Sunny nonexpansive retractions in nonlinear spaces

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Outline

1. Generalization of proofs
2. Smooth W-hyperbolic spaces
3. Results

Proof mining

Inspired by Kreisel's program of unwinding of proofs (1950s),

> *"(...) what more we know about a formally derived theorem* F *than if we merely know that* F *is true?"*

Application of proof interpretations to study *a priori* noneffective mathematical proofs as a way to obtain:

- effective bounds, algorithms;
- uniformities in the parameters;
- weakening of premisses, generalization of proofs.

This talk is focused on the *"generalization of proofs"*.

Recent examples of generalizations

- "Lion-Man" game – weakening of compactness assumption;[1]
- Suzuki's theorem reducing the convergence of a generalized iterative schema to that of its original version;[2]
- Halpern-type abstract proximal algorithm in CAT(0);[3]
- Strong convergence of a general new iterative schema.[4]

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[2] U.Kohlenbach, and P.Pinto. Quantitative translations for viscosity approximation methods in hyperbolic spaces. J. Math. Anal. Appl., 507(2): 33pp, 2022.

[3] A.Sipoș. Abstract strongly convergent variants of the proximal point algorithm. Comput. Optim. Appl., 83(1):349–380, 2022.

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Successful in generalizations from the linear to the nonlinear setting.

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. . . in Hilbert spaces

Browder (1967)

Let X be a Hilbert space, $C \subseteq X$ be a nonempty closed convex subset of X. Let $T : C \to C$ be a nonexpansive map on C, and $u \in C$. For each $t \in (0, 1)$, consider $z_t \in C$ characterized by

$$z_t = (1 - t)T(z_t) + tu. \tag{B}$$

If C is bounded and $t \to 0$, then $(z_t)_t$ converges strongly towards $P(u)$ where $P : C \to \mathrm{Fix}(T)$ is the metric projection onto $\mathrm{Fix}(T)$.

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If C is bounded and $t \to 0$, then $(z_t)_t$ converges strongly towards $P(u)$ where $P : C \to \mathrm{Fix}(T)$ is the metric projection onto $\mathrm{Fix}(T)$.

For each $u \in C$, we have that $P(u)$ is the unique fixed point s.t.

$$\forall y \in \mathrm{Fix}(T)\, (\|u - P(u)\| \leqslant \|u - y\|)$$

which is equivalent to having for all $y \in \mathrm{Fix}(T)$

$$\langle u - P(u), y - P(u)\rangle = \langle y - P(u), u - P(u)\rangle \leqslant 0$$

. . . beyond Hilbert spaces

Let X be a (real) normed space. The duality map, $J : X \to 2^{X^*}$ is defined for all $x \in X$ by

$$J(x) := \{f \in X^* : \langle x, f \rangle = \|x\|^2 = \|f\|^2\},$$

where X^* is the dual space of X and $\langle y, f \rangle$ denotes the functional application $f(y)$. J is homogeneous, i.e. $J(\alpha x) = \alpha J(x)$, for $\alpha \in \mathbb{R}$.

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where X^* is the dual space of X and $\langle y, f \rangle$ denotes the functional application $f(y)$. J is homogeneous, i.e. $J(\alpha x) = \alpha J(x)$, for $\alpha \in \mathbb{R}$. The space X is *smooth* if for any x, y with $\|x\| = \|y\| = 1$, the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{$\star$}$$

exists. We know that X is smooth iff J is single-valued. Moreover, X is *uniformly smooth* if the limit $(\star)$ is attained uniformly in x, y, in which case the duality map is also norm-to-norm uniformly continuous on bounded subsets.

... beyond Hilbert spaces

A normed space X is uniformly convex if

$$\forall \varepsilon > 0\ \exists \delta > 0\ \forall x, y \in \overline{B}_1(0) \left(\|x - y\| \geqslant \varepsilon \to \left\| \frac{x+y}{2} \right\| \leqslant 1 - \delta \right).$$

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A normed space X is <u>uniformly convex</u> if

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Let X be unif. smooth and unif. convex, and C, E subsets of X with $E \neq \emptyset$ convex.

Metric projection $P : C \to E$

$$\forall y \in E \left(\langle y - P(u), J(u - P(u)) \rangle \leqslant 0 \right)$$

however . . .

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Sunny nonexpansive retraction $Q : C \to E$

$$\forall y \in E \left(\langle u - Q(u), J(y - Q(u)) \rangle \leqslant 0 \right)$$

A celebrated result due to Reich extends Browder's theorem, proving in particular:

Reich (1980)

Let X be a unif. smooth and unif. convex Banach space, $C \subseteq X$ be a nonempty closed convex subset of X. Let $T : C \to C$ be a nonexpansive map on C and $u \in C$. For each $t \in (0, 1)$, consider $z_t \in C$ satisfying (B). If C is bounded and $t \to 0$, then $(z_t)_t$ converges strongly towards $Q(u)$ where Q is the unique sunny nonexpansive retraction $Q : C \to \mathrm{Fix}(T)$.

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The proof-theoretical analysis of this result was obtained by Kohlenbach and Sipoș in 2021, probably the most complex proof mining analysis to date.[5]

[5] U.Kohlenbach and A.Sipoș. The finitary content of sunny nonexpansive retractions. Communications in Contemporary Mathematics, 23(1),63pp, 2021.

A triple (X, d, W) is a hyperbolic space (Kohlenbach) if (X, d) is a metric space and $W : X \times X \times [0,1] \to X$ satisfies

W1 $d(W(x,y,\lambda), z) \leqslant (1-\lambda)d(x,z) + \lambda d(y,z)$
W2 $d(W(x,y,\lambda), W(x,y,\lambda')) = |\lambda - \lambda'|d(x,y)$
W3 $W(x,y,\lambda) = W(y,x,1-\lambda)$
W4 $d(W(x,y,\lambda), W(z,w,\lambda)) \leqslant (1-\lambda)d(x,z) + \lambda d(y,w)$.

We write $(1-\lambda)x \oplus \lambda y$ for $W(x,y,\lambda)$.

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We write $(1-\lambda)x \oplus \lambda y$ for $W(x,y,\lambda)$. A hyperbolic space is unif. convex if there is a function $\eta : (0,\infty) \times (0,2] \to (0,1]$ s.t.

$$\left.\begin{aligned} d(x,a) &\leqslant r \\ d(y,a) &\leqslant r \\ d(x,y) &\geqslant \varepsilon \cdot r \end{aligned}\right\} \to d\left(\frac{1}{2}x \oplus \frac{1}{2}y, a\right) \leqslant (1-\eta(r,\varepsilon))\, r$$

X is a *UCW* hyperbolic space (Leuştean) if η is nonincreasing in r.

In any metric space, the quasi-linearization function

$$\langle \overrightarrow{xy}, \overrightarrow{uv} \rangle := \frac{1}{2}\left(d^2(x,v) + d^2(y,u) - d^2(x,u) - d^2(y,v)\right)$$

is the unique function satisfying:

(1) $\langle \overrightarrow{xy}, \overrightarrow{xy} \rangle = d^2(x,y)$,
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A hyperbolic space is a CAT(0) space if

$$\text{(CS)} \quad \langle \overrightarrow{xy}, \overrightarrow{uv} \rangle \leqslant d(x,y) \cdot d(u,v)$$

[6]Any CAT(0) space is a UCW hyperbolic space w/ $\eta(r,\varepsilon) := \varepsilon^2/8$.

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Think of $\langle \overrightarrow{\cdot\cdot}, \overrightarrow{\cdot\cdot} \rangle$ as a nonlinear counterpart to an inner-product.

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Smooth hyperbolic spaces

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We say that X is a smooth hyperbolic space if there is a function $\pi : X^2 \times X^2 \to \mathbb{R}$ satisfying

P1 $\pi(\overrightarrow{xy}, \overrightarrow{xy}) = d^2(x, y)$
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P5 $d^2(W(x, y, \lambda), z) \leqslant (1 - \lambda)^2 d^2(x, z) + 2\lambda\pi\left(\overrightarrow{yz}, \overrightarrow{W(x, y, \lambda)z}\right)$

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Think of $\pi(\overrightarrow{\cdot}, \overrightarrow{\cdot})$ as a nonlinear counterpart to the duality map.

Uniformly smooth

The space X is a uniformly smooth hyperbolic space if additionally

P6
$$\forall \varepsilon > 0\ \forall r > 0\ \exists \delta > 0\ \forall a \in X\ \forall u, v \in \overline{B}_r(a)$$
$$d(u,v) \leqslant \delta \to \forall x, y \in X\ \left(\left|\pi(\overrightarrow{xy}, \overrightarrow{ua}) - \pi(\overrightarrow{xy}, \overrightarrow{va})\right| \leqslant \varepsilon \cdot d(x,y)\right).$$

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- Extending the formal system $\mathcal{A}^\omega[X, d, W]$ with a new constant $\pi : 1(X)(X)(X)(X)$ satisfying (the universal) P1–P5, allows for a bound extraction theorem for results in smooth hyperbolic spaces.

- If we additionally include a modulus of uniform continuity for π, ω_X, providing a witnesses for δ in P6, we can also analyse results in unif. smooth hyperbolic spaces.

More than CAT(0) spaces

Proposition

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Any (uniformly) smooth normed space is a (uniformly) smooth hyperbolic space, with $\pi(\overrightarrow{xy}, \overrightarrow{uv}) := \langle x - y, J(u - v)\rangle$.

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Proposition

Any (uniformly) smooth normed space is a (uniformly) smooth hyperbolic space, with $\pi(\overrightarrow{xy}, \overrightarrow{uv}) := \langle x - y, J(u - v)\rangle$.

- Therefore, the class of (unif.) smooth hyperbolic spaces properly extends the class of CAT(0) spaces, and we regard it as a nonlinear counterpart to (unif.) smooth normed spaces.

π-sunny nonexpansive retractions

Definition

Let X be a smooth hyperbolic space and $E \subseteq C$ subsets of X. A retraction $Q : C \to E$ is a <u>(π-)sunny nonexpansive retraction</u> if

$$\forall x \in C \; \forall y \in E \left(\pi\left(\overrightarrow{xQ(x)}, \overrightarrow{yQ(x)}\right) \leqslant 0 \right).$$

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Lemma

(1) *Any π-sunny nonexpansive retraction is a π-firmly n.e. map, i.e.*

$$\forall x, y \in C \left(d^2(Q(x), Q(y)) \leqslant \pi\left(\overrightarrow{xy}, \overrightarrow{Q(x)Q(y)} \right) \right),$$

and so, in particular, it is a nonexpansive map.

(2) *There exists at most one sunny nonexpansive retraction from C onto E.*

Nonlinear generalization of Reich's theorem

Relying on the proof-theoretically simpler proof due to Kohlenbach and Sipoș in the linear case, we obtained

Theorem (P. 2023)

Let X be a complete uniformly smooth UCW hyperbolic space, C a closed nonempty bounded convex subset, and $u \in C$. Consider $T : C \to C$ a nonexpansive map on C. For any $t \in (0, 1]$, let z_t denote the unique point in C satisfying $z_t = (1-t)T(z_t) \oplus tu$. Then, for all $(t_n) \subseteq (0, 1]$ such that $\lim t_n = 0$, we have that (z_{t_n}) converges to a fixed point of T.

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If we set $Q(u) := \lim z_t$ (is well-defined and a retraction), then

Proposition

The map Q is the unique π-sunny nonexpansive retraction from C onto $\mathrm{Fix}(T)$.

Other convergence results I

In unif. smooth and unif. convex Banach spaces, consider the Halpern-type proximal point algorithm

$$\text{(HPPA)} \quad x_{n+1} := (1-\alpha_n)J_{\gamma_n}(x_n) + \alpha_n u,$$

where $J_{\gamma_n} := (\mathrm{Id} + \gamma_n A)^{-1}$ are resolvent functions associated with an accretive map $A : X \to 2^X$ subject to the range condition.

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From a result by Aoyama and Toyoda (2017), we know that (HPPA) converges to a zero of A, provided that $A^{-1} \neq \emptyset$ and

$$\text{(i)} \;\; \lim \alpha_n = 0, \quad \text{(ii)} \;\; \sum \alpha_n = \infty, \quad \text{(iii)} \;\; \inf \gamma_n > 0.$$

The central argument is a reduction to Reich's theorem.

- Quantitative analysis by Kohlenbach in 2020.
- Proof of corresponding result in CAT(0) space by Sipoș in 2022.

Let X be a hyperbolic space, and $C \neq \emptyset$ a subset of X.
We say that a family $\{T_n : C \to C\}_{n \in \mathbb{N}}$ of nonexpansive maps is resolvent-like if for some/all $n \in \mathbb{N}$, $\mathrm{Fix}(T_n) \neq \emptyset$ and:

(1) There exists a monotone function $\mu : \mathbb{N} \to [1, \infty)$ satisfying

$$\forall n, m \in \mathbb{N}\ \forall x \in C\ (d(x, T_n(x)) \leqslant \mu(n) \cdot d(x, T_m(x)))\,;$$

(2) There exists a function $\Delta : \mathbb{N} \times (0, \infty) \to (0, \infty)$ satisfying

$$\forall \varepsilon > 0\ \forall r \in \mathbb{N}\ \forall n \in \mathbb{N}\ \forall p \in C\ \forall x \in C \cap \overline{B}_r(p)$$
$$\left.\begin{array}{r} d(p, T_n(p) \leqslant \Delta(r, \varepsilon) \\ d(x, p) - d(T_n(x), p) \leqslant \Delta(r, \varepsilon) \end{array}\right\} \to d(x, T_n(x)) \leqslant \varepsilon.$$

These are the central features of $\{J_{\gamma_n}\}$ needed to prove the convergence of HPPA (due to the previous proof mining work).

Theorem (P. 2023)

Let X be a complete uniformly smooth hyperbolic space and C be a nonempty closed convex subset. Consider $\{T_n\}$ to be a resolvent-like family of nonexpansive maps on C. Let $(\alpha_n) \subseteq (0,1]$ be a sequence of positive real numbers satisfying (i) and (ii), and for $u \in C$ let (z_m) be the sequence characterized by the equation

$$z_m = \frac{m}{m+1} T_0(z_m) \oplus \frac{1}{m+1} u, \text{ for all } m \in \mathbb{N}$$

and assume it be a Cauchy sequence. If for a given initial $x_0 \in C$, (x_n) is recursively generated by the schema

$$(\mathrm{H}_{\mathrm{ppa}}) \quad x_{n+1} := (1-\alpha_n) T_n(x_n) \oplus \alpha_n u,$$

then (x_n) converges to a common fixed point of $\{T_n\}$.

We moreover obtained quantitative information. In particular, an effective function:

$$\begin{cases} \omega_X \text{ modulus of unif. continuity for } \pi \\ \mu,\, \Delta \text{ for } \{T_n\} \\ \sigma_1 \text{ convergence rate for (i)} \\ \sigma_2 \text{ divergence rate for (ii)} \\ \xi \text{ metastability rate for } (z_m) \\ b \geqslant \max\{d(x_0,p), d(u,p)\} \text{ for } p \in \mathrm{Fix}(T_0) \\ \tilde{\alpha} \text{ witnessing that } \alpha_n > 0 \end{cases} \Rightarrow \Omega \text{ metastability for } (x_n),$$

$$\forall \varepsilon > 0\ \forall f : \mathbb{N} \to \mathbb{N}\ \exists n \leqslant \Omega\ \forall i,j \in [n; n+f(n)]\ (d(x_i,x_j) \leqslant \varepsilon)\,.$$

- Extends and unifies the previous quantitative results obtained by Kohlenbach (in normed spaces) and by Sipoș (in CAT(0) spaces).

Other convergence results II

Chang (2006)

Let X be a unif. smooth Banach space, $C \neq \emptyset$ a closed convex subset, and $T_0, \cdots, T_{\ell-1}$, be $\ell \geqslant 1$ nonexpansive maps on C s.t.

$$\emptyset \neq \bigcap_{j=0}^{\ell-1} \mathrm{Fix}(T_j) = \mathrm{Fix}(T_0 T_{\ell-1} \cdots T_1) = \cdots = \mathrm{Fix}(T),$$

where $T := T_{\ell-1} \cdots T_0$. Let $\phi : C \to C$ be a strict contraction, and $(\alpha_n) \subseteq [0,1]$ a sequence of real numbers satisfying (i) and (ii). Let (x_n) be recursively generated by

$$(\mathrm{H}^{\phi}_{T_n}) \quad x_{n+1} := (1-\alpha_n) T_n(x_n) + \alpha_n \phi(x_n),$$

with $\{T_n\}$ defined cyclically from $\{T_j\}_{j=0}^{\ell-1}$. Then,

$$\lim \|x_n - T(x_n)\| = 0 \;\Rightarrow\; x_n \to p \in \bigcap \mathrm{Fix}(T_j).$$

We generalize Chang's result:

Theorem (P. 2023)

Let X be a complete unif. smooth hyperbolic space, and $C \subseteq X$ a nonempty closed convex set. Consider T a nonexpansive map on C, $\{T_n\}$ an infinite family of nonexpansive maps on C, and ϕ a strict contraction on C. Let (z_m) be the sequence characterized by

$$z_m = \frac{m}{m+1} T(z_m) \oplus \frac{1}{m+1} \phi(z_m), \text{ for all } m \in \mathbb{N}.$$

With $(\alpha_n) \subseteq [0,1]$ satisfying (i) and (ii), assume the following conditions

(1) $\emptyset \neq \mathrm{Fix}(T) \subseteq \bigcap_{n \in \mathbb{N}} \mathrm{Fix}(T_n)$

(2) *(z_m) is a Cauchy sequence.*

Let (x_n) be recursively defined by $(H^{\phi}_{T_n})$.
If $\lim d(x_n, T(x_n)) = 0$, then (x_n) converges to a fixed point of T.

We moreover obtained quantitative information. In particular, an effective function:

$$\begin{cases} \omega_X \text{ modulus of unif. continuity for } \pi \\ \sigma_1 \text{ convergence rate for (i)} \\ \sigma_2 \text{ divergence rate for (ii)} \\ \tau \text{ quant. data regarding (1)} \\ b \geqslant \max\left\{d(x_0, p), \frac{d(\phi(p),p)}{1-\alpha}\right\}, \text{for } p \in \mathrm{Fix}(T) \\ \xi \text{ metastability rate for } (z_m) \\ \Phi \text{ rate of } T\text{-asymptotic regularity for } (x_n) \end{cases} \Rightarrow \Omega \text{ metastability for } (x_n),$$

As corollaries, we also generalized central results in the linear setting: e.g. the extension of von Neumann's Mean Ergodic Theorem to nonlinear maps due to Wittmann (single map, ϕ constant) and its extension to a finite number of maps by Bauschke.

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- Introduced the notion of smooth hyperbolic space: more general than CAT(0) spaces as well as smooth Banach spaces. The function π is a nonlinear version of the duality map of smooth Banach spaces. We have a formal system suitable for bound extraction in these spaces.

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- Generalized a result by Chang: the necessary conditions (i),(ii) are sufficient for convergence, when one has asymptotic regularity.
- Further results extending Wittmann, Bauschke, and even with viscosity terms in the general sense of Meir-Keeler.

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- Are firmly nonexpansive maps (metrically characterized) always π-firmly nonexpansive?
- Are π-sunny nonexpansive retractions actually 'sunny'?
- ...

Further details and references can be found in my homepage:

- P. Pinto. Nonexpansive maps in nonlinear smooth spaces. 2023

Thank you for your attention!