Quantitative results on a Halpern-type proximal point algorithm

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Abstract

We apply proof mining methods to analyse a result of Boikanyo and Moroşanu on the strong convergence of a Halpern-type proximal point algorithm. As a consequence, we obtain quantitative versions of this result, providing uniform effective rates of asymptotic regularity and metastability.

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1 Introduction

Let H be a real Hilbert space and $A : H \to 2^H$ be a maximally monotone operator such that the set $zer(A)$ of zeros of A is nonempty. For every $\gamma > 0$, the resolvent $J_{\gamma A}$ of γA is defined by $J_{\gamma A} = (id_H + \gamma A)^{-1}$. It is well-known (see, e.g., [2]) that $J_{\gamma A} : H \to H$ is a single-valued firmly nonexpansive (hence, nonexpansive) mapping and that $Fix(J_{\gamma A}) = zer(A)$ for every $\gamma > 0$. Furthermore, $zer(A)$ is a closed convex subset of H and $P_{zer(A)}$ denotes the projection onto $zer(A)$.

A major problem in convex optimization is finding zeros of maximally monotone operators. A classical method for solving this problem is the proximal point algorithm, defined by Rockafellar [32] as follows:

$$PPA \qquad x_0 \in H, \quad x_{n+1} := J_{\beta_n A} x_n + e_n, \tag{1}$$

where $(\beta_n)_{n\in\mathbb{N}}$ is a sequence of positive real numbers and $(e_n)_{n\in\mathbb{N}} \subseteq H$ is a sequence of errors. Special cases of (1) have been previously studied by Martinet [29]. Rockafellar proved, under the assumptions that (β_n) is bounded away from zero and $(\|e_n\|)$ is a summable sequence, that (x_n) is weakly convergent to a zero of A and he posed the question whether the weak convergence can be improved, in general, to strong convergence. This question was answered in the negative by Güler [11].

This being the case, the following problem is very natural:

modify PPA such that strong convergence is guaranteed.

This problem has attracted a lot of research, many new algorithms based on PPA were introduced and proved to be strongly convergent (just to give a few examples, see [9, 33, 39, 37, 6]). Since the set of zeros of a maximally monotone operator coincides with the fixed point set of the resolvent, one idea to obtain new algorithms is to combine PPA with nonlinear iterations studied in metric fixed point theory. One such iteration is the well-known Halpern iteration, defined, for any nonexpansive mapping $T : H \to H$ and any sequence $(\alpha_n)_{n\in\mathbb{N}}$ in $[0, 1]$, by

$$x_0, u \in H, \quad x_{n+1} := \alpha_n u + (1 - \alpha_n)Tx_n. \tag{2}$$

The iteration was introduced by Halpern [12] for the special case $u = 0$. A classical result is Wittmann's theorem [38], which proves the strong convergence of (x_n) towards a fixed point of T, under some assumptions on (α_n) that are satisfied for the natural choice $\alpha_n = \frac{1}{n+1}$. Since, for T linear and $\alpha_n = \frac{1}{n+1}$, the Halpern iteration becomes the ergodic average, Wittmann's result is a nonlinear generalization of the von Neumann mean ergodic theorem.

By combining PPA with the Halpern iteration we obtain the so-called *Halpern-type proximal point algorithms*. One such algorithm was introduced independently by Kamimura and Takahashi [13] and Xu [39]:

$$HPPA \qquad x_0, u \in H \quad x_{n+1} := \alpha_n u + (1 - \alpha_n)J_{\beta_n A}x_n + e_n, \tag{3}$$

where (α_n) is a sequence in $(0, 1]$, (β_n) is a sequence of positive real numbers and $(e_n) \subseteq H$ is the error sequence.

We consider in the sequel the following conditions on the sequences (α_n), (β_n), (e_n):

$$(C0)\ \lim_{n\to\infty} \alpha_n = 0, \qquad (C1)\ \sum_{n=0}^{\infty} \alpha_n = \infty, \qquad (C2)\ \prod_{n=0}^{\infty} (1 - \alpha_n) = 0,$$
$$(C3)\ \lim_{n\to\infty} \frac{|\alpha_{n+1} - \alpha_n|}{\alpha_n^2} = 0, \qquad (C4)\ \lim_{n\to\infty} \beta_n = \beta > 0, \qquad (C5)\ \sum_{n=0}^{\infty} \|e_n\| < \infty,$$
$$(C6)\ \lim_{n\to\infty} \frac{\|e_n\|}{\alpha_n} = 0.$$

The following strong convergence result was proved by Boikanyo and Moroşanu [4].

Theorem 1.1. *Let H be a Hilbert space, $A : H \to 2^H$ be a maximally monotone operator such that $zer(A) \neq \emptyset$ and (x_n) be defined by (3). Assume that the following hold:*

(i) $(C0)$, $(C3)$ and $(C4)$;

(ii) $(C1)$ or, equivalently, $(C2)$;

(iii) $(C5)$ or $(C6)$.

Then (x_n) converges strongly to $P_{zer(A)}u$.

The main results of this paper are effective bounds on the asymptotic behaviour of the HPPA (x_n). We compute rates of asymptotic regularity of (x_n), which turn out to be polynomial for the example we give in Section 5. Furthermore, we prove quantitative versions of Theorem 1.1, providing uniform rates of metastability (in the sense of Tao [34, 35]). As pointed out by Kohlenbach [19], Neumann [30] proves that one cannot obtain computable rates of convergence even for the simple case $H = \mathbb{R}$. Results from proof mining (a research field in mathematical logic) show that one can extract effective rates of metastability of (x_n), defined as mapping $\Phi : \mathbb{N} \times \mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ satisfying

$$\forall k \in \mathbb{N}\, \forall g : \mathbb{N} \to \mathbb{N}\, \exists N \leq \Phi(k, g)\, \forall i, j \in [N, N + g(N)] \left(\|x_i - x_j\| \leq \frac{1}{k+1} \right).$$

As metastability of a sequence is non-effectively equivalent with the Cauchy property, an effective rate of metastability is the best quantitative information one can obtain when rates of convergence are not to be expected, as it is the case with the HPPA (x_n). Metastability was used by Tao [35] and Walsh [36] to obtain far-reaching generalizations of the von Neumann mean ergodic theorem.

By letting $g(n) = L \in \mathbb{N}$, we obtain a mapping $\Phi_L : \mathbb{N} \to \mathbb{N}$ such that

$$\forall k \in \mathbb{N}\ \exists N \leq \Phi_L(k)\ \forall i, j \in [N, N+L]\ \left(\|x_i - x_j\| \leq \frac{1}{k+1} \right).$$

We call such a function Φ_L a rate of L-metastability of (x_n). By varying L, we get that (x_n) is stable on arbitrarily long time-intervals.

The effective bounds for the HPPA (x_n) are computed by using methods from proof mining, transforming the arguments from Boikanyo and Moroşanu's proof of Theorem 1.1 into new ones, providing the computational information which was previously hidden. We refer to Kohlenbach's book [15] for a comprehensive introduction to proof mining and to [16, 18] for surveys on more recent applications. Quantitative results on Halpern-type proximal point algorithms have been only recently obtained by Kohlenbach [19] and the second author [31]. However, proof mining has been applied in a series of papers to obtain rates of asymptotic regularity and metastability for the Halpern iteration [24, 17, 21, 25, 23, 10] and rates of metastability and convergence for the Proximal Point Algorithm [22, 26, 27, 28, 20]. These rates are computed both for Hilbert spaces and for more general classes of spaces: uniformly convex (and uniformly smooth) Banach spaces, $CAT(0)$ spaces and CAT(κ) spaces (with $\kappa > 0$).

We recall in the following some quantitative notions. Let $(a_n)_{n\in\mathbb{N}}$ be a sequence in H. If (a_n) converges to $a \in H$, then a rate of convergence for (a_n) is a mapping $\gamma : \mathbb{N} \to \mathbb{N}$ such that

$$\forall k \in \mathbb{N}\, \forall n \geq \gamma(k)\ \left(\|a_n - a\| \leq \frac{1}{k+1} \right).$$

If (a_n) is Cauchy, then a Cauchy modulus of (a_n) is a mapping $\chi : \mathbb{N} \to \mathbb{N}$ satisfying

$$\forall k \in \mathbb{N}\, \forall n \in \mathbb{N}\ \left(\|a_{\chi(k)+n} - a_{\chi(k)}\| \leq \frac{1}{k+1} \right).$$

As in the case of the Cauchy property, one has also a metastable version of the convergence of a sequence. If (a_n) converges to $a \in H$, we say, following [17], that a quasi-rate of convergence of (a_n) is a mapping $\widetilde{\gamma} : \mathbb{N} \times \mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ satisfying

$$\forall k \in \mathbb{N}\, \forall g : \mathbb{N} \to \mathbb{N}\, \forall n \in [\widetilde{\gamma}(k,g), \widetilde{\gamma}(k,g) + g(\widetilde{\gamma}(k,g))]\ \left(\|a_n - a\| \leq \frac{1}{k+1} \right).$$

Obviously, if γ is a rate of convergence of (a_n), then $\widetilde{\gamma}(k,g) := \gamma(k)$ (for all k, g) is a quasi-rate of convergence of (a_n).

Let (b_n) be a sequence of nonnegative real numbers. If the series $\sum_{n=0}^{\infty} b_n$ diverges, then a function $\theta : \mathbb{N} \to \mathbb{N}$ is called a rate of divergence of the series if $\sum_{i=0}^{\theta(n)} b_i \geq n$ for all $n \in \mathbb{N}$. If $\lim_{n\to\infty} b_n = 0$, then γ is a rate of convergence of (b_n) if and only if for any $k \in \mathbb{N}$, we have that $b_n \leq \frac{1}{k+1}$ for all $n \geq \gamma(k)$.

2 Some useful results on (x_n)

In this section, H is a real Hilbert space, $A : H \to 2^H$ is a maximally monotone operator with $zer(A) \neq \emptyset$, $J_{\gamma A}$ is the resolvent of γA ($\gamma > 0$) and the sequence (x_n) is given by (3). The following well-known resolvent identity will be useful: for any $\beta, \gamma > 0$ and $x \in H$,

$$J_{\beta A}(x) = J_{\gamma A}\left(\frac{\gamma}{\beta}x + \left(1 - \frac{\gamma}{\beta}\right) J_{\beta A}(x)\right) \tag{4}$$

2.1 Upper bounds on (x_n)

Lemma 2.1. *Let p be a zero of A. Then, for all $n \in \mathbb{N}$,*

$$\|x_{n+1} - p\| \leq \alpha_n\|u - p\| + (1 - \alpha_n)\|x_n - p\| + \|e_n\|, \tag{5}$$

$$\|x_n - p\| \leq \max\{\|u - p\|, \|x_0 - p\|\} + \sum_{i=0}^{n-1} \|e_i\|. \tag{6}$$

Proof. We have that

$$\begin{aligned} \|x_{n+1} - p\| &= \|\alpha_n u + (1 - \alpha_n)J_{\beta_n A}x_n + e_n - p\| \\ &= \|\alpha_n(u - p) + (1 - \alpha_n)(J_{\beta_n A}x_n - J_{\beta_n A}p) + e_n\| \\ &\quad \text{since } p \text{ is a fixed point of } J_{\beta_n A} \\ &\leq \alpha_n\|u - p\| + (1 - \alpha_n)\|x_n - p\| + \|e_n\| \\ &\quad \text{since } J_{\beta_n A} \text{ is nonexpansive.} \end{aligned}$$

Thus, (5) holds. We obtain (6) by an easy induction on n. □

In [3] it is shown that the sequence (x_n) is bounded if $\sum_{n=0}^{\infty} \|e_n\| < \infty$ or the sequence $\left(\frac{\|e_n\|}{\alpha_n}\right)$ is bounded. Next it's a quantitative version of this result.

Lemma 2.2. *Let p be a zero of A and D be a natural number. Define*

$$D_1 := \max\{\|u - p\|, \|x_0 - p\|\} + D, \quad D_2 := \max\{2(\|u - p\| + D), \|x_0 - p\|\}.$$

Then

(i) if D is an upper bound on $\left(\sum_{i=0}^{n} \|e_i\|\right)$, then D_1 is an upper bound on the sequence $(\|x_n - p\|)$.

(ii) if D is an upper bound on $\left(\frac{\|e_n\|}{\alpha_n}\right)$, then D_2 is an upper bound on the sequence $(\|x_n - p\|)$.

Proof. (i) Apply (6).

(ii) The proof is by induction on n. The case $n = 0$ is trivial.

$n \Rightarrow n+1$: We get that

$$\begin{aligned}
\|x_{n+1}-p\|^2 &= \|\alpha_n u + (1-\alpha_n)J_{\beta_n A}x_n + e_n - p\|^2 \\
&= \left\|\alpha_n\left(u-p+\frac{e_n}{\alpha_n}\right) + (1-\alpha_n)\left(J_{\beta_n A}x_n - J_{\beta_n A}p\right)\right\|^2 \\
&\leq (1-\alpha_n)^2\|J_{\beta_n A}x_n - J_{\beta_n A}p\|^2 + 2\alpha_n\left\langle u-p+\frac{e_n}{\alpha_n}, x_{n+1}-p\right\rangle \\
&\qquad \text{since } \|x+y\|^2 \leq \|y\|^2 + 2\langle x, x+y\rangle \text{ for all } x,y\in H \\
&\leq (1-\alpha_n)^2\|x_n-p\|^2 + 2\alpha_n\left(\|u-p\|+\frac{\|e_n\|}{\alpha_n}\right)\|x_{n+1}-p\| \\
&\leq (1-\alpha_n)^2 D_2^2 + \alpha_n D_2\|x_{n+1}-p\| \\
&\qquad \text{by the induction hypothesis and the definition of } D_2.
\end{aligned}$$

It follows that

$$\left(\|x_{n+1}-p\| - \frac{D_2}{2}\alpha_n\right)^2 \leq (1-\alpha_n)^2 D_2^2 + \frac{D_2^2}{4}\alpha_n^2,$$

hence

$$\begin{aligned}
\|x_{n+1}-p\| &\leq \frac{D_2}{2}\alpha_n + \sqrt{(1-\alpha_n)^2 D_2^2 + \frac{D_2^2}{4}\alpha_n^2} \\
&\leq \frac{D_2}{2}\alpha_n + (1-\alpha_n)D_2 + \frac{D_2}{2}\alpha_n = D_2.
\end{aligned}$$

□

2.2 An approximate fixed point sequence

One of the main ingredients of Boikanyo and Moroşanu's proof of Theorem 1.1 is a classical theorem of Browder [7] on the strong convergence of a sequence of approximants to fixed points of nonexpansive mappings. Kohlenbach [17] applied proof mining methods both to Browder's original proof and to a simplified proof of this theorem, due to Halpern [12]. In the sequel, we apply Kohlenbach's quantitative version of Browder's theorem obtained by the logical analysis of Halpern's proof.

For each $n \in \mathbb{N}$, let us define

$$S_n : H \to H, \quad S_n(x) = \alpha_n u + (1-\alpha_n)J_{\beta A}x,$$

where $\beta > 0$.

Then for $\alpha_n \in (0,1]$, S_n is a contraction, hence, by the Banach contraction principle, S_n has a unique fixed point z_n. Thus,

$$z_n = \alpha_n u + (1-\alpha_n)J_{\beta A}z_n \quad \text{for all } n \in \mathbb{N}. \tag{7}$$

Lemma 2.3. *Let $p \in zer(A)$. Then, for all $n \in \mathbb{N}$,*

$$\begin{aligned}
\|z_n - p\| &\leq 2\|u-p\|, &(8)\\
\|z_n - u\| &\leq 3\|u-p\|, &(9)\\
\|J_{\beta A}z_n - u\| &\leq 3\|u-p\|. &(10)
\end{aligned}$$

Proof. We have that

$$\begin{aligned}\|z_n - p\|^2 &= \|\alpha_n(u-p) + (1-\alpha_n)(J_{\beta A}z_n - p)\|^2 \\ &\leq (1-\alpha_n)^2\|J_{\beta A}z_n - p)\|^2 + 2\langle \alpha_n(u-p), z_n - p\rangle \\ &= (1-\alpha_n)^2\|J_{\beta A}z_n - J_{\beta A}p)\|^2 + 2\langle \alpha_n(u-p), z_n - p\rangle \\ &\leq (1-\alpha_n)^2\|z_n - p\|^2 + 2\alpha_n\|u-p\|\|z_n - p\|.\end{aligned}$$

We used above the fact that $\|x+y\|^2 \leq \|y\|^2 + 2\langle x, x+y\rangle$ for all $x, y \in H$. It follows that

$$\alpha_n(2-\alpha_n)\|z_n - p\| \leq 2\alpha_n\|u-p\|,$$

hence (8). One obtains immediately (9). To get (10), remark that

$$\|J_{\beta A}z_n - u\| \leq \|J_{\beta A}z_n - p\| + \|u-p\| \leq \|z_n - p\| + \|u-p\|.$$

□

The inequalities below are obtained in the proof of [4, Theorem 2].

Lemma 2.4. *For all $n \in \mathbb{N}$,*

$$\|x_{n+1} - z_n\| \leq (1-\alpha_n)\|x_n - z_n\| + \frac{\alpha_n|\beta-\beta_n|}{\beta}\|u - J_{\beta A}z_n\| + \|e_n\|, \tag{11}$$

$$\|z_n - z_{n+1}\| \leq \frac{|\alpha_n - \alpha_{n+1}|}{\alpha_n}\|u - J_{\beta A}z_{n+1}\|. \tag{12}$$

The following quantitative result on the behaviour of (z_n) is a special case of [17, Theorem 4.2].

Proposition 2.5. *Let $d \in \mathbb{N}^*$ be such that $d \geq 3\|u-p\|$ for some zero p of A.*

(i) Assume that (α_n) is a nonincreasing sequence. Then (z_n) is Cauchy with rate of metastability Ω_d, given by

$$\Omega_d(k,g) := \tilde{g}^{(d^2(k+1)^2)}(0), \tag{13}$$

with $\tilde{g}(n) := n + g(n)$.

(ii) Assume that $\lim_{n\to\infty}\alpha_n = 0$ with quasi-rate of convergence χ and let $h : \mathbb{N} \to \mathbb{N}$ be such that $\alpha_n \geq \frac{1}{h(n)+1}$ for all $n \in \mathbb{N}$. Then (z_n) is Cauchy with rate of metastability $\widetilde{\Omega}_{\chi,h,d}$, given by

$$\widetilde{\Omega}_{\chi,h,d}(k,g) := \chi_g^M\left(g_{h,\chi_g}^{(4d^2(k+1)^2)}(0)\right) \tag{14}$$

with $\chi_g(n) := \chi(n,g)$, $\chi_g^M(n) := \max\{\chi_g(i) \mid i \leq n\}$ and $g_{h,\chi_g}(n) := \max\{h(i) \mid i \leq \chi_g(n) + g(\chi_g(n))\}$.

Proof. Apply [17, Theorem 4.2] with $v_0 := u$, $U := J_{\beta A}$, $s_n := 1-\alpha_n$, $\varepsilon := \frac{1}{k+1}$ and d, h, χ_g as above. □

3 Quantitative lemmas on sequences of real numbers

In the sequel, $(a_n)_{n\in\mathbb{N}}$ is a sequence in $[0,1]$, $(b_n)_{n\in\mathbb{N}}$ is a sequence of real numbers and $(c_n)_{n\in\mathbb{N}}, (s_n)_{n\in\mathbb{N}}$ are sequences of nonnegative real numbers satisfying, for all $n \in \mathbb{N}$,

$$s_{n+1} \leq (1-a_n)s_n + a_n b_n + c_n. \tag{15}$$

The following lemma from [39] is one of the main tools used in the proof of Theorem 1.1.

Lemma 3.1. *Assume that* $\sum_{n=0}^{\infty} a_n$ *diverges (or equivalently,* $\prod_{n=0}^{\infty}(1-a_n) = 0$ *if* $a_n < 1$ *for all* $n \in \mathbb{N}$*),* $\limsup_{n\to\infty} b_n \leq 0$ *and* $\sum_{n=0}^{\infty} c_n$ *converges. Then* $\lim_{n\to\infty} s_n = 0$.

We give in the sequel quantitative versions of Lemma 3.1. We remark that for a particular case of this lemma, obtained by letting $c_n := 0$, the first author and Kohlenbach have already proved quantitative versions in [21]. The proofs of the following results are similar with those of [21, Lemmas 5.2, 5.3]. However, for the sake of completeness, we give them in this paper.

Lemma 3.2. *Let* $p, N \in \mathbb{N}$ *be such that*

$$b_n \leq \frac{1}{p+1} \quad \textit{for all } n \geq N. \tag{16}$$

Then for all $m, n \in \mathbb{N}$ *with* $n \geq N$*,*

$$s_{n+m+1} \leq \left(\prod_{i=n}^{n+m}(1-a_i)\right) s_n + \left(1 - \prod_{i=n}^{n+m}(1-a_i)\right)\frac{1}{p+1} + \sum_{i=n}^{n+m} c_i.$$

Proof. The proof is by induction on m. The case $m = 0$ is trivial.
$m \Rightarrow m+1$: For simplicity, let us denote

$$A := \prod_{i=n}^{n+m}(1-a_i).$$

We get that

$$\begin{aligned}
s_{n+m+2} &\leq (1-a_{n+m+1})s_{n+m+1} + a_{n+m+1}b_{n+m+1} + c_{n+m+1} \\
&\leq (1-a_{n+m+1})\left[As_n + (1-A)\frac{1}{p+1} + \sum_{i=n}^{n+m} c_i\right] + a_{n+m+1}\frac{1}{p+1} \\
&\quad + c_{n+m+1} \qquad \text{by the induction hypothesis and (16)} \\
&= \left(\prod_{i=n}^{n+m+1}(1-a_i)\right) s_n + \left((1-a_{n+m+1})(1-A) + a_{n+m+1}\right)\frac{1}{p+1} \\
&\quad + (1-a_{n+m+1})\sum_{i=n}^{n+m} c_i + c_{n+m+1} \\
&\leq \left(\prod_{i=n}^{n+m+1}(1-a_i)\right) s_n + \left(1 - \prod_{i=n}^{n+m+1}(1-a_i)\right)\frac{1}{p+1} + \sum_{i=n}^{n+m+1} c_i.
\end{aligned}$$

$\square$

Lemma 3.3. *Let* $M \in \mathbb{N}^*$ *and* $\psi, \chi : \mathbb{N} \to \mathbb{N}$ *be such that*

(i) M *is an upper bound on* (s_n)*;*

(ii) for all $k \in \mathbb{N}$ *and all* $n \in \mathbb{N}$ *with* $n \geq \psi(k)$*,* $b_n \leq \frac{1}{k+1}$*;*

(iii) χ *is a Cauchy modulus for* $\left(\tilde{c}_n := \sum\limits_{i=0}^{n} c_i\right)$.

Define

$$\delta : \mathbb{N} \to \mathbb{N}, \quad \delta(k) := \max\{\psi(3k+2), \chi(3k+2)+1\}. \tag{17}$$

Then for all $k, m \in \mathbb{N}$ *and all* $n \in \mathbb{N}$ *with* $n \geq \delta(k)$,

$$s_{n+m+1} \leq M \prod_{i=n}^{n+m}(1-a_i) + \frac{2}{3(k+1)}.$$

Proof. Let $k, m, n \in \mathbb{N}$ be such that $n \geq \delta(k)$. Since $\delta(k) \geq \psi(3k+2)$, we get from (ii) that $b_n \leq \frac{1}{3(k+1)}$. We apply Lemma 3.2 to obtain that

$$\begin{aligned} s_{n+m+1} &\leq \left(\prod_{i=n}^{n+m}(1-a_i)\right) s_n + \left(1 - \prod_{i=n}^{n+m}(1-a_i)\right)\frac{1}{3(k+1)} + \sum_{i=n}^{n+m} c_i \\ &\leq M \prod_{i=n}^{n+m}(1-a_i) + \frac{1}{3(k+1)} + \sum_{i=n}^{n+m} c_i \quad \text{by (i)}. \end{aligned}$$

As $n \geq \delta(k) \geq \chi(3k+2)+1$, we apply (iii) to get, by letting $r := n+m-\chi(3k+2)$, that

$$\begin{aligned} \sum_{i=n}^{n+m} c_i &\leq \sum_{i=\chi(3k+2)+1}^{n+m} c_i = \sum_{i=0}^{n+m} c_i - \sum_{i=0}^{\chi(3k+2)} c_i = \tilde{c}_{\chi(3k+2)+r} - \tilde{c}_{\chi(3k+2)} \\ &\leq \frac{1}{3(k+1)}. \end{aligned}$$

The conclusion follows. ∎

Proposition 3.4. *In the hypothesis of Lemma 3.3, assume, moreover, that* $\sum\limits_{n=0}^{\infty} a_n$ *diverges with rate of divergence* $\theta : \mathbb{N} \to \mathbb{N}$. *Define* $\Sigma := \Sigma_{M,\theta,\psi,\chi}$ *by*

$$\Sigma : \mathbb{N} \to \mathbb{N}, \quad \Sigma(k) = \theta(\delta(k) + \lceil \ln(3M(k+1)) \rceil) + 1, \tag{18}$$

where δ *is given by* (17).

Then $\lim\limits_{n\to\infty} s_n = 0$ *with rate of convergence* Σ.

Proof. Let $k \in \mathbb{N}$ be arbitrary. Denote

$$K := \Sigma(k) - \delta(k) - 1 = \theta(\delta(k) + \lceil \ln(3M(k+1)) \rceil) - \delta(k).$$

As $a_n \leq 1$, we have that $\theta(n+1) \geq n$ for all n. Thus, $K \in \mathbb{N}$. For all $m \geq K$, we get that

$$\begin{aligned} \sum_{i=\delta(k)}^{\delta(k)+m} a_i &\geq \sum_{i=\delta(k)}^{\delta(k)+K} a_i = \sum_{i=0}^{\theta(\delta(k)+\lceil \ln(3M(k+1))\rceil)} a_i - \sum_{i=0}^{\delta(k)-1} a_i \\ &\geq \delta(k) + \lceil \ln(3M(k+1)) \rceil - \sum_{i=0}^{\delta(k)-1} a_i \\ &\geq \ln(3M(k+1)), \end{aligned}$$

hence,

$$\begin{aligned}
\prod_{i=\delta(k)}^{\delta(k)+m}(1-a_i) &\leq \exp\left(-\sum_{i=\delta(k)}^{\delta(k)+m} a_i\right) \quad \text{since } 1-x \leq \exp(-x) \text{ for } x \geq 0 \\
&\leq \frac{1}{3M(k+1)}.
\end{aligned}$$

Let $n \geq \Sigma(k)$. Applying Lemma 3.3 with $m := n - \delta(k) - 1 \geq K$, it follows that

$$s_n \;=\; s_{\delta(k)+m+1} \leq M \prod_{i=\delta(k)}^{\delta(k)+m}(1-a_i) + \frac{2}{3(k+1)} \leq \frac{1}{k+1}.$$

□

In the sequel, we give another quantitative version of Lemma 3.1, when the hypothesis $\prod_{n=0}^{\infty}(1-a_n) = 0$ is used. Let us denote $P_n := \prod_{j=0}^{n}(1-a_j)$ for all $n \in \mathbb{N}$. Then $\prod_{n=0}^{\infty}(1-a_n) = 0$ if and only if $\lim_{n\to\infty} P_n = 0$. A rate of convergence of $\prod_{n=0}^{\infty}(1-a_n)$ towards 0 will be a rate of convergence of (P_n) towards 0.

Proposition 3.5. *In the hypothesis of Lemma 3.3, assume, furthermore, that*

(i) $a_n < 1$ *for all* $n \in \mathbb{N}$;

(ii) $\prod_{n=0}^{\infty}(1-a_n) = 0$ *with rate of convergence* θ;

(iii) $\delta_0 : \mathbb{N} \to \mathbb{N}^*$ *is such that for all* $k \in \mathbb{N}$, $\frac{1}{\delta_0(k)} \leq P_{\delta(k)-1}$.

Define $\widetilde{\Sigma} := \widetilde{\Sigma}_{M,\theta,\psi,\chi,\delta_0}$ *by*

$$\widetilde{\Sigma} : \mathbb{N} \to \mathbb{N}, \quad \widetilde{\Sigma}(k) = \max\left\{\theta\left(3M\delta_0(k)(k+1)-1\right), \delta(k)\right\} + 1, \tag{19}$$

Then $\lim_{n\to\infty} s_n = 0$ *with rate of convergence* $\widetilde{\Sigma}$.

Proof. Let $k \in \mathbb{N}$ and $n \geq \widetilde{\Sigma}(k)$. Applying Lemma 3.3 with $m := n - \delta(k) - 1 \in \mathbb{N}$, we get that

$$\begin{aligned}
s_n &= s_{\delta(k)+m+1} \leq M \prod_{i=\delta(k)}^{\delta(k)+m}(1-a_i) + \frac{2}{3(k+1)} = \frac{MP_{\delta(k)+m}}{P_{\delta(k)-1}} + \frac{2}{3(k+1)} \\
&\leq M\delta_0(k)P_{\delta(k)+m} + \frac{2}{3(k+1)} \quad \text{by (iii)} \\
&\leq \frac{1}{k+1},
\end{aligned}$$

as $\delta(k) + m = n - 1 \geq \theta(3M\delta_0(k)(k+1)-1)$, hence, by (ii), $P_{\delta(k)+m} \leq \frac{1}{3M\delta_0(k)(k+1)}$. □

In the proof of our second main theorem, we shall need a particular case of the inequality (15), obtained by letting $c_n := 0$ for all $n \in \mathbb{N}$. By an easy adaptation of the previous proofs, we obtain the following quantitative result.

Proposition 3.6. *Let $(a_n)_{n\in\mathbb{N}}$ be a sequence in $(0,1)$, $(b_n)_{n\in\mathbb{N}}$ be a sequence of real numbers and $(s_n)_{n\in\mathbb{N}}$ be a sequence of nonnegative real numbers satisfying, for all $n \in \mathbb{N}$,*

$$s_{n+1} \leq (1-a_n)s_n + a_n b_n. \tag{20}$$

Assume that $M \in \mathbb{N}^*$ is an upper bound on (s_n) and that $\psi^* : \mathbb{N} \to \mathbb{N}$ is such that $b_n \leq \frac{1}{p+1}$ for all $p, n \in \mathbb{N}$ with $n \geq \psi^*(p)$. Define

$$\delta^* : \mathbb{N} \to \mathbb{N}, \quad \delta^*(k) = \psi^*(2k+1).$$

The following hold:

(i) If $\sum\limits_{n=0}^{\infty} a_n$ diverges with rate of divergence θ, then $\lim\limits_{n\to\infty} s_n = 0$ with rate of convergence

$$\Sigma^*(k) = \theta(\delta^*(k) + \lceil \ln(2M(k+1)) \rceil) + 1.$$

(ii) If $\prod\limits_{n=0}^{\infty}(1-a_n) = 0$ with rate of convergence θ and $\delta_0^* : \mathbb{N} \to \mathbb{N}^*$ is such that $\dfrac{1}{\delta_0^*(k)} \leq P_{\delta^*(k)-1}$ for all $k \in \mathbb{N}$, then $\lim\limits_{n\to\infty} s_n = 0$ with rate of convergence

$$\widetilde{\Sigma^*}(k) = \max\left\{\theta\left(2M\delta_0^*(k)(k+1) - 1\right), \delta^*(k)\right\} + 1.$$

4 Main results

In this section we compute uniform effective bounds on the asymptotic behaviour of the HPPA (x_n) and of the approximate fixed point sequence (z_n). These culminate with quantitative versions of Theorem 1.1, providing uniform effective rates of metastability for the HPPA (x_n). The fact that one can obtain such effective bounds is guaranteed by general logical metatheorems for Hilbert spaces proved by Kohlenbach [14].

We need quantitative versions of the hypotheses (C0) - (C6) of Theorem 1.1:

- $(C0_q)$ $\lim\limits_{n\to\infty} \alpha_n = 0$ with rate of convergence σ_0,
- $(C0_q^*)$ $\lim\limits_{n\to\infty} \alpha_n = 0$ with quasi-rate of convergence σ_0^*,
- $(C1_q)$ $\sum\limits_{n=0}^{\infty} \alpha_n$ diverges with rate of divergence σ_1,
- $(C2_q)$ $\prod\limits_{n=0}^{\infty}(1-a_n) = 0$ with rate of convergence σ_2,
- $(C3_q)$ $\lim\limits_{n\to\infty} \dfrac{|\alpha_{n+1}-\alpha_n|}{\alpha_n^2} = 0$ with rate of convergence σ_3,
- $(C4_q)$ $\lim\limits_{n\to\infty} \beta_n = \beta > 0$ with rate of convergence σ_4,
- $(C5_q)$ $\sum\limits_{n=0}^{\infty} \|e_n\| < \infty$ with Cauchy modulus σ_5,
- $(C6_q)$ $\lim\limits_{n\to\infty} \dfrac{\|e_n\|}{\alpha_n} = 0$ with rate of convergence σ_6.

We assume for the rest of this section that H is a Hilbert space, $A : H \to 2^H$ is a maximally monotone operator such that $zer(A) \neq \emptyset$, (x_n) is defined by (3) and $b \in \mathbb{N}^*$ is such that

$$b \geq \max\{\|x_0 - p\|,\ \|u-p\|\} \text{ for some zero } p \text{ of } A. \tag{21}$$

We suppose, moreover, that $(C4_q)$ holds and (z_n) is defined by (7) with $\beta = \lim\limits_{n\to\infty} \beta_n$.

4.1 Rates of convergence for $(\|x_n - z_n\|)$

One of the main steps in the proof of Theorem 1.1 is to obtain that

$$\lim_{n\to\infty} \|x_n - z_n\| = 0. \tag{22}$$

In the sequel we give quantitative versions of (22), consisting of uniform effective rates of convergence.

The first quantitative result is the following.

Proposition 4.1. *Assume that $(C3_q)$, $(C5_q)$ hold and $\ell \in \mathbb{N}, D \in \mathbb{N}^*$ satisfy*

$$\beta \geq \frac{1}{\ell+1}, \quad D \geq \sum_{i=0}^{\sigma_5(0)} \|e_i\| + 1. \tag{23}$$

Define

$$\begin{array}{rcl} \psi(k) & := & \max\{\sigma_4(6b(\ell+1)(k+1)-1), \sigma_3(6b(k+1)-1)\}, \\ \delta(k) & := & \max\{\psi(3k+2), \sigma_5(3k+2)+1\}. \end{array}$$

The following hold:

(i) If $(C1_q)$ holds, then $\lim\limits_{n\to\infty} \|x_n - z_n\| = 0$ with rate of convergence Θ given by

$$\Theta(k) := \sigma_1(\delta(k) + \lceil \ln(3(D+5b)(k+1)) \rceil) + 1. \tag{24}$$

(ii) If $(C2_q)$ holds and $\delta_0 : \mathbb{N} \to \mathbb{N}^*$ is such that

$$\frac{1}{\delta_0(k)} \leq \prod_{j=0}^{\delta(k)-1} (1-\alpha_j) \text{ for all } k \in \mathbb{N}, \tag{25}$$

then $\lim\limits_{n\to\infty} \|x_n - z_n\| = 0$ with rate of convergence $\widetilde{\Theta}$ defined by

$$\widetilde{\Theta}(k) := \max\left\{\sigma_2\left(3(D+5b)\delta_0(k)(k+1)-1\right), \delta(k)\right\} + 1. \tag{26}$$

Proof. By (11), (12), (10) and the hypothesis on b, we get that for all $n \in \mathbb{N}$,

$$\begin{array}{rcl} \|x_{n+1} - z_{n+1}\| & \leq & \|x_{n+1} - z_n\| + \|z_n - z_{n+1}\| \\ & \leq & (1-\alpha_n)\|x_n - z_n\| + \alpha_n b_n + \|e_n\|, \end{array}$$

where $b_n := 3b\left(\dfrac{|\beta - \beta_n|}{\beta} + \dfrac{|\alpha_n - \alpha_{n+1}|}{\alpha_n^2}\right)$. We verify in the sequel that, by letting

$$s_n := \|x_n - z_n\|,\ a_n := \alpha_n,\ b_n \text{ as above and } c_n := \|e_n\|,$$

the hypotheses of Lemma 3.3 are satisfied.

Applying $(C5q)$, one can easily see that, for all $n \in \mathbb{N}$, $\sum\limits_{i=0}^{n} \|e_i\| \leq 1 + \sum\limits_{i=0}^{\sigma_5(0)} \|e_i\| \leq D$. Hence, D is an upper bound on $\left(\sum\limits_{i=0}^{n} \|e_i\|\right)$.

Let $p \in zer(A)$ satisfy the hypothesis on b. We get that

$$\begin{array}{rcl} \|x_n - z_n\| & \leq & \|x_n - p\| + \|p - u\| + \|u - z_n\| \\ & \leq & D + 5b \quad \text{by Lemma 2.2.(i) and (9).} \end{array}$$

Let $n \geq \psi(k)$ be arbitrary. Applying $(C3_q)$, $(C4_q)$ and the hypothesis on ℓ, we get that

$$\frac{|\alpha_n - \alpha_{n+1}|}{\alpha_n^2} \leq \frac{1}{6b(k+1)} \quad \text{and} \quad \frac{|\beta - \beta_n|}{\beta} \leq (\ell+1)|\beta - \beta_n| \leq \frac{1}{6b(k+1)}.$$

It follows that $b_n \leq \frac{1}{k+1}$ for all $n \geq \psi(k)$.

Finally, by $(C5_q)$, we have that σ_5 is a Cauchy modulus for $\sum\limits_{n=0}^{\infty} \|e_n\|$.

Thus, the hypotheses of Lemma 3.3 hold. We get (i) by applying Proposition 3.4 with $M := D + 5b$, $\chi := \sigma_5$, $\theta := \sigma_1$, and ψ, δ as above. Furthermore, (ii) is obtained by applying Proposition 3.5 with $M := D + 5b$, $\chi := \sigma_5$, $\theta := \sigma_2$, and ψ, δ, δ_0 as above. $\square$

A result similar to Proposition 4.1 can be obtained by replacing, in the hypothesis, $(C5_q)$ with $(C6_q)$.

Proposition 4.2. *Assume that $(C3_q)$, $(C6_q)$ hold and $\ell \in \mathbb{N}, D^* \in \mathbb{N}^*$ satisfy*

$$\beta \geq \frac{1}{\ell+1}, \quad D^* \geq \max_{i \leq \sigma_6(0)} \left\{ \frac{\|e_i\|}{\alpha_i}, 1 \right\}. \tag{27}$$

Define

$$\begin{aligned} \psi^*(k) &:= \max\{\sigma_4(9b(\ell+1)(k+1)-1), \sigma_3(9b(k+1)-1), \sigma_6(3k+2)\}, \\ \delta^*(k) &:= \psi^*(2k+1). \end{aligned}$$

The following hold:

(i) If $(C1_q)$ holds, then $\lim\limits_{n\to\infty} \|x_n - z_n\| = 0$ with rate of convergence Θ^* defined by

$$\Theta^*(k) := \sigma_1(\delta^*(k) + \lceil \ln(2(2D^* + 6b)(k+1)) \rceil) + 1. \tag{28}$$

(ii) If $(C2_q)$ holds and and $\delta_0^* : \mathbb{N} \to \mathbb{N}^*$ is such that

$$\frac{1}{\delta_0^*(k)} \leq \prod_{j=0}^{\delta^*(k)-1} (1 - \alpha_j) \text{ for all } k \in \mathbb{N}, \tag{29}$$

then $\lim\limits_{n\to\infty} \|x_n - z_n\| = 0$ with rate of convergence with rate of convergence $\widetilde{\Theta^*}$ defined by

$$\widetilde{\Theta^*}(k) := \max\left\{\sigma_2\left(2(2D^* + 6b)\delta_0^*(k)(k+1) - 1\right), \delta^*(k)\right\} + 1. \tag{30}$$

Proof. As in the proof of Proposition 4.2, we apply (11), (12), (10) and the hypothesis on b to obtain, for all $n \in \mathbb{N}$,

$$\|x_{n+1} - z_{n+1}\| \quad \leq \quad \|x_{n+1} - z_n\| + \|z_n - z_{n+1}\| \leq (1 - \alpha_n)\|x_n - z_n\| + \alpha_n b_n,$$

where

$$b_n := 3b\left(\frac{|\beta - \beta_n|}{\beta} + \frac{|\alpha_n - \alpha_{n+1}|}{\alpha_n^2}\right) + \frac{\|e_n\|}{\alpha_n}.$$

We shall apply Proposition 3.6 with $s_n := \|x_n - z_n\|$, $a_n := \alpha_n$, b_n as above.

By the hypothesis on D^* and $(C6_q)$, one can see immediately that D^* is an upper bound on $\left(\frac{\|e_n\|}{\alpha_n}\right)$.

If p is a zero of A as in the hypothesis, we get that

$$\begin{aligned}
\|x_n - z_n\| &\leq \|x_n - p\| + \|p - u\| + \|u - z_n\| \\
&\leq \|x_n - p\| + 4b \quad \text{by (9)} \\
&\leq 2(b + D^*) + 4b \quad \text{by Lemma 2.2.(ii)} \\
&= 2D^* + 6b.
\end{aligned}$$

Let $n \geq \psi^*(k)$. We get that

$$\begin{aligned}
b_n &= 3b\left(\frac{|\beta - \beta_n|}{\beta} + \frac{|\alpha_n - \alpha_{n+1}|}{\alpha_n^2}\right) + \frac{\|e_n\|}{\alpha_n} \\
&\leq \frac{2}{3(k+1)} + \frac{\|e_n\|}{\alpha_n} \quad \text{by the definition of } \psi^*, (C3_q) \text{ and } (C4_q) \\
&\leq \frac{1}{k+1} \quad \text{by } (C6_q) \text{ and the fact that } \psi^*(k) \geq \sigma_6(3k+2).
\end{aligned}$$

Thus, we can can apply Proposition 3.6.(i) with $M := 2D^* + 6b$, $\theta := \sigma_1$ and ψ^*, δ^* as above to get (i), and Proposition 3.6.(ii) with $M := 2D^* + 6b$, $\theta := \sigma_2$ and ψ^*, δ^*, δ_0^* as above to prove (ii). ☐

4.2 Rates of asymptotic regularity

One of the most useful concepts in metric fixed point theory and convex optimization is the asymptotic regularity [8, 5]: if $T : H \to H$ is a mapping and (x_n) is a sequence in H, then (x_n) is asymptotically regular with respect to T if $\lim\limits_{n\to\infty} \|x_n - Tx_n\| = 0$. We can extend this notion to countable families of mappings: (x_n) asymptotically regular with respect to a a family $(T_n : H \to H)_{n\in\mathbb{N}}$ of mappings if $\lim\limits_{n\to\infty} \|x_n - T_n x_n\| = 0$. Rates of convergence of $(\|x_n - Tx_n\|)$, $(\|x_n - T_n x_n\|)$ towards 0 are said to be rates of asymptotic regularity.

The following result shows that rates of asymptotic regularity can be computed, in the presence of $(C0_q)$, from rates of convergence of the sequence $(\|x_n - z_n\|)$.

Theorem 4.3. *Assume that $(C0_q)$ holds, $\ell \in \mathbb{N}^*$ is such that $\beta \geq \frac{1}{\ell+1}$ and that $\lim\limits_{n\to\infty} \|x_n - z_n\| = 0$ with rate of convergence Λ.*

Then

(i) (x_n) is asymptotically regular with respect to $J_{\beta A}$ with rate of asymptotic regularity Σ defined by

$$\Sigma(k) := \max\{\sigma_0(6b(k+1) - 1), \Lambda(4k+3)\}.$$

(ii) (x_n) is asymptotically regular with respect to $(J_{\beta_n A})$ with rate of asymptotic regularity Σ_2 defined by

$$\Sigma^*(k) := \max\{\sigma_4(\ell), \Sigma(2k+1)\}.$$

(iii) For every $i \in \mathbb{N}$, (x_n) is asymptotically regular with respect to $J_{\beta_i A}$ with rate of asymptotic regularity Σ_i defined by

$$\Sigma_i(k) := \Sigma((1 + (\ell+1)M_i)(k+1) - 1),$$

where $M_i \in \mathbb{N}$ is such that $M_i \geq |\beta - \beta_i|$.

Proof. (i) Let $k \in \mathbb{N}$ be arbitrary. For $n \geq \sigma_0(6b(k+1)-1)$, using (10) and the fact that b satisfies (21), we get that

$$\|z_n - J_{\beta A} z_n\| = \alpha_n \|u - J_{\beta A} z_n\| \leq \alpha_n \cdot 3b \leq \frac{1}{2(k+1)}.$$

On the other hand, for $n \geq \Lambda(4k+3)$, we have that $\|x_n - z_n\| \leq \frac{1}{4(k+1)}$. It follows that for all $n \geq \Sigma(k)$,

$$\begin{aligned}
\|x_n - J_{\beta A} x_n\| &\leq \|x_n - z_n\| + \|z_n - J_{\beta A} z_n\| + \|J_{\beta A} z_n - J_{\beta A} x_n\| \\
&\leq 2\|x_n - z_n\| + \|z_n - J_{\beta A} z_n\| \\
&\leq \frac{2}{4(k+1)} + \frac{1}{2(k+1)} = \frac{1}{k+1}.
\end{aligned}$$

(ii) We first note that for all $n \in \mathbb{N}$,

$$\begin{aligned}
\|J_{\beta A} x_n - J_{\beta_n A} x_n\| &\overset{(4)}{=} \left\| J_{\beta_n A}\left(\frac{\beta_n}{\beta} x_n + \left(1 - \frac{\beta_n}{\beta}\right) J_{\beta A} x_n\right) - J_{\beta_n A} x_n \right\| \\
&\leq \left|1 - \frac{\beta_n}{\beta}\right| \|x_n - J_{\beta A} x_n\| \\
&\leq (\ell+1)|\beta - \beta_n| \|x_n - J_{\beta A} x_n\|.
\end{aligned}$$

Let $k \in \mathbb{N}$ be arbitrary and $n \geq \Sigma^*(k)$. It follows that

$$\begin{aligned}
\|x_n - J_{\beta_n A} x_n\| &\leq \|x_n - J_{\beta A} x_n\| + \|J_{\beta A} x_n - J_{\beta_n A} x_n\| \\
&\leq \|x_n - J_{\beta A} x_n\| + (\ell+1)|\beta - \beta_n| \|x_n - J_{\beta A} x_n\| \\
&\leq 2\|x_n - J_{\beta A} x_n\| \quad \text{since } n \geq \sigma_4(\ell), \text{ so } |\beta - \beta_n| \leq \frac{1}{\ell+1} \\
&\leq \frac{1}{k+1} \quad \text{since } n \geq \Sigma(2k+1).
\end{aligned}$$

(iii) The proof is similar with the one of (ii). Let $k \in \mathbb{N}$ and $n \geq \Sigma_i(k)$. Then

$$\begin{aligned}
\|x_n - J_{\beta_i A} x_n\| &\leq \|x_n - J_{\beta A} x_n\| + \|J_{\beta A} x_n - J_{\beta_i A} x_n\| \\
&\leq \|x_n - J_{\beta A} x_n\| + \left|1 - \frac{\beta_i}{\beta}\right| \|x_n - J_{\beta A} x_n\| \\
&\leq \|x_n - J_{\beta A} x_n\| + (\ell+1) M_i \|x_n - J_{\beta A} x_n\| \\
&= (1 + (\ell+1) M_i) \|x_n - J_{\beta A} x_n\| \\
&\leq \frac{1}{k+1} \quad \text{since } n \geq \Sigma((1 + (\ell+1) M_i)(k+1) - 1).
\end{aligned}$$

□

4.3 Quantitative versions of Theorem 1.1

We prove first a useful general result.

Proposition 4.4. *Let $(u_n)_{n\in\mathbb{N}}, (v_n)_{n\in\mathbb{N}}$ be sequences in H such that*

(i) (u_n) is Cauchy with rate of metastability Ω;

(ii) $\lim_{n\to\infty} \|u_n - v_n\| = 0$ *with rate of convergence* φ.

Then (v_n) *is Cauchy with rate of metastability* Γ *given by*

$$\Gamma(k, g) = \max\{\varphi(3k+2),\ \Omega(3k+2, \widehat{g}_k)\},$$

with $\widehat{g}_k(n) := \max\{\varphi(3k+2), n\} - n + g(\max\{\varphi(3k+2), n\})$.

Proof. First, let us remark that for all $m, n \in \mathbb{N}$,

$$\|v_m - v_n\| \leq \|v_m - u_m\| + \|u_m - u_n\| + \|u_n - v_n\|. \tag{31}$$

Let $k \in \mathbb{N}$ and $g : \mathbb{N} \to \mathbb{N}$ be arbitrary. Since Ω is a rate of metastability for (u_n), there exists $N_0 \leq \Omega(3k+2, \widehat{g}_k)$ such that

$$\forall i, j \in [N_0, N_0 + \widehat{g}_k(N_0)] \left(\|u_i - u_j\| \leq \frac{1}{3(k+1)}\right).$$

Define

$$N := \max\{\varphi(3k+2), N_0\}.$$

Since $N \geq N_0$ and $N_0 + \widehat{g}_k(N_0) = N + g(N)$, we have that $[N, N+g(N)] \subseteq [N_0, N_0 + \widehat{g}_k(N_0)]$. Hence,

$$\forall i, j \in [N, N+g(N)] \left(\|u_i - u_j\| \leq \frac{1}{3(k+1)}\right). \tag{32}$$

On the other hand, since $N \geq \varphi(3k+2)$, it follows that

$$\forall n \geq N \left(\|u_n - v_n\| \leq \frac{1}{3(k+1)}\right). \tag{33}$$

Finally, apply (31), (32) and (33) to get that

$$\forall i, j \in [N, N+g(N)] \left(\|v_i - v_j\| \leq \frac{1}{k+1}\right).$$

Since, obviously, $N \leq \Gamma(k, g)$, we conclude that Γ is a rate of metastability for (v_n). $\square$

The first quantitative version of Theorem 1.1 is the following.

Theorem 4.5. *Let* H *be a Hilbert space,* $A : H \to 2^H$ *be a maximally monotone operator such that* $zer(A) \neq \emptyset$, (x_n) *be defined by* (3) *and* $b \in \mathbb{N}^*$ *be such that* $b \geq \max\{\|x_0 - p\|,\ \|u - p\|\}$ *for some zero* p *of* A. *Suppose that* $(C3_q)$, $(C4_q)$, $(C5_q)$ *hold,* (α_n) *is nonincreasing and that* $\ell \in \mathbb{N}, D \in \mathbb{N}^*$ *satisfy* (23). *Define*

$$\Omega(k, g) := \tilde{g}^{(9b^2(k+1)^2)}(0), \qquad \text{where } \tilde{g}(n) := n + g(n). \tag{34}$$

The following hold:

(i) If $(C1_q)$ *holds, then* (x_n) *is a Cauchy sequence with rate of metastability* Φ *defined by*

$$\Phi(k, g) = \max\left\{\Theta(3k+2),\ \Omega\left(3k+2, h_k\right)\right\}, \tag{35}$$

where Θ *is given by* (24) *and*

$$h_k(n) := \max\{\Theta(3k+2), n\} - n + g(\max\{\Theta(3k+2), n\}).$$

(ii) If $(C2_q)$ holds and $\delta_0 : \mathbb{N} \to \mathbb{N}^*$ satisfies (25), then (x_n) is Cauchy with rate of metastability $\widetilde{\Phi}$ defined by

$$\widetilde{\Phi}(k,g) = \max\left\{\widetilde{\Theta}(3k+2),\ \Omega\left(3k+2, \widetilde{h}_k\right)\right\}, \tag{36}$$

where $\widetilde{\Theta}$ is given by (26) and

$$\widetilde{h}_k(n) := \max\left\{\widetilde{\Theta}(3k+2), n\right\} - n + g\left(\max\left\{\widetilde{\Theta}(3k+2), n\right\}\right).$$

Proof. Since (α_n) is nonincreasing, we can apply Proposition 2.5.(i) with $d := 3b$ to get that (z_n) is Cauchy with rate of metastability Ω. Then, (i) is obtained by an application of Proposition 4.4 and Proposition 4.1.(i), while (ii) follows from Proposition 4.4 and Proposition 4.1.(ii). □

The second quantitative version of Theorem 1.1 is obtained by considering $(C6_q)$ instead of $(C5_q)$.

Theorem 4.6. *Let H be a Hilbert space, $A : H \to 2^H$ be a maximally monotone operator such that $zer(A) \neq \emptyset$, (x_n) be defined by (3) and $b \in \mathbb{N}^*$ be such that $b \geq \max\{\|x_0 - p\|,\ \|u - p\|\}$ for some zero p of A. Suppose that $(C3_q)$, $(C4_q)$, $(C5_q)$ hold, (α_n) is nonincreasing and that $\ell \in \mathbb{N}, D^* \in \mathbb{N}^*$ satisfy (27). Let Ω be given by (34).*

The following hold:

(i) If $(C1_q)$ holds, then (x_n) is a Cauchy sequence with rate of metastability Φ^* defined by

$$\Phi^*(k,g) = \max\left\{\Theta^*(3k+2),\ \Omega\left(3k+2, h_k^*\right)\right\}, \tag{37}$$

where Θ^* is given by (28) and

$$h_k^*(n) := \max\{\Theta^*(3k+2), n\} - n + g(\max\{\Theta^*(3k+2), n\}).$$

(ii) If $(C2_q)$ holds and δ_0^* satisfies (29), then (x_n) is Cauchy with rate of metastability $\widetilde{\Phi}$ defined by

$$\widetilde{\Phi^*}(k,g) = \max\left\{\widetilde{\Theta^*}(3k+2),\ \Omega\left(3k+2, \widetilde{h_k^*}\right)\right\}, \tag{38}$$

where Θ^* is given by (30) and

$$\widetilde{h_k^*}(n) := \max\left\{\widetilde{\Theta^*}(3k+2), n\right\} - n + g\left(\max\left\{\widetilde{\Theta^*}(3k+2), n\right\}\right),$$

Proof. Proceed as in the proof of Theorem 4.5. The only difference is that we use Proposition 4.2 instead of Proposition 4.1. □

As one can see from the proofs of the previous results, the hypothesis that (α_n) is nonincreasing, appearing in both our main theorems, is used only for computing a rate of metastability for (z_n). However, the statement of Proposition 2.5 shows that such a rate of metastability can be also computed if we use $(C0_q^*)$. As a consequence, we can obtain slightly changed versions of Theorems 4.5, 4.6 using $(C0_q^*)$ as a hypothesis instead of (α_n) being nonincreasing.

Let us give some consequences of Theorem 4.5. One can obtain in a similar way corollaries of Theorem 4.6.

By letting $e_n = 0$, we get that

$$x_n = \alpha_n u + (1 - \alpha_n) J_{\beta_n A}(x_n). \tag{39}$$

Furthermore, both $(C5_q)$ and $(C6_q)$ hold trivially with $\sigma_5 = \sigma_6 = 0$.

Corollary 4.7. *Let H, A, b be as in Theorem 4.5 with (x_n) defined by (39). Suppose that $(C3_q)$, $(C4_q)$, hold, (α_n) is nonincreasing and that $\ell \in \mathbb{N}^*$ is such that $\beta \geq \frac{1}{\ell+1}$.*

The following hold:

(i) If $(C1_q)$ holds, then (x_n) is a Cauchy sequence with rate of metastability Φ defined by (35), with Ω, h_k, ψ as in Theorem 4.5 and

$$\begin{aligned}\Theta(k) &:= \sigma_1(\delta(k) + \lceil \ln(3(1+5b)(k+1)) \rceil) + 1, \quad (40)\\ \delta(k) &:= \max\{\psi(3k+2), 1\}. \quad (41)\end{aligned}$$

(ii) If $(C2_q)$ holds and $\delta_0 : \mathbb{N} \to \mathbb{N}^*$ satisfies (25), then (x_n) is Cauchy with rate of metastability $\widetilde{\Phi}$ defined by (36), with Ω, $\widetilde{h}_k$, ψ as in Theorem 4.5, δ as in (i) and

$$\widetilde{\Theta}(k) := \max\left\{\sigma_2\left(3(1+5b)\delta_0(k)(k+1) - 1\right), \delta(k)\right\} + 1. \quad (42)$$

Proof. Apply Theorem 4.5 with $D = 1$ and $\sigma_5 = 0$. □

By letting $g(n) = L$, we get uniform rates of L-metastability for every $L \in \mathbb{N}$.

Corollary 4.8. *Assume the hypothesis of Corollary 4.7 and let $L \in \mathbb{N}$. Define*

$$\begin{aligned}\Delta_L : \mathbb{N} \to \mathbb{N}, \quad \Delta_L(k) &:= \Theta(3k+2) + R(k)L\\ \widetilde{\Delta}_L : \mathbb{N} \to \mathbb{N}, \quad \widetilde{\Delta}_L(k) &:= \widetilde{\Theta}(3k+2) + R(k)L,\end{aligned}$$

where Θ is given by (40), $\widetilde{\Theta}$ is given by (42) and $R : \mathbb{N} \to \mathbb{N}$ is defined by $R(k) := 81b^2(k+1)^2$.

The following hold:

(i) If $(C1_q)$ holds, then Δ_L is a rate of L-metastability of (x_n).

(ii) If $(C2_q)$ holds, then $\widetilde{\Delta}_L$ is a rate of L-metastability of (x_n).

Proof. We apply Corollary 4.7 with $g(n) := L$ for all $n \in \mathbb{N}$.

(i) Remark, using the notations from Theorem 4.5.(i), that

$$\begin{aligned}h_k(n) &= \max\{\Theta(3k+2), n\} - n + L,\\ \tilde{h_k}(n) &= \max\{\Theta(3k+2), n\} + L.\end{aligned}$$

One can easily see by induction that $\tilde{h_k}^{(n)}(0) = \Theta(3k+2) + nL$ for all $n \geq 1$. It follows that

$$\begin{aligned}\Omega\left(3k+2, h_k\right) &= \tilde{h_k}^{(R(k))}(0) = \Theta(3k+2) + R(k)L,\\ \Phi(k, g) &= \max\left\{\Theta(3k+2),\ \Omega\left(3k+2, h_k\right)\right\} = \Theta(3k+2) + R(k)L\\ &= \Delta_L(k).\end{aligned}$$

(ii) The proof is similar.

□

Recently, Kohlenbach [19] computed rates of metastability for the particular version of the HPPA (x_n) obtained by letting $e_n := 0$ for all $n \in \mathbb{N}$ (as in (39)), in the more general setting of accretive operators in uniformly convex and uniformly smooth Banach spaces. The strong convergence proof analyzed in [19], due to Aoyama and Toyoda [1], uses conditions $(C0)$, $(C1)$ on (α_n) and the hypothesis $\inf_n \beta_n > 0$ on (β_n) and is different from the one we analyze in this paper. In order to compute the rate of metastability for (x_n), Kohlenbach uses the quantitative form $(C0_q)$, requiring a rate of convergence of (α_n). We compute rates of metastability for (x_n) by using, instead of $(C0_q)$, either the hypothesis that (α_n) is nonincreasing or $(C0_q^*)$, a weaker form of $(C0_q)$ which needs only a quasi-rate of convergence of (α_n). Furthermore, the rate of metastability computed in [19] depends on an extra-sequence $(\widetilde{\alpha}_n)$ satisfying $0 < \widetilde{\alpha}_n \leq \alpha_n$ for all $n \in \mathbb{N}$. As a consequence, as Kohlenbach also points out, the bounds obtained by him for the sequence (x_n), given by (39), are more complicated than the ones we obtain in this paper.

5 An example

We finish with an example of parameters $(\alpha_n), (\beta_n), (e_n)$ satisfying the hypotheses of Theorems 4.5 and 4.6.

Let

$$\alpha_n := (n+2)^{-3/4}, \quad \beta_n := 1 + \frac{(-1)^n}{n+1}, \quad e_n := 0 \quad \text{for all } n \in \mathbb{N}.$$

One can verify that

(i) (α_n) is nonincreasing.

(ii) $(C0_q)$ holds with $\sigma_0(k) = (k+1)^2$.

(iii) $(C1_q)$ holds with $\sigma_1(k) = (k+1)^4$.

(iv) $(C2_q)$ holds with $\sigma_2(k) = k$.

(v) $(C3_q)$ holds with $\sigma_3(k) = (k+1)^4 + 1$.

(vi) $(C4_q)$ holds with $\beta = 1$ and $\sigma_4(k) = k$.

(vii) $(C5_q)$ and $(C6_q)$ hold with $\sigma_5(k) = \sigma_6(k) = 0$.

Furthermore, $\ell = 0$ and $D = 1$ satisfy (23) and, with the notations of Proposition 4.1.(i),

$$\begin{aligned}
\psi(k) &= (6b(k+1))^4 + 1 = 6^4 b^4 (k+1)^4 + 1, \\
\delta(k) &= 18^4 b^4 (k+1)^4 + 1, \\
\Theta(k) &= \left(18^4 b^4 (k+1)^4 + 1 + \lceil \ln(3(1+5b)(k+1)) \rceil\right)^4 + 1.
\end{aligned}$$

Proposition 5.1. $\lim\limits_{n\to\infty} \|x_n - z_n\| = 0$ *with rate of convergence*

$$\Theta_0(k) = \left(18^4 b^4 (k+1)^4 + 18b(k+1) + 1\right)^4 + 1.$$

Proof. By Proposition 4.1.(i) and the fact that $\Theta(k) \leq \Theta_0(k)$. □

Applying Theorem 4.3 with $\Lambda = \Theta_0$ and $M_i = 1$ for every $i \in \mathbb{N}$, we get effective rates of asymptotic regularity.

Proposition 5.2. *Let $C := 72b$ and define $\overline{\Sigma}, \overline{\Sigma}^* : \mathbb{N} \to \mathbb{N}$ by*

$$\begin{aligned}\overline{\Sigma}(k) &:= \left(C^4(k+1)^4 + C(k+1) + 1\right)^4 + 1, \\ \overline{\Sigma}^*(k) &:= \overline{\Sigma}(2k+1) = \left(16C^4(k+1)^4 + 2C(k+1) + 1\right)^4 + 1.\end{aligned}$$

Then

(i) $\overline{\Sigma}$ is a rate of asymptotic regularity of (x_n) with respect to $J_{\beta A}$.

(ii) $\overline{\Sigma}^*$ is a rate of asymptotic regularity of (x_n) with respect to $(J_{\beta_n A})$ and $J_{\beta_i A}$ $(i \in \mathbb{N})$.

Finally, an application of Corollary 4.8.(i) gives us the following result.

Proposition 5.3. *Let $L \in \mathbb{N}$ and define $\overline{\Delta}_L : \mathbb{N} \to \mathbb{N}$ by*

$$\overline{\Delta}_L(k) := \left(54^4 b^4 (k+1)^4 + 54b(k+1) + 1\right)^4 + 81b^2(k+1)^2 L + 1.$$

Then $\overline{\Delta}_L$ is a rate of L-metastability for (x_n), that is: for all $k \in \mathbb{N}$, there exists $N \leq \overline{\Delta}_L(k)$ such that

$$\|x_i - x_j\| \leq \frac{1}{k+1} \qquad \textit{for all } i, j \in [N, N+L].$$

Proof. Let $k \in \mathbb{N}$ be arbitrary. By Corollary 4.8.(i), we get that there exists $N \leq \Theta(3k+2) + R(k)L$ such that

$$\|x_i - x_j\| \leq \frac{1}{k+1} \qquad \text{for all } i, j \in [N, N+L].$$

Remark that $\Theta(3k+2) + R(k)L \leq \Theta_0(3k+2) + R(k)L = \overline{\Delta}_L(k)$. □

Thus, we obtain for this example, as a consequence of the quantitative results from Section 4, polynomial rates of convergence for $(\|x_n - z_n\|)$, polynomial rates of asymptotic regularity for (x_n) and polynomial rates of L-metastability of (x_n) for every $L \in \mathbb{N}$. Furthermore, the rates are highly uniform: they depend on the Hilbert space H, the maximally monotone operator A and the sequence (x_n) only via b, an upper bound on $\|x_0 - p\|, \|u - p\|$ for some zero p of A.

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