Proof-theoretical unwinding of mathematics

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Encontro Nacional da SPM 2024
Universidade do Minho, Braga

12 July

Outline

What is Proof Mining?

Functional Interpretations

Informally, it is a mapping F that takes a formula A of a theory $\mathcal{A}$ and maps it to a formula $A^F \equiv \exists \underline{u} A_F(\underline{u})$ of a theory $\mathcal{B}$

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Proof interpretations: (proofs of) theorems of $\mathcal{A}$ are mapped to (proofs of) theorems of $\mathcal{B}$:

$$\mathcal{A} \vdash A \quad \rightsquigarrow \quad \mathcal{B} \vdash A^F.$$

in a way that one obtains a term $\underline{t}$ witnessing the $\exists$-quantification in A^F:

$$\mathcal{A} \vdash A \quad \Rightarrow \quad \text{there is a term} \underline{t} \text{ such that } \mathcal{B} \vdash A_F(\underline{t}).$$

Main Applications

Functional interpretations are used to show results of:

- **Relative consistency**: $\bot^F \equiv \bot$, so

$$\mathcal{A} \vdash \bot \Rightarrow \mathcal{B} \vdash \bot, \text{ i.e., } Con(\mathcal{B}) \Rightarrow Con(\mathcal{A});$$

- **Conservation**: if $A^F \equiv A$, for $A \in \Gamma$, then

$$\text{for any } A \in \Gamma, \qquad \mathcal{A} \vdash A \Rightarrow \mathcal{B} \vdash A;$$

- **Unprovability**: if $\mathcal{B} \nvdash A_F(t)$, then $\mathcal{A} \nvdash A$;

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- **Unprovability**: if $\mathcal{B} \nvdash A_F(t)$, then $\mathcal{A} \nvdash A$;
- **Kreisel's shift of paradigm:** Functional interpretations allow for the extraction of the **computational content** hidden in the proof of the theorem – the term $\underline{t}$ captures the computational content of (the proof of) A.

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  - General Logical Metatheorems (2003-05)
- F.Ferreira and P.Oliva: bounded functional interpretation (2005)
  - Completely new translation of formulas
  - Independence on bounded parameters is intrinsic/explicit

Peano Arithmetic in all finite types

The finite types $\mathcal{T}^\omega$ are defined inductively by:

$$0 \in \mathcal{T}^\omega \quad \text{(natural numbers)},$$

$$\rho, \sigma \in \mathcal{T}^\omega \Rightarrow \rho \to \sigma \in \mathcal{T}^\omega \quad \text{(functions from } \rho \text{ to } \sigma\text{)}.$$

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In the language of Peano arithmetic in all finite types, we consider an appropriate notion of majorizability:

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\begin{cases}
n \leqslant_0^* m & \leftrightarrow \quad n \leqslant_0 m\\
x \leqslant_{\rho\to\sigma}^* y & \leftrightarrow \quad \forall u^\rho, v^\rho \big(u \leqslant_\rho^* v \to xu \leqslant_\sigma^* yv \wedge yu \leqslant_\sigma^* yv\big)
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- E.g., for $f, g : 0 \to 0$, we have $f \leqslant_1^* g$ iff f is less or equal than g pointwise and that g is nondecreasing. We say that f is *monotone* if $f \leqslant_1^* f$, which just means it is nondecreasing.

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$$\begin{cases} n \trianglelefteq_0 m & \leftrightarrow \quad n \leqslant_0 m \\ x \trianglelefteq_{\rho\to\sigma} y & \to \quad \forall u^\rho, v^\rho \big(u \trianglelefteq_\rho v \to xu \trianglelefteq_\sigma yv \wedge yu \trianglelefteq_\sigma yv\big) \end{cases}$$

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Together with the usual logical and arithmetical constants, and the combinators from λ-abstraction, we also have:
Bounded quantification: $\forall x \trianglelefteq t$ and $\exists x \trianglelefteq t$, where $x \notin \mathrm{FV}(t)$.
Monotone quantification: $\tilde{\forall} x \equiv \forall x \trianglelefteq x$ and $\tilde{\exists} x \equiv \exists x \trianglelefteq x$.

Bounded formulas are formulas that don't contain unbounded quantifiers, for example

$$A_{\mathrm{qf}} \mid \forall n' \leqslant n\, A_{\mathrm{bd}} \mid \forall t \in [0,1]\, A_{\mathrm{bd}}.$$

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$\mathrm{PA}^{\omega}_{\trianglelefteq}$ is Peano Arithmetic in all finite types with the (intensional) majorizability relation $\trianglelefteq$.

Characteristic Principles

- MAJ^ω – **Majorizability Axioms**:

$$\forall x \, \exists y \, (x \trianglelefteq y)$$

- $\mathrm{mAC}^\omega_{\mathrm{bd}}$ – **Monotone Bounded Choice**:

$$\tilde{\forall} x \, \tilde{\exists} y \, A_{\mathrm{bd}}(x,y) \to \tilde{\exists} f \, \tilde{\forall} x \, \tilde{\exists} y \trianglelefteq f(x) \, A_{\mathrm{bd}}(x,y)$$

- $\mathrm{bC}^\omega_{\mathrm{bd}}$ – **Bounded Collection Principle**:

$$\forall x \trianglelefteq t \, \exists y \, A_{\mathrm{bd}}(x,y) \to \exists z \, \forall x \trianglelefteq t \, \exists y \trianglelefteq z \, A_{\mathrm{bd}}(x,y)$$

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Characterization Theorem

$$\mathrm{PA}^\omega_\trianglelefteq + \mathrm{MAJ}^\omega + \mathrm{mAC}^\omega_{\mathrm{bd}} + \mathrm{bC}^\omega_{\mathrm{bd}} \vdash A \leftrightarrow A^B\,.$$

Soundness of the interpretation

Theorem (Soundness)

Let Δ be a set of universal sentences (with bounded matrices). If

$$\mathrm{PA}^{\omega}_{\trianglelefteq} + \mathrm{MAJ}^{\omega} + \mathrm{mAC}^{\omega}_{\mathrm{bd}} + \mathrm{bC}^{\omega}_{\mathrm{bd}} + \Delta \vdash A,$$

then there are closed monotone terms $\underline{t}$ such that

$$\mathrm{PA}^{\omega}_{\trianglelefteq} + \Delta \vdash \tilde{\forall} \underline{x}\, \tilde{\exists} \underline{y} \trianglelefteq \underline{tx}\, A_B(\underline{x}, \underline{y}).$$

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$$\mathrm{PA}^{\omega}_{\trianglelefteq} + \mathrm{MAJ}^{\omega} + \mathrm{mAC}^{\omega}_{\mathrm{bd}} + \mathrm{bC}^{\omega}_{\mathrm{bd}} + \Delta \vdash \forall x\, \exists y\, A_{\mathrm{bd}}(x, y)$$

$$\Rightarrow \mathrm{PA}^{\omega}_{\trianglelefteq} + \Delta \vdash \tilde{\forall} z\, \forall x \trianglelefteq z\, \exists y \trianglelefteq tz\, A_{\mathrm{bd}}(x, y)$$

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$$\mathrm{PA}^{\omega}_{\leqslant^*} + \Delta \vdash \tilde{\forall} \underline{x}\, \tilde{\exists} \underline{y} \leqslant^* \underline{tx}\, A_B(\underline{x}, \underline{y}).$$

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Formal Theories

To apply this proof-theoretical technique is necessary to work with (semi-)formal proofs, which in general is not the case for usual mathematical theorems. We need to extend our formal setting...

- Finite types with base types 0 and X, denoted $\mathcal{T}^{\omega,X}$, are constructed in the usual way, where X is the type of objects in an abstract (metric, Banach, Hilbert, etc.) space.
- Extend the majorizability notion to $\mathcal{T}^{\omega,X}$ in an appropriate way.
- Add axioms characterizing the abstract space and all the required new constants.

We want that a soundness theorem still exists for the extended theory:

- New constants must be majorizable;
- Add moduli (of convergence, of Cauchyness, of asymptotic regularity, of metastability, etc.) witnessing problematic existential quantifiers ('computational gaps');
- Universal axiomatic (possible with bounded matrices and using the new moduli).

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Let $\mathrm{PA}^{\omega}_{\trianglelefteq}[X, \cdots]$ be one such theory. Then,

Soundness for extended theories

If $\mathrm{PA}^{\omega}_{\trianglelefteq}[X, \cdots] + \text{Principles}[X] \vdash A$,

then there are closed monotone terms $\underline{t}$ such that

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What is Metastability?

Metastability

Let (x_n) be a sequence of real numbers.

- (x_n) satisfies the **Cauchy property** if

$$(\mathrm{I}) \quad \forall k \in \mathbb{N}\, \exists n \in \mathbb{N}\, \forall i, j \geq n \left(|x_i - x_j| \leq \frac{1}{k+1} \right)$$

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- (x_n) satisfies the **metastability property** if

$$(\mathrm{II}) \quad \forall k \in \mathbb{N}\, \forall f \in \mathbb{N}^{\mathbb{N}}\, \exists n \in \mathbb{N}\, \forall i, j \in [n; f(n)] \left(|x_i - x_j| \leqslant \frac{1}{k+1} \right)$$

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(I) is equivalent to (II):

Clearly (I) $\rightarrow$ (II). The reverse direction, (II) $\rightarrow$ (I), is shown by contradiction: (II) states that there is no counterexample to (I).

The Cauchy (I) and the metastability (II) properties are equivalent, so... *Why look at* (II)*?*

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- In many cases it is, instead, possible to extract an uniform computable metastability rate Φ

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(The term **metastability** was coined by T.Tao in 2007 while studying the Mean Ergodic Theorem.)

An easy example

Infinite Convergence Principle

Every decreasing sequence of nonnegative real numbers is convergent.

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We have the following metastability result for any such sequence:

Quantitative version (Kreisel, 1952)

Let (x_n) be a decreasing sequence of real numbers and $D \in \mathbb{N}$ be such that $\forall n \in \mathbb{N}\, (0 \leqslant x_n \leqslant D)$. Then

$$\forall k \in \mathbb{N}\, \forall f \in \mathbb{N}^{\mathbb{N}}\, \exists n \leqslant \Phi(k, f)\, \forall i, j \in [n; f(n)] \left(|x_i - x_j| \leqslant \frac{1}{k+1} \right),$$

where $\Phi(k, f) := \max\{f^{(i)}(0) : i < R\}$, with $R = D(k+1)$.

Proof:

Towards a contradiction, assume the result to be false. Then,

$n = 0$ fails:

$$x_0 - x_{f(0)} \geqslant x_{i_0} - x_{j_0} > \frac{1}{k+1}$$

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$\dots$

$n = f^{(R-1)}(0)$ fails:

$$x_{f^{(R-1)}(0)} - x_{f^{(R)}(0)} \geqslant x_{i_R} - x_{j_R} > \frac{1}{k+1}$$

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Hence,

$$x_0 \geqslant x_0 - x_{f^{(R)}(0)} > \tfrac{R}{k+1} = D,$$

which gives the contradiction. □

Unwinding mathematical proofs

Proof Mining

G.Kreisel (1951) was the first to formulate the program of **unwinding proofs** under the general question:

> *"What more do we know if we have proved a theorem by restricted means than if we merely know that it is true?"*

By suggestion of D. Scott, "unwinding proofs" evolved into the more appealing term "**proof mining**".

Goals of proof mining

The main idea is to analyse an ineffective mathematical proof in order to obtain additional information:

- effective bounds, algorithms (rates of convergence, of metastability, of asymptotic regularity, ...);
- results of independence of certain parameters;
- generalization of the proof by weakening of premises.

The good choices for study

The quality of these applications of proof theory seems to rest on two distinct aspects:

1st The chosen result must be **of interest** to the community with its finitary information wanted;

2nd The extracted information must be of a **simple** nature if not new

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1st The chosen result must be **of interest** to the community with its finitary information wanted;

2nd The extracted information must be of a **simple** nature if not new

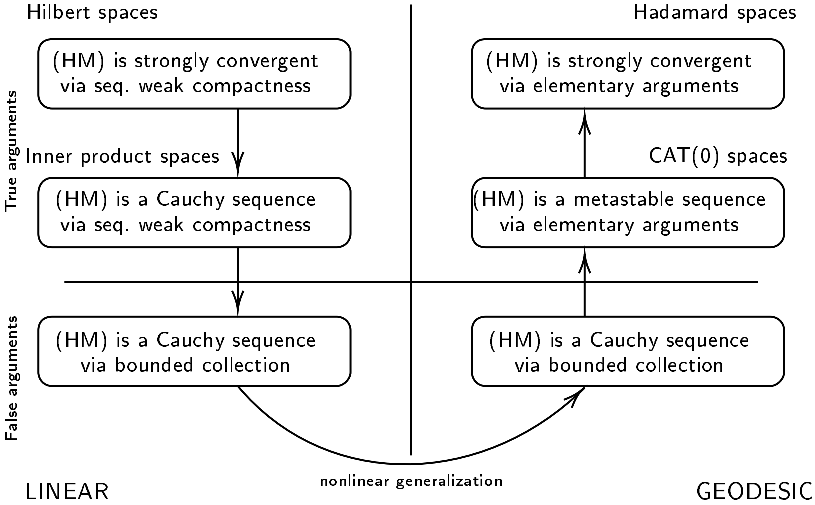
- The analysis of arithmetical comprehension (lurking in common place mathematical arguments, e.g. compactness principles) would require the use of Spector's bar-recursive functionals.
- However, that goes against the goal of "simple information" and would not be appreciated by the general mathematician.
- In most cases the use of such comprehension principles can actually be avoided: it may happen via an 'arithmetization' of the argument, ε-weakening, or by other simplifications to the proof.

Removal by False principles

- In [1], a macro was developed in which certain sequential weak compactness arguments (arithmetical comprehension) were bypassed by instead relying on a (set-theoretically false) bounded collection argument.
- Moreover, the bounded collection argument is computationally tame, in the sense that it does not contribute to an increase in the complexity of the final finitary information.
- This perspective was applied in the study of strong convergence for several Halpern-type iterations and, recently, in the study of the convergence of Dykstra's algorithm (which follows a completely different proof strategy).

[1] F.Ferreira, L.Leuştean, P.Pinto. On the removal of weak compactness arguments in proof mining. *Advances in Mathematics*, 354:106728, 55pp, 2019.

The alternating Halpern-Mann



[2] B.Dinis, P.Pinto. Strong convergence for the alternating Halpern-Mann iteration in CAT(0) spaces. *SIAM Journal on Opt.*, 33(2), 785-815, 2023.

Generalizations

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- In the previous slide, we already saw an example of a generalization from inner product spaces to CAT(0) spaces.

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- In the previous slide, we already saw an example of a generalization from inner product spaces to CAT(0) spaces.
- Successful in generalizations from a linear to nonlinear setting.
  - "Lion-Man" game – weakening of compactness assumption;[2]
  - Suzuki's theorem reducing the convergence of a generalized iterative schema to that of its original version;[3]
  - Abstract versions of proximal algorithm in CAT(0);[4]

[2] U.Kohlenbach, G.López-Acedo, and A.Nicolae. A uniform betweenness property in metric spaces and its role in the quantitative analysis of the "Lion-Man" game. Pacific J. Math., 310(1):181–212, 2021.

[3] U.Kohlenbach, and P.Pinto. Quantitative translations for viscosity approximation methods in hyperbolic spaces. J. Math. Anal. Appl., 507, 2022.

[4] A.Sipoş. Abstract strongly convergent variants of the proximal point algorithm. Comput. Optim. Appl., 83(1):349–380, 2022.

Nonlinear spaces

Let us briefly recall certain settings of geodesic spaces:

$$\begin{aligned} \text{W-hyperbolic spaces} &\Rightarrow \text{normed spaces} \\ \text{UCW-hyperbolic spaces} &\Rightarrow \text{uniformly convex normed spaces} \\ CAT(0)\text{spaces} &\Rightarrow \text{inner product spaces} \end{aligned}$$

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- Several results that hold in Hilbert spaces, still hold in a more general setting of (uniformly) **smooth normed spaces** where ‘inner product’-like arguments are still available. In this instance, one assumes some additional conditions on the norm which are more general than assuming it to arise from an inner-product.
- Despite the relevance of these spaces, no geodesic counterpart existed in the literature.

Smooth Hyperbolic spaces

In [5], the notion of nonlinear smooth space was introduced as a space (X, d, W, π) satisfying:

(P1) $\pi(\overrightarrow{xy}, \overrightarrow{xy}) = d^2(x, y)$

(P2) $\pi(\overrightarrow{xy}, \overrightarrow{uv}) = -\pi(\overrightarrow{yx}, \overrightarrow{uv}) = -\pi(\overrightarrow{xy}, \overrightarrow{vu})$

(P3) $\pi(\overrightarrow{xy}, \overrightarrow{uv}) + \pi(\overrightarrow{yz}, \overrightarrow{uv}) = \pi(\overrightarrow{xz}, \overrightarrow{uv})$

(P4) $\pi(\overrightarrow{xy}, \overrightarrow{uv}) \leqslant d(x, y)d(u, v)$

(P5) $d^2(W(x, y, \lambda), z) \leqslant (1-\lambda)^2 d^2(x, z) + 2\lambda\pi(\overrightarrow{yz}, \overrightarrow{W(x, y, \lambda)z})$,

where $\pi : X \times X \to \mathbb{R}$ and $\overrightarrow{xy}$ denotes the pair $(x, y) \in X \times X$.
The function π is a nonlinear analogue of the normalized duality map in the normed setting.

[5] P.Pinto. Nonexpansive maps in nonlinear smooth spaces. To appear in: *Transactions of the American Mathematical Society*, 48pp, 2024.

Smooth Hyperbolic spaces

In [5], the notion of nonlinear smooth space was introduced as a space (X, d, W, π) satisfying:

(P1) $\pi(\overrightarrow{xy}, \overrightarrow{xy}) = d^2(x, y)$

(P2) $\pi(\overrightarrow{xy}, \overrightarrow{uv}) = -\pi(\overrightarrow{yx}, \overrightarrow{uv}) = -\pi(\overrightarrow{xy}, \overrightarrow{vu})$

(P3) $\pi(\overrightarrow{xy}, \overrightarrow{uv}) + \pi(\overrightarrow{yz}, \overrightarrow{uv}) = \pi(\overrightarrow{xz}, \overrightarrow{uv})$

(P4) $\pi(\overrightarrow{xy}, \overrightarrow{uv}) \leqslant d(x, y)d(u, v)$

(P5) $d^2(W(x, y, \lambda), z) \leqslant (1-\lambda)^2 d^2(x, z) + 2\lambda\pi(\overrightarrow{yz}, \overrightarrow{W(x, y, \lambda)z})$,

where $\pi : X \times X \to \mathbb{R}$ and $\overrightarrow{xy}$ denotes the pair $(x, y) \in X \times X$. The function π is a nonlinear analogue of the normalized duality map in the normed setting.

- In [5], it was shown that this notion extends both CAT(0) spaces as well as smooth normed spaces, providing an unifying framework for several important results in functional analysis.

[5] P.Pinto. Nonexpansive maps in nonlinear smooth spaces. To appear in: *Transactions of the American Mathematical Society*, 48pp, 2024.

Theorem 6.6. *Let X be a uniformly smooth hyperbolic space with ω_X a modulus of uniform continuity for π. Assume that (z_m) is a Cauchy sequence with rate of metastability ξ, and let (x_n) be generated by ($\mathrm{H}_{\mathrm{ppa}}$). Then, for all $\varepsilon > 0$ and $f \in \mathbb{N}^{\mathbb{N}}$,*

$$\exists n \leq \Omega\ \exists w \in C\ \forall i \in [n; n + f(n)] \left(d(w, T_i(w)) \leq \frac{\varepsilon}{2} \wedge d(x_i, w) \leq \frac{\varepsilon}{2} \right),$$

with $\Omega := \Omega(\varepsilon, f, \sigma_1, \sigma_2, \widetilde{\alpha}, b, \mu, \Delta, \xi, \omega_X) := \hat{\chi}(\theta^M(\hat{\xi}))$, where $\hat{\xi} := \tilde{\xi}(\varepsilon_0, f_0, N_0)$ as per the construction in Lemma 2.2,

$$\varepsilon_0 := \min\left\{ \frac{\tilde{\varepsilon}}{2}, \omega_X\left(b, \frac{\tilde{\varepsilon}}{2}\right) \right\},\ f_0(k) := \left\lceil \frac{b}{\delta_1(k)} \right\rceil,\ N_0 := \left\lfloor \frac{3b}{2\tilde{\varepsilon}} \right\rfloor,\ \tilde{\varepsilon} := \frac{\varepsilon^2}{48b},$$

and, taking χ from Lemma 2.6 and writing g^M for $g^M(k) := \max\{g(k') : k' \leq k\}$,

$$\delta_1(k) := \frac{\min\left\{ \frac{\varepsilon}{2}, \Delta\left(b, \frac{\eta_k}{\mu(0)}\right), \Delta\left(b, \frac{\omega_X(b,\tilde{\varepsilon})}{2}\right), 3\widetilde{\alpha}(K(k)) \cdot \tilde{\varepsilon}, \frac{2\tilde{\varepsilon}}{K(k)} \right\}}{\mu(K(k))},$$

$$\eta_k := \frac{\tilde{\varepsilon}}{3(k+1)},\ K(k) := \max\{k' + f_\chi(k') : k' \leq \theta(k)\},$$

$$f_\chi(k) \underset{10}{:=} \hat{\chi}(k) - k + f^M(\hat{\chi}(k)) + 2,\ \hat{\chi}(k) := \chi\left[\sigma_2, b^2\right]\left(\frac{\varepsilon^2}{4}, k\right),$$

and also, with ψ as in Lemma 6.4,

$$\theta(k) := \psi(\varepsilon_1(k), f_\chi, N_1(k), b) := f_\chi^{+\left(\left\lceil \frac{b}{\varepsilon_1(k)} \right\rceil\right)}(N_1(k))$$

$$\varepsilon_1(k) := \frac{1}{2}\Delta\left(b, \frac{\eta_k}{\mu(0)}\right),\ N_1(k) := \sigma_1\left(\frac{\delta_2(k)}{b}\right),$$

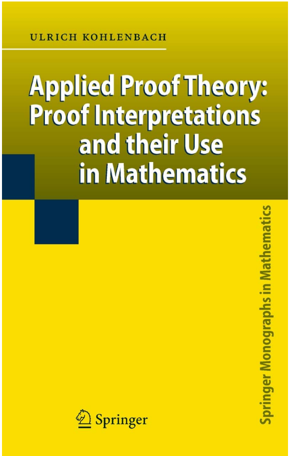
$$\delta_2(k) := \min\left\{ \varepsilon_1(k), \Delta\left(b, \frac{\omega_X(b,\tilde{\varepsilon})}{2}\right), \frac{\omega_X(b,\tilde{\varepsilon})}{2} \right\}.$$

In particular, (x_n) is a Cauchy sequence with rate of metastability Ω.

... which, by easy ('elementary') mathematical arguments, entails the following infinitary result:

Corollary 6.7. *Let X be a complete uniformly smooth UCW hyperbolic space, C a nonempty closed convex subset of X, and $\{T_n\}$ a resolvent-like family of nonexpansive maps on C. For given $x_0, u \in C$, if (x_n) is generated by $(\mathrm{H}_{\mathrm{ppa}})$ with $\{T_n\}$ and a sequence $(\alpha_n) \subseteq (0,1]$ satisfying $\lim \alpha_n = 0$ and $\sum \alpha_n = \infty$, then (x_n) converges towards $Q(u)$, where Q is the sunny nonexpansive retraction of C onto $\mathrm{Fix}(\{T_n\})$.*

The book



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  notes available in his homepage.
- My homepage:
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Thank you