THE ALTERNATING HALPERN-MANN ITERATION FOR FAMILIES OF MAPS

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Abstract. We generalize the alternating Halpern-Mann iteration to countably infinite families of nonexpansive maps and prove its strong convergence towards a common fixed point in the general nonlinear setting of Hadamard spaces. Our approach is based on a quantitative perspective which allowed to circumvent prevalent troublesome arguments and in the end provide a simple convergence proof. In that sense, discussing both the asymptotic regularity and the strong convergence of the iteration in quantitative terms, we furthermore provide low complexity uniform rates of convergence and of metastability (in the sense of T. Tao). In CAT(0) spaces, we obtain linear and quadratic uniform rates of convergence. Our results are made possible by proof-theoretical insights of the research program proof mining and extend several previous theorems in the literature.

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1. Introduction

In the general nonlinear context of a hyperbolic space (X, d, W), the alternating Halpern-Mann iteration, introduced by Dinis and the second author in [17], can be formulated as

$$\text{(HM)} \qquad x_0 \in C \quad \text{and} \quad \begin{cases} y_n = (1-\alpha_n)T(x_n) \oplus \alpha_n u \\ x_{n+1} = (1-\beta_n)U(y_n) \oplus \beta_n y_n \end{cases}$$

where T, U are nonexpansive maps on a nonempty convex set $C \subseteq X$, $u \in C$ and $(\alpha_n), (\beta_n) \subset [0,1]$. In this paper, we employ the novel finitary approach used in [17] and extend it in order to establish the strong convergence of a generalized version of (HM) allowing for countable families of nonexpansive maps. While usual convergence proofs in nonlinear spaces of this kind of fixed point iterations rely on mathematically complicated arguments, like Banach limits and reductions to other iterative methods, the finitistic perspective considered in [17] manages to avoid those

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arguments and the strong convergence result was established via elementary methods. We show that the same approach can also be used in this broader setting.

The algorithm (HM) provides an unified treatment of several well-known iterative schemes:

(1) $U = \mathrm{Id}$, then (x_n) is the Halpern iteration

$$x_{n+1} = y_n = (1-\alpha_n)T(x_n) \oplus \alpha_n u.$$

In Hilbert spaces, Halpern [23] established sufficient conditions and necessary conditions for the strong convergence of this method towards the projection of the anchor point u onto the set of fixed points of T, $\mathrm{Fix}(T)$. These conditions however prevented the natural choice $\alpha_n = \frac{1}{n+1}$, which was later overcome in a celebrated result by Wittmann [48].

(2) If $T = \mathrm{Id}$, then (x_n) is the Tikhonov-Mann iteration introduced in [12] by Cheval and Leuştean, and (y_n) is the modified Halpern iteration introduced in normed linear spaces by Kim and Xu [24]. The work in [24] showed the strong convergence of the modified Halpern iteration (y_n) in uniformly smooth Banach spaces. Under milder conditions, [13] showed the strong convergence of (y_n) in the nonlinear setting of CAT(0) spaces. The quantitative study of the asymptotic behaviour of (y_n) in CAT(0) spaces featured in [42]. In the recent [11], Cheval, Kohlenbach and Leuştean provide an effective algorithm to translate quantitative data from (y_n) to (x_n) and vice-versa, in the general nonlinear setting of hyperbolic spaces.

(3) If $T = \mathrm{Id}$ and $\alpha_n \equiv 0$, then (x_n) is the well-known Krasnoselski-Mann iteration [29, 36]

$$x_{n+1} = (1-\beta_n)U(x_n) \oplus \beta_n x_n.$$

(4) In normed linear spaces, if $T = \mathrm{Id}$ and $u = 0$, then it reduces to the iteration introduced by Yao, Zho and Liou [50], which featured in recent work by Boţ, Csetnek and Meier [4] (in connection with strongly convergent versions of the Douglas-Rachford and the forward-backwards splitting algorithms),

$$x_{n+1} = (1-\beta_n)U((1-\alpha_n)x_n) + \beta_n(1-\alpha_n)x_n,$$

and for which a quantitative analysis was carried out in [16].

In [17], Dinis and the second author prove the strong convergence of the alternating Halpern-Mann iteration in CAT(0) spaces while providing a quantitative analysis of the iteration. A generalization to the asymptotic regularity of the sequence was subsequently obtained in [34].

In this paper, motivated by the recent work of Boţ and Meier [5] and Cheval [9], we consider the following generalization to countably infinite families $\{T_n\}$ and $\{U_n\}$ of nonexpansive selfmaps on a nonempty, closed, convex subset C of a hyperbolic space X,

$$(\mathrm{HM}_\infty) \qquad x_0 \in C \text{ and } \begin{cases} y_n = (1-\alpha_n)T_n(x_n) \oplus \alpha_n u \\ x_{n+1} = (1-\beta_n)U_n(y_n) \oplus \beta_n y_n \end{cases}$$

Our goal is to study the asymptotic behaviour of (HM_∞) under general geodesic settings. We consider the following conditions on the parameters:

(C1) $\lim \alpha_n = 0$,
(C2) $\sum \alpha_n = \infty$,

(C3) $\sum |\alpha_{n+1} - \alpha_n| < \infty$,
(C4) $\sum |\beta_{n+1} - \beta_n| < \infty$,
(C5) $\sum d(T_{n+1}(x_n), T_n(x_n)) < \infty$,
(C6) $\sum d(U_{n+1}(y_n), U_n(y_n)) < \infty$,
(C7) $0 < \liminf \beta_n \leq \limsup \beta_n < 1$.[1]

The central result of the paper, which is also new in normed linear spaces, is the following (the proof of a slightly stronger version is given in section 5).

Theorem 1.1. *Let X be a UCW–hyperbolic space, $C \subseteq X$ a nonempty convex subset, and $u \in C$. Consider $\{T_n\}$, $\{U_n\}$ families of nonexpansive maps in C such that $\mathrm{F} := \mathrm{Fix}(\{T_n\}) \cap \mathrm{Fix}(\{U_n\}) \neq \emptyset$, and sequences $(\alpha_n), (\beta_n) \subset [0,1]$. For arbitrary $x_0 \in C$, let (x_n) be the iteration generated by the scheme (HM_∞). Assume (C1)–(C7) are satisfied. Then, (x_n) is asymptotically regular and $\{T_n\}$– and $\{U_n\}$–asymptotically regular, i.e.*

$$\lim d(x_n, x_{n+1}) = \lim d(x_n, T_n(x_n)) = \lim d(x_n, U_n(x_n)) = 0.$$

Moreover, when X is an Hadamard space, C is closed, and both families satisfy the NST condition (II) with NST_2–moduli, then (x_n) converges strongly to a common fixed point of the maps (the closest one to the anchor point u).

Despite the more convoluted arguments, the overall strategy is the same as in [17]. Indeed, motivated by a proof mining perspective [26], we first establish the metastability property of the iteration, i.e.

$$(\dagger) \qquad \forall \varepsilon > 0\, \forall f : \mathbb{N} \to \mathbb{N}\, \exists n \in \mathbb{N}\, \forall i, j \in [n; f(n)]\, (d(x_i, x_j) \leq \varepsilon)\,.$$

The terminology is due to Tao [47] and this property is an equivalent reformulation of the Cauchy property although computationally weaker. In fact, we are able to obtain an uniform computable so-called rate of metastability, while it is known that computable Cauchy rates are in general excluded. Be that as it may, having the metastability property therefore entails that (x_n) is a Cauchy sequence and strong convergence follows under a Cauchy completeness assumption. The onus of the argument is thus placed on proving the property (†). We note that in the particular case where $T_n \equiv \mathrm{Id}$ and $\{U_n\}$ satisfies condition (C1) from [32], a rate of metastability was recently obtained in [10] also in a manner similar to [17].

The paper is organized as follows. In the next section, we recall all the relevant notions and technical lemmas. In section 3, we show the asymptotic regularity of (HM_∞) in general UCW–hyperbolic spaces. We furthermore obtain linear and quadratic rates of convergence for a particular choice of parameters. A short discussion of the metric projection and useful quantitative results is carried out in section 4. Finally, in section 5, we establish the metastability property of (HM_∞) which then entails the strong convergence of the algorithm in complete CAT(0) spaces.

2. Preliminaries and Lemmas

2.1. **Nonlinear Spaces.** Let (X, d, W) be a metric space together with a function $W : X \times X \times [0,1] \to X$ satisfying for all $x, y, z, w \in X$ and $\lambda, \lambda' \in [0,1]$,

(W1) $d(z, W(x, y, \lambda)) \leq (1-\lambda)d(z, x) + \lambda d(z, y)$
(W2) $d(W(x, y, \lambda), W(x, y, \lambda')) = |\lambda - \lambda'| d(x, y)$
(W3) $W(x, y, \lambda) = W(y, x, 1-\lambda)$

[1]By (C4), we may equivalently replace (C7) with $0 < \lim \beta_n < 1$.

(W4) $d(W(x,y,\lambda),W(z,w,\lambda)) \leq (1-\lambda)d(x,z)+\lambda d(y,w)$.

With these conditions, we say that X is a W–hyperbolic space. The reader can easily convince himself that these are the basic properties that one expects to have from a convex combination and so we shall use the friendlier notation $(1-\lambda)x \oplus \lambda y$ to denote $W(x,y,\lambda)$. The idea of considering a function W to study convexity in a general metric setting is due to Takahashi [45] who introduced the notion of convex metric space as a triple (X,d,W) for a function W satisfying (W1). The notion we consider here was introduce by Kohlenbach [25] and has since become one of the most used settings to discuss fixed point methods in nonlinear geodesic spaces. We remark that this class of hyperbolic spaces is slightly more restrictive than that of Goebel and Kirk [20], but more general than the class of hyperbolic spaces as introduced by Reich and Shafrir [40].

Using (W1), it is easy to verify that for all $x,y \in X$ and $\lambda \in [0,1]$,

$$d(x,(1-\lambda)x \oplus \lambda y) = \lambda d(x,y) \text{ and } d(y,(1-\lambda)x \oplus \lambda y) = (1-\lambda)d(x,y).$$

Naturally, the notion of convex set still makes sense in this framework. We say that a subset $C \subseteq X$ is a convex set if for all $x,y \in C$ and $\lambda \in [0,1]$, the point $(1-\lambda)x \oplus \lambda y$ still lies in C. The space is uniformly convex if for all $r>0$ and $\varepsilon \in (0,2]$ there exists $\delta \in (0,1]$ such that for all $x,y,a \in X$

$$\left.\begin{array}{l} d(x,a) \leq r \\ d(y,a) \leq r \\ d(x,y) \geq \varepsilon r \end{array}\right\} \to d\left(\frac{1}{2}x \oplus \frac{1}{2}y, a\right) \leq (1-\delta)r$$

We say that (X,η) is a UCW–hyperbolic space (as introduced by Leuştean in [30], inspired by [21, p.105]), if X is a W–hyperbolic space and $\eta : (0,\infty)\times(0,2] \to (0,1]$ is a function witnessing δ in the property above which is monotone in r, i.e.

$$\forall \varepsilon \in (0,2]\ \forall r,s>0\,(r \leq s \to \eta(s,\varepsilon) \leq \eta(r,\varepsilon)).$$

Such a function η is called a monotone modulus of uniform convexity. We shall use the following property of UCW–hyperbolic spaces.

Lemma 2.1 ([31, Lemma 2.1(iv)]). *Let (X,η) be a UCW–hyperbolic space. Then for all $s \geq r > 0$, $\varepsilon \in (0,2]$, $x,y,a \in X$ and $\lambda \in [0,1]$*

$$\left.\begin{array}{l} d(x,a) \leq r \\ d(y,a) \leq r \\ d(x,y) \geq \varepsilon r \end{array}\right\} \to d((1-\lambda)x \oplus \lambda y, a) \leq (1-2\lambda(1-\lambda)\eta(s,\varepsilon))r.$$

An important class of hyperbolic spaces is that of CAT(0) spaces, introduced by Alexandrov [1] and named as such by Gromov in [22] (for further details see e.g. [6]). A CAT(0) space is a W–hyperbolic space that satisfies the CN^- property (which, in the presence of the (W1)–(W4) axioms, is equivalent to Bruhat-Tits CN-inequality [8])

$$(\mathrm{CN}^-) \qquad \forall x,y,z \in X\left(d^2\left(z,\frac{1}{2}x \oplus \frac{1}{2}y\right) \leq \frac{1}{2}d^2(z,x)+\frac{1}{2}d^2(z,y)-\frac{1}{4}d^2(x,y)\right)$$

The CN^- property extends to arbitrary convex combinations in the following way (see [15, Lemma 2.5]): for all $x,y,z \in X$ and $\lambda \in [0,1]$,

$$(\mathrm{CN}^+) \qquad d^2(z,(1-\lambda)x \oplus \lambda y) \leq (1-\lambda)d^2(z,x)+\lambda d^2(z,y)-\lambda(1-\lambda)d^2(x,y)$$

Leuştean proved that CAT(0) spaces have a monotone modulus of uniform convexity, and so CAT(0) spaces are in particular UCW–hyperbolic spaces.

Proposition 2.2 ([30, Proposition 8]). *Every* CAT(0) *space is a* UCW*–hyperbolic space with monotone modulus of uniform convexity* $\eta(\varepsilon) = \varepsilon^2/8$.

Examples of W–hyperbolic spaces include the linear normed spaces as well as their convex subsets, and the Hilbert ball [21]. Obviously, UCW–hyperbolic spaces generalize to a nonlinear setting the notion of uniformly convex normed space. Indeed, the hyperbolic setting is considered the nonlinear counterpart to normed spaces and, in that same sense, CAT(0) spaces are the nonlinear counterpart of inner product spaces. Let us make this parallel clearer.

In a metric space, we have the *quasi-linearization function* $\langle \cdot, \cdot \rangle : X^2 \times X^2 \to \mathbb{R}$ defined by

$$\langle \overrightarrow{xy}, \overrightarrow{uv} \rangle := \frac{1}{2}\left(d^2(x,v) + d^2(y,u) - d^2(x,u) - d^2(y,v)\right),$$

where $\overrightarrow{xy}$ denotes the pair (x,y). Note that $\langle \overrightarrow{xy}, \overrightarrow{uv} \rangle = \langle x-y, u-v \rangle$ in inner product spaces. In fact, this function exhibits properties akin to the inner product, and it was shown in [3, Proposition 14] that in metric spaces it is the unique function satisfying simultaneously:

(*i*) $\langle \overrightarrow{xy}, \overrightarrow{xy} \rangle = d^2(x,y)$
(*ii*) $\langle \overrightarrow{xy}, \overrightarrow{uv} \rangle = \langle \overrightarrow{uv}, \overrightarrow{xy} \rangle$
(*iii*) $\langle \overrightarrow{xy}, \overrightarrow{uv} \rangle = -\langle \overrightarrow{yx}, \overrightarrow{uv} \rangle$
(*iv*) $\langle \overrightarrow{xy}, \overrightarrow{uv} \rangle + \langle \overrightarrow{xy}, \overrightarrow{vw} \rangle = \langle \overrightarrow{xy}, \overrightarrow{uw} \rangle$

Moreover, [3] proved that CAT(0) spaces are characterized by a corresponding nonlinear version of the Cauchy-Schwarz inequality,

$$\langle \overrightarrow{xy}, \overrightarrow{uv} \rangle \leq d(x,y)d(u,v). \tag{CS}$$

Using (CN$^+$), one derives the following useful inequality.

Lemma 2.3. *Let X be a* CAT(0) *space, and consider $x, y, z \in X$ and $\lambda \in [0,1]$. Then,*

$$d^2((1-\lambda)x \oplus \lambda y, z) \leq (1-\lambda)^2 d^2(x,z) + 2\lambda(1-\lambda)\langle \overrightarrow{xz}, \overrightarrow{yz} \rangle + \lambda^2 d^2(y,z).$$

2.2. **Quantitative notions and useful lemmas.** Let (x_n) be a sequence in a metric space (X,d) and $x \in X$. We denote $[a;b]$ for the set of natural numbers between a and b, i.e. $[a;b] := [a,b] \cap \mathbb{N}$.

Definition 2.4. *We say that*

(1) $\varphi : (0,\infty) \to \mathbb{N}$ *is a rate of convergence for* (x_n) *(towards* x*) if*
$$\forall \varepsilon > 0\ \forall m \geq \varphi(\varepsilon)\ (d(x_m, x) \leq \varepsilon),$$
(2) $\varphi : (0,\infty) \to \mathbb{N}$ *is a Cauchy rate for* (x_n) *if*
$$\forall \varepsilon > 0\ \forall i,j \geq \varphi(\varepsilon)\ (d(x_i, x_j) \leq \varepsilon),$$
(3) $\Phi : (0,\infty) \times \mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ *is a rate of metastability for* (x_n) *if*
$$\forall \varepsilon > 0\ \forall f \in \mathbb{N}^{\mathbb{N}}\ \exists n \leq \Phi(\varepsilon, f)\ \forall i,j \in [n; f(n)]\ (d(x_i, x_j) \leq \varepsilon).$$

The following result is easily argued by contradiction (see e.g. [17, Lemma 2.5]).

Lemma 2.5. *Let (X, d) be a metric space and (x_n) a sequence in X. Then, (x_n) has the metastability property, i.e.*

$$\forall \varepsilon > 0\ \forall f \in \mathbb{N}^{\mathbb{N}}\ \exists n \in \mathbb{N}\ \forall i, j \in [n; f(n)]\, (d(x_i, x_j) \leq \varepsilon)$$

if and only if (x_n) is a Cauchy sequence.

Note that the despite being equivalent, unsurprisingly metastability is computationally weaker than the Cauchy property, as the proof that metastability implies the Cauchy property is non-effective. Therefore, the existence of a computable rate of metastability usually does not entail a computable Cauchy rate. Moreover, considerations from computability theory exclude the existence of general computable Cauchy rates, while proof-theoretical metatheorems guarantee the availability of computable uniform rates of metastability in broad circumstances. On the other hand, a function φ is a Cauchy rate if and only if $\Phi(\varepsilon, f) = \varphi(\varepsilon)$ is a rate of metastability (e.g. [28, Proposition 2]).

We will also require the following notions of asymptotic regularity (cf. [2, 7]) as well as their corresponding quantitative characterizations.

Definition 2.6. *Consider a family of maps $\{T_n : C \to C\}$ and a map $T : C \to C$, for some nonempty $C \subseteq X$. We say that a sequence $(x_n) \subseteq C$ is*

1. *asymptotically regular if $\lim d(x_n, x_{n+1}) = 0$. A rate of asymptotic regularity for (x_n) is a rate of convergence for $(d(x_n, x_{n+1}))$ towards 0.*
2. *(T_n)–asymptotically regular if $\lim d(x_n, T_n(x_n)) = 0$. A rate of (T_n)–asymptotic regularity for (x_n) is a rate of convergence for $(d(x_n, T_n(x_n))$ towards 0.*
3. *T–asymptotically regular if $\lim d(x_n, T(x_n)) = 0$. A rate of T–asymptotic regularity for (x_n) is a rate of convergence for $(d(x_n, T(x_n)))$ towards 0.*

We recall a well-known lemma by Xu [49] which is particularly useful in the study of fixed point iterative methods.

Lemma 2.7. *Let (s_n) be a sequence of nonnegative real numbers such that*

$$\text{for all } n \in \mathbb{N},\ s_{n+1} \leq (1 - a_n)s_n + a_n b_n + c_n,$$

with sequences $(a_n) \subseteq [0, 1]$, $(b_n) \subseteq \mathbb{R}$, and $(c_n) \subseteq [0, +\infty)$ satisfying

$$(i)\ \sum a_n = +\infty, \quad (ii)\ \limsup b_n \leq 0, \quad (iii) \sum c_n < +\infty.$$

Then $\lim s_n = 0$.

We say that $\theta : \mathbb{N} \to \mathbb{N}$ is a rate of divergence for a sequence of real numbers (r_n) satisfying $\lim r_n = +\infty$, whenever

$$\forall K \in \mathbb{N}\ \forall m \geq \theta(K)\, (r_m \geq K).$$

Quantitative versions of Lemma 2.7 were given before, for example in [27] for the case $c_n \equiv 0$, and for the general case in [33, Section 3]. We shall require first a quantitative version of this lemma in the particular case when $b_n \equiv 0$, which we give below and whose proof can be found for example in [17, Lemma 2.9(1)].

Proposition 2.8. *Let (s_n), $(c_n) \subseteq [0, +\infty)$ and $(a_n) \subseteq [0, 1]$ be such that*

$$\text{for all } n \in \mathbb{N},\ s_{n+1} \leq (1 - a_n)s_n + c_n.$$

Assume that $L \in \mathbb{N}$ is an upper bound on (s_n), that $\sum a_n = +\infty$ with rate of divergence θ, and that $\sum c_n$ converges with a Cauchy rate χ. Then $\lim s_n = 0$ with rate of convergence

$$\Sigma(\varepsilon) := \theta\left(\chi\left(\frac{\varepsilon}{2}\right) + 1 + \left\lceil \ln\left(\frac{2L}{\varepsilon}\right)\right\rceil\right) + 1.$$

Sabach and Shtern [41, Lemma 3] proved an interesting version of Xu's lemma that, with a particular choice of the sequence (a_n), allowed them to obtain linear rates of asymptotic regularity for the sequential averaging method (SAM), itself a generalization of the Halpern iteration. The following version is from [34].

Lemma 2.9. *Let $L > 0$, $J \geq N \geq 2$, and $\gamma \in (0,1]$. Assume that $a_n = \frac{N}{\gamma(n+J)}$ and $c_n \leq L$ for all $n \in \mathbb{N}$. Consider a sequence of nonnegative real numbers (s_n) satisfying $s_0 \leq L$ and $s_{n+1} \leq (1-\gamma a_{n+1})s_n + (a_n - a_{n+1})c_n$, for all $n \in \mathbb{N}$. Then,*

$$s_n \leq \frac{JL}{\gamma(n+J)}, \text{ for all } n \in \mathbb{N}.$$

A slightly more complicate version of Lemma 2.8 will be needed in establishing the metastability property of (HM_∞). A proof can be found in [17, Lemma 2.11] (see also similar quantitative versions in the previous [27]).

Lemma 2.10. *Let (s_n) be a bounded sequence of non-negative real numbers and $L \in \mathbb{N}\backslash\{0\}$ an upper bound on (s_n). Consider sequences of real numbers $(a_n) \subseteq [0,1]$, $(b_n) \subseteq \mathbb{R}$ and assume that $\sum a_n = \infty$ with rate of divergence θ. Let $\varepsilon > 0$, $K, P \in \mathbb{N}$ be given. If for all $m \in [K; P]$*

$$(i)\ s_{m+1} \leq (1-a_m)s_m + a_m b_m + \frac{\varepsilon}{3(P+1)} \quad \text{and} \quad (ii)\ b_m \leq \frac{\varepsilon}{3},$$

then $\forall m \in [\sigma; P]\,(s_m \leq \varepsilon)$, where

$$\sigma := \theta\left(K + \left\lceil \ln\left(\frac{3L}{\varepsilon}\right)\right\rceil\right) + 1.$$

3. Rates of convergence and asymptotic regularity

In this section we study the asymptotic behaviour of the sequence with regards to the family of nonexpansive maps. We obtain uniform rates of asymptotic regularity of low complexity in general nonlinear geodesic settings.

3.1. **Initial considerations and the NST conditions.** A frequent first step in proving the convergence of sequences approximating a fixed point is to establish the asymptotic regularity of the sequence. When stating that a sequence (u_n) is asymptotically regular with respect to a map T, i.e. $\lim d(u_n, T(u_n)) = 0$, what one means is that such sequence behaves asymptotically as a fixed point of T. When generalizing some iterative scheme to families of maps, it is not hard to find simple conditions in order to extend the original arguments and in the end prove $\lim d(u_n, T_n(u_n)) = 0$. However, in this case, the corresponding 'asymptotic meaning' is no longer fully clear. Indeed, the asymptotic regularity result for a sequence wanting to approximate a common fixed point of a family of maps $\{T_n\}$ should entails that the sequence behaves asymptotically as a common fixed point. In this perspective, what one would like to have is that the sequence is asymptotically regular with respect to each of the individual maps, i.e. $\lim d(u_n, T_j(u_n)) = 0$, for all $j \in \mathbb{N}$. In order to bridge the gap between this "diagonal-type" of asymptotic regularity and the desired result, several notions have been introduced in the literature. A pair of conditions which have played a crucial role in discussing the asymptotic regularity in the context of families of maps are the so-called NST conditions (originally considered by Nakajo, Shimoji and Takahashi in Banach spaces, cf. [37, 46]):

Let T be a map and $\{T_n\}$ be a countable family of maps, all defined on a set $C \subseteq X$ and such that $\mathrm{Fix}(\{T_n\}) \neq \emptyset$.[2] Then, $\{T_n\}$ satisfies the NST condition (I) with respect to the map T if $\mathrm{Fix}(T) \subseteq \mathrm{Fix}(\{T_n\})$ and for any bounded sequence $(u_n) \subseteq C$,

$$\lim d(u_n, T_n(u_n)) = 0 \Rightarrow \lim d(u_n, T(u_n)) = 0.$$

$\{T_n\}$ is also said to satisfy the NST-condition (II) if for each bounded sequence $(u_n) \subseteq C$

$$\text{if} \quad \lim d(u_n, T_n(u_n)) = 0, \quad \text{then} \quad \forall j \in \mathbb{N} \left(\lim d(u_n, T_j(u_n)) = 0\right).$$

For the sequel, consider a reference point $p \in X$ fixed. A quantitative perspective on the NST condition (I) gives rise to the following definition.

Definition 3.1. *We say that (γ, Γ) is an NST_1–modulus for the pair $(\{T_n\}, T)$ if*

$$\forall \varepsilon > 0\, \forall b, J \in \mathbb{N}\, \forall u \in \overline{B}_b(p) \cap C \left(d(u, T(u)) \leq \gamma(b, J, \varepsilon) \to \forall j \leq J \left(d(u, T_j(u)) \leq \varepsilon\right)\right)$$

and, for any $b \in \mathbb{N}$, $\varphi : (0, \infty) \times \mathbb{N}^{\mathbb{N}} \to \mathbb{N}$, and sequence $(u_n) \subseteq \overline{B}_b(p) \cap C$,

$$\textit{if } d(u_n, T_n(u_n)) \to 0 \textit{ with rate } \varphi, \textit{ then } d(u_n, T(u_n)) \to 0 \textit{ with rate } \Gamma(b, \varphi)$$

When we write 'rate', it is intended in its general form of 'rate of metastability'. Thus, in the case above, we have that if

$$\forall \varepsilon > 0\, \forall f \in \mathbb{N}^{\mathbb{N}}\, \exists n \leq \varphi(\varepsilon, f)\, \forall m \in [n; f(n)]\, (d(u_m, T_m(u_m)) \leq \varepsilon),$$

then

$$\forall \varepsilon > 0\, \forall f \in \mathbb{N}^{\mathbb{N}}\, \exists n \leq \Gamma(b, \varphi)(\varepsilon, f)\, \forall m \in [n; f(n)]\, (d(u_m, T(u_m)) \leq \varepsilon).$$

Remark 3.2. *Note that, in finite dimensional metric spaces the existence of an NST_1–modulus is guaranteed for any family $\{T_n\}$ satisfying the NST condition (I) with respect to some map T. In general however, the existence of such a modulus is a stronger requirement (due to the uniformization of the property) but a necessary one for the proof-theoretical techniques underlying these quantitative studies.*

Example 3.3. *Let $(c_n) \subseteq (0, \infty)$ be a sequence of positive real numbers such that $\inf c_n \geq \hat{c} > 0$. Let X be a Banach space and consider T_n to be the single-valued resolvent function $J^A_{c_n} = (\mathrm{Id} + c_n A)^{-1}$ of an accretive operator A subject to the range condition,*

$$\overline{\mathrm{D}(A)} \subseteq C \subseteq \mathrm{R}(\mathrm{Id} + cA), \quad \textit{for all } c > 0,$$

where $\overline{\mathrm{D}(A)}$ is the closure of the domain of A, and C is a nonempty closed subset of X. One moreover assumes that $\mathrm{zer}(A) \neq \emptyset$. From the well-known resolvent identity, it easily follows that

$$\forall n, m \in \mathbb{N}\, \forall x \in C \left(\|x - J^A_{c_n}(x)\| \leq \left(2 + \frac{c_n}{c_m}\right) \|x - J^A_{c_m}(x)\|\right).$$

[2] We denote $\mathrm{Fix}(\{T_n\}) := \bigcap_{n \in \mathbb{N}} \mathrm{Fix}(T_n)$.

Therefore, the families of maps $\{J^A_{c_n}\}$ *satisfies the NST condition (I) with respect to the map* J^A_1 *with an* NST_1*–modulus given by*

$$\gamma(b,J,\varepsilon) := \frac{\varepsilon}{2+\max\{c_j \mid j \le J\}} \quad \textit{and} \quad \Gamma(b,\varphi)(\varepsilon,f) := \varphi\left(\frac{\varepsilon}{2+1/\hat{c}}, f\right).$$

Example 3.4. *In a metric space, a resolvent-like family of nonexpansive maps* [38] *is a family of nonexpansive maps* $\{T_n : C \to C\}$ *satisfying* $\mathrm{Fix}(T_n) \neq \emptyset$ *for some* $n \in \mathbb{N}$*, and*

(1) There exists a function $\mu : \mathbb{N} \to [1,\infty)$ *satisfying*

$$\forall n,m \in \mathbb{N}\ \forall x \in C\big(d(x,T_n(x)) \le \mu(n)\cdot d(x,T_m(x))\big);$$

(2) There exists a function $\Delta : \mathbb{N}\times(0,\infty) \to (0,\infty)$ *satisfying*

$$\forall \varepsilon > 0\ \forall r\ \forall n \in \mathbb{N}\ \forall p \in C\ \forall x \in C \cap \overline{B}_r(p)$$
$$\Big(\Big(d(p,T_n(p)) \le \Delta(r,\varepsilon) \wedge d(x,p) - d(T_n(x),p) \le \Delta(r,\varepsilon)\Big) \to d(x,T_n(x)) \le \varepsilon\Big).$$

This notion was introduced in [38] *by the second author as an abstraction of central properties of both resolvent functions of accretive operators in Banach spaces as well as families of maps that are jointly (P2) in the nonlinear setting of* $\mathrm{CAT}(0)$ *spaces – see e.g.* [32, 44]*. Furthermore,* [38] *shows that this general notion already entails crucial properties useful in the study of Halpern-type iterative methods. Similar to the previous example, the condition (1) above entails that* $\{T_n\}$ *satisfies the NST condition (I) with respect to the map* T_1 *with an* NST_1*–modulus given by*[3]

$$\gamma(b,J,\varepsilon) := \frac{\varepsilon}{\max\{\mu(j) \mid j \le J\}} \quad \textit{and} \quad \Gamma(b,\varphi)(\varepsilon,f) := \varphi\left(\frac{\varepsilon}{\mu(1)}, f\right).$$

A quantitative perspective on the NST condition (II) gives rise to the following definition.

Definition 3.5. *We say that* ζ *is an* NST_2*–modulus for* $\{T_n\}$ *if for any function* $\varphi : (0,\infty)\times\mathbb{N}^{\mathbb{N}} \to \mathbb{N}$*, natural numbers* $b, J \in \mathbb{N}$ *and sequence* $(u_n) \subseteq \overline{B}_b(p) \cap C$*,*

$$\textit{if } d(u_n,T_n(u_n)) \to 0 \textit{ with rate } \varphi,$$
$$\textit{then } \max_{j\le J}\{d(u_n,T_j(u_n))\} \to 0 \textit{ with rate } \zeta(b,\varphi,J).$$

Proposition 3.6. *If a pair* $(\{T_n\},T)$ *has an* NST_1*–modulus* (γ,Γ)*, then*

$$\zeta(b,\varphi,J)(\varepsilon,f) := \Gamma(b,\varphi)(\gamma(b,J,\varepsilon),f)$$

is an NST_2*–modulus for* $\{T_n\}$*.*

Proof. Let (u_n) be a sequence in $\overline{B}_b(p) \cap C$ such that $d(u_n,T_n(u_n)) \to 0$ with rate of metastability φ. Since (γ,Γ) is an NST_1-modulus for the pair $(\{T_n\},T)$, we have that $\Gamma(b,\varphi)$ is a rate of metastability for $d(u_n,T(u_n)) \to 0$. Therefore, for any $J \in \mathbb{N}$

$$\forall \varepsilon > 0\ \forall f \in \mathbb{N}^{\mathbb{N}}\ \exists n \le \zeta(b,\varphi,J)(\varepsilon,f)\ \forall m \in [n;f(n)]\,(d(u_m,T(u_m)) \le \gamma(b,J,\varepsilon)),$$

and, again by the fact that (γ,Γ) is an NST_1-modulus, the result follows. ∎

[3]In fact, $\{T_n\}$ satisfies the NST condition (I) with respect to T_k for any $k \in \mathbb{N}$, with an NST_1–modulus given by $\gamma(b,J,\varepsilon) := \dfrac{\varepsilon}{\max\{\mu(j) \mid j \le J\}}$ and $\Gamma(b,\varphi)(\varepsilon,f) := \varphi\left(\frac{\varepsilon}{\mu(k)}, f\right)$.

Remark 3.7. *The reverse direction to Proposition 3.6 is a more delicate matter. Even for a pair* $(\{T_n\}, T_{j_0})$*, the existence of an* NST_2*–modulus does not, in general, imply the existence of an* NST_1*–modulus, since the first condition typically fails. Note, however, that if there exists a function* γ *such that for all* $\varepsilon > 0$ *and* $b, J \in \mathbb{N}$*,*

$$\forall u \in \overline{B}_b(p) \cap C \left(d(u, T_{j_0}(u)) \leq \gamma(b, J, \varepsilon) \to \forall j \leq J \left(d(u, T_j(u)) \leq \varepsilon\right)\right),$$

then an NST_2*–modulus is immediately sufficient to construct the second component of an* NST_1*–modulus. Indeed, in this case one may simply define*

$$\Gamma(b, \varphi) := \zeta(b, \varphi, j_0).$$

In Theorem 1.1, we state that the families $\{T_n\}$ and $\{U_n\}$ must come equipped with NST$_2$–moduli. However, as we shall see in section 5, a weaker requirement suffices where we only ask for a function outputting a rate of metastability for the conclusion, from the stronger input of a convergence rate for the premise.

3.2. **Conditions on the parameters.** We now consider a nonempty convex subset C of X, two families of nonexpansive maps $\{T_n : C \to C\}$ and $\{U_n : C \to C\}$. For $x_0 \in C$, $u \in C$ and sequences $(\alpha_n), (\beta_n) \subseteq [0, 1]$, let (x_n) be the iteration generated by (HM$_\infty$). We work with the following conditions:

(Q1) $\lim \alpha_n = 0$ with rate of convergence $\sigma_1 : (0, \infty) \to \mathbb{N}$.
(Q2) $\sum \alpha_n = \infty$ with rate of divergence $\sigma_2 : \mathbb{N} \to \mathbb{N}$.
(Q3) $\sum |\alpha_{n+1} - \alpha_n| < \infty$ with Cauchy rate $\sigma_3 : (0, \infty) \to \mathbb{N}$.
(Q4) $\sum |\beta_{n+1} - \beta_n| < \infty$ with Cauchy rate $\sigma_4 : (0, \infty) \to \mathbb{N}$.
(Q5) $\sum d(T_{n+1}(x_n), T_n(x_n)) < \infty$ with a Cauchy rate $\sigma_5 : (0, \infty) \to \mathbb{N}$.
(Q6) $\sum d(U_{n+1}(y_n), U_n(y_n)) < \infty$ with a Cauchy rate $\sigma_6 : (0, \infty) \to \mathbb{N}$.
(Q7) $\beta \in (0, 1/2]$ is such that $\beta \leq \beta_n \leq 1 - \beta$, for all $n \in \mathbb{N}$.

We further assume the feasibility of finding a common fixed point to the families of maps $\{T_n\}$ and $\{U_n\}$, i.e. we assume that $\mathrm{F} := \mathrm{Fix}(\{T_n\}) \cap \mathrm{Fix}(\{U_n\}) \neq \emptyset$ and consider a natural number $b \geq 1$ such that $b \geq \max\{d(u, p), d(x_0, p)\}$, for some point $p \in \mathrm{F}$.

Lemma 3.8. *The sequence* (x_n) *is bounded, and for all* $n \in \mathbb{N}$

$$d(x_n, p), d(y_n, p) \leq b.$$

Proof. The result is established by an easy induction. We have $d(x_0, p) \leq b$ by hypothesis, and

$$d(y_0, p) \leq (1 - \alpha_0) d(T_0(x_0), p) + \alpha_0 d(u, p) \leq b$$

For the induction step,

$$d(x_{n+1}, p) \leq (1 - \beta_n) d(U_n(y_n), p) + \beta_n d(y_n, p) \leq d(y_n, p) \overset{\text{IH}}{\leq} b,$$

and

$$d(y_{n+1}, p) \leq (1 - \alpha_{n+1}) d(T_{n+1}(x_{n+1}), p) + \alpha_{n+1} d(u, p) \overset{\text{IH}}{\leq} b. \qquad \square$$

3.3. **Rates in UCW–hyperbolic spaces.** We begin with the following easy result.

Proposition 3.9. *We have the following,*

(*i*) $\lim d(x_n, x_{n+1}) = 0$, *and* (x_n) *has a rate of asymptotic regularity defined by*

$$\psi_1(\varepsilon) := \sigma_2\left(\chi\left(\frac{\varepsilon}{2}\right) + 2 + \left\lceil \ln\left(\frac{4b}{\varepsilon}\right)\right\rceil\right),$$

where $\chi(\varepsilon) := \max\left\{\sigma_3\left(\frac{\varepsilon}{8b}\right), \sigma_4\left(\frac{\varepsilon}{8b}\right), \sigma_5\left(\frac{\varepsilon}{4}\right), \sigma_6\left(\frac{\varepsilon}{4}\right)\right\}$.

(*ii*) $\lim d(y_n, y_{n+1}) = 0$, *and* (y_n) *has a rate of asymptotic regularity defined by*

$$\psi_2(\varepsilon) := \max\left\{\psi_1\left(\frac{\varepsilon}{3}\right), \sigma_3\left(\frac{\varepsilon}{6b}\right) + 1, \sigma_5\left(\frac{\varepsilon}{3}\right) + 1\right\}.$$

Proof. (*i*): We have

$$\begin{aligned} d(y_{n+1}, y_n) &\leq d(y_{n+1}, (1-\alpha_{n+1})T_n(x_n) \oplus \alpha_{n+1}u) \\ &\quad + d((1-\alpha_{n+1})T_n(x_n) \oplus \alpha_{n+1}u, y_n) \\ &\leq (1-\alpha_{n+1})d(T_{n+1}(x_{n+1}), T_n(x_n)) + |\alpha_{n+1} - \alpha_n| d(T_n(x_n), u) \\ &\leq (1-\alpha_{n+1})d(x_{n+1}, x_n) + 2b \cdot |\alpha_{n+1} - \alpha_n| + d(T_{n+1}(x_n), T_n(x_n)) \end{aligned}$$

and so

$$\begin{aligned} d(x_{n+2}, x_{n+1}) &\leq d(x_{n+2}, (1-\beta_{n+1})U_n(y_n) \oplus \beta_{n+1}y_n) \\ &\quad + d((1-\beta_{n+1})U_n(y_n) \oplus \beta_{n+1}y_n, x_{n+1}) \\ &\leq (1-\beta_{n+1})d(U_{n+1}(y_{n+1}), U_n(y_n)) + \beta_{n+1}d(y_{n+1}, y_n) \\ &\quad + |\beta_{n+1} - \beta_n| d(U_n(y_n), y_n) \\ &\leq d(y_{n+1}, y_n) + 2b \cdot |\beta_{n+1} - \beta_n| + d(U_{n+1}(y_n), U_n(y_n)) \\ &\leq (1-\alpha_{n+1})d(x_{n+1}, x_n) + 2b\left(|\alpha_{n+1} - \alpha_n| + |\beta_{n+1} - \beta_n|\right) \\ &\qquad + d(T_{n+1}(x_n), T_n(x_n)) + d(U_{n+1}(y_n), U_n(y_n)). \end{aligned}$$

With $s_n := d(x_{n+1}, x_n)$, $a_n := \alpha_{n+1}$, and

$c_n := 2b\left(|\alpha_{n+1} - \alpha_n| + |\beta_{n+1} - \beta_n|\right) + d(T_{n+1}(x_n), T_n(x_n)) + d(U_{n+1}(y_n), U_n(y_n))$,

we can rewrite the inequality in the compact form

$$s_{n+1} \leq (1-a_n)s_n + c_n$$

From the conditions (Q3)–(Q6), we have for all $m \in \mathbb{N}$,

$$\sum_{k=\chi(\varepsilon)+1}^{m} c_k \leq 2b\left(\frac{\varepsilon}{8b} + \frac{\varepsilon}{8b}\right) + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \varepsilon$$

thus χ is a Cauchy rate for $(\sum_0^n c_k)$.

Writing $\theta(K) := \max\{\sigma_2(K+1) - 1, 0\}$, for any $L \in \mathbb{N}$, using (Q2) and the fact that $\alpha_0 \leq 1$, we have

$$\sum_{k=0}^{\theta(K)} a_k = \sum_{k=0}^{\theta(K)} \alpha_{k+1} = \sum_{k=0}^{\theta(K)+1} \alpha_k - \alpha_0 \geq \sum_{k=0}^{\sigma_2(K+1)} \alpha_k - \alpha_0 \geq K + 1 - \alpha_0 \geq K.$$

Thus, θ is a rate of divergence for $\left(\sum_0^n a_k\right)$. Since $d(x_{n+1}, x_n) \leq 2b$ for all $n \in \mathbb{N}$, by Lemma 2.8 we conclude that ψ_1 is a rate of asymptotic regularity for (x_n).

(ii): For $n \geq \psi_2(\varepsilon)$, we have

$$\begin{aligned} d(y_{n+1}, y_n) &\leq d(x_{n+1}, x_n) + 2b|\alpha_{n+1} - \alpha_n| + d(T_{n+1}(x_n), T_n(x_n)) \\ &\leq \frac{\varepsilon}{3} + 2b\frac{\varepsilon}{6b} + \frac{\varepsilon}{3} = \varepsilon, \end{aligned}$$

and thus, ψ_2 is a rate of asymptotic regularity for (y_n). □

Assume now that X is a UCW–hyperbolic space with a monotone modulus of uniform convexity $\eta : (0, \infty) \times (0, 2] \to (0, 1]$. The next result establishes the central result on the asymptotic regularity of the iterative schema (HM_∞).[4]

Proposition 3.10. *We have the following*

(i) $\lim d(y_n, U_n(y_n)) = 0$, *and* (y_n) *has a rate of* (U_n)*–asymptotic regularity defined by*

$$\psi_3(\varepsilon) := \max\left\{\psi_1\left(\frac{\varepsilon}{2P}\right), \sigma_1\left(\frac{\varepsilon}{2Pb}\right)\right\}, \quad \textit{with} \;\; P := \frac{1}{\beta^2\eta\left(b, \frac{\varepsilon}{b}\right)}.$$

(ii) *If* $\eta(r, \varepsilon) = \varepsilon \cdot \tilde{\eta}(r, \varepsilon)$ *for some* $\tilde{\eta}$ *that increases with* ε, *then we can replace* P *by*

$$\tilde{P} := \frac{1}{\beta^2\tilde{\eta}\left(b, \frac{\varepsilon}{b}\right)}.$$

Proof. (i): For a given $\varepsilon > 0$, consider $n \geq \psi_3(\varepsilon)$. Assume towards a contradiction that $d(y_n, U_n(y_n)) > \varepsilon$. Since $d(y_n, U_n(y_n)) \leq d(y_n, p) + d(p, U_n(y_n)) \leq 2d(y_n, p)$, we get that

$$\frac{\varepsilon}{2} < d(y_n, p) \tag{1}$$

Furthermore, we have that $d(y_n, p) \leq b$, thus $d(U_n(y_n), p) \leq b$ and

$$d(y_n, U_n(y_n)) > \frac{\varepsilon}{b}d(y_n, p).$$

We also deduce that $\frac{\varepsilon}{b} < 2$. We can thus apply Lemma 2.1 with $\varepsilon = \frac{\varepsilon}{b}$ and

$$x := U_n(y_n),\; y := y_n,\; a := p,\; r := d(y_n, p),\; \lambda := \beta_n,\; s := b$$

to conclude that

$$\begin{aligned} d(x_{n+1}, p) &= d(\beta_n y_n + (1 - \beta_n)U_n(y_n), p) \\ &\leq \left(1 - 2\beta_n(1 - \beta_n)\eta\left(b, \frac{\varepsilon}{b}\right)\right) d(y_n, p) \\ &= d(y_n, p) - 2d(y_n, p)\beta_n(1 - \beta_n)\eta\left(b, \frac{\varepsilon}{b}\right) \\ &\leq d(y_n, p) - 2d(y_n, p)\beta^2\eta\left(b, \frac{\varepsilon}{b}\right) \quad \text{by (Q7)} \\ &< d(y_n, p) - \varepsilon\beta^2\eta\left(b, \frac{\varepsilon}{b}\right) \quad \text{by (1)} \end{aligned}$$

Since, by (W1), $d(y_n, p) \leq (1 - \alpha_n)d(T(x_n), p) + \alpha_n d(u, p) \leq d(x_n, p) + \alpha_n b$, we get that

$$d(x_{n+1}, p) < d(x_n, p) + \alpha_n b - \varepsilon\beta^2\eta\left(b, \frac{\varepsilon}{b}\right)$$

[4]See Remark 5.2.

Thus,

$$\begin{aligned}\frac{\varepsilon}{P} &= \varepsilon\beta^2\eta\left(b,\frac{\varepsilon}{b}\right)\\ &< d(x_n,p)-d(x_{n+1},p)+\alpha_n b\\ &\le d(x_{n+1},x_n)+\alpha_n b\\ &\le \frac{\varepsilon}{2P}+\frac{\varepsilon}{2Pb}b=\frac{\varepsilon}{P} \quad \text{as } n\ge\psi_3(\varepsilon)\end{aligned}$$

which is a contradiction. Thus ψ_3 is a rate of (U_n)-asymptotic regularity of (y_n).

(ii): For a given $\varepsilon>0$, consider $n\ge\psi_3(\varepsilon)$ (with P replaced by $\tilde{P}$). We follow the proof of (i), but apply Lemma 2.1 with $\varepsilon=\dfrac{\varepsilon}{d(y_n,p)}$ instead of $\varepsilon=\dfrac{\varepsilon}{b}$. Then,

$$\begin{aligned}d(x_{n+1},p) &\le \left(1-2\beta^2\eta\left(b,\frac{\varepsilon}{d(y_n,p)}\right)\right)\cdot d(y_n,p)\\ &= d(y_n,p)-2\varepsilon\beta^2\tilde{\eta}\left(b,\frac{\varepsilon}{d(y_n,p)}\right)\\ &< d(y_n,p)-\varepsilon\beta^2\tilde{\eta}\left(b,\frac{\varepsilon}{d(y_n,p)}\right)\end{aligned}$$

Since $d(y_n,p)\le b$ and $\tilde{\eta}$ increases with ε, we have

$$\tilde{\eta}\left(b,\frac{\varepsilon}{d(y_n,p)}\right)\ge\tilde{\eta}\left(b,\frac{\varepsilon}{b}\right),$$

and it follows that

$$d(x_{n+1},p)<d(y_n,p)-\varepsilon\beta^2\tilde{\eta}\left(b,\frac{\varepsilon}{b}\right).$$

The argument now continues as in (i) with P and η replaced by $\tilde{P}$ and $\tilde{\eta}$, respectively. □

We can now easily compute the remaining rates of asymptotic regularity.

Proposition 3.11. *We have the following*

(i) $\lim d(x_{n+1},y_n)=0$ *with rate of convergence* ψ_3.

(ii) $\lim d(x_n,y_n)=0$ *with rate of convergence*

$$\psi_4(\varepsilon):=\max\left\{\psi_1\left(\frac{\varepsilon}{2}\right),\psi_3\left(\frac{\varepsilon}{2}\right)\right\}.$$

(iii) $\lim d(x_n,U_n(x_n))=0$, *and* (x_n) *has a rate of* (U_n)*–asymptotic regularity given by*

$$\psi_5(\varepsilon):=\max\left\{\psi_3\left(\frac{\varepsilon}{3}\right),\psi_4\left(\frac{\varepsilon}{3}\right)\right\}.$$

(iv) $\lim d(y_n,T_n(y_n))=0$, *and* (y_n) *has a rate of* (T_n)*–asymptotic regularity given by*

$$\psi_6(\varepsilon):=\max\left\{\sigma_1\left(\frac{\varepsilon}{4b}\right),\psi_4\left(\frac{\varepsilon}{2}\right)\right\}.$$

(v) $\lim d(x_n,T_n(x_n))=0$, *and* (x_n) *has a rate of* (T_n)*–asymptotic regularity given by*

$$\psi_7(\varepsilon):=\max\left\{\psi_4\left(\frac{\varepsilon}{3}\right),\psi_6\left(\frac{\varepsilon}{3}\right)\right\}.$$

Proof. (i): Follows from $d(x_{n+1}, y_n) = (1-\beta_n)d(y_n, U_n(y_n)) \leq d(y_n, U_n(y_n))$.

(ii): Follows easily by triangle inequality, from (i) and Proposition 3.9(i).

(iii): For a given $\varepsilon > 0$, if $n \geq \psi_5(\varepsilon)$ then, using the nonexpansivity of U_n, we get

$$\begin{aligned} d(x_n, U_n(x_n)) &\leq d(x_n, y_n) + d(y_n, U_n(y_n)) + d(U_n(y_n), U_n(x_n)) \\ &\leq 2d(x_n, y_n) + d(y_n, U_n(y_n)) \leq \frac{2\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

(iv): With $n \geq \psi_6(\varepsilon)$, using (W1) and the definition of y_n, we have

$$\begin{aligned} d(y_n, T_n(y_n)) &\leq (1-\alpha_n)d(T_n(x_n), T_n(y_n)) + \alpha_n d(u, T_n(y_n)) \\ &\leq d(x_n, y_n) + 2b \cdot \alpha_n \leq 2b\frac{\varepsilon}{4b} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

(v): Similarly to (iii), for $n \geq \psi_7(\varepsilon)$, we have

$$\begin{aligned} d(x_n, T_n(x_n)) &\leq d(x_n, y_n) + d(y_n, T_n(y_n)) + d(T_n(y_n), T_n(x_n)) \\ &\leq 2d(x_n, y_n) + d(y_n, T_n(y_n)) \leq \frac{2\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

□

3.4. **Rates in CAT(0) spaces.** Here we briefly discuss rates of asymptotic regularity in the particular case when X is a CAT(0) space. As per Lemma 2.2, CAT(0) spaces are UCW–hyperbolic spaces and $\eta(\varepsilon) = \varepsilon^2/8$ is a monotone modulus of uniform convexity. Therefore, $\eta(\varepsilon) = \varepsilon \cdot \tilde{\eta}(\varepsilon)$ and we can use Proposition 3.10(ii) to immediately obtain a rate of (U_n)–asymptotic regularity for (y_n),

$$\psi_3(\varepsilon) := \max\left\{\psi_1\left(\frac{\varepsilon^2\beta^2}{16b}\right), \sigma_1\left(\frac{\varepsilon^2\beta^2}{16b^2}\right)\right\}.$$

A small improvement on the constants is possible by a short argument (similar to the one used in [17]), which we include for completeness.

Proposition 3.12. *If X is a* CAT(0) *space, then (y_n) has a rate of (U_n)-asymptotic regularity*

$$\psi_3(\varepsilon) := \max\left\{\psi_1\left(\frac{\varepsilon^2\beta^2}{4b}\right), \sigma_1\left(\frac{\varepsilon^2\beta^2}{2b^2}\right)\right\}.$$

Proof. Using the CN$^+$-inequality, we have for any $n \in \mathbb{N}$

$$d^2(y_n, p) \leq d^2(x_n, p) + \alpha_n d^2(u, p),$$

and also

$$\begin{aligned} d^2(x_{n+1}, p) &\leq (1-\beta_n)d^2(U_n(y_n), p) + \beta_n d^2(y_n, p) - \beta_n(1-\beta_n)d^2(y_n, U_n(y_n)) \\ &\leq d^2(y_n, p) - \beta^2 d^2(y_n, U_n(y_n)) \\ &\leq d^2(x_n, p) + \alpha_n d^2(u, p) - \beta^2 d^2(y_n, U_n(y_n)). \end{aligned}$$

For a given $\varepsilon > 0$ and $n \geq \psi_3(\varepsilon)$, the result follows from

$$\begin{aligned} d^2(y_n, U_n(y_n)) &\leq \frac{1}{\beta^2}\left(d^2(x_n, p) - d^2(x_{n+1}, p) + \alpha_n d^2(u, p)\right) \\ &\leq \frac{1}{\beta^2}d(x_{n+1}, x_n)\left(d(x_n, p) + d(x_{n+1}, p)\right) + \frac{\alpha_n}{\beta^2}d^2(u, p) \\ &\leq \frac{2b}{\beta^2} \cdot \frac{\varepsilon^2\beta^2}{4b} + \frac{b^2}{\beta^2} \cdot \frac{\varepsilon^2\beta^2}{2b^2} = \varepsilon^2. \end{aligned}$$

□

3.5. **Linear and quadratic rates.** Take $\beta_n :\equiv \beta \in (0, 1/2]$ and $\alpha_n := \dfrac{2}{n+2}$. Consider families of nonexpansive maps $\{T_n\}$ and $\{U_n\}$ subject to the condition

$$d(U_{n+1}(y_n), U_n(y_n)) + d(T_{n+1}(x_n), T_n(x_n)) \leq \frac{2c}{(n+2)(n+3)}$$

for some nonnegative constant $c \in \mathbb{R}$, in particular $\{T_n\}$ and $\{U_n\}$ satisfy the conditions (Q5) and (Q6).

Proposition 3.13. *The sequence (x_n) has a linear rate of asymptotic regularity*

$$\xi_c(\varepsilon) := \left\lfloor \frac{2(2b+c)}{\varepsilon} \right\rfloor.$$

Proof. For all $n \in \mathbb{N}$, as in the proof of Proposition 3.9(i), we have

$$\begin{aligned} d(x_{n+2}, x_{n+1}) \leq (1 - \alpha_{n+1}) &d(x_{n+1}, x_n) + 2b|\alpha_{n+1} - \alpha_n| \\ &+ d(T_{n+1}(x_n), T_n(x_n)) + d(U_{n+1}(y_n), U_n(y_n)) \end{aligned}$$

Hence, we can apply Lemma 2.9 with $N = J = 2$, $\gamma = 1$, $L = 2b + c$, and sequences

$$s_n := d(x_{n+1}, x_n),\ \ a_n := \alpha_n = \frac{2}{n+2},$$

$$\text{and}\ \ c_n := 2b + \frac{(n+2)(n+3)}{2}(d(U_{n+1}(y_n), U_n(y_n)) + d(T_{n+1}(x_n), T_n(x_n))).$$

Since $s_0 \leq 2b \leq L$ and $c_n \leq 2b + c = L$, we conclude $d(x_{n+1}, x_n) \leq \dfrac{2(2b+c)}{n+2}$, from which the result immediately follows. $\square$

When X is a CAT(0) space, by Propositions 3.12 and 3.11 with ξ_c in place of ψ_1, noting that we can use $\sigma_1(\varepsilon) := \lfloor 2/\varepsilon \rfloor$, we immediately obtain the following result.

Proposition 3.14. *We have the following quadratic rates of convergence:*[5]

(i) *(y_n) has a rate of (U_n)-asymptotic regularity*

$$\psi_3(\varepsilon) := \left\lfloor \frac{8b(2b+c)}{\varepsilon^2\beta^2} \right\rfloor.$$

(ii) $\lim d(x_n, y_n) = 0$ *with rate of convergence*

$$\psi_4(\varepsilon) := \left\lfloor \frac{2^5 b(2b+c)}{\varepsilon^2\beta^2} \right\rfloor.$$

(iii) *(x_n) has a rate of (U_n)–asymptotic regularity*

$$\psi_5(\varepsilon) := \left\lfloor \frac{9 \cdot 2^5 b(2b+c)}{\varepsilon^2\beta^2} \right\rfloor.$$

(iv) *(y_n) has a rate of (T_n)–asymptotic regularity*

$$\psi_6(\varepsilon) := \left\lfloor \frac{2^7 b(2b+c)}{\varepsilon^2\beta^2} \right\rfloor.$$

(v) *(x_n) has a rate of (T_n)–asymptotic regularity*

$$\psi_7(\varepsilon) := \left\lfloor \frac{9 \cdot 2^7 b(2b+c)}{\varepsilon^2\beta^2} \right\rfloor.$$

[5]For simplicity, we assume here that $\varepsilon \in (0, 1]$.

4. Metric projection

In this section we discuss our treatment of the metric projection in a geodesic setting. Consider a nonempty closed convex subset S of a complete CAT(0) space X. The metric projection onto S is the mapping $\mathrm{P}_S : X \to S$ defined for each $u \in X$ as the unique point $\mathrm{P}_S(u) \in S$ satisfying $d(u, \mathrm{P}_S(u)) = \min_{x\in S} d(u, x)$ (see e.g. [6]). As the desired conclusion is the Cauchy property of the iteration, it will suffice to work with an ε-weakening of the metric projection, namely

$$(\star) \quad \forall \varepsilon > 0\ \exists x \in S\ \forall y \in S\, (d(u, x) \leq d(u, y) + \varepsilon)\,.$$

Notation 4.1. *Consider families of nonexpansive maps $\{T_n\}$ and $\{U_n\}$ as before. For $N, r \in \mathbb{N}$ and $a \in X$, we write*

$$\mathrm{F}_N(a, r) := \left\{x \in C : \forall n \leq r \left(d(x, T_n(x)), d(x, U_n(x)) \leq \frac{1}{r+1}\right)\right\} \cap \overline{B}_N(a),$$

for the set of "almost"-common fixed points intersected with the closed ball centered at the point a with radius N.

The next result corresponds to a quantitative version of the statement $(\star)$ when S is the set of common fixed points $\mathrm{F} := \mathrm{Fix}(\{T_n\}) \cap \mathrm{Fix}(\{U_n\})$ which is assumed to be nonempty.

Proposition 4.2. *Given $u \in C$, let $N \in \mathbb{N}$ be such that $N \geq d(u, p)$, for some $p \in \mathrm{F}$. Then, for every $\varepsilon > 0$ and $f : \mathbb{N} \to \mathbb{N}$, there exist $n \leq \phi(N, \varepsilon, f)$ and $x \in \mathrm{F}_N(p, f(n))$ such that*

$$\forall y \in \mathrm{F}_N(p, n) \left(d^2(u, x) \leq d^2(u, y) + \varepsilon\right),$$

where $\phi(N, \varepsilon, f) := \max\{f^{(i)}(0) : i \leq r\}$ with $r := r(N, \varepsilon) := \lceil \frac{N^2}{\varepsilon} \rceil$.

Proof. If the result does not hold, than there exists a sequence $x_0, \ldots, x_r$ such that that $x_0 = p$ and $d^2(u, x_{i+1}) < d^2(u, x_i) - \varepsilon$, entailing the contradiction

$$d^2(u, x_r) < d^2(u, p) - r\varepsilon \leq N^2 - \frac{N^2}{\varepsilon}\varepsilon = 0. \qquad \square$$

We have the following useful characterization of the metric projection in terms of the quasi-linearization function.

Lemma 4.3 ([14])**.** *Let S be a nonempty convex closed subset of a complete* CAT(0) *space X. For any $u \in X$, it holds*

$$\forall y \in S \left(\langle \overrightarrow{\mathrm{P}_S(u)u}, \overrightarrow{\mathrm{P}_S(u)y} \rangle \leq 0\right).$$

We have a quantitative version corresponding to this characterization (this is similar to [17]; see also the discussion in section 3 of the recent [39]).

Proposition 4.4. *Let $u \in X$ and consider $N \in \mathbb{N}$ such that $N \geq d(u, p)$ for some $p \in \mathrm{F}$. For every $\varepsilon > 0$ and function $f : \mathbb{N} \to \mathbb{N}$, there exist $n \leq \Phi(N, \varepsilon, f)$ and $x \in \mathrm{F}_N(p, f(n))$ such that*

$$\forall y \in \mathrm{F}_N(p, n)\, (\langle \overrightarrow{xu}, \overrightarrow{xy} \rangle \leq \varepsilon)\,,$$

where $\Phi(N, \varepsilon, f) := 24N(\phi(N, \varepsilon_0, h_f) + 1)^2$, with $\varepsilon_0 := \dfrac{\varepsilon^2}{4N^2}$ and

$$h_f(m) := \max\{f(24N(m+1)^2), 24N(m+1)^2\}.$$

5. Metastability and Convergence

This section pertains to the study of the metastable behaviour of the (HM_∞) iteration in the setting of CAT(0) spaces, and we establish its strong convergence.

In the setting of the previous conditions (Q1)–(Q7), assume that the families $\{T_n\}$ and $\{U_n\}$ satisfy the NST condition (II) with an NST_2-modulus. It is then easy to argue that there exists a function $\widehat{\zeta}$ such that for any function $\varphi : (0, \infty) \to \mathbb{N}$, natural numbers $b, J \in \mathbb{N}$ and bounded sequence $(u_n) \subseteq \overline{B}_b(p) \cap C$,

$$(+)\quad \begin{cases} \text{if} \quad \forall \varepsilon > 0\, \forall m \geq \varphi(\varepsilon)\, (d(u_m, T_m(u_m)), d(u_m, U_m(u_m)) \leq \varepsilon)\,, \\ \text{then} \quad \forall \varepsilon > 0\, \forall f \in \mathbb{N}^{\mathbb{N}}\, \exists n \leq \widehat{\zeta}(b, J, \varphi)(\varepsilon, f) \\ \qquad\qquad \forall m \in [n; f(n)]\, \forall j \leq J\, (d(u_m, T_j(u_m)), d(u_m, U_j(u_m)) \leq \varepsilon)\,. \end{cases}$$

Indeed, since a rate of convergence is in particular a rate of metastability, having NST_2–moduli for the families entails that we have rates of metastability for both $\max_{j \leq J}\{d(u_n, T_j(u_n))\} \to 0$ and $\max_{j \leq J}\{d(u_n, U_j(u_n))\} \to 0$, whenever (u_n) is a bounded sequence in C. The conjugation of metastability statements is slightly more convoluted than the conjugation of convergence statements but is nevertheless possible (see e.g. [18, Proposition 2.10]), and gives rise to the statement (+).

In a setting where the asymptotic behaviour of the families of maps entails the existence of such a function $\widehat{\zeta}$, we can use Proposition 4.4 to obtain the following result. We remark that, this quantitative result provides a different avenue to the sequential weak compactness argument used by Boţ and Meier [5] in Hilbert spaces – the proof-theoretical justification is akin to [19]; see also the recent overview [39].

Proposition 5.1. *Consider a function $\widehat{\zeta}$ satisfying (+). For every $\varepsilon > 0$ and function $f : \mathbb{N} \to \mathbb{N}$, there exist $n \leq \beta(\varepsilon, f)$ and $x \in \mathrm{F}_b(p, f(n))$ such that*

$$\forall m \in [n; f(n)]\, (\langle \overrightarrow{xu}, \overrightarrow{xx_m} \rangle \leq \varepsilon)\,,$$

where

$$\beta(\varepsilon, f) := \max \left\{ \widehat{\zeta}(b, n, \varphi_0) \left(\frac{1}{n+1}, f \right) \,\middle|\, n \leq \Phi(b, \varepsilon, f_0) \right\}$$

with $\varphi_0(\varepsilon) := \max\{\psi_5(\varepsilon), \psi_7(\varepsilon)\}$ and

$$f_0(n) := \max \left\{ f(n') \,\middle|\, n' \leq \widehat{\zeta}(b, n, \varphi_0) \left(\frac{1}{n+1}, f \right) \right\}.$$

Proof. Let $\varepsilon > 0$ and a function $f : \mathbb{N} \to \mathbb{N}$ be given. Applying Proposition 4.4 we conclude the existence of $n_0 \leq \Phi(b, \varepsilon, f_0)$ and $x \in \mathrm{F}_b(p, f_0(n_0))$ such that

$$\forall y \in \mathrm{F}_b(p, n_0)\, (\langle \overrightarrow{xu}, \overrightarrow{xy} \rangle \leq \varepsilon)\,.$$

By Proposition 3.11, we know that

$$\forall \varepsilon > 0\, \forall m \geq \varphi_0(\varepsilon)\, (d(x_m, T_m(x_m)), d(x_m, U_m(x_m)) \leq \varepsilon)\,,$$

and since $(x_n) \subseteq \overline{B}_b(p) \cap C$, by the assumption (+) on $\widehat{\zeta}$ it follows that

$$\exists n_1 \leq \widehat{\zeta}(b, n_0, \varphi_0) \left(\frac{1}{n_0+1}, f \right)$$
$$\forall m \in [n_1; f(n_1)]\, \forall j \leq n_0 \left(d(x_m, T_j(x_m)), d(x_m, U_j(x_m)) \leq \frac{1}{n_0+1} \right),$$

i.e. $\forall m \in [n_1; f(n_1)]\,(x_m \in \mathrm{F}_b(p, n_0))$. Therefore, a fortiori $n_1 \leq \beta(\varepsilon, f)$, and

$$\forall m \in [n_1; f(n_1)]\,(\langle \overrightarrow{xu}, \overrightarrow{xx_m} \rangle \leq \varepsilon)\,.$$

Moreover, by the definition of f_0 implies that $f(n_1) \leq f_0(n_0)$ and so $x \in \mathrm{F}_b(p, f(n_1))$, concluding the proof. □

Remark 5.2. *Suppose that σ_1 in (Q1) is monotone in the sense that*

$$0 < \varepsilon \leq \varepsilon' \to \sigma_1(\varepsilon') \leq \sigma_1(\varepsilon),$$

i.e. that σ_1 is decreasing in ε. Similarly assume that ψ_1 is monotone (which itself follows from natural monotonicity assumptions on the remaining input functions σ_i) and also that η is increasing in ε. Then, ψ_3 from Proposition 3.10 (and even the optimized version for CAT(0) *spaces in Proposition 3.12) is also monotone and, under the restriction to $\varepsilon \in (0, 1]$, the following simplifications to Proposition 3.11 hold*

$$\psi_4(\varepsilon) = \psi_3\left(\frac{\varepsilon}{2}\right), \quad \psi_5(\varepsilon) = \psi_3\left(\frac{\varepsilon}{6}\right), \quad \psi_6(\varepsilon) = \psi_3\left(\frac{\varepsilon}{4}\right), \quad \psi_7(\varepsilon) = \psi_3\left(\frac{\varepsilon}{12}\right).$$

In particular, in Theorem 5.1 we can take $\varphi_0(\varepsilon) := \psi_3\left(\frac{\varepsilon}{12}\right)$. This reflects the essential role that the rate ψ_3 has in our argument and why it was singled out in Propositions 3.10 and 3.12.

We are now ready to establish the metastability property of (HM_∞).

Theorem 5.3. *Let $u \in C$, $x_0 \in C$, and consider a natural number $b \in \mathbb{N} \setminus \{0\}$ such that $b \geq \max\{d(u, p), d(x_0, p)\}$, for some $p \in \mathrm{F}$. Let (x_n) be the iteration generated by (HM_∞) with u, x_0, sequences $(\alpha_n), (\beta_n) \subseteq [0, 1]$, and families $\{T_n\}$, $\{U_n\}$ satisfying the NST condition (II) with a function $\widehat{\zeta}$ satisfying $(+)$. Further assume that the conditions (Q1)–(Q7) hold. Then,*

$$\forall \varepsilon \in (0, 1]\; \forall f \in \mathbb{N}^{\mathbb{N}}\; \exists n \leq \Omega(\varepsilon, f)\; \forall i, j \in [n; n + f(n)]\,(d(x_i, x_j) \leq \varepsilon)\,,$$

where

$$\Omega(\varepsilon, f) := \max\left\{\sigma(n) \,\middle|\, n \leq \max\{\beta(\varepsilon_1, f_1), \varphi_1(\varepsilon)\}\right\}$$

with

$$\varepsilon_1 := \frac{\varepsilon^2}{72}, \;\; f_1(n) := \left\lfloor \frac{144b\,(f_2\,(\max\{n, \varphi_1(\varepsilon)\}) + 1)}{\varepsilon^2} \right\rfloor$$

$$\varphi_1(\varepsilon) := \max\left\{\sigma_1\left(\frac{\varepsilon^2}{144b^2}\right), \psi_7\left(\frac{\varepsilon^2}{144b}\right)\right\}, \;\; f_2(n) := f(\sigma(n))$$

$$\textit{and} \;\; \sigma(n) := \sigma_2\left(n + \left\lceil \ln\left(\frac{48b^2}{\varepsilon^2}\right)\right\rceil\right) + 1.$$

Proof. Let $\varepsilon \in (0, 1]$ and $f : \mathbb{N} \to \mathbb{N}$ be given. By Proposition 5.1, we may consider $n_1 \leq \beta(\varepsilon_1, f_1)$ and $x \in \overline{B}_b(p)$ such that

$$\forall m \leq f_1(n_1)\left(d(x, T_m(x)), d(x, U_m(x)) \leq \frac{1}{f_1(n_1) + 1}\right) \tag{2}$$

and

$$\forall m \in [n_1; f_1(n_1)]\,(\langle \overrightarrow{xu}, \overrightarrow{xx_m} \rangle \leq \varepsilon_1)\,. \tag{3}$$

Consider $n_2 := \max\{n_1, \varphi_1(\varepsilon)\}$. Then,

$$f_1(n_1) = \left\lfloor \frac{144b\left(f_2\left(n_2\right)+1\right)}{\varepsilon^2} \right\rfloor,$$

which, since $\varepsilon \in (0,1]$, entails

$$f_1(n_1)+1 \geq \frac{144b\left(f_2\left(n_2\right)+1\right)}{\varepsilon^2} \quad \text{and} \quad f_1(n_1) \geq f_2(n_2).$$

Therefore, from (2) and (3), it follows that

$$\forall m \leq f_2(n_2) \left(d(x, T_m(x)), d(x, U_m(x)) \leq \frac{\varepsilon^2}{144b\left(f_2(n_2)+1\right)} \right) \tag{4}$$

and

$$\forall m \in [n_2; f_2(n_2)] \left(\langle \overrightarrow{xu}, \overrightarrow{xx_m} \rangle \leq \frac{\varepsilon^2}{72} \right). \tag{5}$$

Note also that, for $m \geq \varphi_1(\varepsilon)$, by (Q1) and Proposition 3.11(v) we have

$$\alpha_m \leq \frac{\varepsilon^2}{144b^2} \quad \text{and} \quad d(x_m, T_m(x_m)) \leq \frac{\varepsilon^2}{144b}. \tag{6}$$

Since $d(x_{m+1}, x) \leq d(y_m, x) + d(x, U_m(x))$, using Lemma 2.3, we have

$$\begin{aligned}
d^2(x_{m+1}, x) &\leq d^2(y_m, x) + d(x, U_m(x))\left(d(x, U_m(x)) + 2d(y_m, x)\right)\\
&\leq (1-\alpha_m)^2 d^2(T_m(x_m), x) + 2\alpha_m(1-\alpha_m)\langle \overrightarrow{xu}, \overrightarrow{xT_m(x_m)} \rangle\\
&\quad + \alpha_m^2 d^2(u, x) + 6b \cdot d(x, U_m(x))\\
&\leq (1-\alpha_m) d^2(x_m, x) + \alpha_m \left(2\langle \overrightarrow{xu}, \overrightarrow{xT_m(x_m)} \rangle + 4b^2\alpha_m \right)\\
&\quad + 6b\left(d(x, T_m(x)) + d(x, U_m(x))\right),
\end{aligned}$$

which can be written in a more compact way as

$$d^2(x_{m+1}, x) \leq (1-\alpha_m) d^2(x_m, x) + \alpha_m r_m + \mathcal{E},$$

where

$$\begin{gathered}
r_m := 2\langle \overrightarrow{xu}, \overrightarrow{xx_m} \rangle + 4bd(x_m, T_m(x_m)) + 4b^2\alpha_m\\
\text{and} \quad \mathcal{E} := 6b\left(d(x, T_m(x)) + d(x, U_m(x))\right).
\end{gathered}$$

By (4), for all $m \leq f_2(n_2)$,

$$\mathcal{E} \leq \frac{12b\varepsilon^2}{144b(f_2(n_2)+1)} \leq \frac{\varepsilon^2}{12(f_2(n_2)+1)},$$

and by (5) and (6), for $m \in [n_2; f_2(n_2)]$,

$$r_m \leq \frac{2\varepsilon^2}{72} + \frac{4b\varepsilon^2}{144b} + \frac{4b^2\varepsilon^2}{144b^2} = \frac{\varepsilon^2}{12}.$$

We can therefore apply Lemma 2.10 to conclude that

$$\forall m \in [\sigma(n_2); f_2(n_2)] \left(d^2(x_m, \tilde{x}) \leq \frac{\varepsilon^2}{4} \right).$$

By triangle inequality, we get

$$\forall i, j \in [\sigma(n_2); f_2(n_2)] \left(d(x_i, x_j) \leq \varepsilon \right),$$

and since $f_2(n_2) = f(\sigma(n_2))$, the result follows with

$$n := \sigma(n_2) \leq \max\left\{\sigma(n') \,\middle|\, n' \leq \max\{\beta(\varepsilon_1, f_1), \varphi_1(\varepsilon)\}\right\} =: \Omega(\varepsilon, f).$$

□

We can now establish our main convergence result.

Theorem 5.4. *Let X be a UCW–hyperbolic space, $C \subseteq X$ a nonempty convex subset, and $u \in C$. Consider $\{T_n\}$, $\{U_n\}$ families of nonexpansive maps in C such that $\mathrm{F} := \mathrm{Fix}(\{T_n\}) \cap \mathrm{Fix}(\{U_n\}) \neq \emptyset$, and sequences $(\alpha_n), (\beta_n) \subset [0,1]$. Let (x_n) be the iteration generated by the scheme (HM_∞). Assume (C1)–(C7) are satisfied. Then, (x_n) is asymptotically regular and $\{T_n\}$– and $\{U_n\}$–asymptotically regular, i.e.*

$$\lim d(x_n, x_{n+1}) = \lim d(x_n, T_n(x_n)) = \lim d(x_n, U_n(x_n)) = 0.$$

Moreover, if X is a Hadamard space, C is closed and both families satisfy the NST condition (II) with a function satisfying (+), then (x_n) converges strongly to $\mathrm{P}_\mathrm{F}(u)$.

Proof. The first part was established in Section 3, and we only need to discuss the second part of the theorem. Assume X to be a complete CAT(0) space and C to be closed. By Theorem 5.3, we conclude in particular that (x_n) is metastable, i.e.

$$\forall \varepsilon > 0\; \forall f \in \mathbb{N}^\mathbb{N}\; \exists n \in \mathbb{N}\; \forall i, j \in [n; f(n)]\, (d(x_i, x_j) \leq \varepsilon).$$

This entails that (x_n) is a Cauchy sequence and, since the space is complete, it converges to some point $z \in X$. Since the two families of nonexpansive maps satisfy the NST condition (II), we conclude that $z \in \mathrm{F}$. It remains to verify that z is indeed $\mathrm{P}_\mathrm{F}(u)$, the projection point of u onto the set F. For all $n \in \mathbb{N}$, we have

$$\begin{aligned} d^2(x_{n+1}, \mathrm{P}_\mathrm{F}(u)) &\leq d^2(y_n, \mathrm{P}_\mathrm{F}(u)) \\ &\leq (1-\alpha_n)^2 d^2(x_n, \mathrm{P}_\mathrm{F}(u)) + 2\alpha_n(1-\alpha_n)\langle \overrightarrow{\mathrm{P}_\mathrm{F}(u)u}, \overrightarrow{\mathrm{P}_\mathrm{F}(u)T_n(x_n)}\rangle \\ &\quad + \alpha_n^2 d^2(u, \mathrm{P}_\mathrm{F}(u)) \\ &\leq (1-\alpha_n) d^2(x_n, \mathrm{P}_\mathrm{F}(u)) + \alpha_n \left(2\langle \overrightarrow{\mathrm{P}_\mathrm{F}(u)u}, \overrightarrow{\mathrm{P}_\mathrm{F}(u)T_n(x_n)}\rangle + b^2\alpha_n\right). \end{aligned}$$

Since $\alpha_n \to 0$, $x_n \to z$, and $z \in \mathrm{F}$, using the characterization of the projection in Lemma 4.3 we see that

$$\limsup \left(2\langle \overrightarrow{\mathrm{P}_\mathrm{F}(u)u}, \overrightarrow{\mathrm{P}_\mathrm{F}(u)T_n(x_n)}\rangle + b^2\alpha_n\right) = 2\langle \overrightarrow{\mathrm{P}_\mathrm{F}(u)u}, \overrightarrow{\mathrm{P}_\mathrm{F}(u)z}\rangle \leq 0.$$

Thus, we can apply Lemma 2.7 and conclude that indeed $x_n \to \mathrm{P}_\mathrm{F}(u)$. □

The following result gives a sufficient condition on the maps $\{T_n\}$ so that condition (Q5) is satisfied (and similarly for $\{U_n\}$ with regards to (Q6)).

Proposition 5.5 ([9]). *Let (γ_n) be a sequence of positive real numbers satisfying*

(i) *$\sum |\gamma_{n+1} - \gamma_n| < \infty$ with Cauchy rate χ,*
(ii) *$\inf \gamma_n \geq \gamma > 0$ for some positive real number γ.*

Assume that the family of maps $\{T_n\}$ satisfies

$$\forall x \in C\; \forall n, m \in \mathbb{N} \left(d(T_m(x), T_n(x)) \leq \frac{|\gamma_m - \gamma_n|}{\gamma_n} d(x, T_n(x)) \right). \tag{7}$$

Then, $\sum d(T_{n+1}(x_n), T_n(x_n)) < \infty$ with Cauchy rate σ_5 defined by

$$\sigma_5(\varepsilon) := \chi\left(\frac{\varepsilon\gamma}{2b}\right).$$

In [32], the authors discuss a generalization to CAT(0) spaces of the proximal point algorithm with resolvent functions associated with monotone operators in Hilbert spaces. They show that the condition (7) is satisfied by any family of maps which is jointly (P2) with respect to (γ_n). As explained in [38] (see also [43]), these families of maps are particular instances of 'resolvent-like' families of nonexpansive maps and so, by Example 3.4 and Proposition 3.6, they satisfy the NST condition (II) with an NST_2-modulus. Therefore, as a particular instance of Theorem 5.4, it follows that the iteration (HM_∞) is strongly convergent when $\{T_n\}$, $\{U_n\}$ are jointly (P2) with respect to sequences (γ_n) and (γ'_n) satisfying (i) and (ii) in Proposition 5.5 (which extends [10] for $T_n \not\equiv \mathrm{Id}$). In particular, we can take families of maps which are jointly (P2) with respect to parameter sequences defined by

$$\gamma_0 := 1,\ \gamma_{n+1} := \frac{1}{(n+2)(n+3)} + \gamma_n,\ \text{and}\ \gamma'_n := \gamma_n$$

and, using the results in section 3.5, obtain a strongly convergent (HM_∞) iteration with linear and quadratic rates of asymptotic regularity.

6. Final Remarks

In this work, we extended the strong convergence of the alternating Halpern-Mann iteration (established in [17]) to countable families of nonexpansive maps in the nonlinear setting of complete CAT(0) spaces. The result is centered in a finitary formulation of the Cauchy property (metastability), following and extending the argument used in [17], and we incidently also obtained rates of metastability which are uniform in the parameters of the problem. Indeed, the rate does not depend on the specific space X, the set C, the families $\{T_n\}$ and $\{U_n\}$, neither on the precise parameter sequences (α_n) and (β_n), and instead only depends on quantitative information on the conditions (the functions σ_i and $\widehat{\zeta}$) and on bounding information on the distance of the points x_0 and u to the target set F. Moreover, even in the broader setting of UCW–hyperbolic spaces, several uniform rates of convergence were obtained regarding the asymptotic regularity of the iteration, extending the quantitative study on asymptotic regularity in [9]. Our argument is reliant on the notion that the iteration is a sequence of almost common fixed points, which doesn't follow immediately from the usual asymptotic regularity result. We bridge the gap between an available 'diagonal-like' asymptotic regularity and the required notion using the so-called NST conditions, namely on a weaker version of (the uniformization of) condition (II). This provides a different condition to strong convergence than the demiclosedness-like condition [35] required on the family of maps in [5] for the Tikhonov-Mann iteration with $u = 0$ generalized to a countable family of maps (which is a particular instance of our schema (HM_∞)) in the setting of Hilbert spaces.

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