A finitistic approach to a generalized viscosity implicit rule in Hadamard spaces

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Abstract. We introduce a generalized viscosity implicit rule and, by adopting a finitistic perspective, establish its convergence in the geodesic setting of Hadamard spaces. Our results are guided by the quantitative study of other convergence proofs in the literature and, in particular, provide computable quantitative data in the form of rates of convergence and of metastability. These results are achieved through the application of proof interpretations within the framework of the proof mining research program, highlighting the program's potential to uncover hidden computational insights embedded in mathematical proofs.

Keywords: Viscosity implicit rules · Rates of convergence · Rates of metastability · Proof mining.

1 Introduction

The implicit midpoint rule is a powerful method for solving ordinary differential equation. Consider the initial value problem $y'(t) = f(y(t))$ and $y(0) = y_0$, for a function $f : \mathbb{R}^d \to \mathbb{R}^d$. The implicit midpoint rule produces a sequence (y_n) via the equation

$$\frac{1}{h}(y_{n+1} - y_n) = f\left(\frac{y_n + y_{n+1}}{2}\right),$$

where h is the time step size, and y_n is the numerical approximation of $y(t_n)$. Under suitable conditions on the function f, the sequence (y_n) will converge to the solution of the problem as $h \to 0$ uniformly over $t \in [0, \bar{t}]$ for any $\bar{t} > 0$. For the case of nonlinear dissipative evolution problems in a Hilbert space X, the function f takes the form $f = Id - T$, for T a nonexpansive map (i.e. $\|T(x) - T(y)\| \leq \|x - y\|$, for all $x, y \in X$). The equilibrium problem is thus reduced to solving the fixed point problem $x = T(x)$. It was under this motivation that the work in [2] extended the method to nonexpansive maps in Hilbert spaces, namely by studying the sequence (x_n) generated by the implicit method

$$x_0 \in X, \quad x_{n+1} = \alpha_n x_n + (1 - \alpha_n) T\left(\frac{x_n + x_{n+1}}{2}\right), \tag{IMR}$$

for $(\alpha_n) \subseteq (0,1)$. The authors show in [2] that the sequence (x_n) is Fejér monotone and that it weakly converges to a fixed point of T.

In 2015, in order to regularize the implicit midpoint rule for nonexpansive maps, Xu et al [30] applied the viscosity approximation technique [3,23] and proposed the viscosity implicit midpoint rule

$$x_0 \in X, \quad x_{n+1} = \alpha_n \phi(x_n) + (1-\alpha_n) T\left(\frac{x_n + x_{n+1}}{2}\right), \tag{VIMR}$$

where ϕ is a strict contraction on X. They established the strong convergence of (VIMR) to the unique fixed point of T solving the variational inequality problem

$$\forall y \in \mathrm{Fix}(T)\left(\langle x - \phi(x), y - x\rangle \geq 0\right),$$

where $\mathrm{Fix}(T)$ is the set of fixed points of T.

Inspired by the work of [30], Ke and Ma [14] extended the viscosity implicit rule beyond the midpoint requirement, by showing in Hilbert spaces the strong convergence of the implicit schema

$$x_0 \in X, \quad x_{n+1} = \alpha_n \phi(x_n) + (1-\alpha_n) T\left(\beta_n x_n + (1-\beta_n) x_{n+1}\right), \tag{VIR}$$

under suitable conditions on the sequences $(\alpha_n), (\beta_n) \subseteq (0,1)$.

The contributions of this paper are threefold. First, we introduce a generalized implicit rule, namely

$$x_0 \in X, \quad x_{n+1} = \alpha_n \phi(x_n) + (1-\alpha_n) T\left(\beta_n U(x_n) + (1-\beta_n) x_{n+1}\right), \tag{GIR}$$

which now incorporates an additional nonexpansive map U. Second, our convergence result is established in the general geodesic setting of Hadamard spaces, the canonical nonlinear generalization of Hilbert spaces. Lastly, our work is guided by a finitistic perspective in the context of the proof mining research program [17]. Using in the background proof-theoretical techniques, namely proof interpretations, one is guided into the computational data regarding the asymptotic behaviour of the iteration. Through a quantitative study of the work in [14] and subsequent generalization (in particular to the nonlinear setting), we begin by proving the so-called metastability property of the sequence (x_n) generated via (GIR),

$$\forall k \in \mathbb{N}\, \forall f : \mathbb{N} \to \mathbb{N}\, \exists N \in \mathbb{N}\, \forall i,j \in [N; f(N)] \left(d(x_i, x_j) \leq \frac{1}{k+1}\right), \tag{$\star$}$$

where d is the metric function on the space and $[a;b]$ denotes $[a,b] \cap \mathbb{N}$. It is easy to see that ($\star$) is equivalently to the usual Cauchy property of the sequence (although in a noneffective way) and will therefore entail the convergence of the sequence. Over the course of the study, we in particular obtain computable and uniform rates of asymptotic regularity as well as of metastability, i.e. a function

μ depending on k and f providing a bound on the value of N in $(\star)$. Moreover, relying on the work in [10], the results are only made possible through the elimination of the sequential weak compactness argument crucial in the linear setting and which is presumably problematic in a nonlinear context where one frequently relies on other complicated arguments like Banach limits. This issue is discussed further in section 5.

2 Preliminaries

2.1 The geodesic setting

A W-hyperbolic space [17] is a triple (X, d, W) where (X, d) is a metric space and $W : X \times X \times [0, 1] \to X$ is a convexity function required to satisfy for all $x, y, z, w \in X$ and $\lambda, \lambda' \in [0, 1]$:

(W1) $d(W(x, y, \lambda), z) \leq (1 - \lambda)d(x, z) + \lambda d(y, z)$,
(W2) $d(W(x, y, \lambda), W(x, y, \lambda')) = |\lambda - \lambda'|d(x, y)$,
(W3) $W(x, y, \lambda) = W(y, x, 1 - \lambda)$,
(W4) $d(W(x, y, \lambda), W(z, w, \lambda)) \leq (1 - \lambda)d(x, z) + \lambda d(y, z)$.

It is easy to verify that

$$d(W(x, y, \lambda), x) = \lambda d(x, y) \quad \text{and} \quad d(W(x, y, \lambda), y) = (1 - \lambda)d(x, y).$$

The reader may quickly see that W enjoys properties one would expect from a convex combination and for that reason we shall use the more intuitive notation $(1-\lambda)x \oplus \lambda y$ for $W(x, y, \lambda)$. A subset C is called convex if $(1-\lambda)x \oplus \lambda y \in C$ for all $\lambda \in [0, 1]$, whenever $x, y \in C$. These spaces provide a nonlinear counterpart to normed spaces. An important subclass of W-hyperbolic spaces is that of CAT(0) spaces [1,13], namely those that satisfy for all $x, y, z \in X$

$$d^2\left(\frac{1}{2}x \oplus \frac{1}{2}y, z\right) \leq \frac{1}{2}d^2(x, z) + \frac{1}{2}d^2(y, z) - \frac{1}{4}d^2(x, y), \tag{CN$^-$}$$

equivalent (in the presence of (W1)–(W4)) to the Bruhat-Tits CN-inequality [5]. This property actually entails (e.g. [7, Lemma 2.5]) that for all $x, y, z \in X$ and $\lambda \in [0, 1]$,

$$d^2(\lambda x \oplus (1 - \lambda)y, z) \leq (1 - \lambda)d^2(x, z) + \lambda d^2(y, z) - \lambda(1 - \lambda)d^2(x, y) \tag{CN$^+$}$$

By the work of Berg and Nikolaev [4], in any metric space, the quasilinearization function $\langle \overrightarrow{\cdot}, \overrightarrow{\cdot} \rangle : X^2 \times X^2 \to \mathbb{R}$ defined by

$$\langle \overrightarrow{xy}, \overrightarrow{uv} \rangle := \frac{1}{2}\left(d^2(x, v) + d^2(y, u) - d^2(x, u) - d^2(y, v)\right)$$

is the unique function satisfying for all $x, y, u, v \in X$

(i) $\langle \overrightarrow{xy}, \overrightarrow{xy} \rangle = d^2(x, y)$,

(ii) $\langle \overrightarrow{xy}, \overrightarrow{uv} \rangle = \langle \overrightarrow{uv}, \overrightarrow{xy} \rangle$,
(iii) $\langle \overrightarrow{xy}, \overrightarrow{uv} \rangle = -\langle \overrightarrow{yx}, \overrightarrow{uv} \rangle$,
(iv) $\langle \overrightarrow{xy}, \overrightarrow{uv} \rangle = \langle \overrightarrow{xy}, \overrightarrow{uw} \rangle + \langle \overrightarrow{xy}, \overrightarrow{wv} \rangle$,

where $\overrightarrow{xy}$ denotes the pair $(x, y) \in X^2$. They moreover demonstrated that CAT(0) spaces are also characterized as the geodesic spaces that satisfy

$$\forall x, y, u, v \in X \left(\langle \overrightarrow{xy}, \overrightarrow{uv} \rangle \leq d(x, y) d(u, v) \right),$$

which can be seen as a nonlinear version of the Cauchy-Schwarz inequality with the quasilinearization function working as a nonlinear version of the inner product. Indeed, CAT(0) spaces are a nonlinear counterpart to inner product spaces and thus Hadamard spaces (i.e. complete CAT(0) spaces) are a nonlinear generalization of Hilbert spaces.

Using (CN^+), it is easy to derive that for every $x, y, z \in X$ and $\lambda \in [0, 1]$,

$$d^2(\lambda x \oplus (1-\lambda) y, z) \leq \lambda^2 d^2(x, z) + (1-\lambda)^2 d^2(y, z) + 2\lambda(1-\lambda) \langle \overrightarrow{xz}, \overrightarrow{yz} \rangle \tag{1}$$

2.2 Technical lemmas

We will make use of the following notions.

Definition 1. *1. For $(a_n) \subseteq \mathbb{R}$, a function $\sigma : \mathbb{N} \to \mathbb{N}$ is a rate of convergence for $\lim a_n = 0$ if*

$$\forall k \in \mathbb{N}\ \forall n \geq \sigma(k) \left(|a_n| \leq \frac{1}{k+1} \right).$$

2. For $(a_n) \subseteq \mathbb{R}$, a function $\sigma : \mathbb{N} \to \mathbb{N}$ is a rate of divergence for $\sum a_n = \infty$ if

$$\forall L \in \mathbb{N} \left(\sum_{i=0}^{\sigma(L)} a_i \geq L \right).$$

3. For $(a_n) \subseteq [0, \infty)$, a function $\sigma : \mathbb{N} \to \mathbb{N}$ is a Cauchy rate for $\sum a_n < \infty$ if

$$\forall k \in \mathbb{N}\ \forall n \in \mathbb{N} \left(\sum_{i=\sigma(k)+1}^{n} a_i \leq \frac{1}{k+1} \right).$$

The following lemma will prove useful.

Lemma 1 (e.g. [29]). *Consider a sequence of nonnegative real numbers (a_n) such that $a_{n+1} \leq (1-\gamma_n) a_n + \gamma_n b_n + c_n$, for all $n \in \mathbb{N}$, where the sequences $(\gamma_n) \subseteq [0, 1]$ and $(b_n), (c_n) \subseteq \mathbb{R}$ satisfy*

$$(i)\ \sum \gamma_n = \infty, \qquad (ii)\ \limsup b_n \leq 0, \qquad (iii)\ \sum c_n < \infty.$$

Then $\lim a_n = 0$.

The following quantitative versions of the previous lemma will be necessary for the quantitative study. The proofs can be found, for example in [8].

Lemma 2. *Let (a_n) be a bounded sequence of nonnegative real numbers and $D \in \mathbb{N} \setminus \{0\}$ an upper bound on (a_n). Consider sequences of real numbers $\gamma_n \subseteq [0,1]$, $(c_n) \subseteq [0,\infty)$, and functions $\varGamma : \mathbb{N} \to \mathbb{N}$, $C : \mathbb{N} \to \mathbb{N}$ such that*

(a) $\varGamma$ is a rate of divergence for $\sum \gamma_n = \infty$,
(b) C is a Cauchy rate for $\sum c_n < \infty$.

Assume that for all $n \in \mathbb{N}$, $a_{n+1} \leq (1-\gamma_n)a_n + c_n$. Then, $\lim a_n = 0$ with rate of convergence

$$\theta(k) := \theta[\varGamma, C, D](k) := \varGamma\left(C\left(2k+1\right) + \lceil \ln\left(2D(k+1)\right)\rceil + 1\right) + 1$$

Lemma 3. *Let (a_n) be a bounded sequence of nonnegative real numbers and $D \in \mathbb{N} \setminus \{0\}$ an upper bound on (a_n). Consider sequences $(\gamma_n) \subseteq [0,1]$, $(b_n) \subseteq \mathbb{R}$, and assume that $\sum \gamma_n = \infty$ with a rate of divergence $\varGamma$. Let $k, K, P \in \mathbb{N}$, and $\mathcal{E} \in \mathbb{R}$ be given. If for all $n \in [K, P]$*

$$(i)\ a_{n+1} \leq (1-\gamma_n)a_n + \gamma_n b_n + \mathcal{E}, \quad (ii)\ b_n \leq \frac{1}{3(k+1)}, \quad (iii)\ \mathcal{E} \leq \frac{1}{3(P+1)(k+1)},$$

then $\forall n \in [\chi; P]\left(a_n \leq \frac{1}{k+1}\right)$, where

$$\chi := \chi(\varGamma, D, k, K) := \varGamma\left(K + \lceil 3D(k+1)\rceil\right) + 1.$$

2.3 On the metric projection

We require a quantitative rendering regarding the existence of a fixed point for the composition of the metric projection with a strict contraction. The existence of the projection point onto the nonempty convex set F of the common fixed points of T and U requires arithmetical comprehension for which the quantitative version would require bar-recursive functionals. Instead, we work with approximations to these projection points. We have the following characterization in terms of the quasilinearization function.

Lemma 4 ([6]). *Let S be a nonempty convex closed subset of a complete $CAT(0)$ space X. For any $u \in X$, let $P_S(u)$ denote the metric projection of u onto S. Then,*

$$P_S(u) \in S \quad \text{and} \quad \forall y \in S \left(\langle \overrightarrow{uP_S(u)}, \overrightarrow{yP_S(u)} \rangle \leq 0\right).$$

Quantitative versions related to the projection argument first appeared in [18,19]. We recall the following result from [8] (see also [10]). In what follows, X is a CAT(0) space and C a nonempty convex subset. Consider two nonexpansive maps $T : U : C \to C$ under the assumption that the convex set of common fixed points $\mathrm{F} := \mathrm{Fix}(T) \cap \mathrm{Fix}(U)$ is nonempty.

Proposition 1. *Given $u \in C$, let $b \in \mathbb{N} \setminus \{0\}$ be such that $b \geq d(u,p)$ for some $p \in \mathrm{F}$. For any $k \in \mathbb{N}$ and function $f : \mathbb{N} \to \mathbb{N}$, there exist $N \leq \Phi(k,f)$ and $z \in C \cap \overline{B}_b(p)$ such that $d(T(z),z), d(U(z),z) \leq \frac{1}{f(N)+1}$ and*

$$\forall y \in C \cap \overline{B}_b(p) \left(d(T(y),y), d(U(y),y) \leq \frac{1}{N+1} \to \langle \overrightarrow{uz}, \overrightarrow{yz} \rangle \leq \frac{1}{k+1} \right),$$

where $\Phi(k,f) := 12b\left(\check{f}^{(R)}(0)+1\right)^2$ with $R := b^4(k+1)^2 + b^2$ and

$$\check{f}(m) := \max\{\tilde{f}(12b(m+1)^2), 12b(m+1)\},$$

where $(\cdot)^{(R)}$ denotes the R-fold composition and $\tilde{f}(m) := \max\{f(m') : m' \leq m\}$.

It is well known that, as in a Hilbert space, the metric projection in a CAT(0) space is nonexpansive, and so its composition with a strict contraction is again a strict contraction. Therefore, by Banach's contraction principle, it has a unique fixed point. It is this argument that we need for the study of the iteration (x_n) and thus it requires a finitary formulation. We point out that a quantitative version pertaining to this property was first established by Körnlein in the context of his doctoral studies [21] using the monotone functional interpretation [15]. Here we are guided by the bounded functional interpretation [9,11]. Let $r \in \mathbb{N}$ be such that $\phi : C \to C$ is a strict contraction with factor $\leq 1 - \frac{1}{r+1}$.

Proposition 2. *Consider $x_0 \in C$ and let $b \in \mathbb{N} \setminus \{0\}$ be such that $b \geq d(x_0,p)$ for some $p \in \mathrm{F}$. For any $k \in \mathbb{N}$ and function $f : \mathbb{N} \to \mathbb{N}$, there exist $N \leq \Omega(k,f)$ and $z \in C \cap \overline{B}_b(p)$ such that $d(T(z),z), d(U(z),z) \leq \frac{1}{f(N)+1}$ and*

$$\forall y \in C \cap \overline{B}_b(p) \left(d(T(y),y), d(U(y),y) \leq \frac{1}{N+1} \to \langle \overrightarrow{\phi(z)z}, \overrightarrow{yz} \rangle \leq \frac{1}{k+1} \right),$$

where $\Omega(\varepsilon,\delta) := \omega_M$, with

$$\omega_0 := \Phi(\tilde{k}, f_M[0]), \quad \omega_{i+1} := \max\left\{\Phi(\tilde{k}, f_{M-i}[s]) \,:\, s \leq \omega_i\right\},$$

$$M := \left\lceil \log_{1-\frac{1}{r+1}}\left(\frac{1}{16b^2(k+1)}\right)\right\rceil + 1, \quad \tilde{k} := 128b^2(k+1)^2(r+1)^2 - 1,$$

$$\text{and for each } n,m \in \mathbb{N}, \begin{cases} f_0[n](m) & := f(m) \\ f_{i+1}[n](m) & := \max\left\{n, \Phi(\tilde{k}, f_i[m])\right\} \end{cases}$$

3 The finitary perspective

Let X be a CAT(0) space and C a nonempty convex subset. Consider a strict contractions $\phi : C \to C$ with contraction factor $\leq 1 - \frac{1}{r+1}$ for some $r \in \mathbb{N}$, and two nonexpansive maps $T, U : C \to C$ such that $\mathrm{F} := \mathrm{Fix}(T) \cap \mathrm{Fix}(U) \neq \emptyset$. For $x_0 \in C$ and sequences of real numbers $(\alpha_n), (\beta_n) \subseteq (0,1)$, we consider the following generalized implicit rule

$$x_{n+1} = \alpha_n \phi(x_n) \oplus T(\beta_n U(x_n) \oplus (1-\beta_n)x_{n+1}) \tag{GIR}$$

For $\alpha, \beta \in (0,1)$ and $v, v' \in C$, consider the map defined by

$$T_{v,v'}(x) := \alpha v \oplus (1-\alpha)T(\beta v' \oplus (1-\beta)x).$$

One easily sees that the map $T_{v,v'}$ is a strict contraction with contraction factor $(1-\alpha)(1-\beta) < 1$. Indeed, since T is nonexpansive,

$$\begin{aligned} d(T_{v,v'}(x), T_{v,v'}(y)) &\leq (1-\alpha)d(T(\beta v' \oplus (1-\beta)x), T(\beta v' \oplus (1-\beta)y)) \\ &\leq (1-\alpha)(1-\beta)d(x,y). \end{aligned}$$

Hence, by Banach's contraction principle, $T_{v,v'}$ has a unique fixed point in C entailing that the iteration (x_n) is well-defined.

We shall work with the following conditions on the sequences of parameters $(\alpha_n), (\beta_n) \subseteq (0,1)$:

(Q1) $\lim \alpha_n = 0$ with rate of convergence σ_1,
(Q2) $\sum \alpha_n = \infty$ with rate of divergence σ_2,
(Q3) $\sum |\alpha_{n+1} - \alpha_n| < \infty$ with Cauchy rate σ_3,
(Q4) $\sum |\beta_{n+1} - \beta_n| < \infty$ with Cauchy rate σ_4,
(Q5) There is $\beta \in \mathbb{N}$ such that $\beta \geq 2$ and $\frac{1}{\beta} \leq \beta_n \leq 1 - \frac{1}{\beta}$, for all $n \in \mathbb{N}$. [1]

3.1 Asymptotic regularity

A common first step in proving the convergence of fixed point methods is to show that it eventually satisfies a regularity property. For historical reasons it frequently comes in two flavours:

$$\begin{aligned} (x_n) \text{ is asymptotically regular } &\Leftrightarrow \lim d(x_{n+1}, x_n) = 0 \\ (x_n) \text{ is } T\text{-asymptotically regular } &\Leftrightarrow \lim d(T(x_n), x_n) = 0. \end{aligned}$$

These two notions are frequently proven in tandem, and note that they coincide in the particular case of the Picard iteration. We begin with the following lemma. To ease notation, we will henceforth write $y_n := \beta_n U(x_n) \oplus (1-\beta)x_{n+1}$.

Lemma 5. *The sequence (x_n) generated via the rule* (GIR) *is bounded. With $b \in \mathbb{N} \setminus \{0\}$ satisfying $b \geq \max\{d(x_0, p), d(\phi(p), p)(r+1)\}$, we moreover have for all $n \in \mathbb{N}$ and $p \in \mathrm{F}$,*

(i) $d(x_n, p), d(y_n, p) \leq b$,
(ii) $d(\phi(x_n), p) \leq 2b$,
(iii) $d(T(x_n), p), d(U(x_n), p) \leq b$.

We now have the following rates of asymptotic regularity, where (iii) and (iv) establishes that (x_n) is a sequence of almost common fixed points of T and U.

[1] Condition (Q4) entails that (β_n) is a Cauchy sequence and thus converges. Naturally, one can instead work with the condition $\lim \beta_n \in [\frac{1}{\beta}, 1 - \frac{1}{\beta}]$.

Proposition 3. *We have the following:*

(i) $\lim d(x_{n+1}, x_n) = 0$ *with rate* θ_1
(ii) $\lim d(y_{n+1}, y_n) = 0$, *with rate* θ_2
(iii) $\lim d(U(x_n), x_n) = 0$, *with rate* θ_3
(iv) $\lim d(T(x_n), x_n) = 0$, *with rate* θ_4,

where

$$\theta_1(k) := \theta[\Gamma, C, D](k) \text{ from Lemma 2 with parameters } D = 2b,$$
$$\Gamma(L) := \sigma_2\left(L(r+1)+1\right), \quad C(k) := \max\left\{\sigma_i\left(6b\beta(k+1)-1\right) \,:\, i \in \{3,4\}\right\},$$
$$\theta_2(k) := \max\left\{\sigma_4\left(6b(k+1)-1\right)+1, \theta_1\left(3k+2\right)\right\},$$
$$\theta_3(k) := \max\left\{\theta_1\left(2\tilde{k}-1\right), \theta_1\left(2k+1\right), \theta_2\left(\tilde{k}-1\right), \sigma_1\left(2b\tilde{k}-1\right)\right\}+1,$$
$$\tilde{k} := 24b\beta^2(k+1)^2$$
$$\theta_4(k) := \max\left\{\theta_1\left(6k+5\right), \theta_3\left(3k+2\right), \sigma_1\left(9b(k+1)-1\right)\right\}.$$

Remark 1. For each $k \in \mathbb{N}$, we define $\rho(k) := \max\{\theta_3(k), \theta_4(k)\}$. Thus, by Proposition 3, ρ is a common rate of T- and U-asymptotic regularity for (x_n), i.e.

$$\forall k \in \mathbb{N}\ \forall n \geq \rho(k) \left(d(U(x_n), x_n), d(T(x_n), x_n) \leq \frac{1}{k+1} \right).$$

3.2 Metastability

The following result is the central argument for the metastability result (see the discussion on section 5).

Proposition 4. *For any* $k \in \mathbb{N}$ *and* $f : \mathbb{N} \to \mathbb{N}$, *there exist* $N \leq \Psi(k, f)$ *and* $z \in C \cap \overline{B}_b(p)$ *such that*

$$d(T(z), z), d(U(z), z) \leq \frac{1}{f(N)+1} \ \wedge\ \forall i \geq n \left(\langle \overrightarrow{\phi(z)z}, \overrightarrow{x_i z} \rangle \leq \frac{1}{k+1} \right),$$

where $\Psi(k, f) := \max\{\rho(N') \,:\, N' \leq \Omega(k, f \circ \rho)\}$, *with* ρ *and* Ω *as in Remark 1 and Proposition 2, respectively.*

We are now ready to establish the metastability of the sequence (x_n).

Theorem 1. *Let* X *be a* $CAT(0)$ *space and* C *a nonempty convex subset. Consider* $\phi : C \to C$ *a strict contraction with contraction factor in* $[\frac{1}{2}, 1 - \frac{1}{r+1}]$ *for some* $r \in \mathbb{N} \setminus \{0\}$, *and nonexpansive maps* $T, U : C \to C$ *such that* $\mathrm{F} \neq \emptyset$. *Let* $(\alpha_n), (\beta_n) \subseteq (0, 1)$ *be sequences of real numbers such that conditions (Q1)–(Q5) hold. Given* $x_0 \in C$, *let* (x_n) *be the sequence defined via the rule* (GIR). *Then,* (x_n) *is a Cauchy sequence and moreover*

$$\forall k \in \mathbb{N}\ \forall f : \mathbb{N} \to \mathbb{N}\ \exists N \leq \mu(k, f)\ \forall i, j \in [N; f(N)] \left(d(x_i, x_j) \leq \frac{1}{k+1} \right),$$

where $\mu(k,f) := \max\left\{\chi\left(\Gamma, 4b^2, 4(k+1)^2-1, M(m)\right) \,:\, m \leq \Psi(\tilde{k}, h_f)\right\}$, *with*

$$\chi \ \text{ as in Lemma 3}, \quad \Psi \ \text{ as in Proposition 4},$$
$$M(m) := \max\left\{m, \sigma_1\left(27b^2P-1\right), \theta_1\left(6bP-1\right)\right\}, \quad \text{for} \ \ P := 36(r+1)\beta(k+1)^2,$$
$$\tilde{k} := 2P-1, \quad \Gamma(L) := \sigma_2\left(L(r+1)\right),$$
$$h_f(m) := 12\beta(k+1)^2(2+20b)(f(\chi\left(\Gamma, 4b^2, 4(k+1)^2-1, M(m)\right)) + 1) - 1$$

4 The infinitary perspective

Having established a finitary result regarding the metastability of the sequence (x_n) in the previous section, we can now in a simple way lift it up to a full infinitary result which is new and extends several previous convergence results.

Theorem 2. *Let X be a Hadamard space and C a nonempty, convex and closed subset. Consider $\phi : C \to C$ a strict contraction, and two nonexpansive maps $T, U : C \to C$ such that* $\mathrm{F} \neq \emptyset$. *Let $(\alpha_n), \beta_n \subseteq (0,1)$ be sequences of real numbers satisfying*

$$(i)\ \lim \alpha_n = 0, \quad (ii)\ \sum \alpha_n = \infty, \quad (iii)\ \sum |\alpha_{n+1} - \alpha_n| < \infty,$$
$$(iv)\ \sum |\beta_{n+1} - \beta_n| < \infty, \quad (v)\ 0 < \inf \beta_n \leq \sup \beta_n < 1.$$

Then, (x_n) generated via the rule (GIR) *converges to a point z which is a common fixed point of T and U and also the unique solution to the variational inequality*

$$\forall y \in \mathrm{F}\left(\langle \overrightarrow{\phi(z)z}, \overrightarrow{yz}\rangle \leq 0\right).$$

In other words, the limit point z is the unique point satisfying $P_{\mathrm{F}}\phi(z) = z$.

5 On the tame use of false principles

Convergence proofs of viscosity implicit methods in the linear setting of Hilbert spaces rely on the use of sequential weak compactness in order to establish that

$$\limsup \langle \phi(z) - z, x_n - z\rangle \leq 0,$$

where z is the unique point satisfying $P_{\mathrm{F}}\phi(z) = z$. Proof-theoretical considerations make it desirable that such arguments are absent from the proof subject to a quantitative analysis. Namely, these arguments are logically substantiated by arithmetical comprehension which results in quantitative data expressed by bar-recursive functionals and is undesired. The work in [10] introduces a macro to bypass such applications of weak compactness in Hilbert spaces by instead relying on a set-theoretically false principle (which through the course of the quantitative analysis disappears from the final finitary result, see the recent discussion in [26], cf. [25]). Interestingly, the same approach can be employed beyond Hilbert spaces (as first shown in [8], and recently in [27,28]). Here, we again employ it in the setting of CAT(0) spaces and now in conjunction with the argument establishing the existence of (an approximation to) the point z satisfying $P_{\mathrm{F}}\phi(z) = z$. Indeed, it can be seen as an application of the macro in [10] with the parameter function

$$\varphi(x, y) := \langle \overrightarrow{\phi(x)x}, \overrightarrow{xy} \rangle.$$

The argument demonstrating Theorem 2 was to first prove that (x_n) is metastable in CAT(0) spaces, and so it is a Cauchy sequence. Subsequently, the result is lifted to the full infinitary convergence statement using only elementary arguments by further assuming the completeness of the space and that C is closed. We now argue that a direct proof of the Cauchy property is possible without the detour through metastability if one is allowed to use the following false principle: for any $b, \ell \in \mathbb{N}$, $p \in X$ and $z \in C$

$$(\dagger) \quad C \cap \overline{B}_b(p) \subseteq \bigcup_{m \in \mathbb{N}} \Omega_m \to \exists m_0 \in \mathbb{N} \left(C \cap \overline{B}_b(p) \subseteq \Omega_{m_0} \right),$$

where

$$\Omega_m := \left\{ y \in X : d(T(y), y) > \frac{1}{m+1} \right\} \cup \left\{ y \in X : d(U(y), y) > \frac{1}{m+1} \right\} \cup \left\{ y \in X : \langle \overrightarrow{\phi(z)z}, \overrightarrow{yz} \rangle < \frac{1}{\ell+1} \right\},$$

corresponding to a particular instance of Heine/Borel compactness which is in general false, for example failing in infinite dimensional Hilbert spaces (see also the discussion in [20]).

We know that (x_n) is bounded and T- and U-asymptotically regular. Consider an arbitrary point $p \in \mathrm{F}$ and $b \in \mathbb{N}$ such that $b \geq d(x_n, p)$, for all $n \in \mathbb{N}$. We have the following three facts

$$\text{(I)} \quad \forall \ell \in \mathbb{N}\, \exists z \in \mathrm{F} \cap \overline{B}_b(p)\, \forall y \in \mathrm{F} \cap \overline{B}_b(p) \left(\langle \overrightarrow{\phi(z)z}, \overrightarrow{yz} \rangle < \frac{1}{\ell+1} \right).$$

Through the combinatorial arguments regarding the convergence of the sequence, there exists an expression $b_i(z)$ satisfying [2]

$$\text{(II)} \quad \forall r \in \mathbb{N}\, \exists \ell \in \mathbb{N}\, \forall z \in \overline{B}_b(p)\, \forall i \geq \ell \left(\langle \overrightarrow{\phi(z)z}, \overrightarrow{x_i z} \rangle < \frac{1}{\ell+1} \to b_i(z) \leq \frac{1}{r+1} \right).$$

Tied with the application of Lemma 1, we have the last fact

$$\text{(III)} \quad \forall k \in \mathbb{N}\, \exists r \in \mathbb{N}\, \forall z \in \mathrm{F} \cap \overline{B}_b(p)$$
$$\left(\exists N_1 \in \mathbb{N}\, \forall i \geq N_1 \, (b_i(z) \leq \frac{1}{r+1}) \to \exists N_2 \in \mathbb{N}\, \forall i \geq N_2 \, (d(x_i, z) \leq \frac{1}{k+1}) \right).$$

[2] In this discussion, $b_i(z)$ is the same as expressed in the proof of the infinitary result.

We now prove that (x_n) is a Cauchy sequence using (†). Let $k \in \mathbb{N}$ be given. Consider $r \in \mathbb{N}$ as in (III) and $\ell \in \mathbb{N}$ as in (II). By (I), consider $z \in \mathrm{F} \cap \overline{B}_b(p)$ such that

$$\forall y \in \mathrm{F} \cap \overline{B}_b(p) \left(\langle \overrightarrow{\phi(z)z}, \overrightarrow{yz} \rangle < \frac{1}{\ell+1} \right).$$

Equivalently,

$$\forall y \in C \cap \overline{B}_b(p) \left(\forall m \in \mathbb{N} \left(d(T(y), y), d(U(y), y) \leq \frac{1}{m+1} \right) \to \langle \overrightarrow{\phi(z)z}, \overrightarrow{yz} \rangle < \frac{1}{\ell+1} \right).$$

Hence, $C \cap \overline{B}_b(p) \subseteq \bigcup_{m \in \mathbb{N}} \Omega_m$. By (†), there exists $m_0 \in \mathbb{N}$ such that

$$\forall y \in C \cap \overline{B}_b(p) \left(d(T(y), y), d(U(y), y) \leq \frac{1}{m_0+1} \to \langle \overrightarrow{\phi(z)z}, \overrightarrow{yz} \rangle < \frac{1}{\ell+1} \right).$$

Since the sequence (x_n) is T- and U-asymptotically regular,

$$\exists N \in \mathbb{N}\ \forall i \geq N \left(d(T(y), y), d(U(y), y) \leq \frac{1}{m_0+1} \right),$$

and thus, relying on the fact that $(x_n) \subseteq C \cap \overline{B}_b(p)$, we get

$$\forall i \geq N \left(\langle \overrightarrow{\phi(z)z}, \overrightarrow{x_i z} \rangle < \frac{1}{\ell+1} \right).$$

We can now use fact (II) and (III) to conclude that

$$\exists N \in \mathbb{N}\ \forall i \geq N \left(d(x_i, z) \leq \frac{1}{k+1} \right),$$

entailing that (x_n) is a Cauchy sequence and concluding the (false) proof. It is this argument that gives rise to Theorem 1. The striking use of (†) turns out to be unproblematic since, by being an instance of bounded collection, it gets trivialized under the functional interpretation, which is to say that it vanishes from the final proof of metastability.

6 Final remarks

We introduce a novel iteration inspired by previous studies on viscosity implicit methods. In the geodesic setting of Hadamard spaces, this iteration converges strongly to a common fixed point of two nonexpansive maps, uniquely characterized by $P_{\mathrm{F}}\phi(z) = z$. Our approach first establishes a finitary result, namely the metastability of the iteration, by eliminating a prevalent weak compactness argument used in the linear setting. We then lift metastability to full infinitary convergence using simple arguments. Finally, we note that the iteration is expected to remain asymptotically regular in more general geodesic settings, particularly in UCW hyperbolic spaces, using arguments similar to those in [22].

Moreover, it can likely be extended to asymptotically nonexpansive maps, generalizing [24], and even to families of nonexpansive maps with arguments similar to those of [12]. These extensions are left for future research.

This work illustrates the significance of metastability and highlights the role of proof-theoretic techniques in general mathematics.

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Proof of Proposition 2

We define a sequence of pairs $(z_i, n_i)_{i\leq M}$ as follows. From Proposition 1 for $u = \phi(x_0)$, $k = \tilde{k}$ and $f = f_M[0]$, there exist $n_0 \leq \Phi(\tilde{k}, f_M[0])$ and $z_0 \in C \cap \overline{B}_b(p)$ such that $d(T(z_0), z_0), d(U(z_0), z_0) \leq \frac{1}{f_M[0](n_0)+1}$ and

$$\forall y \in C \cap \overline{B}_b(p) \left(d(T(y), y), d(U(y), y) \leq \frac{1}{n_0+1} \to \langle \overrightarrow{\phi(x_0)z_0}, \overrightarrow{yz_0} \rangle \leq \frac{1}{\tilde{k}+1} \right).$$

For $i < M$, assume now that z_i, n_i are already defined. Again by Proposition 1, there exist $n_{i+1} \leq \Phi(\tilde{k}, f_{M-i-1}[n_i])$ and $z_{i+1} \in C \cap \overline{B}_b(p)$ such that

$$d(T(z_{i+1}), z_{i+1}), d(U(z_{i+1}), z_{i+1}) \leq \frac{1}{f_{M-i-1}[n_i](n_{i+1})+1}$$

and

$$\forall y \in C \cap \overline{B}_b(p) \left(d(T(y), y), d(U(y), y) \leq \frac{1}{n_{i+1}+1} \to \langle \overrightarrow{\phi(z_i)z_{i+1}}, \overrightarrow{yz_{i+1}} \rangle \leq \frac{1}{\tilde{k}+1} \right).$$

Observe that for $i \in [1; M-1]$, since $z_i, z_{i+1} \in C \cap \overline{B}_b(p)$,

$$\begin{aligned} d(T(z_i), z_i), d(U(z_i), z_i) &\leq \frac{1}{f_{M-i}[n_{i-1}](n_i)+1} \\ &\leq \frac{1}{\Phi(\tilde{k}, f_{M-i-1}[n_i])+1} \leq \frac{1}{n_{i+1}+1} \end{aligned}$$

and

$$d(T(z_{i+1}), z_{i+1}), d(U(z_{i+1}), z_{i+1}) \leq \frac{1}{f_{M-i-1}[n_i](n_{i+1})+1} \leq \frac{1}{n_i+1},$$

we have

$$\langle \overrightarrow{\phi(z_i)z_{i+1}}, \overrightarrow{z_i z_{i+1}} \rangle \leq \frac{1}{\tilde{k}+1} \quad \text{and} \quad \langle \overrightarrow{\phi(z_{i-1})z_i}, \overrightarrow{z_{i+1}z_i} \rangle \leq \frac{1}{\tilde{k}+1}.$$

Combining, we get for all $i \in [1; M-1]$,

$$\begin{aligned} d^2(z_{i+1}, z_i) &= \langle \overrightarrow{z_{i+1}z_i}, \overrightarrow{z_{i+1}z_i} \rangle \\ &\leq \langle \overrightarrow{\phi(z_i)z_i}, \overrightarrow{z_{i+1}z_i} \rangle + \frac{1}{\tilde{k}+1} \\ &\leq \langle \overrightarrow{\phi(z_i)\phi(z_{i-1})}, \overrightarrow{z_{i+1}z_i} \rangle + \frac{2}{\tilde{k}+1} \\ &\leq d(\phi(z_i), \phi(z_{i-1})) d(z_{i+1}, z_i) + \frac{2}{\tilde{k}+1} \\ &\leq \left(1 - \frac{1}{r+1}\right) d(z_i, z_{i-1}) d(z_{i+1}, z_i) + \frac{2}{\tilde{k}+1}. \end{aligned}$$

If $d(z_{i+1}, z_i) \geq \sqrt{\frac{2}{\tilde{k}+1}}$, dividing by $d(z_{i+1}, z_i)$ we conclude that

$$d(z_{i+1}, z_i) \leq \left(1 - \frac{1}{r+1}\right) d(z_i, z_{i-1}) + \sqrt{\frac{2}{\tilde{k}+1}}.$$

If $d(z_{i+1}, z_i) < \sqrt{\frac{2}{\tilde{k}+1}}$, the inequality also holds a fortiori. By an inductive argument, using the definition of M and $\tilde{k}$, we see that

$$\begin{aligned} d(z_M, z_{M-1}) &\leq \left(1 - \frac{1}{r+1}\right)^{M-1} d(z_1, z_0) + \sqrt{\frac{2}{\tilde{k}+1}} \sum_{i=0}^{M-1} \left(1 - \frac{1}{r+1}\right)^i \\ &\leq 2b \left(1 - \frac{1}{r+1}\right)^{M-1} + \sqrt{\frac{2}{\tilde{k}+1}} (r+1) \\ &\leq 2b \frac{1}{16b^2(k+1)} + \sqrt{\frac{2}{128b^2(k+1)^2(r+1)^2}} (r+1) = \frac{1}{4b(k+1)}. \end{aligned}$$

We now show that the theorem holds with $N = n_M$ and $z = z_M$. By the definition of f_0, we have

$$d(T(z), z), d(U(z), z) \leq \frac{1}{f(N)+1}.$$

By construction, for any $y \in C \cap \overline{B}_b(p)$ such that $d(T(y), y), d(U(y), y) \leq \frac{1}{N+1}$,

$$\langle \overrightarrow{\phi(z_{M-1})z}, \overrightarrow{yz} \rangle \leq \frac{1}{\tilde{k}+1} \leq \frac{1}{2(k+1)}.$$

Thus, using the fact that ϕ is in particular nonexpansive, we get for any such y,

$$\begin{aligned} \langle \overrightarrow{\phi(z)z}, \overrightarrow{yz} \rangle &= \langle \overrightarrow{\phi(z_{M-1})z}, \overrightarrow{yz} \rangle + \langle \overrightarrow{\phi(z)\phi(z_{M-1})}, \overrightarrow{yz} \rangle \\ &\leq \frac{1}{2(k+1)} + 2b \cdot d(z, z_{M-1}) \\ &\leq \frac{1}{2(k+1)} + 2b \frac{1}{4b(k+1)} = \frac{1}{k+1}. \end{aligned}$$

As inductively we have $n_i \leq \omega_i$, we conclude that $N \leq \Omega(k, f)$, as desired. $\square$

Proof of Lemma 5

For $p \in \mathrm{Fix}(T)$, we have

$$\begin{aligned}
d(x_{n+1},p) &\leq \alpha_n d(\phi(x_n),p) + (1-\alpha_n)d(T(y_n),p)\\
&\leq \alpha_n d(\phi(x_n),\phi(p)) + \alpha_n d(\phi(p),p) + (1-\alpha_n)d(y_n,p)\\
&\leq \alpha_n\left(1-\frac{1}{r+1}\right)d(x_n,p) + \alpha_n d(\phi(p),p) + (1-\alpha_n)\beta_n d(U(x_n),p)\\
&\quad + (1-\alpha_n)(1-\beta_n)d(x_{n+1},p)\\
&\leq \alpha_n\left(1-\frac{1}{r+1}\right)d(x_n,p) + \alpha_n d(\phi(p),p) + (1-\alpha_n)\beta_n d(x_n,p)\\
&\quad + (1-\alpha_n)(1-\beta_n)d(x_{n+1},p)\\
&= \left(\alpha_n\left(1-\frac{1}{r+1}\right) + \beta_n - \alpha_n\beta_n\right)d(x_n,p) + \alpha_n d(\phi(p),p)\\
&\quad + (1-\alpha_n-\beta_n+\alpha_n\beta_n)d(x_{n+1},p).
\end{aligned}$$

This entails that

$$\begin{aligned}
d(x_{n+1},p) &\leq \frac{\alpha_n\left(1-\frac{1}{r+1}\right)+\beta_n-\alpha_n\beta_n}{\alpha_n+\beta_n-\alpha_n\beta_n}d(x_n,p) + \frac{\alpha_n}{\alpha_n+\beta_n-\alpha_n\beta_n}d(\phi(p),p)\\
&= \left(1-\frac{\alpha_n}{(\alpha_n+\beta_n-\alpha_n\beta_n)(r+1)}\right)d(x_n,p)\\
&\quad + \frac{\alpha_n}{(\alpha_n+\beta_n-\alpha_n\beta_n)(r+1)}d(\phi(p),p)(r+1).
\end{aligned}$$

Item (i) now follows with a trivial induction and the fact that

$$d(y_n,p) \leq \beta_n d(x_n,p) + (1-\beta_n)d(x_{n+1},p) \leq b.$$

Item (ii) follows by triangle inequality and the fact that ϕ is in particular non-expansive. Item (iii) is immediate from item (i) and the fact that $p \in \mathrm{F}$. □

Proof of Proposition 3

For $n \in \mathbb{N}$, we have

$$\begin{aligned}
d(x_{n+2},x_{n+1}) &\leq d(x_{n+2},\alpha_{n+1}\phi(x_n)\oplus(1-\alpha_{n+1})T(y_n))\\
&\quad + d(\alpha_{n+1}\phi(x_n)\oplus(1-\alpha_{n+1})T(y_n),x_{n+1})\\
&\leq \alpha_{n+1}d(\phi(x_{n+1}),\phi(x_n)) + (1-\alpha_n)d(T(y_{n+1}),T(y_n))\\
&\quad + |\alpha_{n+1}-\alpha_n|d(\phi(x_n),T(y_n))\\
&\leq \alpha_{n+1}\left(1-\frac{1}{r+1}\right)d(x_{n+1},x_n) + (1-\alpha_{n+1})d(y_{n+1},y_n)\\
&\quad + |\alpha_{n+1}-\alpha_n|d(\phi(x_n),T(y_n)).
\end{aligned}$$

Similarly, we can see that

$$\begin{aligned} d(y_{n+1}, y_n) &\le d(y_{n+1}, \beta_{n+1}U(x_n) \oplus (1-\beta_{n+1})x_{n+1}) \\ &\quad + d(\beta_{n+1}U(x_n) \oplus (1-\beta_{n+1})x_{n+1}, y_n) \\ &\le \beta_{n+1} d(U(x_{n+1}), U(x_n)) + (1-\beta_{n+1}) d(x_{n+2}, x_{n+1}) \\ &\quad + |\beta_{n+1} - \beta_n| d(U(x_n), x_{n+1}) \\ &\le \beta_{n+1} d(x_{n+1}, x_n) + (1-\beta_{n+1}) d(x_{n+2}, x_{n+1}) \\ &\quad + |\beta_{n+1} - \beta_n| \cdot 2b. \end{aligned}$$

Since $d(\phi(x_n), T(y_n)) \le d(\phi(x_n), p) + d(y_n, p) \le 3b$, together we conclude

$$\begin{aligned} d(x_{n+2}, x_{n+1}) &\le \left(\alpha_{n+1} \left(1 - \frac{1}{r+1}\right) + (1-\alpha_{n+1})\beta_{n+1} \right) d(x_{n+1}, x_n) \\ &\quad + (1-\alpha_{n+1})(1-\beta_{n+1}) d(x_{n+2}, x_{n+1}) \\ &\quad + (|\alpha_{n+1} - \alpha_n| + |\beta_{n+1} - \beta_n|) \cdot 3b. \end{aligned}$$

This now entails,

$$\begin{aligned} d(x_{n+2}, x_{n+1}) &\le \left(\frac{\alpha_{n+1}\left(1 - \frac{1}{r+1}\right) + \beta_{n+1} - \alpha_{n+1}\beta_{n+1}}{1 - (1-\alpha_{n+1})(1-\beta_{n+1})} \right) d(x_{n+1}, x_n) \\ &\quad + (|\alpha_{n+1} - \alpha_n| + |\beta_{n+1} - \beta_n|) \cdot \frac{3b}{1 - (1-\alpha_{n+1})(1-\beta_{n+1})} \\ &= \left(1 - \frac{\alpha_{n+1}}{(1 - (1-\alpha_{n+1})(1-\beta_{n+1}))(r+1)}\right) d(x_{n+1}, x_n) \\ &\quad + (|\alpha_{n+1} - \alpha_n| + |\beta_{n+1} - \beta_n|) \cdot \frac{3b}{1 - (1-\alpha_{n+1})(1-\beta_{n+1})} \\ &\le \left(1 - \frac{\alpha_{n+1}}{r+1}\right) d(x_{n+1}, x_n) + 3b\beta(|\alpha_{n+1} - \alpha_n| + |\beta_{n+1} - \beta_n|), \end{aligned}$$

since, using (Q5),

$$\frac{1}{\beta} \le 1 - (1-\alpha_{n+1})(1-\beta_{n+1}) \le 1.$$

Since $\sum \alpha_n = \infty$ and $\sum |\alpha_{n+1} - \alpha_n| + |\beta_{n+1} - \beta_n| < \infty$, by Lemma 1, we conclude that $\lim d(x_{n+1}, x_n) = 0$. We now apply the quantitative version in Lemma 2. Clearly, $D = 2b$ is an upper bound on $(d(x_{n+1}, x_n))$. By (Q2), it follows that $\Gamma(L)$ is a rate of divergence for $\sum \frac{\alpha_{n+1}}{r+1}$,

$$\begin{aligned} \sum_{i=0}^{\Gamma(L)} \gamma_i = \sum_{i=0}^{\Gamma(L)} \frac{\alpha_{i+1}}{r+1} &= \frac{1}{r+1} \left(\sum_{i=0}^{\sigma_2(L(r+1)+1)} \alpha_i - \alpha_0 \right) \\ &\ge \frac{1}{r+1} (L(r+1) + 1 - \alpha_0) \ge L. \end{aligned}$$

Lastly, C is a Cauchy rate for $\sum 3b\beta(|\alpha_{n+1}-\alpha_n|+|\beta_{n+1}-\beta_n|)$, as for all $n \in \mathbb{N}$

$$\begin{aligned}\sum_{i=C(k)+1}^{n} c_i &\leq 3b\beta\left(\sum_{i=\sigma_3(6b\beta(k+1)-1)+1}^{n}|\alpha_{i+1}-\alpha_i| + \sum_{i=\sigma_4(6b\beta(k+1)-1)+1}^{n}|\beta_{i+1}-\beta_i|\right)\\ &\leq 3b\beta\frac{2}{6b\beta(k+1)} = \frac{1}{k+1}.\end{aligned}$$

It thus follows that θ_1 is a rate of convergence for $\lim d(x_{n+1}, x_n) = 0$.

For item (ii), first observe that by (Q4), for all $k \in \mathbb{N}$ and $n \geq \sigma_4(k)+1$

$$|\beta_{n+1}-\beta_n| \leq \sum_{i=\sigma_4(k)+1}^{n}|\beta_{i+1}-\beta_i| \leq \frac{1}{k+1},$$

and thus σ_4+1 is a rate of convergence towards zero for $(|\beta_{n+1}-\beta_n|)$. We have,

$$\begin{aligned}d(y_{n+1}, y_n) &\leq d(y_{n+1}, \beta_{n+1}U(x_n) \oplus (1-\beta_{n+1})x_{n+1})\\ &\quad + d(\beta_{n+1}U(x_n) \oplus (1-\beta_{n+1})x_{n+1}, y_n)\\ &\leq \beta_{n+1}d(U(x_{n+1}), U(x_n)) + (1-\beta_{n+1})d(x_{n+2}, x_{n+1})\\ &\quad + |\beta_{n+1}-\beta_n| d(U(x_n), x_{n+1})\\ &\leq d(x_{n+1}, x_n) + d(x_{n+2}, x_{n+1}) + |\beta_{n+1}-\beta_n| \cdot 2b.\end{aligned}$$

The result now follows immediately from (i).

In order to prove item (iii), we make use of (CN^+) as follows: for any $n \in \mathbb{N}$ and some $p \in \mathrm{F}$,

$$\begin{aligned}d^2(y_{n+1}, p) &\leq \beta_{n+1}d^2(U(x_{n+1}), p) + (1-\beta_{n+1})d^2(x_{n+2}, p)\\ &\quad - \beta_{n+1}(1-\beta_{n+1})d^2(U(x_{n+1}), x_{n+2})\\ &\leq \beta_{n+1}d^2(x_{n+1}, p) + (1-\beta_{n+1})d^2(x_{n+1}, p)\\ &\quad + d(x_{n+2}, x_{n+1})(d(x_{n+2}, x_{n+1}) + 2d(x_{n+1}, p))\\ &\quad - \beta_{n+1}(1-\beta_{n+1})d^2(U(x_{n+1}), x_{n+2})\\ &\leq d^2(x_{n+1}, p) + d(x_{n+2}, x_{n+1}) \cdot 4b - \frac{d^2(U(x_{n+1}), x_{n+2})}{\beta^2},\end{aligned}$$

using (Q5) in the last step. On the other hand, again by (CN^+), we have

$$d^2(x_{n+1}, p) \leq \alpha_n d^2(\phi(x_n), p) + (1-\alpha_n)d^2(T(y_n), p) \leq d^2(y_n, p) + \alpha_n \cdot 4b^2.$$

Hence,

$$\begin{aligned}\frac{d^2(U(x_{n+1}), x_{n+2})}{\beta^2} &\leq d^2(y_n, p) - d^2(y_{n+1}, p) + \alpha_n \cdot 4b^2 + d(x_{n+2}, x_{n+1}) \cdot 4b\\ &= (d(y_n, p) - d(y_{n+1}, p))\,(d(y_n, p) + d(y_{n+1}, p))\\ &\quad + \alpha_n \cdot 4b^2 + d(x_{n+2}, x_{n+1}) \cdot 4b\\ &\leq d(y_{n+1}, y_n) \cdot 2b + \alpha_n \cdot 4b^2 + d(x_{n+2}, x_{n+1}) \cdot 4b.\end{aligned}$$

Now consider, for $k \in \mathbb{N}$

$$\theta_3'(k) := \max\{\theta_1\left(12b\beta^2(k+1)^2-1\right), \theta_2\left(6b\beta^2(k+1)^2-1\right), \\ \sigma_1\left(12b^2\beta^2(k+1)^2-1\right)\}.$$

For $n \geq \theta_3'(k)$, as

$$\begin{aligned} n \geq \theta_1\left(12b\beta^2(k+1)^2-1\right) \quad &\Rightarrow \quad d(x_{n+2},x_{n+1})\cdot 4b \leq \frac{1}{3\beta^2(k+1)^2} \\ n \geq \theta_2\left(6b\beta^2(k+1)^2-1\right) \quad &\Rightarrow \quad d(y_{n+1},y_n)\cdot 2b \leq \frac{1}{3\beta^2(k+1)^2} \\ n \geq \sigma_1\left(12b^2\beta^2(k+1)^2-1\right) \quad &\Rightarrow \quad \alpha_n\cdot 4b^2 \leq \frac{1}{3\beta^2(k+1)^2}, \end{aligned}$$

we conclude that $d(U(x_{n+1}),x_{n+2}) \leq \frac{1}{k+1}$. Since

$$d(U(x_{n+1}),x_{n+1}) \leq d(U(x_{n+1}),x_{n+2}) + d(x_{n+2},x_{n+1}),$$

it follows that $\max\{\theta_3'(2k+1),\theta_1(2k+1)\}+1 = \theta_3(k)$, is a rate of convergence for $\lim d(U(x_n),x_n)=0$.

Item (iv) now follows easily from

$$\begin{aligned} d(T(x_n),x_n) &\leq d(T(x_n),T(y_n)) + d(T(y_n),x_{n+1}) + d(x_{n+1},x_n) \\ &\leq d(x_n,y_n) + \alpha_n d(\phi(x_n),T(y_n)) + d(x_{n+1},x_n) \\ &\leq d(U(x_n),x_n) + \alpha_n\cdot 3b + 2d(x_{n+1},x_n). \end{aligned}$$

□

Proof of Proposition 4

From Proposition 2, there exist $N' \leq \Omega(k, f\circ\rho)$ and $z \in C \cap \overline{B}_b(p)$ such that $d(T(z),z), d(U(z),z) \leq \frac{1}{f(\rho(N'))+1}$ and

$$\forall y \in C \cap \overline{B}_b(p) \left(d(T(y),y), d(U(y),y) \leq \frac{1}{N'+1} \to \langle \overrightarrow{\phi(z)z}, \overrightarrow{yz}\rangle \leq \frac{1}{k+1}\right).$$

Consider $N := \rho(N')$. Then, $N \leq \Psi(k,f)$ and z is in the desired conditions. By the definition of ρ and the fact that $d(x_i,p) \leq b$ for all $i \in \mathbb{N}$, we have

$$\forall i \geq N \left(x_i \in C \cap \overline{B}_b(p) \ \wedge \ d(T(x_i),x_i), d(U(x_i),x_i) \leq \frac{1}{N'+1}\right),$$

and the result follows immediately. □

Proof of Theorem 1

Let $k \in \mathbb{N}$ and $f : \mathbb{N} \to \mathbb{N}$ be given. From Proposition 4, there exist $n_0 \leq \Psi(\tilde{k}, h_f)$ and $z \in C \cap \overline{B}_b(p)$ such that

$$d(T(z), z), d(U(z), z) \leq \frac{1}{h_f(n_0)+1} \quad \text{and} \quad \forall i \geq n_0 \left(\langle \overrightarrow{\phi(z)z}, \overrightarrow{x_i z} \rangle \leq \frac{1}{\tilde{k}+1} \right).$$

For $i \in \mathbb{N}$, by (1), we have

$$d^2(x_{i+1}, z) \leq \alpha_i^2 d^2(\phi(x_i), z) + (1-\alpha_i)^2 d^2(T(y_i), z) + 2\alpha_i(1-\alpha_i)\langle \overrightarrow{\phi(x_i)z}, \overrightarrow{T(y_i)z} \rangle.$$

Since,

$$\begin{aligned} d^2(T(y_i), z) &\leq d^2(T(y_i), T(z)) + d(T(z), z)\left(d(T(z), z) + 2d(T(y_i), Tz)\right) \\ &\leq d^2(y_i, z) + \frac{1+4b}{h_f(n_0)+1} \end{aligned}$$

and

$$\begin{aligned} \langle \overrightarrow{\phi(x_i)z}, \overrightarrow{T(y_i)z} \rangle &= \langle \overrightarrow{\phi(x_i)\phi(z)}, \overrightarrow{T(y_i)z} \rangle + \langle \overrightarrow{\phi(z)z}, \overrightarrow{T(y_i)z} \rangle \\ &\leq \left(1 - \frac{1}{r+1}\right) d(x_i, z) d(y_i, z) + \frac{2b}{h_f(n_0)+1} + \langle \overrightarrow{\phi(z)z}, \overrightarrow{T(y_i)z} \rangle, \end{aligned}$$

we conclude that

$$\begin{aligned} d^2(x_{i+1}, z) \leq (1-\alpha_i)^2 d^2(y_i, z) &+ 2\alpha_i(1-\alpha_i)\left(1 - \frac{1}{r+1}\right) d(x_i, z) d(y_i, z) \\ &+ R_i + \frac{1+8b}{h_f(n_0)+1}, \end{aligned}$$

with $R_i := \alpha_i^2 d^2(\phi(x_i), z) + 2\alpha_i(1-\alpha_i)\langle \overrightarrow{\phi(z)z}, \overrightarrow{T(y_i)z} \rangle$. Solving the inequality $aX^2 + bX + c \geq 0$ for $X = d^2(y_i, z)$ where

$$\begin{aligned} a = (1-\alpha_i)^2, \quad b = 2\alpha_i(1-\alpha_i)\left(1 - \frac{1}{r+1}\right) d(x_i, z), \\ c = R_i - d^2(x_{i+1}, z) + \frac{1+8b}{h_f(n_0)+1}, \end{aligned}$$

we get

$$
\begin{aligned}
d(y_i, z) \geq & -\frac{2\alpha_i(1-\alpha_i)\left(1-\frac{1}{r+1}\right) d(x_i, z)}{2(1-\alpha_i)^2} \\
& + \frac{\sqrt{4\alpha_i^2(1-\alpha_i)^2\left(1-\frac{1}{r+1}\right)^2 d^2(x_i, z) - 4(1-\alpha_i)^2 \cdot c}}{2(1-\alpha_i)^2} \\
= & \frac{-\alpha_i\left(1-\frac{1}{r+1}\right) d(x_i, z)}{1-\alpha_i} \\
& + \frac{\sqrt{\alpha_i^2\left(1-\frac{1}{r+1}\right)^2 d^2(x_i, z) - R_i + d^2(x_{i+1}, z) - \frac{1+8b}{h_f(n_0)+1}}}{1-\alpha_i}.
\end{aligned}
$$

Since $d(y_i, z) \leq \beta_i d(x_i, z) + (1-\beta_i) d(x_{i+1}, z) + \frac{1}{h_f(n_0)+1}$, it follows that

$$
\begin{aligned}
& \sqrt{\alpha_i^2\left(1-\frac{1}{r+1}\right)^2 d^2(x_i, z) - R_i + d^2(x_{i+1}, z) - \frac{1+8b}{h_f(n_0)+1}} \\
& \leq (1-\alpha_i)(1-\beta_i) d(x_{i+1}, z) + \left((1-\alpha_i)\beta_i + \alpha_i\left(1-\frac{1}{r+1}\right)\right) d(x_i, z) \\
& \quad + \frac{1}{h_f(n_0)+1}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
& \alpha_i^2\left(1-\frac{1}{r+1}\right)^2 d^2(x_i, z) - R_i + d^2(x_{i+1}, z) - \frac{1+8b}{h_f(n_0)+1} \\
& \leq (1-\alpha_i)^2(1-\beta_i)^2 d^2(x_{i+1}, z) + \left((1-\alpha_i)\beta_i + \alpha_i\left(1-\frac{1}{r+1}\right)\right)^2 d^2(x_i, z) \\
& \quad + 2(1-\alpha_i)(1-\beta_i)\left((1-\alpha_i)\beta_i + \alpha_i\left(1-\frac{1}{r+1}\right)\right) d(x_i, z) d(x_{i+1}, z) \\
& + \frac{2(1-\alpha_i)(1-\beta_i) d(x_{i+1}, z) + 2\left((1-\alpha_i)\beta_i + \alpha_i\left(1-\frac{1}{r+1}\right)\right) d(x_i, z) + \frac{1}{h_f(n_0)+1}}{h_f(n_0)+1} \\
& \leq (1-\alpha_i)^2(1-\beta_i)^2 d^2(x_{i+1}, z) + \left((1-\alpha_i)\beta_i + \alpha_i\left(1-\frac{1}{r+1}\right)\right)^2 d^2(x_i, z) \\
& \quad + (1-\alpha_i)(1-\beta_i)\left((1-\alpha_i)\beta_i + \alpha_i\left(1-\frac{1}{r+1}\right)\right)\left(d^2(x_i, z) + d^2(x_{i+1}, z)\right) \\
& \quad + \frac{1+12b}{h_f(n_0)+1},
\end{aligned}
$$

which entails

$$\left(1-(1-\alpha_i)^2(1-\beta_i)^2-(1-\alpha_i)(1-\beta_i)\left((1-\alpha_i)\beta_i+\alpha_i\left(1-\frac{1}{r+1}\right)\right)\right)d^2(x_{i+1},z)$$
$$\leq\left(\left((1-\alpha_i)\beta_i+\alpha_i\left(1-\frac{1}{r+1}\right)\right)^2\right.$$
$$\left.+(1-\alpha_i)(1-\beta_i)\left((1-\alpha_i)\beta_i+\alpha_i\left(1-\frac{1}{r+1}\right)\right)-\alpha_i^2\left(1-\frac{1}{r+1}\right)^2\right)d^2(x_i,z)$$
$$+R_i+\frac{2+20b}{h_f(n_0)+1}.$$

This can be simplified to

$$\left(1-(1-\alpha_i)(1-\beta_i)\left(1-\frac{\alpha_i}{r+1}\right)\right)d^2(x_{i+1},z)$$
$$\leq\left(\left((1-\alpha_i)\beta_i+\alpha_i\left(1-\frac{1}{r+1}\right)\right)\left(1-\frac{\alpha_i}{r+1}\right)-\alpha_i^2\left(1-\frac{1}{r+1}\right)^2\right)d^2(x_i,z)$$
$$+R_i+\frac{2+20b}{h_f(n_0)+1},$$

and thus

$$d^2(x_{i+1},z)\leq\frac{\left((1-\alpha_i)\beta_i+\alpha_i\left(1-\frac{1}{r+1}\right)\right)\left(1-\frac{\alpha_i}{r+1}\right)-\alpha_i^2\left(1-\frac{1}{r+1}\right)^2}{1-(1-\alpha_i)(1-\beta_i)\left(1-\frac{\alpha_i}{r+1}\right)}d^2(x_i,z)$$
$$+\frac{R_i}{1-(1-\alpha_i)(1-\beta_i)\left(1-\frac{\alpha_i}{r+1}\right)}$$
$$+\frac{2+20b}{(h_f(n_0)+1)\left(1-(1-\alpha_i)(1-\beta_i)\left(1-\frac{\alpha_i}{r+1}\right)\right)}.$$

One can easily see that,

$$\frac{\left((1-\alpha_i)\beta_i+\alpha_i\left(1-\frac{1}{r+1}\right)\right)\left(1-\frac{\alpha_i}{r+1}\right)-\alpha_i^2\left(1-\frac{1}{r+1}\right)^2}{1-(1-\alpha_i)(1-\beta_i)\left(1-\frac{\alpha_i}{r+1}\right)}$$
$$=1-\frac{\left(1-\frac{2}{r+1}\right)\alpha_i+\frac{2}{r+1}}{1-(1-\alpha_i)(1-\beta_i)\left(1-\frac{\alpha_i}{r+1}\right)}\alpha_i\leq1-\frac{\alpha_i}{r+1},$$

using the fact that $1-\frac{2}{r+1}\geq 0$ [3] and that the big denominator is ≤ 1. Note that from (Q5),

$$1-(1-\alpha_i)(1-\beta_i)(1-\frac{\alpha_i}{r+1})\geq\frac{1}{\beta},$$

[3] This makes use of the assumption that $r\geq 1$.

and so we conclude

$$d^2(x_{i+1}, z) \leq \left(1 - \frac{\alpha_i}{r+1}\right) d^2(x_i, z) + \frac{\alpha_i}{r+1} b_i + \mathcal{E},$$

with

$$b_i := \frac{r+1}{\alpha_i} \cdot \frac{R_i}{1 - (1-\alpha_i)(1-\beta_i)\left(1 - \frac{\alpha_i}{r+1}\right)} \text{ and } \mathcal{E} := \frac{(2+20b)\beta}{h_f(n_0)+1}.$$

We want to apply Lemma 3. Clearly $D = 4b^2$ is an upper bound on the sequence $(d^2(x_i, z))$ and the function Γ is a rate of divergence for $\sum \frac{\alpha_n}{r+1}$.

For $i \geq n_0$, we have

$$\begin{aligned}
&\frac{2(r+1)\alpha_i(1-\alpha_i)\langle\overrightarrow{\phi(z)z}, \overrightarrow{T(y_i)z}\rangle}{\alpha_i\left[1 - (1-\alpha_i)(1-\beta_i)\left(1 - \frac{\alpha_i}{r+1}\right)\right]} \\
&\qquad = \frac{2(r+1)(1-\alpha_i)\langle\overrightarrow{\phi(z)z}, \overrightarrow{x_i z}\rangle}{1 - (1-\alpha_i)(1-\beta_i)\left(1 - \frac{\alpha_i}{r+1}\right)} \\
&\qquad\qquad + \frac{2(r+1)(1-\alpha_i)\langle\overrightarrow{\phi(z)z}, \overrightarrow{T(y_i)x_i}\rangle}{1 - (1-\alpha_i)(1-\beta_i)\left(1 - \frac{\alpha_i}{r+1}\right)} \\
&\qquad \leq \frac{2(r+1)\beta}{\tilde{k}+1} + 2(r+1)\beta d(\phi(z), z) d(T(y_i), x_i) \\
&\qquad \leq (r+1)\beta\left(\frac{2}{\tilde{k}+1} + 6b \cdot d(T(y_i), x_{i+1}) + 6b \cdot d(x_{i+1}, x_i)\right) \\
&\qquad = (r+1)\beta\left(\frac{2}{\tilde{k}+1} + 6b\alpha_i d(T(y_i), \phi(x_i)) + 6b \cdot d(x_{i+1}, x_i)\right) \\
&\qquad \leq (r+1)\beta\left(\frac{2}{\tilde{k}+1} + 18b^2\alpha_i + 6b \cdot d(x_{i+1}, x_i)\right)
\end{aligned}$$

On the other hand,

$$\frac{(r+1)\alpha_i^2 d^2(\phi(x_i), z)}{\alpha_i\left[1 - (1-\alpha_i)(1-\beta_i)\left(1 - \frac{\alpha_i}{r+1}\right)\right]} \leq 9b^2(r+1)\beta\alpha_i.$$

Hence, for $i \geq M(n_0)$,

$$\begin{aligned}
b_i &\leq (r+1)\beta\left(\frac{2}{\tilde{k}+1} + 27b^2\alpha_i + 6b \cdot d(x_{i+1}, x_i)\right) \\
&\leq (r+1)\beta\left(\frac{2}{2P} + \frac{27b^2}{27b^2 P} + \frac{6b}{6bP}\right) \\
&= \frac{3(r+1)\beta}{P} = \frac{1}{12(k+1)^2}.
\end{aligned}$$

From the definition of h_f, writing $\chi(M(n_0))$ for $\chi\left(\Gamma, 4b^2, 4(k+1)^2-1, M(n_0)\right)$, we have

$$\mathcal{E} = \frac{(2+20b)\,\beta}{12\beta(k+1)^2(2+20b)(f(\chi(M(n_0)))+1)} = \frac{1}{12(k+1)^2(f(\chi(M(n_0)))+1)}.$$

Hence, Lemma 3 entails

$$\forall i \in [\chi(M(n_0)); f(\chi(M(n_0)))] \left(d^2(x_i, z) \leq \frac{1}{4(k+1)^2}\right),$$

and therefore, by removing the square and using triangle inequality,

$$\forall i, j \in [\chi(M(n_0)); f(\chi(M(n_0)))] \left(d(x_i, x_j) \leq \frac{1}{k+1}\right).$$

This shows that the result holds with $N := \chi(M(n_0)) \leq \mu(k, f)$. $\square$

Proof of Theorem 2

By Theorem 1, (x_n) is a Cauchy sequence and, since the space is complete, it must converge. Let $x = \lim x_n$, which is in C, as the set is closed. By the fact that (x_n) is simultaneously T- and U-asymptotically regular, it follows that x is a common fixed point of T and U, i.e. $x \in \mathrm{F}$. To see that $x = z$ where $P_{\mathrm{F}}\phi(z) = z$, follow the proof of Theorem 1 to prove that

$$d^2(x_{i+1}, z) \leq \left(1 - \frac{\alpha_i}{r+1}\right) d^2(x_i, z) + \frac{\alpha_i}{r+1} b_i,$$

with

$$b_i := \frac{(r+1)\alpha_i d^2(\phi(x_i), z) + 2(r+1)(1-\alpha_i)\langle\overrightarrow{\phi(z)z}, \overrightarrow{T(y_i)z}\rangle}{1-(1-\alpha_i)(1-\beta_i)\left(1-\frac{\alpha_i}{r+1}\right)}$$

Since $x \in \mathrm{F}$, we have $\langle\overrightarrow{\phi(z)z}, \overrightarrow{xz}\rangle \leq 0$, and so

$$\begin{aligned}\langle\overrightarrow{\phi(z)z}, \overrightarrow{T(y_i)z}\rangle &= \langle\overrightarrow{\phi(z)z}, \overrightarrow{T(y_i)x}\rangle + \langle\overrightarrow{\phi(z)z}, \overrightarrow{xz}\rangle \\ &\leq d(\phi(z), z) d(T(y_i), x).\end{aligned}$$

Since $\lim x_n = x$, we get $\lim d(T(y_n), x) = 0$ which together with $\alpha_n \to 0$ entails that $\limsup b_n \leq 0$. Now Lemma 1 entails $\lim d(x_n, z) = 0$ and so $x = z$. $\square$