Metastability of the proximal point algorithm with multi-parameters *

Bruno Dinis[†] Pedro Pinto[‡]

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Abstract

In this article we use techniques of proof mining to analyse a result, due to Yonghong Yao and Muhammad Aslam Noor, concerning the strong convergence of a generalized proximal point algorithm which involves multiple parameters. Yao and Noor's result ensures the strong convergence of the algorithm to the nearest projection point onto the set of zeros of the operator. Our quantitative analysis, guided by Fernando Ferreira and Paulo Oliva's bounded functional interpretation, provides a primitive recursive bound on the metastability for the convergence of the algorithm, in the sense of Terence Tao. Furthermore, we obtain quantitative information on the asymptotic regularity of the iteration. The results of this paper are made possible by an arithmetization of the lim sup.

1 Introduction

In the theory of maximal monotone operators, many problems, such as problems of minimization of a function, variational inequalities, etc., can be formulated as finding a zero of a maximal monotone operator (see e.g. [17] and the references therein). The proximal point algorithm (PPA) [39] is a powerful and successful algorithm in finding a solution of maximal monotone operators. Starting from any initial guess $x_0 \in H$, the (PPA) generates a sequence (x_n) which approximates the solution, defined by

$$x_{n+1} = \mathsf{J}_{c_n}(x_n) + e_n, \tag{PPA}$$

where J_{c_n} are resolvent functions of a maximal monotone operator with parameter $c_n > 0$, and (e_n) an error sequence. Ralph Rockafellar showed that (PPA) converges weakly towards a zero of the operator, provided that (c_n) is bounded away from zero and $\|e_n\|$ is summable, i.e. $\sum_{n=0}^{\infty} \|e_n\| < \infty$. Osman Güler showed in [13], by providing a counter-example, that in general (PPA) does not converge strongly. For this reason, several modifications of the algorithm were studied (see for example [8, 15, 35]).

The Krasnosel'skiĭ-Mann (KM) iteration is an important algorithm for the approximation of fixed points of nonexpansive maps [34]. One relevant feature that makes the KM iteration attractive is the fact that it is Fejér monotone relative to the set of fixed points [6, 25]. In general, one can only guarantee weak convergence for the KM iteration. Of course, if in practical optimization problems we restrict ourselves to a finite-dimensional context, then this limitation disappears. On the other hand, with the so-called Halpern variant iteration [14] one can actually establish a general strong convergence result. The fact that the Halpern iteration seems to be less sought after for optimization practice may rest on the fact that this iteration is no longer Fejér monotone. However, recent results brought a renewed interest to the Halpern type iteration (see e.g. [33, 7]). Through speedup techniques (as in [36]), improvements on the rate of asymptotic regularity were obtained for the Halpern iteration.

The impact of these iterations in fixed point theory motivated the following variants of (PPA):

$$x_{n+1} = \lambda_n x_n + (1-\lambda_n)\mathsf{J}_{c_n}(x_n) + e_n, \tag{KM-PPA}$$

$$x_{n+1} = \lambda_n u + (1-\lambda_n)\mathsf{J}_{c_n}(x_n) + e_n, \tag{H-PPA}$$

with $(\lambda_n) \subset [0,1]$, and for all $n \in \mathbb{N}$, $c_n > 0$ and e_n is an error term. While, in general, (KM-PPA) is only weakly convergent [16], the algorithm (H-PPA) is strongly convergent [16, 46]. Other strongly convergent variants of

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†Departamento de Matemática, Faculdade de Ciências da Universidade de Lisboa, Campo Grande, Edifício C6, 1749-016 Lisboa, Portugal. bmdinis@fc.ul.pt.

‡Department of Mathematics, Technische Universität Darmstadt, Schlossgartenstrasse 7, 64289 Darmstadt, Germany. pinto@mathematik.tu-darmstadt.de.

the proximal point algorithm are, for example, the hybrid projection PPA [40], the shrinking projection PPA [5] and the viscosity PPA [43].

In this paper, we will focus on the following multi-parameter version of the proximal point algorithm, considered by Yonghong Yao and Muhammad Aslam Noor in [47], which is an hybrid between (KM-PPA) and (H-PPA). Let $z_0 \in H$ be an initial guess and define

$$z_{n+1} = \lambda_n u + \gamma_n z_n + \delta_n \mathsf{J}_{c_n}(z_n) + e_n, \tag{mPPA}$$

where $u \in H$ is given, and for all $n \in \mathbb{N}$ it holds that $c_n > 0$, $\lambda_n, \gamma_n, \delta_n \in (0, 1)$ and $\lambda_n + \gamma_n + \delta_n = 1$.

Yao and Noor showed in [47] that the algorithm (mPPA) is strongly convergent to the nearest projection point onto the set of zeros of the operator (see Theorem 3). The goal of this paper is to study quantitative information regarding the strong convergence of the iteration (mPPA).

Using proof-theoretical techniques, we analyse Yao and Noor's proof, and are able to prove the strong convergence of (mPPA) using only a weaker version of the metric projection principle, an arithmetization of the lim sup, and bypassing the use of a sequential weak compactness argument. Theorem 29 provides an effective bound on the metastability – in the sense of Terence Tao [45, 44] – of (mPPA), i.e. we obtain a computable function $\phi : \mathbb{N} \times \mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ such that

$$\forall k \in \mathbb{N}\, \forall f : \mathbb{N} \to \mathbb{N}\, \exists n \leq \phi(k, f)\, \forall i, j \in [n, n + f(n)] \left(\|z_i - z_j\| \leq \frac{1}{k+1} \right). \tag{$\star$}$$

Note that ($\star$) is equivalent (in a non-effective way) to the Cauchy property of the sequence (z_n). For a general sequence (z_n) this is the best one can hope for, in the sense that it is not possible to obtain a computable rate for the Cauchy property. Furthermore, we obtain quantitative information on the asymptotic regularity of the iteration (Proposition 21 and Corollary 22). The complexity of the extracted information follows directly from the strength of the logical principles required for the proof. The fact that these results are proved using the weaker principles mentioned above is reflected in the bounds obtained, which are primitive recursive (in the sense of Kleene). The way to deal with the projection and sequential weak compactness arguments was already understood [23, 10], so the remaining obstacle was on how to deal with the lim sup. The problem is solved by an arithmetization of the lim sup (Lemma 17) tailored to deal with a key argument (Lemma 5) in Yao and Noor's proof (see also [29] for an alternative way to deal with the lim sup).

The methods used in this paper are set in the framework of proof mining, a program that describes the process of using proof-theoretical techniques to analyse mathematical proofs with the aim of extracting new information. This program was developed mainly by the works of Ulrich Kohlenbach and his collaborators, producing a vast number of results; analysing proofs from areas such as approximation theory, ergodic theory, fixed point theory and optimization theory (see e.g. [24, 28, 22]). In the proof mining program, proof interpretations are used as tools to extract constructive (i.e. computational) information from noneffective proofs. The output of the interpretation, which we refer to as *quantitative version*, gives explicit information that previously was implicit and hidden behind the use of quantifiers. The standard technique in proof mining is Kohlenbach's *monotone functional interpretation* [18, 22] which is based on Kurt Gödel's *Dialectica* interpretation [12, 1] that works with upper bounds for witnessing terms instead of precise witnesses. Recently, Fernando Ferreira and Paulo Oliva's *bounded functional interpretation* (BFI) [11] (see also the classical version [9]) has proven to be a valid alternative for the proof mining program, providing new insight to some theoretical questions concerning the elimination of sequential weak compactness [10].

The quantitative results, as well as their proofs, obtained by the proof mining program do not presuppose any particular knowledge of logical tools because the latter are only used as an intermediate step and are not visible in the final product. Apart from the end of Section 5, where we make some remarks on the logical aspects of the analysis, our work is no exception and so, knowledge of tools from logic in general and familiarity with the BFI in particular are not necessary to read this paper.

We would like to point out that our work comes as a natural generalization of [30, 32, 31].

The structure of the paper is the following. In Section 2 we recall some notions concerning Hilbert spaces and monotone operators. We also state the result by Yao and Noor for which we give a quantitative analysis in the subsequent sections and explain its original proof. In Section 3 we give a quantitative treatment of the principles used in the original proof: the projection argument, sequential weak compactness, and the arithmetization of the lim sup. The quantitative analysis of Yao and Noor's proof is carried out in Section 4. We start by obtaining a partial result which depends on an additional condition (Subsection 4.1). This condition is then shown to be satisfied by a concrete functional (Subsection 4.2). Finally, we prove the main result regarding the metastability for (mPPA) (Subsection 4.3). We leave some final remarks that allow to better understand some theoretical aspects of the analysis to Section 5.

2 Preliminaries

We work in a real Hilbert space H with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$. We recall that a multi-valued operator $T : H \to 2^H$ is said to be *monotone* if and only if whenever (x, y) and (x', y') are elements of the graph of T, it holds that $\langle x - x', y - y' \rangle \geq 0$. A monotone operator T is *maximal monotone* if the graph of T is not properly contained in the graph of any other monotone operator on H. We denote by S the set of all *zeros* of T, i.e. $S = T^{-1}(0)$. We fix T a maximal monotone operator and assume henceforth S to be nonempty. For $c > 0$, we use J_c to denote the *resolvent function* of T with parameter c, i.e. the single-valued function defined by

$$\mathsf{J}_c = (I + cT)^{-1}.$$

A mapping $f : H \to H$ is called *nonexpansive* if

$$\forall x, y \in H(\|f(x) - f(y)\| \leq \|x - y\|).$$

The set of fixed points of a mapping f is the set $\mathrm{Fix}(f) := \{x \in H : f(x) = x\}$. The resolvent mapping J_c is nonexpansive, and for every $c > 0$, the set of fixed points of J_c is S. If f is nonexpansive, then $\mathrm{Fix}(f)$ is a closed and convex subset of H. For a comprehensive introduction to convex analysis and the theory of monotone operators in Hilbert spaces we refer to [2].

The following lemmas are well-known.

Lemma 1 (Resolvent identity)**.** *For every $a, b > 0$ and every $x \in H$ it holds that*

$$\mathsf{J}_a(x) = \mathsf{J}_b\left(\frac{b}{a}x + \left(1 - \frac{b}{a}\right)\mathsf{J}_a(x)\right).$$

Lemma 2 ([35])**.** *If $0 < a \leq b$, then for all $x \in H$ it holds that $\|\mathsf{J}_a(x) - x\| \leq 2\|\mathsf{J}_b(x) - x\|$.*

The quantitative analysis carried out in this paper makes use of the notion of monotone functional. For that matter we consider the notion of strong majorizability $\leq^*$ from [3] in the following two particular cases. In the case of functions $f, g : \mathbb{N} \to \mathbb{N}$, we have

$$g \leq^* f := \forall n \, \forall m \leq n \; (g(m) \leq f(n) \wedge f(m) \leq f(n)),$$

and in the case of functionals $\varphi, \psi : \mathbb{N} \times \mathbb{N}^{\mathbb{N}} \to \mathbb{N}$, we have

$$\varphi \leq^* \psi := \forall n \, \forall f : \mathbb{N} \to \mathbb{N} \, \forall m \leq n \, \forall g \leq^* f \; (\varphi(m, g) \leq \psi(n, f) \wedge \psi(m, g) \leq \psi(n, f)).$$

We say that f is monotone if $f \leq^* f$ and similarly for a functional φ. For functions $f : \mathbb{N} \to \mathbb{N}$, this monotone property coincides with the usual property $\forall n \in \mathbb{N} \; (f(n) \leq f(n+1))$. Quantifications over monotone functions $\forall f(f \leq^* f \to \cdots)$ will be abbreviated by $\tilde{\forall} f(\cdots)$. Due to the particularities of the BFI, our quantitative results quantify over such monotone functions. Note however that for all $f : \mathbb{N} \to \mathbb{N}$, one has $f \leq^* f^{\mathrm{maj}}$, where $f^{\mathrm{maj}}(n) := \max_{i \leq n}\{f(i)\}$. Hence there is no real restriction in working with monotone quantifications.

2.1 A result by Yao and Noor

We present the result by Yao and Noor concerning the strong convergence of the (mPPA) (Theorem 3) and give a detailed description of its proof. This will hopefully guide the reader through our quantitative analysis and the several steps that it requires.

Consider the following set of conditions.

(H_1) $\lim_{n \to \infty} \lambda_n = 0$.

(H_2) $\sum_{n=0}^{\infty} \lambda_n = \infty$.

(H_3) $0 < \liminf_{n \to \infty} \gamma_n \leq \limsup_{n \to \infty} \gamma_n < 1$.

(H_4) $c_n \geq c$, where c is some positive constant.

(H_5) $c_{n+1} - c_n \to 0$.

(H_6) $\sum_{n=1}^{\infty} \|e_n\| < \infty$.

Theorem 3. ([47, Theorem 3.3]) *Let (z_n) be generated by* (mPPA)*. Assume that (H_1)-(H_6) hold. Then (z_n) converges strongly to a point $z \in S$, the nearest to u.*

The proof of Theorem 3 relies on Lemma 4 and Lemma 5 below, due to Tomonari Suzuki [42], for which we give quantitative versions in Subsection 4.2 (Lemma 25 and Lemma 26).[1]

Lemma 4. ([42, Lemma 2.1]) *Let (z_n) and (w_n) be sequences in a Banach space X and let (α_n) be a sequence in $[0,1]$ such that $\limsup \alpha_n < 1$. Suppose that $z_{n+1} = \alpha_n w_n + (1-\alpha_n) z_n$ for all $n \in \mathbb{N}$, $\limsup(\|w_{n+1} - w_n\| - \|z_{n+1} - z_n\|) \leq 0$ and $d := \limsup \|w_n - z_n\| < \infty$. Then*

$$\forall t \in \mathbb{N} \, (\liminf | \, \|w_{n+t} - z_n\| - (1 + \alpha_n + \cdots + \alpha_{n+t-1}) d \, | = 0) \, .$$

Lemma 5. ([42, Lemma 2.2]) *Let (z_n) and (w_n) be bounded sequences in a Banach space X and let (α_n) be a sequence in $[0,1]$ with $0 < \liminf \alpha_n \leq \limsup \alpha_n < 1$. Suppose that $z_{n+1} = \alpha_n w_n + (1-\alpha_n) z_n$ for all $n \in \mathbb{N}$, and $\limsup(\|w_{n+1} - w_n\| - \|z_{n+1} - z_n\|) \leq 0$. Then $\lim \|w_n - z_n\| = 0$.*

The proof of Theorem 3 is divided in the following steps:

(1) Show that (z_n) is bounded. This is just a simple proof by induction and some easy computations.

(2) $\lim \|z_{n+1} - z_n\| = 0$. Letting $z_{n+1} = \gamma_n z_n + (1-\gamma_n) w_n$, it is shown first that $\limsup(\|w_{n+1} - w_n\| - \|z_{n+1} - z_n\|) \leq 0$. Using Lemma 5 one concludes that $\lim \|z_n - w_n\| = 0$ which, by the definition of w_n, is enough to conclude this step.

(3) $\lim \|\mathsf{J}_c(z_n) - z_n\| = 0$. From step (2) and the hypothesis of the theorem one concludes that $\lim \|\mathsf{J}_{c_n}(z_n) - z_n\| = 0$. The conclusion follows from Lemma 2.

(4) Projection argument. With $\tilde{p}$ the projection point of u onto S it is shown that $\forall q \in S(\langle \tilde{p} - u, \tilde{p} - q \rangle \leq 0)$.

(5) Sequential weak compactness and demiclosedness. Pick a subsequence (z_{n_j}) of (z_n) such that $\limsup \langle \tilde{p} - u, \tilde{p} - z_n \rangle = \lim_j \langle \tilde{p} - u, \tilde{p} - z_{n_j} \rangle$ and (z_{n_j}) converges weakly to some $q \in S$. Here the following demiclosedness principle is used.

   Lemma 6 (Demiclosedness principle [4])**.** *Let C be a closed convex subset of H and let $f : C \to C$ be a nonexpansive mapping such that $\mathrm{Fix}(f) \neq \emptyset$. Assume that (x_n) is a sequence in C such that (x_n) weakly converges to $x \in C$ and $((I-f)(x_n))$ converges strongly to $y \in H$. Then $(I-f)(x) = y$.*

   By step (4) it follows that $\limsup \langle \tilde{p} - u, \tilde{p} - z_n \rangle \leq 0$.

(6) Main combinatorial part. In this final step it is shown that the conditions of Lemma 7 below are satisfied. The application of this lemma is enough to conclude the result.

   Lemma 7. ([46, Lemma 2.5]) *Let (a_n) be a sequence of nonnegative real numbers such that for all $n \in \mathbb{N}$*

$$a_{n+1} \leq (1 - s_n) a_n + s_n t_n + \delta_n, \tag{1}$$

   where $(s_n) \subset [0,1]$ and $(t_n), (\delta_n)$ are such that $\sum_{n=0}^{\infty} s_n = \infty$, $\limsup_{n \to \infty} t_n \leq 0$ and $\sum_{n=0}^{\infty} \delta_n < \infty$. Then (a_n) converges to zero.

3 Avoiding roadblocks

From the point of view of proof mining the analysis of the original proof of Theorem 3 presents some difficulties that prevent the extraction of simple bounds. These concern the projection argument in step (4), the weak compactness in step (5) and the assumed existence of the lim sup in Suzuki's lemmas. In Subsections 3.1 and 3.2 we adapt the way to deal with projection [23] and sequential weak compactness [10] to our context. Furthermore, in Subsection 3.2 we give a quantitative version of Lemma 7. In Subsection 3.3 we give an arithmetization of the lim sup that allows to obtain a quantitative version of Lemma 5. A more detailed explanation to why these principles are problematic will be given in Section 5.

3.1 Projection argument

In this section we deal with the following projection argument which is used in the original proof

$$\exists p \in S \, \forall k \in \mathbb{N} \, \forall q \in S \left(\|p - u\|^2 \leq \|q - u\|^2 + \frac{1}{k+1} \right), \tag{2}$$

stating that there is a zero of T that is the nearest to a given $u \in H$. The squares are added here only for an easier connection to the inner product of the space that will be required below.

[1] In fact, Lemma 4 is only used to prove Lemma 5.

As noticed by Kohlenbach [23], instead of (2), it is enough for the quantitative analysis to consider the weaker statement

$$\forall k \in \mathbb{N}\, \exists p \in S\, \forall q \in S \left(\|p-u\|^2 \leq \|q-u\|^2 + \frac{1}{k+1} \right). \tag{3}$$

While the proof of (2) requires the use of countable choice, the statement (3) can be proved by a simple induction argument and this fact is reflected on the extracted bounds, which are then recursively defined.

An analysis of the projection argument via the monotone functional interpretation was previously carried out by Kohlenbach [23]. A detailed explanation of the analysis of the projection argument using the bounded functional interpretation was shown in [10]. In the latter, the assumption that one is working in a bounded set plays an important role in simplifying the interpretation. Although we do not have that assumption here, we can equivalently consider the projection statement in (2) restricted to a big enough ball centered at some zero point $s \in S$. This restriction was considered in [38], giving rise to the quantitative version in Proposition 9 bellow.

Notation 8. *From now on, we will write* J *instead of* $\mathsf{J}_{\frac{1}{c}}$ *and* J_n *instead of* J_{c_n}*, under the assumption that* $c \in \mathbb{N} \setminus \{0\}$ *satisfies* $c_n \geq \frac{1}{c}$ *for all* $n \in \mathbb{N}$*. For each* $r \in \mathbb{N}$ *and function* $f : \mathbb{N} \to \mathbb{N}$ *we denote the* r*-fold iteration of* f *by* $f^{(r)}$*. I.e.* $f^{(0)} \equiv \mathrm{Id}$ *and* $f^{(r+1)} = f(f^{(r)})$*. Let* s *be a zero of* T*. For any* $N \in \mathbb{N}$*, let* $\mathrm{B}_N := \{x \in H : \|x-s\| \leq N\}$ *denote the closed ball centered at* s *with radius* N*. The zero point is always made clear by the context.*

Proposition 9 ([38])**.** *Let* $N \in \mathbb{N} \setminus \{0\}$ *be such that* $N \geq 2\|u-s\|$ *for some* $s \in S$*.*
For any natural number k *and monotone function* $f : \mathbb{N} \to \mathbb{N}$*, there are* $n \leq f^{(r)}(0)$ *and* $p \in \mathrm{B}_N$ *such that*

$$\|\mathsf{J}(p) - p\| \leq \frac{1}{f(n)+1}$$

and

$$\forall q \in \mathrm{B}_N \left(\|\mathsf{J}(q) - q\| \leq \frac{1}{n+1} \to \|p-u\|^2 \leq \|q-u\|^2 + \frac{1}{k+1} \right),$$

where $r := N^2(k+1)$*.*

In Proposition 9, we considered only the (almost) fixed points of J since the fixed point set of all resolvent functions coincide. We prove the following quantitative version which requires majorizing information on the sequence of real numbers (c_n).

Lemma 10. *Let* $(c_n) \subset \mathbb{R}^+$ *and* $c \in \mathbb{N}\setminus\{0\}$ *satisfying* (Q_4)*. Consider* $\mathcal{C} : \mathbb{N} \to \mathbb{N}$ *a monotone function satisfying* $\forall n \in \mathbb{N}\, (c_n \leq \mathcal{C}(n))$*. Let* $\zeta : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ *be the monotone function defined by* $\zeta(k,n) := \mathcal{C}(n)c(k+1) - 1$*. For any* $k, n \in \mathbb{N}$ *and any* $p \in H$*,*

$$\|\mathsf{J}(p) - p\| \leq \frac{1}{\zeta(k,n)+1} \;\to\; \forall n' \leq n \left(\|\mathsf{J}_{n'}(p) - p\| \leq \frac{1}{k+1} \right). \tag{4}$$

Proof. Let $e = \mathsf{J}(p) - p$ and assume $\|e\| \leq \dfrac{1}{\zeta(k,n)+1}$. For $n' \leq n$, we have

$$\begin{aligned}
\mathsf{J}(p) = p + e &\leftrightarrow p \in p + e + \frac{1}{c}T(p+e) \\
&\leftrightarrow -e \cdot c \in T(p+e) \\
&\leftrightarrow p + (1 - c \cdot c_{n'})e \in p + e + c_{n'}T(p+e) \\
&\leftrightarrow \mathsf{J}_{n'}(p + (1 - c \cdot c_{n'})e) = p + e.
\end{aligned}$$

Hence

$$\begin{aligned}
\|\mathsf{J}_{n'}(p) - p\| &\leq \|\mathsf{J}_{n'}(p) - \mathsf{J}_{n'}(p + (1 - c \cdot c_{n'})e)\| + \|\mathsf{J}_{n'}(p + (1 - c \cdot c_{n'})e) - p\| \\
&\leq (1 + |1 - c \cdot c_{n'}|)\|e\| = c \cdot c_{n'}\|e\| \leq \frac{1}{k+1},
\end{aligned}$$

since $c_{n'} \leq \mathcal{C}(n)$. □

3.2 Sequential weak compactness

In the proof by Yao and Noor, sequential weak compactness, together with the demiclosedness principle, is used to show that $\limsup \langle \tilde{p} - u, \tilde{p} - z_m \rangle \leq 0$, where $\tilde{p}$ is the projection point of u onto S and (z_n) is generated by (mPPA). This means that there exists $\tilde{p} \in S$ (the projection point) such that

$$\forall k \in \mathbb{N}\, \exists n \in \mathbb{N}\, \forall m \geq n \left(\langle \tilde{p} - u, \tilde{p} - z_m \rangle \leq \frac{1}{k+1} \right). \tag{5}$$

Without having access to the projection point, and working only with approximations, we switch from (5) to the following weakening

$$\forall k \in \mathbb{N}\, \exists p \in S\, \exists n \in \mathbb{N}\, \forall m \geq n \left(\langle p - u, p - z_m \rangle \leq \frac{1}{k+1} \right).$$

As it turns out, this weaker form is still enough to carry out the proof, avoiding the use of sequential weak compactness. This is similar to the arguments in the beginning of section 2 in [10].

Proposition 11 ([38]). *Let $N \in \mathbb{N} \setminus \{0\}$ be a natural number satisfying $N \geq 2\|u - s\|$ for some point $s \in S$. For any $k \in \mathbb{N}$ and monotone function $f : \mathbb{N} \to \mathbb{N}$, there are $n \leq 24N(\check{f}^{(R)}(0) + 1)^2$ and $p \in \mathrm{B}_N$ such that*

$$\|J(p) - p\| \leq \frac{1}{f(n)+1} \wedge \forall q \in B_N \left(\|J(q) - q\| \leq \frac{1}{n+1} \to \langle u - p, q - p \rangle \leq \frac{1}{k+1} \right),$$

with $R := 4N^4(k+1)^2$ and $\check{f} := \max\{f(24N(m+1)^2), 24N(m+1)^2\}$.

In Proposition 21(iii) below (under an additional assumption (Q_S) on a functional χ) we compute a monotone functional ξ_χ satisfying

$$\forall k \in \mathbb{N}\, \tilde{\forall} f : \mathbb{N} \to \mathbb{N}\, \exists n \leq \xi_\chi(k, f)\, \forall m \in [n, n + f(n)] \left(\|\mathsf{J}(z_m) - z_m\| \leq \frac{1}{k+1} \right). \tag{6}$$

Furthermore, in Lemma 18, we show that the sequence (z_n) is bounded and compute an explicit natural number N_0 such that $(z_n) \subset \mathrm{B}_{N_0}$.

Proposition 12 corresponds to the elimination of the sequential weak compactness argument. It can be seen as an application of the general principle in [10, Proposition 4.3] with $\alpha(k, f) = \xi_\chi(k, f+1)$, where the sequence being considered is (z_{m+1}), β is given by Proposition 11 and $\varphi(x, y) = \langle x - u, y \rangle$.

Proposition 12. *Let $\xi_\chi : \mathbb{N} \times \mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ be a monotone function satisfying (6). For some $s \in S$, let $N_0 \in \mathbb{N}$ be such that $(z_n) \subset \mathrm{B}_{N_0}$, and $N \geq \max\{2\|u - s\|, N_0\}$. For any $k \in \mathbb{N}$ and monotone function $f : \mathbb{N} \to \mathbb{N}$, there are $n \leq \psi_\chi(k, f)$ and $p \in \mathrm{B}_N$ such that $\|\mathsf{J}(p) - p\| \leq \frac{1}{f(n)+1}$ and*

$$\forall m \in [n, n + f(n)] \left(\langle p - u, p - z_{m+1} \rangle \leq \frac{1}{k+1} \right),$$

where $\psi_\chi(k, f) := \xi_\chi(24N(\breve{g}^{(R)}(0)+1)^2, f+1)$ with $R := N^4(k+1)^2$, $\breve{g}(m) := \max\{g(24N(m+1)^2), 24N(m+1)^2\}$ and $g(m) := f(\xi_\chi(m, f+1))$.

Proof. Let $k \in \mathbb{N}$ and a monotone function f be given. By Proposition 11 applied to k and the function g, we get $n' \leq 24N\left(\breve{g}^{(R)}(0) + 1\right)^2$ and $p \in \mathrm{B}_N$ such that $\|\mathsf{J}(p) - p\| \leq \frac{1}{g(n')+1}$ and

$$\forall q \in \mathrm{B}_N \left(\|\mathsf{J}(q) - q\| \leq \frac{1}{n'+1} \to \langle p - u, p - q \rangle \leq \frac{1}{k+1} \right). \tag{7}$$

By (6), there is $n \leq \xi_\chi(n', f+1) \leq \psi_\chi(k, f)$ such that

$$\forall m \in [n, n + f(n) + 1] \left(\|\mathsf{J}(z_m) - z_m\| \leq \frac{1}{n'+1} \right).$$

Hence, $\forall m \in [n, n + f(n)] \left(\|\mathsf{J}(z_{m+1}) - z_{m+1}\| \leq \frac{1}{n'+1} \right)$. Since $(z_n) \subset \mathrm{B}_N$, by (7) we conclude that

$$\forall m \in [n, n + f(n)] \left(\langle p - u, p - z_{m+1} \rangle \leq \frac{1}{k+1} \right).$$

Finally, by the monotonicity of the function f,

$$\|\mathsf{J}(p) - p\| \leq \frac{1}{g(n')+1} = \frac{1}{f(\xi_\chi(n', f+1))+1} \leq \frac{1}{f(n)+1}. \qquad \square$$

As mentioned in Subsection 2.1, the final step of the proof of Theorem 3 is an application of Lemma 7. There $s_n = \|z_n - p\|$, with p the projection point of u onto S. However, using approximations to the projection point instead of the projection point itself, the inequality (1) only holds with $s_n + v_n$ in place of s_n, for (v_n) a certain sequence of errors. The following result from [38] corresponds to a quantitative version of this statement.

Lemma 13 ([38])**.** *Let (s_n) be a bounded sequence of non-negative real numbers and $D \in \mathbb{N}$ a positive upper bound on (s_n). Consider sequences of real numbers $(\lambda_n) \subset (0,1)$, (r_n), (v_n) and $(\gamma_n) \subset [0, +\infty)$ and assume the existence of a monotone function L satisfying $\sum_{i=1}^{L(k)} \lambda_i \geq k$, for all $k \in \mathbb{N}$. For natural numbers k, n and p assume*

(i) $\forall m \in [n,p] \left(v_m \leq \frac{1}{4(k+1)(p+1)} \wedge r_m \leq \frac{1}{4(k+1)} \right)$.

(ii) $\forall m \in \mathbb{N} \left(\sum_{i=n}^{n+m} \gamma_i \leq \frac{1}{4(k+1)} \right)$.

(iii) $\forall m \in \mathbb{N} (s_{m+1} \leq (1-\lambda_m)(s_m + v_m) + \lambda_m r_m + \gamma_m)$.

Then

$$\forall m \in [\sigma(k,n), p] \left(s_m \leq \frac{1}{k+1} \right),$$

with $\sigma(k,n) := L(n + \lceil \ln(4D(k+1)) \rceil) + 1$.

A direct application of Lemma 13 gives the following result which is more suitable for our analysis.

Lemma 14. *Let Ω be a bounded subset of H. Let $(\lambda_n) \subset (0,1)$ be given and, for each $p \in \Omega$, consider the sequences of real numbers $(s_{n,p})$, $(v_{n,p})$, $(r_{n,p})$ and $(\gamma_{n,p})$ with $(s_{n,p})$, $(\gamma_{n,p}) \subset [0, +\infty)$ and such that, for all $p \in \Omega$,*

$$\forall m \in \mathbb{N} \left(s_{m+1,p} \leq (1-\lambda_m)(s_{m,p} + v_{m,p}) + \lambda_m r_{m,p} + \gamma_{m,p} \right).$$

For a natural number $D \in \mathbb{N}$ and monotone functions L, $G : \mathbb{N} \to \mathbb{N}$ and $\Psi : \mathbb{N} \times \mathbb{N}^{\mathbb{N}} \to \mathbb{N}$, assume that:

(i) *L is a rate of divergence for $(\sum \lambda_n)$, i.e.* $\forall k \in \mathbb{N} \left(\sum_{i=1}^{L(k)} \lambda_i \geq k \right)$.

(ii) *For all $p \in \Omega$, D is a positive upper bound on $(s_{n,p})$.*

(iii) *For all $p \in \Omega$, G is a Cauchy rate for $(\sum \gamma_{n,p})$, i.e.* $\forall k, n \in \mathbb{N} \left(\sum_{i=G(k)+1}^{G(k)+n} \gamma_{i,p} \leq \frac{1}{k+1} \right)$.

(iv) $\forall k \in \mathbb{N} \tilde{\forall} f : \mathbb{N} \to \mathbb{N} \, \exists p \in \Omega \, \exists n \leq \Psi(k,f) \, \forall m \in [n, f(n)] \left(v_{m,p} \leq \frac{1}{f(n)+1} \wedge r_{m,p} \leq \frac{1}{k+1} \right)$.

Then, for any natural number k and monotone function $f : \mathbb{N} \to \mathbb{N}$, there are $p \in \Omega$ and $n \leq \Theta(k,f)$ such that

$$\forall m \in [n, f(n)] \left(s_{m,p} \leq \frac{1}{k+1} \right),$$

where $\Theta(k,f) = \Theta(k,f,L,\Psi,G,D) := L(h(\Psi(4k+3,g))) + 1$ with $h(m) = \max\{m, G(4k+3)+1\} + \lceil \ln(4D(k+1)) \rceil$ and $g(m) := 4(k+1)(f(L(h(m))) + 1) + 1)$.

Proof. Let $k \in \mathbb{N}$ and a monotone function f be given. By condition (iv), consider $p_0 \in \Omega$ and $n_1 \leq \Psi(4k+3, g)$ such that for $m \in [n_1, g(n_1)]$

$$v_{m,p_0} \leq \frac{1}{g(n_1)+1} \text{ and } r_{m,p_0} \leq \frac{1}{4(k+1)}.$$

Define $n_2 := \max\{n_1, G(4k+3)+1\}$. By condition (iii), for all $m \in \mathbb{N}$, $\sum_{i=n_2}^{n_2+m} \gamma_{m,p_0} \leq \frac{1}{4(k+1)}$. We have $n_1 \leq n_2$ and

$$g(n_1) \geq f(L(h(n_1)) + 1) = f(\sigma(k, n_2)),$$

where σ is as in Lemma 13. Hence, for $m \in [n_2, f(\sigma(k,n_2))]$,

$$v_{m,p_0} \leq \frac{1}{4(k+1)(f(\sigma(k,n_2))+1)} \text{ and } r_{m,p_0} \leq \frac{1}{4(k+1)}.$$

We are in the conditions of Lemma 13 with $n = n_2$ and $p = f(\sigma(k,n_2))$, and so

$$\forall m \in [\sigma(k,n_2), f(\sigma(k,n_2))] \left(s_{m,p_0} \leq \frac{1}{k+1} \right).$$

Noticing that, by the monotonicity of L, we have $\sigma(k,n_2) \leq \Theta(k,f)$, we conclude the proof. ∎

We recall that the last step in the proof by Yao and Noor is an application of Lemma 7. Similarly, in our quantitative analysis, the final step to prove metastability for (mPPA) is an application of Lemma 14. As such, we need to verify each of the conditions of that result (for a specific choice of parameters). Conditions (i) and (ii) are easy to check. The existence of a function G as in condition (iii) follows from a quantitative version of (H_6). The next result ensures that condition (iv) holds.

Lemma 15. *Assume the conditions of Proposition 12, and that there exist $c \in \mathbb{N} \setminus \{0\}$ satisfying (Q_4) and $\mathcal{C} : \mathbb{N} \to \mathbb{N}$ a monotone function such that $\forall n \in \mathbb{N}(c_n \leq \mathcal{C}(n))$. For any $k \in \mathbb{N}$ and monotone function $f : \mathbb{N} \to \mathbb{N}$, there are $p \in \mathrm{B}_N$ and $n \leq \Psi_\chi(k, f)$ such that for all $m \in [n, f(n)]$,*

$$v_{m,p} \leq \frac{1}{f(n)+1} \wedge r_{m,p} \leq \frac{1}{k+1},$$

where $v_{m,p} = \|\mathsf{J}_m(p) - p\| (\|\mathsf{J}_m(p) - p\| + 2\|z_m - p\|)$, $r_{m,p} = 2\langle u - p, z_{m+1} - p\rangle$, $\Psi_\chi(k, f) := \psi_\chi(2k+1, h)$, where ψ_χ is the function defined in Proposition 12 and $h : \mathbb{N} \to \mathbb{N}$ is the monotone function defined by

$$h(m) := \zeta\left((1+4N)(f(m)+1) - 1, f(m)\right),$$

with ζ as in Lemma 10.

Proof. Let $k \in \mathbb{N}$ and monotone f be given. Applying Proposition 12 to $2k+1$ and to the monotone function h we obtain $p \in \mathrm{B}_N$ and $n \leq \psi_\chi(2k+1, h)$ such that

$$\|\mathsf{J}(p) - p\| \leq \frac{1}{h(n)+1} \tag{8}$$

and

$$\forall m \in [n, n+f(n)] \left(\langle u - p, z_{m+1} - p\rangle \leq \frac{1}{2(k+1)}\right). \tag{9}$$

Clearly (9) implies that for $m \in [n, f(n)]$ one has $r_{m,p} \leq \frac{1}{k+1}$.

Now, by (8)

$$\|\mathsf{J}(p) - p\| \leq \frac{1}{\zeta\left((1+4N)(f(n)+1) - 1, f(n)\right) + 1}.$$

Hence, by (4), for $m \leq f(n)$,

$$\|\mathsf{J}_m(p) - p\| \leq \frac{1}{(1+4N)(f(n)+1)}.$$

Also $\|\mathsf{J}_n(p) - p\| \leq 1$ so, for $m \leq f(n)$,

$$v_{m,p} = \|\mathsf{J}_m(p) - p\| (\|\mathsf{J}_m(p) - p\| + 2\|z_m - p\|) \leq \frac{1+4N}{(1+4N)(f(n)+1)} = \frac{1}{f(n)+1},$$

which concludes the proof. □

3.3 Rational approximation of the lim sup

In this section we show that the assumption of the existence of the lim sup, as in Lemma 4, can be replaced by a rational approximation. A detailed explanation on the origin of these lemmas is given in Section 5.

The idea is that, by working with approximated notions, one can relax the properties of the lim sup to something which is already satisfied by a suitable rational number. We start with the following easy result.

Lemma 16. *Let $N \in \mathbb{N}$ and (x_n) be a sequence of real numbers such that $\forall n \in \mathbb{N}(0 \leq x_n \leq N)$. Then*

$$\forall k, n \in \mathbb{N} \tilde{\forall} f : \mathbb{N} \to \mathbb{N} \, \exists p < N(k+1) \\ \left(\exists m \in [n, n+f(n)] \left(x_m \geq \frac{p}{k+1}\right) \wedge \forall m' \in [n, n+f(n)] \left(x_{m'} \leq \frac{p+1}{k+1}\right)\right). \tag{10}$$

Proof. Suppose towards a contradiction that (10) does not hold. Then there exist $k, n \in \mathbb{N}$ and a monotone function f such that for all $p < N(k+1)$ it holds that

$$\forall m \in [n, n+f(n)] \left(x_m < \frac{p}{k+1}\right) \vee \exists m' \in [n, n+f(n)] \left(x_{m'} > \frac{p+1}{k+1}\right). \tag{11}$$

This implies

$$\forall p < N(k+1) \left(A(p) \vee \neg A(p+1)\right),$$

where $A(p) :\equiv \forall m \in [n, n+f(n)] \left(x_m < \frac{p}{k+1}\right)$. One easily shows by induction on $M \in \mathbb{N}$ that

$$\forall M \left(\forall p \leq M \left(A(p) \vee \neg A(p+1)\right) \rightarrow \left(A(0) \vee \neg A(M+1)\right)\right).$$

Hence, with $M = N(k+1) - 1$ we conclude that

$$\forall m \in [n, n+f(n)]\,(x_m < 0) \vee \exists m \in [n, n+f(n)]\,(x_m \geq N).$$

Hence $\exists m \in [n, n+f(n)]\,(x_m \geq N)$. Now, by (11), for $p = N(k+1) - 1$ and the hypothesis, we have that for all $m \in [n, n+f(n)]$ it holds that $x_m < \frac{N(k+1)-1}{k+1} = N - \frac{1}{k+1}$, which gives a contradiction. We conclude that (10) holds. $\square$

The proof of Lemma 4 requires the following property of the lim sup

$$\forall k, M, t \in \mathbb{N}\, \exists m \geq M\, \forall n \geq m \left(x_{m+t} \geq \limsup x_n - \frac{1}{k+1} \wedge x_n \leq \limsup x_n + \frac{1}{k+1}\right). \tag{12}$$

We adapt Lemma 16 to this property, obtaining a result that corresponds to a quantitative version of (12).

Lemma 17. *Let $N \in \mathbb{N}$ and (x_n) be a sequence of real numbers such that $\forall n \in \mathbb{N}(0 \leq x_n \leq N)$. Let $k, M, t \in \mathbb{N}$ and $f : \mathbb{N} \to \mathbb{N}$ be monotone, let $P := N(k+1)$. For $i \in \{0, \ldots, P\}$ define $n_i = M + it$ and $r_i := \begin{cases} 0, & i = P \\ t + r_{i+1} + f(n_{i+1} + r_{i+1}), & i < P. \end{cases}$ Then*

$$\exists p < P\, \exists m \in [M, \theta] \left(x_{m+t} \geq \frac{p}{k+1} \wedge \forall n \in [m, m+f(m)] \left(x_n \leq \frac{p+1}{k+1}\right)\right), \tag{13}$$

where $\theta = \theta(k, M, t, N, f) := M + (P-1)t + r_0$.

Proof. Let $k, M, t \in \mathbb{N}$ and f be a given monotone function. We define, for each $i \leq P$, the monotone functions $g_i := \lambda m.r_i$. We apply (10) with $k = k$, $f = g_i$ and $n = n_i$, for $i \leq P$. Then, we find, for each $i \leq P$, $m_i \in [n_i, n_i + r_i]$ and $p_i < P$ such that $x_{m_i} \geq \frac{p_i}{k+1}$ and $\forall n \in [n_i, n_i + r_i] \left(x_n \leq \frac{p_i+1}{k+1}\right)$. Now, there exists $i_0 < P$ such that $p_{i_0} \leq p_{i_0+1}$, otherwise there would be a sequence of length $P+1$ of natural numbers such that $p_P < p_{P-1} < \cdots < p_1 < p_0 < P$, which is absurd. Define the natural numbers $m := m_{i_0+1} - t$ and $p := p_{i_0+1}$. Clearly $m \in [M, \theta]$ and $p < P$. We have that $x_{m+t} \geq \frac{p}{k+1}$. To conclude the result it is enough to show that $[m, m+f(m)] \subseteq [n_{i_0}, n_{i_0} + r_{i_0}]$. Indeed, we would get, for $n \in [m, m+f(m)]$ that $x_n \leq \frac{p_{i_0}+1}{k+1} \leq \frac{p_{i_0+1}+1}{k+1} = \frac{p+1}{k+1}$. We have that $m = m_{i_0+1} - t \geq n_{i_0+1} - t = n_{i_0}$, and since f is monotone, $m + f(m) \leq m_{i_0+1} + f(m_{i_0+1}) \leq n_{i_0+1} + r_{i_0+1} + f(n_{i_0+1} + r_{i_0+1}) = n_{i_0} + t + r_{i_0+1} + f(n_{i_0+1} + r_{i_0+1}) = n_{i_0} + r_{i_0}$. Hence $[m, m+f(m)] \subseteq [n_{i_0}, n_{i_0} + r_{i_0}]$. $\square$

4 Quantitative analysis

In this section we carry out the quantitative analysis of Theorem 3. In Subsection 4.1 we obtain intermediate results regarding asymptotic regularity and metastability depending on an additional condition. This additional condition is studied in Subsection 4.2 through the analysis of Suzuki's lemmas (Lemmas 4 and 5). In Subsection 4.3 we prove our main result establishing the metastability for (mPPA).

We start our quantitative analysis of Theorem 3 by giving quantitative versions of the hypothesis of the theorem. We assume that there exist $a, c \in \mathbb{N} \setminus \{0\}$ and monotone functions $\ell, L, \Gamma, E : \mathbb{N} \to \mathbb{N}$ such that

(Q_1) $\forall k \in \mathbb{N}\, \forall n \geq \ell(k) \left(\lambda_n \leq \frac{1}{k+1}\right)$.

(Q_2) $\forall k \in \mathbb{N} \left(\sum_{i=1}^{L(k)} \lambda_i \geq k\right)$.

(Q_3) $\forall m \in \mathbb{N} \left(\frac{1}{a} \leq \gamma_m \leq 1 - \frac{1}{a}\right)$.

(Q_4) $\forall n \in \mathbb{N} \left(c_n \geq \frac{1}{c}\right)$.

(Q_5) $\forall k \in \mathbb{N}\, \forall n \geq \Gamma(k) \left(|c_{n+1} - c_n| \leq \frac{1}{k+1}\right)$.

(Q_6) $\forall k \in \mathbb{N}\, \forall n \in \mathbb{N} \left(\sum_{i=E(k)+1}^{E(k)+n} \|e_i\| \leq \frac{1}{k+1}\right)$.

The conditions $(Q_1) - (Q_6)$ are quantitative versions of, respectively, the hypothesis (H_1)–(H_6). Indeed, condition (Q_1) states that ℓ is a rate of convergence for the sequence (λ_n); condition (Q_2) postulates that L is a rate of divergence for $(\sum \lambda_n)$; condition (Q_3) is the quantitative version of (H_3) together with the fact that $(\gamma_n) \subset (0, 1)$; condition (Q_4) expresses the fact that the terms of the sequence (c_n) are above some positive quantity; condition (Q_5) states that Γ is a rate of convergence for the difference of terms of the sequence (c_n) and condition (Q_6) expresses quantitatively that the sequence of the partial sums of the errors e_n is a Cauchy sequence with Cauchy rate E.

In our main result (Theorem 29) we compute an explicit bound on the metastability of the iteration (mPPA) under the assumptions $(Q_1) - (Q_6)$.

4.1 Metastability of the mPPA

We show an intermediate metastability result depending on an additional condition (Q_S) in Theorem 23. Moreover, Proposition 21 and Corollary 22 give quantitative information on the asymptotic regularity of the iteration. We start by showing that the sequence (z_n) generated by (mPPA) is bounded and give in (14) the computational information corresponding to $\limsup (\|w_{n+1} - w_n\| - \|z_{n+1} - z_n\|) \leq 0$.

Lemma 18. *Let (z_n) be generated by (mPPA). Assume that there exist $a, c \in \mathbb{N} \setminus \{0\}$ and monotone functions ℓ, Γ, E such that $(Q_1), (Q_3) - (Q_6)$ hold. Let $N_1, N_2, N_3 \in \mathbb{N}$ be such that $N_1 \geq \|u\|$, $N_2 \geq \sum_{i=0}^{E(0)} \|e_i\| + 1$, and for some $s \in S$ one has $N_3 \geq \max\{\|u - s\|, \|z_0 - s\|\}$. Then $\|z_n - s\| \leq N_0$, where $N_0 := N_2 + N_3$. Moreover, with $z_{n+1} = \gamma_n z_n + (1 - \gamma_n) w_n$, we have $\|w_n - s\| \leq 2aN_0$ and*

$$\forall k \in \mathbb{N}\, \forall n \geq \nu(k) \left(\|w_{n+1} - w_n\| - \|z_{n+1} - z_n\| \leq \frac{1}{k+1} \right), \tag{14}$$

where $\nu(k) := \max\{\Gamma(10acN_0(k + 1)), \ell(10a(N_0 + N_1 + N_3)(k + 1)), E(5a(k + 1)) + 1\}$.

Proof. Observe that $\sum_{i=0}^{n} \|e_i\| \leq N_2$ for all $n \in \mathbb{N}$. By the fact that $\lambda_n + \gamma_n + \delta_n = 1$, for all $n \geq 0$ and observing that each resolvent J_c is nonexpansive, we have

$$\begin{aligned}
\|z_{n+1} - s\| &= \|\lambda_n(u - s) + \gamma_n(z_n - s) + \delta_n(\mathsf{J}_n(z_n) - s) + e_n\| \\
&\leq \lambda_n \|u - s\| + \gamma_n \|z_n - s\| + \delta_n \|z_n - s\| + \|e_n\| \\
&= \lambda_n \|u - s\| + (1 - \lambda_n) \|z_n - s\| + \|e_n\|.
\end{aligned}$$

One easily shows by induction on $n \in \mathbb{N}$ that $\|z_n - s\| \leq \max\{\|u - s\|, \|z_0 - s\|\} + \sum_{i=0}^{n-1} \|e_i\| \leq N_0$, from which we deduce that (z_n) is bounded. We have that $\|w_n - s\| \leq 2aN_0$. Indeed, by (Q_3) we have

$$\|w_n - s\| = \frac{\|z_{n+1} - s - \gamma_n(z_n - s)\|}{1 - \gamma_n} \leq \frac{2N_0}{1 - \gamma_n} \leq 2aN_0.$$

We have

$$\begin{aligned}
w_{m+1} - w_m &= \frac{\lambda_{m+1}u + \delta_{m+1}\mathsf{J}_{m+1}(z_{m+1}) + e_{m+1}}{1 - \gamma_{m+1}} - \frac{\lambda_m u + \delta_m \mathsf{J}_m(z_m) + e_m}{1 - \gamma_m} \\
&= \left(\frac{\lambda_{m+1}}{1 - \gamma_{m+1}} - \frac{\lambda_m}{1 - \gamma_m} \right) u + \frac{\delta_{m+1}}{1 - \gamma_{m+1}} \left(\mathsf{J}_{m+1}(z_{m+1}) - \mathsf{J}_m(z_m) \right) \\
&\quad + \left(\frac{\delta_{m+1}}{1 - \gamma_{m+1}} - \frac{\delta_m}{1 - \gamma_m} \right) \mathsf{J}_m(z_m) + \frac{e_{m+1}}{1 - \gamma_{m+1}} - \frac{e_m}{1 - \gamma_m}.
\end{aligned}$$

We claim that

$$\|\mathsf{J}_{m+1}(z_{m+1}) - \mathsf{J}_m(z_m)\| \leq \|z_{m+1} - z_m\| + 2cN_0|c_{m+1} - c_m|. \tag{15}$$

To prove the claim observe that for every $n, m \in \mathbb{N}$ it holds that $\|\mathsf{J}_m(z_n) - s\| \leq \|z_n - s\| \leq N_0$. If $c_m \leq c_{m+1}$, by the resolvent identity we have

$$\begin{aligned}
\|\mathsf{J}_{m+1}(z_{m+1}) - \mathsf{J}_m(z_m)\| &= \left\| \mathsf{J}_m \left(\frac{c_m}{c_{m+1}} z_{m+1} + \left(1 - \frac{c_m}{c_{m+1}} \right) \mathsf{J}_{m+1}(z_{m+1}) \right) - \mathsf{J}_m(z_m) \right\| \\
&\leq \frac{c_m}{c_{m+1}} \|z_{m+1} - z_m\| + \left(1 - \frac{c_m}{c_{m+1}} \right) \|\mathsf{J}_{m+1}(z_{m+1}) - z_m\| \\
&\leq \|z_{m+1} - z_m\| + c|c_{m+1} - c_m| \, \|\mathsf{J}_{m+1}(z_{m+1}) - z_m\| \\
&\leq \|z_{m+1} - z_m\| + 2cN_0|c_{m+1} - c_m|.
\end{aligned}$$

If $c_{m+1} < c_m$, again by the resolvent identity we have

$$\begin{aligned}\|\mathsf{J}_m(z_m) - \mathsf{J}_{m+1}(z_{m+1})\| &= \left\|\mathsf{J}_{m+1}\left(\frac{c_{m+1}}{c_m} z_m + \left(1 - \frac{c_{m+1}}{c_m}\right)\mathsf{J}_m(z_m)\right) - \mathsf{J}_{m+1}(z_{m+1})\right\| \\ &\leq \frac{c_{m+1}}{c_m}\|z_{m+1} - z_m\| + \left(1 - \frac{c_{m+1}}{c_m}\right)\|\mathsf{J}_m(z_m) - z_{m+1}\| \\ &\leq \|z_{m+1} - z_m\| + c|c_{m+1} - c_m|\,\|\mathsf{J}_m(z_m) - z_{m+1}\| \\ &\leq \|z_{m+1} - z_m\| + 2cN_0|c_{m+1} - c_m|.\end{aligned}$$

Hence (15) holds. Then

$$\begin{aligned}\|w_{n+1} - w_n\| - \|z_{m+1} - z_m\| \leq& \left|\frac{\lambda_{m+1}}{1-\gamma_{m+1}} - \frac{\lambda_m}{1-\gamma_m}\right|\|u\| + \left|\frac{\delta_{m+1}}{1-\gamma_{m+1}} - 1\right|\|z_{m+1} - z_m\| \\ &+ \left|\frac{\delta_{m+1}}{1-\gamma_{m+1}}\right| 2cN_0|c_{m+1} - c_m| + \left|\frac{\delta_{m+1}}{1-\gamma_{m+1}} - \frac{\delta_m}{1-\gamma_m}\right|\|\mathsf{J}_m(z_m)\| + \frac{\|e_{m+1}\|}{1-\gamma_{m+1}} + \frac{\|e_m\|}{1-\gamma_m}.\end{aligned} \tag{16}$$

Let $k \in \mathbb{N}$ and $m \geq \nu(k)$. We will see that each of the terms in (16) is less than or equal to $\frac{1}{5(k+1)}$. Since ℓ satisfies (Q_1), we have that, for $m \geq \ell(10a(N_0 + N_1 + N_3)(k+1))$

$$\begin{aligned}\left|\frac{\lambda_{m+1}}{1-\gamma_{m+1}} - \frac{\lambda_m}{1-\gamma_m}\right|\|u\| &\leq \left(\frac{\lambda_{m+1}}{1-\gamma_{m+1}} + \frac{\lambda_m}{1-\gamma_m}\right)N_1 \\ &\leq (\lambda_{m+1} + \lambda_m)\,aN_1 \\ &\leq \frac{2aN_1}{10a(N_0 + N_1 + N_3)(k+1)} \\ &\leq \frac{1}{5(k+1)}\end{aligned} \tag{17}$$

and

$$\begin{aligned}\left|1 - \frac{\delta_{m+1}}{1-\gamma_{m+1}}\right|\|z_{m+1} - z_m\| &\leq \left|\frac{1 - \gamma_{m+1} - \delta_{m+1}}{1-\gamma_{m+1}}\right| 2N_0 \\ &\leq \lambda_{m+1} 2aN_0 \\ &\leq \frac{2aN_0}{10a(N_0 + N_1 + N_3)(k+1)} \\ &\leq \frac{1}{5(k+1)}.\end{aligned} \tag{18}$$

Observe that $\|\mathsf{J}_m(z_m)\| \leq \|\mathsf{J}_m(z_m) - s\| + \|u - s\| + \|u\| \leq N_0 + N_1 + N_3$. We then have

$$\begin{aligned}\left|\frac{\delta_{m+1}}{1-\gamma_{m+1}} - \frac{\delta_m}{1-\gamma_m}\right|\|\mathsf{J}_m(z_m)\| &\leq \left(\frac{\lambda_{m+1}}{1-\gamma_{m+1}} + \frac{\lambda_m}{1-\gamma_m}\right)(N_0 + N_1 + N_3) \\ &\leq \frac{2a(N_0 + N_1 + N_3)}{10a(N_0 + N_1 + N_3)(k+1)} = \frac{1}{5(k+1)}.\end{aligned} \tag{19}$$

Since Γ satisfies (Q_5), for $m \geq \Gamma(10acN_0(k+1))$ it holds that

$$\left|\frac{\delta_{m+1}}{1-\gamma_{m+1}}\right| 2cN_0|c_{m+1} - c_m| \leq \frac{2acN_0}{10acN_0(k+1)} = \frac{1}{5(k+1)}. \tag{20}$$

Since E satisfies (Q_6), for $m \geq E(5a(k+1)) + 1$ we have

$$\frac{\|e_{m+1}\|}{1-\gamma_{m+1}} + \frac{\|e_m\|}{1-\gamma_m} \leq a\left(\|e_m\| + \|e_{m+1}\|\right) \leq a\left(\sum_{i=E(5a(k+1))+1}^{m+1}\|e_i\|\right) \leq \frac{a}{5a(k+1)+1} \leq \frac{1}{5(k+1)}. \tag{21}$$

Combining (16)-(21) we conclude that (14) holds. ☐

Definition 19. Let $(z_n), (w_n)$ be sequences in H. We say that a monotone function $\chi : \mathbb{N} \times \mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ satisfies (Q_S) if

$$\forall k \in \mathbb{N}\,\tilde{\forall} f : \mathbb{N} \to \mathbb{N}\,\exists n \leq \chi(k, f)\,\forall m \in [n, n + f(n)]\left(\|w_m - z_m\| \leq \frac{1}{k+1}\right). \tag{Q_S}$$

Remark 20. The hypothesis (Q_S) corresponds to the quantitative information from Suzuki's lemma (Lemma 5). With this assumption we will compute in Theorem 23 metastability for (mPPA) as well as some intermediate results regarding asymptotic regularity (Proposition 21 and Corollary 22). An explicit function satisfying (Q_S) is computed in Remark 27, using Lemma 26.

The next two results give quantitative information on asymptotic regularity for the sequence (z_n).

Proposition 21. *Let (z_n) be generated by (mPPA). Assume that there exist $a, c \in \mathbb{N} \setminus \{0\}$ and monotone functions ℓ, E such that $(Q_1), (Q_3), (Q_4)$ and (Q_6) hold. Let $N_1, N_2, N_3 \in \mathbb{N}$ be such that $N_1 \geq \|u\|$, $N_2 \geq \sum_{i=0}^{E(0)} \|e_i\| + 1$, and for some $s \in S$ one has $N_3 \geq \max\{\|u - s\|, \|z_0 - s\|\}$. Define $N_0 := N_2 + N_3$. Let (w_n) be such that $z_{n+1} = \gamma_n z_n + (1 - \gamma_n) w_n$ and assume that there is a monotone function χ satisfying (Q_S). Then*

(i) $\forall k \in \mathbb{N} \tilde{\forall} f : \mathbb{N} \to \mathbb{N} \exists n \leq \chi(k, f) \forall m \in [n, n + f(n)] \left(\|z_{m+1} - z_m\| \leq \frac{1}{k+1} \right)$;

(ii) $\forall k \in \mathbb{N} \tilde{\forall} f : \mathbb{N} \to \mathbb{N} \exists n \leq \max\left\{\mu(k), \chi\left(2a(k+1), \tilde{f}_k\right)\right\} \forall m \in [n, n + f(n)] \left(\|\mathsf{J}_m(z_m) - z_m\| \leq \frac{1}{k+1} \right)$;

(iii) $\forall k \in \mathbb{N} \tilde{\forall} f : \mathbb{N} \to \mathbb{N} \exists n \leq \xi_\chi(k, f) \forall m \in [n, n + f(n)] \left(\|\mathsf{J}(z_m) - z_m\| \leq \frac{1}{k+1} \right)$;

where $\mu(k) := \max\{\ell(4a(k+1)(N_0 + N_3)), E(4a(k+1) + 1\}$ and $\xi_\chi(k, f) := \max\{\mu(2k+1), \chi(4a(k+1), \tilde{f}_{2k+1})\}$, with $\tilde{f}_k : \mathbb{N} \to \mathbb{N}$ the monotone function defined by $\tilde{f}_k(m) = \mu(k) + f(\max\{\mu(k), m\})$.

Proof. (i). The result is a consequence of the fact that

$$\|z_m - z_{m+1}\| = \|z_m - \gamma_m z_m - (1 - \gamma_m) w_m\| = (1 - \gamma_m) \|z_m - w_m\| \leq \|z_m - w_m\|.$$

(ii). Observe that

$$\begin{aligned} \|\mathsf{J}_m(z_m) - z_m\| &\leq \|\mathsf{J}_m(z_m) - z_{m+1}\| + \|z_{m+1} - z_m\| \\ &\leq \|z_{m+1} - z_m\| + \lambda_m \|\mathsf{J}_m(z_m) - u\| + \gamma_m \|\mathsf{J}_m(z_m) - z_m\| + \|e_m\|. \end{aligned}$$

Then

$$\|\mathsf{J}_m(z_m) - z_m\| \leq \frac{\|z_{m+1} - z_m\|}{1 - \gamma_m} + \frac{\lambda_m \|\mathsf{J}_m(z_m) - u\|}{1 - \gamma_m} + \frac{\|e_m\|}{1 - \gamma_m}. \tag{22}$$

We have that

$$\forall k \in \mathbb{N} \forall m \geq \mu(k) \left(\frac{\lambda_m \|\mathsf{J}_m(z_m) - u\|}{1 - \gamma_m} + \frac{\|e_m\|}{1 - \gamma_m} \leq \frac{1}{2(k+1)} \right). \tag{23}$$

Indeed, for $m \geq \mu(k)$ we have that

$$\begin{aligned} \frac{\lambda_m \|\mathsf{J}_m(z_m) - u\|}{1 - \gamma_m} + \frac{\|e_m\|}{1 - \gamma_m} &\leq \frac{a\,(N_0 + N_3)}{4a(k+1)\,(N_0 + N_3)} + a \left(\sum_{i=E(4a(k+1))+1}^{m} \|e_i\| \right) \\ &\leq \frac{1}{4(k+1)} + \frac{1}{4(k+1)} = \frac{1}{2(k+1)}. \end{aligned}$$

Applying Part (i) to $2a(k+1)$ and $\tilde{f}_k$ we find $n' \leq \chi(2a(k+1), \tilde{f}_k)$ such that

$$\forall m \in [n', n' + \tilde{f}_k(n')] \left(\frac{\|z_{m+1} - z_m\|}{1 - \gamma_m} \leq \frac{a}{(2a(k+1)) + 1} \leq \frac{1}{2(k+1)} \right). \tag{24}$$

Put $n = \max\{\mu(k), n'\}$. Now $[n, n + f(n)] \subseteq [n', n' + \tilde{f}_k(n')]$ because clearly $n' \leq n$ and $n + f(n) \leq n' + \mu(k) + f(\max\{\mu(k), n'\}) = n' + \tilde{f}_k(n')$. Then, from (23) and (24) we conclude that Part (ii) holds.

(iii). By Lemma 2 and (Q_4) we have $\|\mathsf{J}(z_m) - z_m\| \leq 2\|\mathsf{J}_m(z_m) - z_m\|$. Hence, Part (iii) follows from Part (ii). ☐

If (Q_S) is satisfied with a rate of convergence, i.e. when χ does not depend on f, then the properties (i)–(iii) in Proposition 21 also hold with rates of convergence.

Corollary 22. Assume the conditions of Proposition 21. Assume also that $\chi(k, f) = \chi(k)$, for all f, i.e.

$$\forall k \in \mathbb{N} \forall m \geq \chi(k) \left(\|w_m - z_m\| \leq \frac{1}{k+1} \right). \tag{25}$$

Then

(*i*) $\forall k \in \mathbb{N}\, \forall m \geq \chi(k) \left(\|z_{m+1} - z_m\| \leq \frac{1}{k+1} \right)$;

(*ii*) $\forall k \in \mathbb{N}\, \forall m \geq \max\{\mu(k), \chi(2a(k+1)\} \left(\|\mathsf{J}_m(z_m) - z_m\| \leq \frac{1}{k+1} \right)$;

(*iii*) $\forall k \in \mathbb{N}\, \forall m \geq \xi_\chi(k) \left(\|\mathsf{J}(z_m) - z_m\| \leq \frac{1}{k+1} \right)$;

where $\xi_\chi(k) := \max\{\mu(2k+1), \chi(4a(k+1))\}$.

Under the additional condition (Q_S) we have the following result concerning the metastability of (mPPA).

Theorem 23. *Let (z_n) be generated by (mPPA). Assume that there exist $a, c \in \mathbb{N}\setminus\{0\}$ and monotone functions ℓ, L, E such that $(Q_1) - (Q_4)$ and (Q_6) hold. Let $\mathcal{C} : \mathbb{N} \to \mathbb{N}$ be a monotone function such that $c_n \leq \mathcal{C}(n)$, for all $n \in \mathbb{N}$. Let $N_2, N_3 \in \mathbb{N}$ be such that $N_2 \geq \sum_{i=0}^{E(0)} \|e_i\| + 1$, and for some $s \in S$ one has $N_3 \geq \max\{\|u - s\|, \|z_0 - s\|\}$. Define $N := \max\{2N_3, N_2 + N_3\}$. Let (w_n) be such that $z_{n+1} = \gamma_n z_n + (1 - \gamma_n) w_n$ and assume that there is a monotone function χ satisfying (Q_S). Then*

$$\forall k \in \mathbb{N}\, \tilde{\forall} f : \mathbb{N} \to \mathbb{N}\, \exists n \leq \phi_\chi(k, f) \forall i, j \in [n, n + f(n)] \left(\|z_i - z_j\| \leq \frac{1}{k+1} \right),$$

where $\phi_\chi(k, f) := \Theta(4(k+1)^2 - 1, \lambda m. (m + f(m)), L, \Psi_\chi, G, 4N^2)$, $G(k) = E(M_2(k+1))$, Θ is as in Lemma 14, Ψ_χ as in Lemma 15, $M_1 := 3N_2 + 4N$ and $M_2 = M_1 + 2(N_3 + N)$.

Remark 24. Note that $\Psi_\chi = \Psi_\chi[N_2, N_3, a, c, \ell, \mathcal{C}]$, i.e. the functional Ψ_χ depends on the value of N_2, N_3, a and c, as well as on the functions ℓ and $\mathcal{C}$ – and obviously on the functional χ.

Proof of Theorem 23. Let $p \in \mathrm{B}_N$. Then

$$\begin{aligned}
\|z_{m+1} - p\|^2 &\leq (\|z_{m+1} - p - e_m\| + \|e_m\|)^2 \\
&= \|z_{m+1} - p - e_m\|^2 + \|e_m\| (\|e_m\| + 2\|z_{m+1} - p - e_m\|) \\
&\leq \|z_{m+1} - p - e_m\|^2 + M_1 \|e_m\| \\
&= \|z_{m+1} - p - e_m - \lambda_m(u - p) + \lambda_m u - p\|^2 + M_1 \|e_m\| \\
&\leq \|z_{m+1} - p - e_m - \lambda_m(u - p)\|^2 + 2\lambda_m \langle u - p, z_{m+1} - p - e_m\rangle + M_1 \|e_m\| \\
&\leq \|\gamma_m(z_m - p) + \delta_m (\mathsf{J}_m(z_m) - p)\|^2 + 2\lambda_m \langle u - p, z_{m+1} - p\rangle + \|e_m\| (M_1 + 2\lambda_m \|u - p\|) \\
&\leq (\gamma_m \|(z_m - p)\| + \delta_m \|\mathsf{J}_m(z_m) - p\|)^2 + 2\lambda_m \langle u - p, z_{m+1} - p\rangle \\
&\quad + \|e_m\| (M_1 + 2\lambda_m \|u - p\|) \\
&\leq (\gamma_m \|(z_m - p)\| + \delta_m \|\mathsf{J}_m(z_m) - \mathsf{J}_m(p)\| + \delta_m \|\mathsf{J}_m(p) - p\|)^2 + 2\lambda_m \langle u - p, z_{m+1} - p\rangle \\
&\quad + \|e_m\| (M_1 + 2\lambda_m \|u - p\|) \\
&\leq ((1 - \lambda_m) \|(z_m - p)\| + (1 - \lambda_m) \|\mathsf{J}_m(p) - p\|)^2 + 2\lambda_m \langle u - p, z_{m+1} - p\rangle \\
&\quad + \|e_m\| (M_1 + 2\lambda_m \|u - p\|) \\
&\leq (1 - \lambda_m) \|z_m - p\|^2 + (1 - \lambda_m) \|\mathsf{J}_m(p) - p\| (\|\mathsf{J}_m(p) - p\| + 2\|z_m - p\|) \\
&\quad + 2\lambda_m \langle u - p, z_{m+1} - p\rangle + \|e_m\| (M_1 + 2\lambda_m \|u - p\|).
\end{aligned}$$

Then, for all $m \in \mathbb{N}$

$$s_{m+1,p} \leq (1 - \lambda_m) s_{m,p} + (1 - \lambda_m) v_{m,p} + \lambda_m r_{m,p} + \gamma_{m,p},$$

where $s_{m,p} = \|z_m - p\|^2$, $v_{m,p} = \|\mathsf{J}_m(p) - p\| (\|\mathsf{J}_m(p) - p\| + 2\|z_m - p\|)$, $r_{m,p} = 2\langle u - p, z_{m+1} - p\rangle$ and $\gamma_{m,p} = \|e_m\| (M_1 + 2\lambda_m \|u - p\|)$.

We verify that the conditions of Lemma 14 are satisfied with $\Omega = \mathrm{B}_N$, $D = 4N^2$, Ψ_χ as in Lemma 15 and $G(k) = E(M_2(k+1))$.

The first condition holds by hypothesis. Since $\|z_n - p\| \leq 2N$, the second condition is true with $D = (2N)^2$. For the third condition, using (Q_6), and the fact that $M_2 \geq M_1 + 2\lambda_m \|u - p\|$ we have

$$\begin{aligned}
\sum_{i=G(k)+1}^{G(k)+n} \gamma_{i,p} &= \sum_{i=E(M_2(k+1))+1}^{E(M_2(k+1))+n} \|e_m\| (M_1 + 2\lambda_m \|u - p\|) \\
&\leq \sum_{i=E(M_2(k+1))+1}^{E(M_2(k+1))+n} M_2 \|e_m\| \\
&\leq \frac{M_2}{M_2(k+1)+1} \leq \frac{1}{k+1}.
\end{aligned}$$

Finally, by Lemma 15 the fourth condition of Lemma 14 is also verified.

By Lemma 14 we conclude that

$$\forall k \in \mathbb{N}\,\tilde{\forall} f : \mathbb{N} \to \mathbb{N}\,\exists p \in \mathrm{B}_N\,\exists n \leq \Theta(k, f, L, \Psi_\chi, G, 4N^2)\,\forall m \in [n, f(n)]\left(\|z_m - p\|^2 \leq \frac{1}{k+1}\right). \tag{26}$$

Let $k \in \mathbb{N}$ and a monotone function f be given. By (26) applied to $4(k+1)^2 - 1$ and to the function $\lambda m.\,(m + f(m))$, we find $p \in \mathrm{B}_N$ and $n \leq \phi_\chi(k, f)$ such that for $m \in [n, n + f(n)]$,

$$\|z_m - p\|^2 \leq \frac{1}{4(k+1)^2}.$$

Hence, for $m \in [n, n + f(n)]$, $\|z_m - p\| \leq \frac{1}{2(k+1)}$ and with $i, j \in [n, n + f(n)]$, we have

$$\|z_i - z_j\| \leq \|z_i - p\| + \|z_j - p\| \leq \frac{1}{k+1},$$

which concludes the proof. □

4.2 A quantitative version of Suzuki's lemmas

We now turn to the two lemmas by Suzuki required in the original proof. The next result is a partial quantitative version of Lemma 4, which is enough for the quantitative version of Lemma 5 given in Lemma 26. As discussed in Remark 27 below, the latter allows us to obtain a concrete functional satisfying the condition (Q_S).

Lemma 25. *Let $(z_n), (w_n)$ be sequences in a normed space X. Let $(\alpha_n) \subset [0, 1]$ be a sequence of real numbers and $a \in \mathbb{N} \setminus \{0\}$ be such that*

$$\forall n \geq a \left(\alpha_n \leq 1 - \frac{1}{a}\right). \tag{27}$$

Suppose that $z_{n+1} = \alpha_n w_n + (1 - \alpha_n) z_n$, for all $n \in \mathbb{N}$ and that there exists a monotone function $\nu : \mathbb{N} \to \mathbb{N}$ satisfying

$$\forall k \in \mathbb{N}\,\forall n \geq \nu(k)\left(\|w_{n+1} - w_n\| - \|z_{n+1} - z_n\| \leq \frac{1}{k+1}\right). \tag{28}$$

Let $N \in \mathbb{N}$ be such that $\forall n \in \mathbb{N}(\|w_n - z_n\| \leq N)$. Then

$$\begin{gathered}\forall k, l \in \mathbb{N}\,\forall t \in \mathbb{N} \setminus \{0\}\,\tilde{\forall} f : \mathbb{N} \to \mathbb{N}\,\exists m \in [l, \varphi(k, f)]\,\exists p < R(a, k, t) \cdot N \\ \left[\left(\|w_{m+t} - z_m\| - \left(1 + \sum_{i=0}^{t-1} \alpha_{m+i}\right)\frac{p+1}{R(a,k,t)} \geq -\frac{1}{k+1}\right) \wedge \|w_{m+t} - z_{m+t}\| \geq \frac{p}{R(a,k,t)}\right. \\ \left.\wedge\ \forall n \in [m, m + t + f(m)]\left(\|w_n - z_n\| \leq \frac{p+1}{R(a,k,t)}\right)\right],\end{gathered}$$

where $R(a, k, t) = t(2t+1)a^t(k+1)$ and $\varphi(k, f) = \varphi(k, f, l, t, a, \nu, N) := \theta(R(a, k, t) - 1, h(R(a, k, t) - 1), t, N, g)$, with g, h functions defined respectively by $g(m) := t + f(m)$ and $h(r) := h(a, l, \nu, r) := \max\{a, l, \nu(r)\}$. The function θ is as in Lemma 17.

Proof. Let $r, l \in \mathbb{N}$, $t \in \mathbb{N} \setminus \{0\}$ be arbitrary and $f : \mathbb{N} \to \mathbb{N}$ be a monotone function. Define $M := h(r) = \max\{a, l, \nu(r)\}$. Considering $x_n := \|w_n - z_n\|$ and $\theta = \theta(r, M, t, N, g)$ we apply Lemma 17 to r, M, t and $g : \mathbb{N} \to \mathbb{N}$ defined by $g(m) = t + f(m)$ to find $p < N(r+1)$ and $m \in [M, \theta]$ such that

$$\|w_{m+t} - z_{m+t}\| \geq \frac{p}{r+1} \wedge \forall n \in [m, m + g(m)]\left(\|w_n - z_n\| \leq \frac{p+1}{r+1}\right). \tag{29}$$

Furthermore, for $n \geq m$ we have

$$\alpha_n \leq 1 - \frac{1}{a} \wedge \|w_{n+1} - w_n\| - \|z_{n+1} - z_n\| \leq \frac{1}{r+1}. \tag{30}$$

From (29), (30) and the fact that $M \geq l$ we conclude that

$$\begin{gathered}\forall r, l \in \mathbb{N}\,\forall t \in \mathbb{N} \setminus \{0\}\,\tilde{\forall} f : \mathbb{N} \to \mathbb{N}\,\exists m \in [l, \theta]\,\exists p < N(r+1) \\ \left[\forall n \geq m\left(\alpha_n \leq 1 - \frac{1}{a} \wedge \|w_{n+1} - w_n\| - \|z_{n+1} - z_n\| \leq \frac{1}{r+1}\right) \wedge \|w_{m+t} - z_{m+t}\| \geq \frac{p}{r+1}\right. \\ \left.\wedge\,\forall n \in [m, m + t + f(m)]\left(\|w_n - z_n\| \leq \frac{p+1}{r+1}\right)\right].\end{gathered} \tag{31}$$

Working with m, p given by (31), we now argue that for all $j \le t-1$.

$$\|w_{m+t} - z_{m+j}\| \ge \left(1 + \sum_{i=j}^{t-1} \alpha_{m+i}\right) \frac{p+1}{r+1} - \frac{(t-j)(2t+1)a^{t-j}}{r+1}. \tag{32}$$

We have

$$\begin{aligned}
\frac{p}{r+1} &\le \|w_{m+t} - z_{m+t}\| \\
&= \|w_{m+t} - \alpha_{m+t-1} w_{m+t-1} - (1-\alpha_{m+t-1}) z_{m+t-1}\| \\
&\le \alpha_{m+t-1} \|w_{m+t} - w_{m+t-1}\| + (1-\alpha_{m+t-1}) \|w_{m+t} - z_{m+t-1}\| \\
&\le \alpha_{m+t-1} \|z_{m+t} - z_{m+t-1}\| + \frac{1}{r+1} + (1-\alpha_{m+t-1}) \|w_{m+t} - z_{m+t-1}\| \\
&= \alpha_{m+t-1}^2 \|w_{m+t-1} - z_{m+t-1}\| + \frac{1}{r+1} + (1-\alpha_{m+t-1}) \|w_{m+t} - z_{m+t-1}\| \\
&\le \alpha_{m+t-1}^2 \frac{p+1}{r+1} + \frac{1}{r+1} + (1-\alpha_{m+t-1}) \|w_{m+t} - z_{m+t-1}\|.
\end{aligned}$$

Hence

$$\begin{aligned}
\|w_{m+t} - z_{m+t-1}\| &\ge \frac{\left(1-\alpha_{m+t-1}^2\right) \frac{p+1}{r+1} - \frac{2}{r+1}}{1-\alpha_{m+t-1}} \\
&= (1+\alpha_{m+t-1}) \frac{p+1}{r+1} - \frac{2}{(r+1)(1-\alpha_{m+t-1})} \\
&\ge (1+\alpha_{m+t-1}) \frac{p+1}{r+1} - \frac{2a}{r+1} \\
&\ge (1+\alpha_{m+t-1}) \frac{p+1}{r+1} - \frac{(2t+1)a}{r+1}.
\end{aligned}$$

So, (32) holds for $j = t-1$. To conclude we assume that (32) holds for some $j \in [1, t-1]$ and want to see that it holds for $j-1$. Since

$$\begin{aligned}
\left(1 + \sum_{i=j}^{t-1} \alpha_{m+i}\right) \frac{p+1}{r+1} - \frac{(t-j)(2t+1)a^{t-j}}{(r+1)} &\le \|w_{m+t} - z_{m+j}\| \\
&= \|w_{m+t} - \alpha_{m+j-1} w_{m+j-1} - (1-\alpha_{m+j-1}) z_{m+j-1}\| \\
&\le \alpha_{m+j-1} \|w_{m+t} - w_{m+j-1}\| + (1-\alpha_{m+j-1}) \|w_{m+t} - z_{m+j-1}\| \\
&\le \alpha_{m+j-1} \sum_{i=j-1}^{t-1} \|w_{m+i+1} - w_{m+i}\| + (1-\alpha_{m+j-1}) \|w_{m+t} - z_{m+j-1}\| \\
&\le \alpha_{m+j-1} \sum_{i=j-1}^{t-1} \|z_{m+i+1} - z_{m+i}\| + \frac{t}{r+1} + (1-\alpha_{m+j-1}) \|w_{m+t} - z_{m+j-1}\| \\
&= \alpha_{m+j-1} \sum_{i=j-1}^{t-1} \alpha_{m+i} \|w_{m+i} - z_{m+i}\| + \frac{t}{r+1} + (1-\alpha_{m+j-1}) \|w_{m+t} - z_{m+j-1}\| \\
&\le \alpha_{m+j-1} \sum_{i=j-1}^{t-1} \alpha_{m+i} \left(\frac{p+1}{r+1}\right) + \frac{t}{r+1} + (1-\alpha_{m+j-1}) \|w_{m+t} - z_{m+j-1}\|,
\end{aligned}$$

we obtain that

$$\begin{aligned}
\|w_{m+t} - z_{m+j-1}\| &\ge \frac{p+1}{r+1} \frac{\left(1 + \sum_{i=j}^{t-1} \alpha_{m+i}\right) - \alpha_{m+j-1} \sum_{i=j-1}^{t-1} \alpha_{m+i}}{1-\alpha_{m+j-1}} - \frac{(t-j)(2t+1)a^{t-j} + t}{(r+1)(1-\alpha_{m+j-1})} \\
&\ge \frac{p+1}{r+1} \frac{1 + \sum_{i=j}^{t-1} \alpha_{m+i} (1-\alpha_{m+j-1}) - \alpha_{m+j-1}^2}{1-\alpha_{m+j-1}} - \frac{(t-j)(2t+1)a^{t-j+1} + ta}{r+1} \\
&\ge \frac{p+1}{r+1} \left(\frac{1-\alpha_{m+j-1}^2}{1-\alpha_{m+j-1}} + \sum_{i=j}^{t-1} \alpha_{m+i}\right) - \frac{(t-j)(2t+1)a^{t-j+1} + (2t+1)a^{t-j+1}}{r+1} \\
&= \left(1 + \sum_{i=j-1}^{t-1} \alpha_{m+i}\right) \frac{p+1}{r+1} - \frac{(t-j+1)(2t+1)a^{t-j+1}}{r+1},
\end{aligned}$$

which implies that (32) holds for all $j \le t-1$ as we wanted. Instantiating j with 0 in (32) we obtain

$$\|w_{m+t} - z_m\| \ge \left(1 + \sum_{i=0}^{t-1} \alpha_{m+i}\right) \frac{p+1}{r+1} - \frac{t(2t+1)a^t}{r+1}.$$

Hence

$$\begin{aligned}&\forall r, l \in \mathbb{N}\, \forall t \in \mathbb{N} \setminus \{0\}\, \tilde{\forall} f : \mathbb{N} \to \mathbb{N}\, \exists m \in [l, \theta(r, h(r), t, N, g)]\, \exists p < N(r+1) \\ &\left[\|w_{m+t} - z_m\| \ge \left(1 + \sum_{i=0}^{t-1} \alpha_{m+i}\right) \frac{p+1}{r+1} - \frac{t(2t+1)a^t}{r+1} \wedge \|w_{m+t} - z_{m+t}\| \ge \frac{p}{r+1} \right. \\ &\left. \quad \wedge\ \forall n \in [m, m+t+f(m)] \left(\|w_n - z_n\| \le \frac{p+1}{r+1} \right) \right].\end{aligned} \tag{33}$$

Given $k \in \mathbb{N}$, we conclude the result by putting $r = R(a, k, t) - 1$ in (33). □

Lemma 26. *Let $(z_n), (w_n)$ be sequences in a normed space X and $N \in \mathbb{N}$ be such that $\|z_n\|, \|w_n\| \le N$, for all $n \in \mathbb{N}$. Let $(\alpha_n) \subset [0,1]$ be a sequence of real numbers and $a \in \mathbb{N} \setminus \{0\}$ be such that $\forall n \ge a \left(\frac{1}{a} \le \alpha_n \le 1 - \frac{1}{a}\right)$. Suppose that $z_{n+1} = \alpha_n w_n + (1-\alpha_n) z_n$, for all $n \in \mathbb{N}$ and that there exists a monotone function $\nu : \mathbb{N} \to \mathbb{N}$ such that*

$$\forall k \in \mathbb{N}\, \forall n \ge \nu(k) \left(\|w_{n+1} - w_n\| - \|z_{n+1} - z_n\| \le \frac{1}{k+1} \right). \tag{34}$$

Then

$$\forall k \in \mathbb{N}\, \tilde{\forall} f : \mathbb{N} \to \mathbb{N}\, \exists n \le \widetilde{\chi}(k, f)\, \forall m \in [n, n+f(n)] \left(\|w_m - z_m\| \le \frac{1}{k+1} \right),$$

where $\widetilde{\chi}(k, f) = \widetilde{\chi}(k, f, a, \nu, N) = \varphi(k, f, a, t, a, \nu, 2N)$, with φ is as in Lemma 25 and $t := \max\{2Na(k+1), 1\}$.

Proof. Suppose towards a contradiction that there exist $k_0 \in \mathbb{N}$ and a monotone function f_0 such that

$$\forall m \le \widetilde{\chi}(k_0, f_0)\, \exists n \in [m, m+f_0(m)] \left(\|w_n - z_n\| > \frac{1}{k_0+1} \right).$$

Define $t_0 := \max\{2Na(k_0+1), 1\}$. We have that $t_0 \ge 1$ and $\left(1 + \frac{t_0}{a}\right) \frac{1}{k_0+1} \ge 2N + \frac{1}{k_0+1}$. Applying Lemma 25 with $k = k_0$, $l = a$, $t = t_0$ and $f = f_0$, we find $p \in \mathbb{N}$ and $m \in [a, \varphi(k_0, f_0, a, t_0, a, \nu, 2N)] = [a, \widetilde{\chi}(k_0, f_0)]$ such that

$$\begin{aligned}&\|w_{m+t_0} - z_m\| \ge \left(1 + \sum_{i=0}^{t_0-1} \alpha_{m+i}\right) \frac{p+1}{R(a, k_0, t_0)} - \frac{1}{k_0+1} \wedge \|w_{m+t_0} - z_{m+t_0}\| \ge \frac{p}{R(a, k_0, t_0)} \\ &\qquad \wedge\ \forall n \in [m, m+t_0+f_0(m)] \left(\|w_n - z_n\| \le \frac{p+1}{R(a, k_0, t_0)} \right).\end{aligned}$$

We have $[m, m+f_0(m)] \subseteq [m, m+t_0+f_0(m)]$, and then

$$\frac{1}{k_0+1} < \|w_m - z_m\| \le \frac{p+1}{R(a, k_0, t_0)}.$$

Hence

$$\begin{aligned} 2N + \frac{1}{k_0+1} &\le \left(1 + \frac{t_0}{a}\right) \frac{1}{k_0+1} \\ &\le \left(1 + \sum_{i=0}^{t_0-1} \alpha_{m+i}\right) \frac{1}{k_0+1} \\ &< \left(1 + \sum_{i=0}^{t_0-1} \alpha_{m+i}\right) \frac{p+1}{R(a, k_0, t_0)} \\ &\le \|w_{m+t_0} - z_m\| + \frac{1}{k_0+1} \le 2N + \frac{1}{k_0+1}, \end{aligned}$$

a contradiction. □

4.3 Metastability revisited

Remark 27. Under the conditions of Lemma 18, the function $\nu(k) := \max\{\Gamma(10acN_0(k+1)), \ell(10a(N_0+N_1+N_3)(k+1)), E(5a(k+1))+1\}$ satisfies (34). Furthermore, $\|z_n\|, \|w_n\| \leq 2aN_0+N_1+N_3$. Hence, Lemma 26 with $N = 2aN_0+N_1+N_3$, $\alpha_n = 1-\gamma_n$ is satisfied with the function ν and outputs the function χ_0 defined by

$$\chi_0(k,f) := \widetilde{\chi}(k,f,a,\nu,2aN_0+N_1+N_3),$$

which satisfies (Q_S). Note that having this function χ_0, the hypothesis (Q_S) can be removed from Proposition 21. Moreover, the bound in Proposition 21(ii) can be simplified to $\chi_0(2a(k+1),\tilde{f}_k)$, and similarly for ξ_{χ_0}.

Notation 28. *We write ξ, ψ and Ψ, instead of $\xi_{\chi_0}, \psi_{\chi_0}$ and Ψ_{χ_0}, respectively, where χ_0 is the functional defined in Remark 27.*

Having an explicit function satisfying (Q_S) we can now prove metastability for (mPPA) with the initial quantitative assumptions $(Q_1)-(Q_6)$.

Theorem 29. *Let (z_n) be generated by (mPPA). Assume that there exist $a, c \in \mathbb{N}\setminus\{0\}$ and monotone functions ℓ, L, Γ, E such that $(Q_1)-(Q_6)$ hold. Let $\mathcal{C}:\mathbb{N}\to\mathbb{N}$ be a monotone function such that $c_n \leq \mathcal{C}(n)$, for all $n \in \mathbb{N}$. Let $N_1, N_2, N_3 \in \mathbb{N}$ be such that $N_1 \geq \|u\|$, $N_2 \geq \sum_{i=0}^{E(0)}\|e_i\|+1$, and for some $s \in S$ one has $N_3 \geq \max\{\|u-s\|, \|z_0-s\|\}$. Define $N := \max\{2N_3, N_2+N_3\}$. Let (w_n) be such that $z_{n+1} = \gamma_n z_n + (1-\gamma_n)w_n$. Then*

$$\forall k \in \mathbb{N}\, \forall f:\mathbb{N}\to\mathbb{N}\, \exists n \leq \phi(k,f)\, \forall i,j \in [n, n+f(n)] \left(\|z_i - z_j\| \leq \frac{1}{k+1}\right),$$

where $\phi(k,f) := \phi_{\chi_0}(k, f^{\mathrm{maj}})$, with $\phi_{\chi_0}(k,f) := \Theta(4(k+1)^2-1, \lambda m.\,(m+f(m)), L, \Psi, G, 4N^2)$, $G(k) = E(M_2(k+1))$, Θ as in Lemma 14, Ψ given by Lemma 15, $M_1 := 3N_2+4N$ and $M_2 = M_1 + 2(N_3+N)$.

As an application we consider the special case where the sequence (c_n) is constant. We note that this case was also considered in Yao and Noor's paper [47, Corollary 3.1].

Let c_0 be a positive real number. Consider the sequence (c_n) constantly equal to c_0, and (z_n) generated by (mPPA). Assume that there exist $a, c, C \in \mathbb{N}\setminus\{0\}$ and monotone functions ℓ, L, E such that $(Q_1)-(Q_3)$ and (Q_6) hold and that $C \geq c_0 \geq \frac{1}{c}$. We consider $\mathcal{C}:\mathbb{N}\to\mathbb{N}$ to be the function identicaly equal to C. Note that we can take either $c = 1$ or $C = 1$. Clearly (Q_4), and (Q_5) with $\Gamma(k) \equiv 0$ are satisfied. The definition of ν in Lemma 18 simplifies to

$$\nu(k) = \max\{\ell(8a(N_0+N_1+N_3)(k+1)), E(4a(k+1))+1\},$$

which causes changes in χ_0 and consequently in ξ, ψ and Ψ. This simplifies the bound ϕ obtained in Theorem 29.

5 Final considerations

In Theorem 29 we obtain a bound on the metastability of (mPPA) which is uniform on the parameters of the algorithm. Let us elaborate on this. The bound is uniform on the choice of the anchor point u, the given initial point z_0 and a point $s \in S$ witnessing the assumption that S is nonempty, depending only on natural numbers N_1 and N_3. The dependence on the sequences $(\lambda_n), (\gamma_n)$ and (c_n) is respectively only in the form of a rate of convergence ℓ satisfying (Q_1) and a rate of divergence L satisfying (Q_2); a natural number a satisfying (Q_3); and a natural number c satisfying (Q_4), a rate of convergence Γ, satisfying (Q_5) and on a majorizing function $\mathcal{C}$. Finally, it is uniform on the error sequence (e_n), as it depends only on a Cauchy rate E satisfying (Q_6) and on a natural number N_2.

It is well-know that for $(\lambda_n) \subset (0,1)$ the condition $\sum \lambda_n = \infty$ is equivalent to the condition $\prod(1-\lambda_n) = 0$. Although we don't do this here, one could have worked with a rate of convergence towards zero L' for $(\prod(1-\lambda_n))$ instead of the rate of divergence L for $(\sum \lambda_n)$. This is done for similar quantitative analyses in [31, 38]. In certain cases, that option may prove useful as a function L' may be of lower complexity than that of a function L, e.g. the sequence $\lambda_n = \frac{1}{n+1}$ has a liner rate L' but an exponential rate L.

Metastability and the Cauchy property are equivalent. So, Theorem 29 and the fact that the space H is complete imply that (mPPA) is strongly convergent. Similarly, from Proposition 21 (with instantiation $\chi = \chi_0$) we conclude that $\lim \|\mathsf{J}(z_n) - z_n\| = 0$ and thus (mPPA) converges to a zero of the maximal monotone operator T, say $\tilde{z}$. To see that $\tilde{z}$ must be the projection point $P_S(u)$ first note that since $\tilde{z} \in S$ we have that $\langle u - P_S(u), \tilde{z} - P_S(u)\rangle \leq 0$. Since $z_n \to \tilde{z}$, the conclusion of Lemma 15 is satisfied with $p = P_S(u)$. Finally, following the arguments in the proof of Theorem 23 with $p = P_S(u)$, we see that (26) holds with the point p always equal to $P_S(u)$, implying that $\tilde{z} = P_S(u)$. This shows that the analysis in this paper indeed corresponds to a quantitative analysis of the original proof by Yao and Noor.

We finish with some considerations concerning the logical aspects of our analysis.

Let us start by pointing out that, in principle, the monotone functional interpretation could be used to analyse the results presented in this paper. However, our elimination of the sequential weak compactness argument can be seen as an application of the general method obtained in [10]. Also, using the BFI enables us to make use of Proposition 9 and Lemma 13 (shown in [38] using the BFI) which makes our analysis easier to carry out.

As already mentioned, the original proof of Theorem 3 requires strong principles. These principles are countable choice as used in the projection argument, sequential weak compactness and the existence of the $\limsup$ in Lemma 4, which require arithmetical comprehension. The analysis of these principles require the use of a stronger form of recursion called *bar recursion* [41, 37]. As shown below, the arguments in Section 3 allow us to avoid the use of bar recursion. There are already many examples in the proof mining literature where it is possible to avoid the use of arithmetical comprehension[2]. For example, in [21, 25] the use of the existence of a limit point for a sequence in a compact geodesic space (which requires arithmetical comprehension) is replaced by a combinatorial argument. In [26, 27], a proof of an asymptotic regularity theorem that was based on countable nested uses of sequential compactness (and hence arithmetic comprehension) is analysed resulting in a simple exponential bound, by elimination of sequential compactness. Moreover, in a series of papers, Kohlenbach showed how the monotone functional interpretation can be used to replace the use of arithmetical comprehension by optimal arithmetic substitutes (see e.g. [19, 20] and [22, Section 17.9]).

The elimination of countable choice required for the projection argument is carried out in Section 3.1. The key observation is that (2) can be replaced by (3). This is in line with earlier analyses (see for example [23, 10, 38]) and it is well-known that this allows for the extracted quantitative information to be expressed in terms of Gödel's primitive recursive functionals.

The way to deal with sequential weak compactness is explained in full detail in [10], in the context of the BFI. The key point is that sequential weak compactness can be replaced by countable Heine/Borel compactness. The content of Section 3.2 shows that it can be adapted to our context in a similar way.

In Section 3.3 we made the modifications necessary to bypass the assumed existence of the real number $d = \limsup \|w_n - z_n\|$ in Lemma 4. In [29], Kohlenbach and Andrei Sipoş give a rational approximation to the $\limsup$ of a certain sequence by interpreting the approximation. As described in detail below, we must deal with a similar issue.

Let $N \in \mathbb{N}$ and let (x_n) be a sequence of real numbers contained in the interval $[0, N]$. The existence of the $\limsup x_n$ can be stated as

$$\exists d \in \mathbb{R}\, \forall k \in \mathbb{N} \left(\forall n \in \mathbb{N}\, \exists m \geq n \left(x_m \geq d - \frac{1}{k+1} \right) \wedge \exists n' \in \mathbb{N}\, \forall m' \geq n' \left(x_{m'} \leq d + \frac{1}{k+1} \right) \right).$$

The main point is that we can weaken this statement by switching the outermost quantifiers

$$\forall k \in \mathbb{N}\, \exists d \in \mathbb{R} \left(\forall n \in \mathbb{N}\, \exists m \geq n \left(x_m \geq d - \frac{1}{k+1} \right) \wedge \exists n' \in \mathbb{N}\, \forall m' \geq n' \left(x_{m'} \leq d + \frac{1}{k+1} \right) \right). \tag{35}$$

In fact, we will show that such d in (35) is already witnessed by a rational number satisfying

$$\forall k \in \mathbb{N}\, \exists p < N(k+1) \left(\forall n \in \mathbb{N}\, \exists m \geq n \left(x_m \geq \frac{p}{k+1} \right) \wedge \exists n' \in \mathbb{N}\, \forall m' \geq n' \left(x_{m'} \leq \frac{p+1}{k+1} \right) \right), \tag{36}$$

which implies that (35) holds with $d = \frac{p}{k+1}$.

The idea behind (36) is the following. For each $k \in \mathbb{N}$, by dividing the interval $[0, N]$ into subintervals of length $\frac{1}{k+1}$, there exists $p < N(k+1)$ such that $\frac{p}{k+1} \leq \limsup x_n \leq \frac{p+1}{k+1}$. If we take $d = \frac{2p+1}{2(k+1)}$, i.e. the middle point, then it should satisfy (35) for $2k+1$ (and hence for k). This results in statement (36).

Lemma 16, which is shown by Π^0_1-induction, can be seen to imply (36) using a collection argument. First note that Lemma 16 implies

$$\begin{aligned} &\forall k, n \in \mathbb{N}\, \tilde{\forall} f : \mathbb{N} \to \mathbb{N}\, \exists p < N(k+1) \\ &\qquad \left(\exists m \geq n \left(x_m \geq \frac{p}{k+1} \right) \wedge \exists n' \in \mathbb{N}\, \forall m' \in [n', n' + f(n')] \left(x_{m'} \leq \frac{p+1}{k+1} \right) \right). \end{aligned} \tag{37}$$

By a collection argument, we conclude

$$\begin{aligned} &\forall k \in \mathbb{N}\, \exists p < N(k+1)\, \forall n \in \mathbb{N}\, \tilde{\forall} f : \mathbb{N} \to \mathbb{N} \\ &\qquad \left(\exists m \geq n \left(x_m \geq \frac{p}{k+1} \right) \wedge \exists n' \in \mathbb{N}\, \forall m' \in [n', n' + f(n')] \left(x_{m'} \leq \frac{p+1}{k+1} \right) \right), \end{aligned} \tag{38}$$

[2] We would like to thank Ulrich Kohlenbach for pointing this out to the first author and for providing the appropriate references.

which by (monotone) choice axiom is equivalent to (36).

Let us elaborate on (38). Clearly it is a sufficient condition to (37). Let see that (37) implies (38). Assuming that (38) fails, we obtain that for some $k \in \mathbb{N}$

$$\forall r \leq N(k+1)\, \exists n\, \tilde{\exists} f : \mathbb{N} \to \mathbb{N}\, \forall p < r \left(\forall m \geq n \left(x_m < \frac{p}{k+1} \right) \vee \forall n' \in \mathbb{N}\, \exists m' \in [n', n' + f(n')] \left(x_{m'} > \frac{p+1}{k+1} \right) \right).$$

By instantiating $r = N(k+1)$, one concludes that (37) must also fail.

Of course, this collection argument is fully justified by a form of induction. The reader can compare this way of proving (36), using Π_1^0-induction and a collection argument, to the similar [29, Proposition 4.2], where Π_2^0-induction was used.

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