(\sum \varepsilon_n)$.

Furthermore, both Propositions 3.19 and 3.21 can be adapted to use a function A' satisfying (QII) instead of the rate of divergence A , in which case one makes use of the function σ_2 from Lemma 2.9, instead of the function σ_1 .

4 Rates of convergence

Let $(X, \|\cdot\|)$ be a normed linear space and $C \subseteq X$ some subset. Suppose that $T : C \rightarrow C$ has at most one fixed point, i.e.

$$\forall p_1, p_2 \in C (p_1 = Tp_1 \wedge p_2 = Tp_2 \rightarrow p_1 = p_2). \quad (\dagger)$$

We say that T has uniformly at most one fixed point with modulus of uniqueness $\omega : (0, \infty) \rightarrow (0, \infty)$ if

$$\forall \varepsilon > 0 \forall p_1, p_2 \in C (\|p_1 - Tp_1\|, \|p_2 - Tp_2\| \leq \omega(\varepsilon) \rightarrow \|p_1 - p_2\| \leq \varepsilon). \quad (\ddagger)$$

If T is continuous and C is compact, (\dagger) implies the existence of a modulus ω satisfying (\ddagger) but in general (\ddagger) is stronger than (\dagger) . However, for large classes of uniqueness proofs one can actually extract from a proof of (\dagger) an explicit effective modulus ω satisfying (\ddagger) . We refer to [28] for discussions on this topic. Suppose now that we have for some iterative algorithm $(x_n) \subseteq C$ with $\lim \|x_n - Tx_n\| = 0$ a rate $\tau : (0, \infty) \rightarrow \mathbb{N}$ of convergence, i.e.

$$\forall \varepsilon > 0 \forall n \geq \tau(\varepsilon) (\|x_n - Tx_n\| \leq \varepsilon).$$

If ω is a modulus of uniqueness for T in the sense above, then $\rho(\varepsilon) := \tau(\omega(\varepsilon))$ is a Cauchy rate for (x_n) . Hence if X is complete, C closed and T is continuous, then (x_n) converges to a (unique) fixed point p of T with rate of convergence ρ .

In the following we describe a class of nonexpansive operators T for which a modulus ω can be computed if X is uniformly convex. Consequently, we can then compute rates of convergence for (x_n) defined by Browder, Krasnoselskii-Mann or Halpern iterations.

Definition 4.1. *An operator $A : D(A) \rightarrow 2^X$ is accretive if for all $u \in Ax$ and $v \in Ay$ there exists some $j \in J(x - y)$ such that $\langle u - v, j \rangle \geq 0$, where J is the normalized duality mapping of X .*

Various strengthened forms of this notion have been considered in the literature which guarantee that A has at most one zero (see e.g. [14]):

Definition 4.2. *(i) Let $\psi : [0, \infty) \rightarrow [0, \infty)$ be a continuous function with $\psi(0) = 0$ and $\psi(x) > 0$ for $x > 0$. Then an operator $A : D(A) \rightarrow 2^X$ is ψ -strongly accretive if*

$$\forall (x, u), (y, v) \in A \exists j \in J(x - y) (\langle u - v, j \rangle \geq \psi(\|x - y\|) \|x - y\|).$$

(ii) Let $\phi : [0, \infty) \rightarrow [0, \infty)$ be a continuous function with $\phi(0) = 0$ and $\phi(x) > 0$ for $x > 0$. Then an operator $A : D(A) \rightarrow 2^X$ is uniformly ϕ -accretive if

$$\forall (x, u), (y, v) \in A \exists j \in J(x - y) (\langle u - v, j \rangle \geq \phi(\|x - y\|)).$$

In the case of ψ -strongly accretive operators, ψ is often assumed to be strictly increasing in addition (see e.g. [59]).

In [31, Section 2.1] it has been exhibited that all what is needed to get a modulus of uniqueness for the property of being a zero of A is the following (which is implied by the aforementioned concepts of ψ, ϕ -accretivity):

Definition 4.3. *An accretive operator $A:D(A) \rightarrow 2^X$ is uniformly accretive if*

$$\forall \varepsilon, K > 0 \exists \delta > 0 \forall (x, u), (y, v) \in A \\ (\|x - y\| \in [\varepsilon, K] \rightarrow \exists j \in J(x - y) (\langle u - v, j \rangle \geq \delta)). \quad (*)$$

Any function $\Theta_{(\cdot)}(\cdot) : (0, \infty) \times (0, \infty) \rightarrow (0, \infty)$ such that $\delta := \Theta_K(\varepsilon)$ satisfies () for all $\varepsilon, K > 0$ is called a modulus of uniform accretivity for A .*

Remark 4.4. *If A is assumed to have a zero q , then the definition above can be modified to ‘uniform accretivity at zero’*

$$\forall \varepsilon, K > 0 \exists \delta > 0 \forall (x, u) \in A (\|x - q\| \in [\varepsilon, K] \rightarrow \exists j \in J(x - q) (\langle u, j \rangle \geq \delta))$$

which still is sufficient to get a modulus of uniqueness for the property of being a zero of A (see [31, Section 2.1]).

In [6], the following concept of uniform accretivity is introduced and shown to imply the uniqueness of zeros if X is a uniformly convex normed space (see also [18] and [59] for results involving this and related notions):

Definition 4.5. *A is uniformly accretive in the sense of [6] if there is a strictly increasing function $\varphi : [0, \infty) \rightarrow \mathbb{R}$ with $\lim_{t \rightarrow \infty} \varphi(t) = \infty$ such that*

$$\forall (x, u), (y, v) \in A \exists j \in J(x - y) (\langle u - v, j \rangle \geq (\varphi(\|x\|) - \varphi(\|y\|)) \cdot (\|x\| - \|y\|)).$$

Consider in the following a nonexpansive selfmapping $T : C \rightarrow C$ of a bounded convex subset $C \subseteq X$ of a normed space X such that $A := I - T$ is an (single-valued) accretive operator with $D(A) = C$. If A admits a modulus of uniqueness for the property of being a zero, then the same modulus is also a modulus of uniqueness for being a fixed point of T . We now extract from the uniqueness proof given in [6] a modulus of uniqueness (again in the case of uniformly convex normed spaces) which only depends on a modulus function $\Omega : (0, \infty)^2 \rightarrow (0, \infty)$ such that

$$\forall (x, u), (y, v) \in A \forall b, \varepsilon > 0 \exists j \in J(x - y) \\ (\max\{\|x\|, \|y\|\} \leq b \wedge \|\|x\| - \|y\|\| \geq \varepsilon \rightarrow \langle u - v, j \rangle > \Omega(\varepsilon, b)). \quad (*)$$

Proposition 4.6. *If A is uniformly accretive in the sense of [6], then there exists a modulus Ω satisfying (\star) .*

Proof. Let φ be as in Definition 4.5. Let $b, \varepsilon > 0$. It suffices to show that

$$\exists \delta > 0 \forall x \in [0, b] (\varphi(x + \varepsilon) - \varphi(x) > \delta)$$

since $\varphi(y) \geq \varphi(x + \varepsilon)$ for $y \geq x + \varepsilon$ and we can then take $\Omega(\varepsilon, b) := \varepsilon \cdot \delta$. Suppose otherwise, i.e.

$$\forall n \in \mathbb{N} \exists x_n \in [0, b] (\varphi(x_n + \varepsilon) - \varphi(x_n) \leq \frac{1}{n+1}).$$

Let x be a cluster point of (x_n) . Then for arbitrary large n we have $x_n \leq x + \frac{\varepsilon}{3}$ and $x + \frac{2\varepsilon}{3} \leq x_n + \varepsilon$ and so

$$\varphi\left(x + \frac{2\varepsilon}{3}\right) - \varphi\left(x + \frac{\varepsilon}{3}\right) \leq \varphi(x_n + \varepsilon) - \varphi(x_n) \leq \frac{1}{n+1}.$$

Thus $\varphi\left(x + \frac{2\varepsilon}{3}\right) - \varphi\left(x + \frac{\varepsilon}{3}\right) \leq 0$ which contradicts φ being strictly increasing. \square

In this section, we will compute a modulus of uniqueness for approximate fixed points of $T = I - A$ in terms of a given modulus of uniform convexity for X and a modulus Ω for A . This then gives us explicit rates of convergence for Browder-type sequences, Krasnoselskii-Mann iterations and Halpern iterations of T and so – by the results in the previous section – also such rates for the viscosity versions of these algorithms.

In the following $C \subseteq X$ is a bounded convex subset of a uniformly convex normed space $(X, \|\cdot\|)$. Let $b \geq \|x\|$ for all $x \in C$ and $\eta : (0, 2] \rightarrow (0, 1]$ be a modulus of uniform convexity for X , i.e.

$$\forall \varepsilon \in (0, 2] \forall x, y \in X \left(\|x\|, \|y\| \leq 1 \wedge \|x - y\| \geq \varepsilon \rightarrow \left\| \frac{1}{2}(x + y) \right\| \leq 1 - \eta(\varepsilon) \right).$$

Note that this implication holds trivially if $\varepsilon > 2$ and so we can extend η to $(0, \infty)$ e.g. by putting $\eta(\varepsilon) := 1$ for $\varepsilon > 2$.

$T : C \rightarrow C$ is a nonexpansive mapping such that $A := I - T$ is uniformly accretive in the sense of (\star) above with a respective modulus Ω . In the following, we write $\Omega(\varepsilon) := \Omega(\varepsilon, b)$.

Lemma 4.7. *Let $x_1, x_2 \in C$ be $\frac{\Omega(\varepsilon)}{4b}$ -approximate fixed points of T . Then $|||x_1|| - ||x_2||| \leq \varepsilon$.*

Proof. By assumption we have $\|A(x_1)\|, \|A(x_2)\| \leq \frac{\Omega(\varepsilon)}{4b}$. W.l.o.g. $\|x_1\| \geq \|x_2\|$. Let $j \in J(x_1 - x_2)$ be as in (\star) . Assume that $\|x_1\| - \|x_2\| > \varepsilon$. Then

$$\begin{aligned} \Omega(\varepsilon) &= \frac{\Omega(\varepsilon)}{2b} \cdot 2b \geq (\|Ax_1\| + \|Ax_2\|) \cdot \|x_1 - x_2\| \\ &\geq \|Ax_1 - Ax_2\| \cdot \|x_1 - x_2\| \geq \langle Ax_1 - Ax_2, j \rangle > \Omega(\varepsilon). \end{aligned}$$

Hence $|||x_1|| - ||x_2||| \leq \varepsilon$. \square

Lemma 4.8. *If $x_1, x_2 \in C$ are $\frac{\varepsilon \cdot \eta(\varepsilon/2b)}{4}$ -approximate fixed points of T , then $\frac{x_1 + x_2}{2}$ is an ε -approximate fixed point of T .*

If $\eta(\varepsilon)$ can be written as $\varepsilon \cdot \tilde{\eta}(\varepsilon)$ with $0 < \varepsilon_1 \leq \varepsilon_2 \leq 2 \rightarrow \tilde{\eta}(\varepsilon_1) \leq \tilde{\eta}(\varepsilon_2)$, then we can replace $\frac{\varepsilon \cdot \eta(\varepsilon/2b)}{4}$ by $\frac{\varepsilon}{2} \cdot \tilde{\eta}(\varepsilon/2b) = b \cdot \eta(\varepsilon/2b)$.

Proof. The proof follows the pattern of the proofs of Lemmas 2.2 and 2.3 in [29]. Claim:

$$\forall \varepsilon > 0 \forall a, x, y \in C \left(\left\| a - \frac{x+y}{2} \right\| > K - \frac{\varepsilon \cdot \eta(\varepsilon/2b)}{2} \rightarrow \|x - y\| < \varepsilon \right),$$

where $K := \max\{\|a - x\|, \|a - y\|\} \leq 2b$.

Proof of claim: we may assume that $K \geq \varepsilon/2$, otherwise $\|x - y\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$. Consider

$$\tilde{x} := \frac{a - x}{K}, \tilde{y} := \frac{a - y}{K}.$$

Then $\tilde{x}, \tilde{y} \in \overline{B}_1(0)$. Assume that

$$\left\| a - \frac{x+y}{2} \right\| > K - \frac{\varepsilon \cdot \eta(\varepsilon/2b)}{2} \geq K - K \cdot \eta(\varepsilon/2b)$$

and so

$$\left\| \frac{\tilde{x} + \tilde{y}}{2} \right\| = \frac{1}{K} \left\| a - \frac{x+y}{2} \right\| > 1 - \eta(\varepsilon/2b).$$

Then by the definition of η

$$\frac{1}{K} \|x - y\| = \|\tilde{x} - \tilde{y}\| < \frac{\varepsilon}{2b} \leq \frac{\varepsilon}{K}.$$

Proof of the lemma: by assumption

$$\bigwedge_{i=1}^2 \|x_i - Tx_i\| \leq \frac{\varepsilon \cdot \eta(\varepsilon/2b)}{4}. \quad (16)$$

Define $x := \frac{x_1 + x_2}{2}$. From $i = 2$ in (16), we get

$$\begin{aligned} \left\| x_2 - \frac{x + Tx}{2} \right\| &\leq \frac{1}{2} \|x_2 - x\| + \frac{1}{2} \|x_2 - Tx\| \\ &\leq \frac{1}{2} \|x_2 - x\| + \frac{1}{2} \|Tx_2 - Tx\| + \frac{\varepsilon \cdot \eta(\varepsilon/2b)}{8} \\ &\leq \frac{1}{2} \|x_2 - x\| + \frac{1}{2} \|x_2 - x\| + \frac{\varepsilon \cdot \eta(\varepsilon/2b)}{8} \\ &= \|x_2 - x\| + \frac{\varepsilon \cdot \eta(\varepsilon/2b)}{8}. \end{aligned}$$

This implies that

$$\left\| x_1 - \frac{x + Tx}{2} \right\| \geq \|x_1 - x\| - \frac{\varepsilon \cdot \eta(\varepsilon/2b)}{8} \quad (17)$$

otherwise, we get the contradiction

$$\begin{aligned} \|x_1 - x_2\| &\leq \left\| x_1 - \frac{x + Tx}{2} \right\| + \left\| x_2 - \frac{x + Tx}{2} \right\| \\ &< \|x_1 - x\| - \frac{\varepsilon \cdot \eta(\varepsilon/2b)}{8} + \|x_2 - x\| + \frac{\varepsilon \cdot \eta(\varepsilon/2b)}{8} \\ &= \frac{1}{2} \|x_1 - x_2\| + \frac{1}{2} \|x_1 - x_2\| = \|x_1 - x_2\|. \end{aligned}$$

By (16) applied to $i = 1$ we also have

$$\|x_1 - Tx\| \leq \|Tx_1 - Tx\| + \frac{\varepsilon \cdot \eta(\varepsilon/2b)}{4} \leq \|x_1 - x\| + \frac{\varepsilon \cdot \eta(\varepsilon/2b)}{4}. \quad (18)$$

With $K := \max\{\|x_1 - x\|, \|x_1 - Tx\|\} \leq 2b$, (17) and (18) yield that

$$\begin{aligned} \left\|x_1 - \frac{x + Tx}{2}\right\| &\geq \|x_1 - x\| - \frac{\varepsilon \cdot \eta(\varepsilon/2b)}{8} \\ &\geq K - \frac{\varepsilon \cdot \eta(\varepsilon/2b)}{4} - \frac{\varepsilon \cdot \eta(\varepsilon/2b)}{8} > K - \frac{\varepsilon \cdot \eta(\varepsilon/2b)}{2}. \end{aligned}$$

The 1st part of the lemma now follows by the claim with x_1, x, Tx as a, x, y . For the 2nd part, one first has to observe that the proof of the claim also establishes the claim (we may again assume that $K \geq \varepsilon/2$ so that $\varepsilon/2b \leq \varepsilon/K \leq 2$) for

$$K \cdot \eta(\varepsilon/K) = \varepsilon \cdot \tilde{\eta}(\varepsilon/K) \geq \varepsilon \cdot \tilde{\eta}(\varepsilon/2b)$$

instead of $\frac{\varepsilon}{2} \cdot \eta(\varepsilon/2b)$. Using this,

$$\bigwedge_{i=1}^2 \|x_i - Tx_i\| \leq \frac{\varepsilon}{2} \cdot \tilde{\eta}(\varepsilon/2b) \quad (16')$$

now implies that $\|x - Tx\| < \varepsilon$ as in the proof above. \square

The next lemma is a slight adaption of Lemma 3.2 in [32] (we include the brief proof for completeness):

Lemma 4.9. *If $x_1, x_2 \in C$ with $\|x_1\| \geq \|x_2\|$ and $\|x_1 - x_2\| > \varepsilon$, then $\left\|\frac{x_1 + x_2}{2}\right\| < \|x_1\| - \beta(b, \varepsilon)$, where*

$$\beta(b, \varepsilon) := \frac{\varepsilon}{2} \cdot \eta(\varepsilon/b).$$

If $\eta(\varepsilon)$ can be written as $\varepsilon \cdot \tilde{\eta}(\varepsilon)$ with $0 < \varepsilon_1 \leq \varepsilon_2 \leq 2 \rightarrow \tilde{\eta}(\varepsilon_1) \leq \tilde{\eta}(\varepsilon_2)$, then we can replace β by $\beta'(b, \varepsilon) := \varepsilon \cdot \tilde{\eta}(\varepsilon/b) = b \cdot \eta(\varepsilon/b)$.

Proof. $\|x_1\| \geq \|x_2\|$ and $\|x_1 - x_2\| > \varepsilon$ implies that

$$\|x_1\| > \frac{\varepsilon}{2} > 0. \quad (19)$$

Define $\tilde{x}_1 := \frac{x_1}{\|x_1\|}$, $\tilde{x}_2 := \frac{x_2}{\|x_1\|}$. Then $\|\tilde{x}_2\| \leq \|\tilde{x}_1\| = 1$ and

$$\|\tilde{x}_1 - \tilde{x}_2\| = \frac{1}{\|x_1\|} \|x_1 - x_2\| > \frac{\varepsilon}{\|x_1\|} \geq \frac{\varepsilon}{b}.$$

Since η is a modulus of uniform convexity we get

$$\left\|\frac{x_1 + x_2}{2}\right\| = \|x_1\| \cdot \left\|\frac{\tilde{x}_1 + \tilde{x}_2}{2}\right\| \leq \|x_1\| - \|x_1\| \cdot \eta(\varepsilon/b) \stackrel{(19)}{<} \|x_1\| - \frac{\varepsilon}{2} \cdot \eta(\varepsilon/b). \quad (20)$$

The 2nd claim follows from

$$\begin{aligned} \left\| \frac{x_1 + x_2}{2} \right\| &= \|x_1\| \cdot \left\| \frac{\tilde{x}_1 + \tilde{x}_2}{2} \right\| < \|x_1\| - \|x_1\| \cdot \eta(\varepsilon/\|x_1\|) \\ &= \|x_1\| - \varepsilon \cdot \tilde{\eta}(\varepsilon/\|x_1\|) \leq \|x_1\| - \varepsilon \cdot \tilde{\eta}(\varepsilon/b). \end{aligned} \quad (20')$$

□

Proposition 4.10. *For $\varepsilon > 0$, let*

$$\omega_b(\varepsilon) := \frac{1}{16b} \Omega\left(\frac{\varepsilon}{2} \cdot \eta\left(\frac{\varepsilon}{b}\right)\right) \cdot \eta\left(\frac{1}{8b^2} \Omega\left(\frac{\varepsilon}{2} \cdot \eta\left(\frac{\varepsilon}{b}\right)\right)\right).$$

If $x_1, x_2 \in C$ are $\omega_b(\varepsilon)$ -approximate fixed points of T , then $\|x_1 - x_2\| \leq \varepsilon$, i.e. ω_b is a modulus of uniqueness for being a fixed point of T .

If $\eta(\varepsilon) = \varepsilon \cdot \tilde{\eta}(\varepsilon)$ with $\tilde{\eta}$ as in Lemma 4.9, then we can improve ω_b to

$$\omega_b(\varepsilon) := \min \left\{ b \cdot \eta\left(\frac{\Omega(b \cdot \eta(\frac{\varepsilon}{b}))}{8b^2}\right), \frac{\Omega(b \cdot \eta(\frac{\varepsilon}{b}))}{4b} \right\}.$$

Proof. W.l.o.g. $\|x_1\| \geq \|x_2\|$. Assume that $\|x_1 - x_2\| > \varepsilon$. By Lemma 4.8 $\frac{x_1 + x_2}{2}$ (and by assumption also x_1 since $\eta(\dots) \leq 1$) is a $\frac{1}{4b} \Omega\left(\frac{\varepsilon}{2} \cdot \eta\left(\frac{\varepsilon}{b}\right)\right)$ -approximate fixed point of T . Hence by Lemma 4.7 (applied to $\frac{x_1 + x_2}{2}$ as x_2)

$$\|x_1\| - \left\| \frac{x_1 + x_2}{2} \right\| \leq \frac{\varepsilon}{2} \cdot \eta\left(\frac{\varepsilon}{b}\right).$$

By Lemma 4.9, on the other hand, we have that

$$\left\| \frac{x_1 + x_2}{2} \right\| < \|x_1\| - \frac{\varepsilon}{2} \cdot \eta\left(\frac{\varepsilon}{b}\right)$$

which gives a contradiction. The proof of the 2nd claim is analogous making use of the 2nd claims in lemmas 4.8 and 4.9. □

A first application of Proposition 4.10 gives a Cauchy rate for the Browder path $(x_\alpha)_{\alpha \in (0,1)}$ of points in C such that $x_\alpha = (1 - \alpha)T(x_\alpha) + \alpha u$, for $u \in C$.

Theorem 4.11. *Let C, X, η, b, T, Ω be as before. Then*

$$\forall \varepsilon > 0 \forall \alpha_1, \alpha_2 \in (0, 1) \left(\bigwedge_{i=1}^2 \alpha_i \leq \frac{\omega_b(\varepsilon)}{2b} \rightarrow \|x_{\alpha_1} - x_{\alpha_2}\| \leq \varepsilon \right).$$

Proof. Let for $i = 1, 2$ be $\alpha_i \in (0, 1)$ be such that $\alpha_i \leq \frac{\omega_b(\varepsilon)}{2b}$. Then

$$\begin{aligned} \|T(x_{\alpha_i}) - x_{\alpha_i}\| &= \|T(x_{\alpha_i}) - T(x_{\alpha_i}) + \alpha_i T(x_{\alpha_i}) - \alpha_i u\| \\ &= \alpha_i \|T(x_{\alpha_i}) - u\| \leq \alpha_i \cdot 2b \leq \omega_b(\varepsilon). \end{aligned}$$

Hence by Proposition 4.10 we get that $\|x_{\alpha_1} - x_{\alpha_2}\| \leq \varepsilon$. □

As another application of Proposition 4.10 we get the following quantitative form of Theorem 1 in [59]:

Theorem 4.12. *Let C, X, η, b, T, Ω be as before. Let $(\beta_n) \subseteq [0, 1]$ be such that $\sum_{n=0}^{\infty} \beta_n(1 - \beta_n) = \infty$ with a rate of divergence γ . Define for $x_0 \in C$*

$$x_{n+1} := (1 - \beta_n)x_n + \beta_n T x_n.$$

Then (x_n) is a Cauchy sequence with rate

$$\forall \varepsilon > 0 \forall n, m \geq \gamma \left(\left\lceil \frac{4b^2}{\pi \cdot (\omega_b(\varepsilon))^2} \right\rceil \right) (\|x_n - x_m\| \leq \varepsilon),$$

where ω_b is the modulus of uniqueness from Proposition 4.10. For X complete and C closed, (x_n) converges with this rate to the unique fixed point of T .

Proof. By [11] we have for all n

$$\|x_n - T x_n\| \leq \frac{2b}{\sqrt{\pi \cdot \sum_{i=0}^n \beta_i(1 - \beta_i)}}. \quad (21)$$

Now let $n \geq \gamma \left(\left\lceil \frac{4b^2}{\pi \cdot (\omega_b(\varepsilon))^2} \right\rceil \right)$, then

$$\sum_{i=0}^n \beta_i(1 - \beta_i) \geq \frac{4b^2}{\pi \cdot \omega_b(\varepsilon)^2}$$

and so by (21)

$$\|x_n - T x_n\| \leq \omega_b(\varepsilon).$$

By Proposition 4.10, the claim on the Cauchy rate follows. If X is complete and C is closed, then by (21) the limit p of (x_n) is a fixed point of T (which by Proposition 4.10 is unique). \square

Theorem 4.13. *Let C, X, η, b, T, Ω be as before. Assume that $\alpha_n \in (0, 1)$ with*

$$\lim_{n \rightarrow \infty} \alpha_n = 0, \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty, \prod_{n=1}^{\infty} (1 - \alpha_n) = 0.$$

Let α be a rate of convergence for $\lim \alpha_n = 0$, β be a Cauchy rate for the sequence $(\sum_{i=1}^n |\alpha_{i+1} - \alpha_i|)$ and θ be a rate of convergence for $\prod_{n=1}^{\infty} (1 - \alpha_{n+1}) = 0$. Define the Halpern iteration of T with starting point $x_0 \in C$ and anchor point $u \in C$ by

$$x_{n+1} := (1 - \alpha_n)T(x_n) + \alpha_n u.$$

Then (x_n) is a Cauchy sequence with rate

$$\forall \varepsilon > 0 \forall n, m \geq \Phi(\omega_b(\varepsilon)) (\|x_n - x_m\| \leq \varepsilon), \text{ where}$$

$$\Phi(\varepsilon, b, \theta, \alpha, \beta, D) = \max \left\{ \theta \left(\frac{D\varepsilon}{4b} \right) + 1, \alpha \left(\frac{\varepsilon}{4b} \right) \right\} + 1,$$

and

$$0 < D \leq \prod_{n=1}^{\beta(\varepsilon/8b)} (1 - \alpha_{n+1}).$$

For the choice $\alpha_n := \frac{1}{n+1}$, we may take

$$\Phi(\varepsilon, b) := \left\lceil \frac{4b}{\varepsilon} + \frac{32b^2}{\varepsilon^2} \right\rceil.$$

Proof. The theorem follows from Proposition 4.10 combined with [33, Proposition 6.2, Corollary 6.3]. \square

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