Rates of asymptotic regularity for the alternating Halpern-Mann iteration

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Abstract

In this paper we extend to UCW-hyperbolic spaces the quantitative asymptotic regularity results for the alternating Halpern-Mann iteration obtained by Dinis and the second author for CAT(0) spaces. These results are new even for uniformly convex normed spaces. Furthermore, for a particular choice of the parameter sequences, we compute linear rates of asymptotic regularity in W-hyperbolic spaces and quadratic rates of T- and U-asymptotic regularity in CAT(0) spaces.

Keywords: Mann iteration; Halpern iteration; Rates of asymptotic regularity; Uniform convexity; Proof mining.

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1 Introduction

Let X be a CAT(0) space, $C \subseteq X$ a convex subset, and $T, U : C \to C$ two nonexpansive mappings. Dinis and the second author introduced recently [7] the alternating Halpern-Mann iteration (x_n) as follows: if $x_0, u \in C$ and $(\alpha_n)_{n\in\mathbb{N}}$, $(\beta_n)_{n\in\mathbb{N}}$ are sequences in $[0, 1]$, then

$$\begin{cases} x_{2n+1} & = (1-\alpha_n)Tx_{2n} + \alpha_n u, \\ x_{2n+2} & = (1-\beta_n)Ux_{2n+1} + \beta_n x_{2n+1}. \end{cases} \tag{1}$$

Thus, the alternating Halpern-Mann iteration is an iterative scheme that alternates between the well-known Halpern [11] and Mann [22, 18, 10] iterations. As pointed out in [7], (x_n) is a generalization of the Tikhonov-Mann iteration, introduced by Cheval and the first author [5] as an extension of a modified Mann iteration studied by Yao, Zhou, and Liou [25] and rediscovered by Boţ, Csetnek, and Meier [3]. The Tikhonov-Mann iteration was recently shown by Cheval, Kohlenbach, and the first author [6] to be essentially the modified Halpern iteration, a generalization of the Halpern iteration due to Kim and Xu [12].

Quantitative theorems, providing rates of asymptotic regularity and rates of metastability for (x_n), were proved in [7] by using techniques developed in [8], under the assumption that T and U have common fixed points and the parameter sequences (α_n), (β_n) satisfy certain hypotheses. In particular, the results in [7] show that, in CAT(0) spaces, the alternating Halpern-Mann iteration converges strongly to a common fixed point of the mappings.

Rates of asymptotic regularity were obtained for the Halpern iteration by Kohlenbach and the first author [17] in the much more general setting of W-hyperbolic spaces [14], and for the Mann iteration by the first author [19] for UCW-hyperbolic spaces [19, 20], a class of geodesic spaces that generalize both CAT(0) spaces and uniformly convex normed spaces. A natural question is then

to extend to UCW-hyperbolic spaces the quantitative asymptotic regularity results obtained in [7] for CAT(0) spaces.

In this paper, we give an answer to this question by computing uniform rates of (T- and U-)asymptotic regularity for the alternating Halpern-Mann iteration. These results are new even for uniformly convex normed spaces.

Moreover, we compute for the first time, for a particular choice of the parameter sequences, linear rates of asymptotic regularity in W-hyperbolic spaces and quadratic rates of T- and U-asymptotic regularity in CAT(0) spaces. In order to obtain the linear rates, we adapt to our setting a lemma on sequences of real numbers due to Sabach and Shtern [23].

Let us define the asymptotic regularity notions that we use. Assume that (X, d) is a metric space. A sequence $(z_n)_{n\in\mathbb{N}}$ in X is asymptotically regular if $\lim\limits_{n\to\infty} d(z_n, z_{n+1}) = 0$. A rate of asymptotic regularity of (z_n) is a rate of convergence of $(d(z_n, z_{n+1}))$ towards 0. Assume, furthermore, that $\emptyset \neq C \subseteq X$, $S : C \to C$ is a mapping and (z_n) is a sequence in C. We say that (z_n) is S-asymptotically regular if $\lim\limits_{n\to\infty} d(z_n, Sz_n) = 0$. A rate of S-asymptotic regularity of (z_n) is a rate of convergence of $(d(z_n, Sz_n))$ towards 0.

We recall in the following the main quantitative notions that appear throughout the paper. If $(a_n)_{n\in\mathbb{N}}$ is a sequence in a metric space (X, d), $a \in X$ and $\varphi : \mathbb{N} \to \mathbb{N}$, then

(i) φ is a rate of convergence of (a_n) (towards a) if $\lim\limits_{n\to\infty} a_n = a$ and

$$\forall k \in \mathbb{N}\, \forall n \geq \varphi(k) \left(d(a_n, a) \leq \frac{1}{k+1} \right).$$

(ii) φ is a Cauchy modulus of (a_n) if (a_n) is Cauchy and

$$\forall k \in \mathbb{N}\, \forall n \geq \varphi(k)\, \forall p \in \mathbb{N} \left(d(a_{n+p}, a_n) \leq \frac{1}{k+1} \right).$$

Consider the series $\sum_{n=0}^{\infty} b_n$, where $(b_n)_{n\in\mathbb{N}}$ is a sequence of nonnegative reals. We say that the series diverges with rate of divergence $\theta : \mathbb{N} \to \mathbb{N}$ if $\sum_{i=0}^{\theta(n)} b_i \geq n$ for all $n \in \mathbb{N}$. Furthermore, if $\sum_{n=0}^{\infty} b_n$ converges, then a Cauchy modulus of the series is a Cauchy modulus of the sequence $\left(\sum_{i=0}^{n} b_i\right)$ of partial sums.

2 Preliminaries

2.1 UCW-hyperbolic spaces

Following [5], a metric space (X, d) endowed with a function $W : X \times X \times [0,1] \to X$ is said to be a W-space. As W is thought as a generalization to metric spaces of the usual convex combination mapping in normed spaces, we use the notation $(1-\lambda)x + \lambda y$ for $W(x, y, \lambda)$. Furthermore, we denote a W-space simply by X. A nonempty subset $C \subseteq X$ is said to be convex if for all $x, y \in C$ and all $\lambda \in [0,1]$, $(1-\lambda)x + \lambda y \in C$.

Motivated by proof-theoretical considerations, Kohlenbach introduced in [14] the class of W-hyperbolic spaces (called by him 'hyperbolic spaces'). This class of W-spaces is now broadly used in the study of nonlinear iterations. Let us recall that a W-hyperbolic space is a W-space X satisfying, for all $x, y, w, z \in X$ and all $\lambda,\ \tilde{\lambda} \in [0,1]$,

(W1) $d(z, (1-\lambda)x + \lambda y) \leq (1-\lambda)d(z, x) + \lambda d(z, y)$,

(W2) $d((1-\lambda)x + \lambda y, (1-\tilde{\lambda})x + \tilde{\lambda}y) = |\lambda - \tilde{\lambda}| d(x, y)$,

(W3) $(1-\lambda)x + \lambda y = \lambda y + (1-\lambda)x$,

(W4) $d((1-\lambda)x + \lambda z, (1-\lambda)y + \lambda w) \leq (1-\lambda)d(x, y) + \lambda d(z, w)$.

The following properties of a W-hyperbolic space X are easy to verify: for all $x, y \in X$ and all $\lambda \in [0,1]$,

(W5) $(1-\lambda)x + \lambda x = x$,

(W6) $1x + 0y = 0y + 1x = x$,

(W7) $d(x, (1-\lambda)x + \lambda y) = \lambda d(x, y)$ and $d(y, (1-\lambda)x + \lambda y) = (1-\lambda)d(x, y)$.

An immediate example of a W-hyperbolic space is (a convex subset of) a normed space. An extensive discussion on the relations between W-hyperbolic spaces and other geodesic spaces can be found in [15, pp. 384-387].

The class of uniformly convex W-hyperbolic spaces was introduced by the first author in [19], inspired by [9, p. 105], as a generalization to the nonlinear setting of the well-known uniformly convex normed spaces. Thus, a uniformly convex W-hyperbolic space is a structure (X, η), where X is a W-hyperbolic space and $\eta : (0, \infty) \times (0, 2] \to (0, 1]$ is a so-called modulus of uniform convexity, satisfying the following: for all $r > 0$ and $\varepsilon \in (0, 2]$, and for any $x, y, a \in X$,

$$\text{if } d(x,a), d(y,a) \leq r, \text{ and } d(x,y) \geq \varepsilon r, \text{ then } d\left(\frac{1}{2}x + \frac{1}{2}y, a\right) \leq (1 - \eta(r,\varepsilon))r.$$

Following [20], an UCW-hyperbolic space is a uniformly convex W-hyperbolic space (X, η) with the property that the modulus of uniform convexity is monotone in the following sense:

$$\text{for any } \varepsilon \in (0, 2], \text{ if } 0 < r \leq s, \text{ then } \eta(s, \varepsilon) \leq \eta(r, \varepsilon).$$

The class of UCW-hyperbolic spaces turns out to be an appropriate setting for obtaining quantitative results on the asymptotic behaviour of the Mann iteration for (asymptotically) nonexpansive mappings (see [19, 20, 16]), as well as of the Picard iteration for firmly nonexpansive mappings (see [2]).

The following property of UCW-hyperbolic spaces will be used in Section 3. We refer to [20, Lemma 2.1(iv)] for its proof.

Lemma 2.1. *Let (X, η) be a UCW-hyperbolic space. Assume that $r > 0$, $\varepsilon \in (0, 2]$, and $x, y, a \in X$ satisfy*

$$d(x, a) \leq r,\ d(y, a) \leq r, \text{ and } d(x, y) \geq \varepsilon r.$$

For all $\lambda \in [0, 1]$ and any $s \geq r$,

$$d((1 - \lambda)x + \lambda y, a) \leq (1 - 2\lambda(1 - \lambda)\eta(s, \varepsilon))\, r.$$

Obviously, uniformly convex normed spaces are UCW-hyperbolic spaces. CAT(0) spaces [1, 4] are a very important class of spaces in geodesic geometry and geometric group theory. The first author showed in [19] that CAT(0) spaces are UCW-hyperbolic spaces with the following modulus of uniform convexity $\eta(r, \varepsilon) = \dfrac{\varepsilon^2}{8}$, that does not depend at all on r and is quadratic in ε.

2.2 Alternating Halpern-Mann iteration

We give, in the very general setting of W-spaces, a reformulation of the iterative scheme (1) introduced in [7]. Let X be a W-space, $C \subseteq X$ a convex subset and $T, U : C \to C$ be nonexpansive mappings. The alternating Halpern-Mann iteration is defined by the following scheme:

$$\begin{aligned} x_{n+1} &= (1 - \beta_n)Uy_n + \beta_n y_n, \\ y_n &= (1 - \alpha_n)Tx_n + \alpha_n u, \end{aligned} \tag{2}$$

where $x_0, u \in C$ and $(\alpha_n)_{n\in\mathbb{N}}$, $(\beta_n)_{n\in\mathbb{N}}$ are sequences in $[0, 1]$.

Notice that for the iteration (1), the even terms correspond to our (x_n) and the odd terms to our (y_n). Indeed, this reformulation is just for convenience of notation.

Remark 2.2. *(i) If $U = Id_C$, then (y_n) is the Halpern iteration.*

(ii) If $T = Id_C$, then (x_n) is the Tikhonov-Mann iteration [5] and (y_n) is the modified Halpern iteration [12].

(iii) If $T = Id_C$ and $\alpha_n = 0$, then (x_n) is the Mann iteration.

Proof. (i) By (W5), we have that $x_{n+1} = y_n$.

(ii) Consider the definitions from [6, p.4] and use [6, Proposition 3.2].

(iii) Apply (W6) to get that $y_n = x_n$. $\square$

Let us recall some useful properties of the alternating Halpern-Mann iteration. In the sequel, X is a W-hyperbolic space.

Lemma 2.3. *The following hold for all $n \in \mathbb{N}$:*

$$d(y_{n+1}, y_n) \leq (1 - \alpha_{n+1})d(x_{n+1}, x_n) + |\alpha_{n+1} - \alpha_n| d(Tx_n, u), \tag{3}$$

$$d(x_{n+2}, x_{n+1}) \leq d(y_{n+1}, y_n) + |\beta_{n+1} - \beta_n| d(Uy_n, y_n). \tag{4}$$

Proof. By the proof of [7, Lemma 3.3]. However, as the notations from [7] are changed in this paper, we give the proofs below.

$$\begin{aligned}
d(y_{n+1}, y_n) &= d((1-\alpha_{n+1})Tx_{n+1} + \alpha_{n+1}u, (1-\alpha_n)Tx_n + \alpha_n u)) \\
&\leq d((1-\alpha_{n+1})Tx_{n+1} + \alpha_{n+1}u, (1-\alpha_{n+1})Tx_n + \alpha_{n+1}u) \\
&\quad + d((1-\alpha_{n+1})Tx_n + \alpha_{n+1}u, (1-\alpha_n)Tx_n + \alpha_n u) \\
&\leq (1-\alpha_{n+1})d(Tx_{n+1}, Tx_n) + |\alpha_{n+1} - \alpha_n| d(Tx_n, u) \\
&\text{by (W4) and (W2)} \\
&\leq (1-\alpha_{n+1})d(x_{n+1}, x_n) + |\alpha_{n+1} - \alpha_n| d(Tx_n, u), \\
d(x_{n+2}, x_{n+1}) &= d((1-\beta_{n+1})Uy_{n+1} + \beta_{n+1}y_{n+1}, (1-\beta_n)Uy_n + \beta_n y_n) \\
&\leq d((1-\beta_{n+1})Uy_{n+1} + \beta_{n+1}y_{n+1}, (1-\beta_{n+1})Uy_n + \beta_{n+1}y_n) \\
&\quad + d((1-\beta_{n+1})Uy_n + \beta_{n+1}y_n, (1-\beta_n)Uy_n + \beta_n y_n) \\
&\leq (1-\beta_{n+1})d(Uy_{n+1}, Uy_n) + \beta_{n+1}d(y_{n+1}, y_n) \\
&\quad + |\beta_{n+1} - \beta_n| d(Uy_n, y_n) \quad \text{by (W4) and (W2)} \\
&\leq d(y_{n+1}, y_n) + |\beta_{n+1} - \beta_n| d(Uy_n, y_n).
\end{aligned}$$ $\square$

We assume, furthermore, that T and U have common fixed points and let p be a common fixed point of T and U. Define

$$M_p = \max\{d(x_0, p), d(u, p)\}. \tag{5}$$

Lemma 2.4. *For all $n \in \mathbb{N}$,*

(i) $d(x_n, p) \leq M_p$ and $d(y_n, p) \leq M_p$.

(ii) $d(x_{n+1}, x_n) \leq 2M_p$ and $d(y_{n+1}, y_n) \leq 2M_p$.

(iii) $d(Tx_n, u) \leq 2M_p$ and $d(Uy_n, y_n) \leq 2M_p$.

Proof. For (i) see [7, Lemma 3.2]. (ii), (iii) follow easily from (i). □

Lemma 2.5. *The following hold for all $n \in \mathbb{N}$:*

$$d(y_{n+1}, y_n) \leq (1-\alpha_{n+1})d(x_{n+1}, x_n) + 2M_p|\alpha_{n+1} - \alpha_n|, \tag{6}$$

$$d(x_{n+2}, x_{n+1}) \leq (1-\alpha_{n+1})d(x_{n+1}, x_n) + 2M_p(|\alpha_{n+1} - \alpha_n| + |\beta_{n+1} - \beta_n|). \tag{7}$$

Proof. Apply (3), (4), and Lemma 2.4.(iii). □

2.3 Quantitative lemmas

The following lemma due to Xu [24] is well-known, as it is one of the main tools in different convergence proofs for nonlinear iterations.

Lemma 2.6. *Let $(s_n)_{n\in\mathbb{N}}$ be a sequence of nonnegative real numbers satisfying, for all $n \in \mathbb{N}$,*

$$s_{n+1} \leq (1-a_n)s_n + a_n b_n + c_n,$$

where $(a_n)_{n\in\mathbb{N}}$ is a sequence in $[0,1]$ such that $\sum\limits_{n=0}^{\infty} a_n$ diverges, $(b_n)_{n\in\mathbb{N}}$ is a sequence of real numbers with $\limsup\limits_{n\to\infty} b_n \leq 0$, and $(c_n)_{n\in\mathbb{N}}$ is a sequence of nonnegative reals such that $\sum\limits_{n=0}^{\infty} c_n$ converges.

Then $\lim\limits_{n\to\infty} s_n = 0$.

Quantitative versions of this lemma were given for the first time in [17] for the particular case $c_n = 0$ and for the general case in [21, Section 3]. We refer also to [7, Section 2.3] for an extensive discussion on different quantitative variants of Xu's lemma. We shall use in this paper a quantitative version of another particular case of Lemma 2.6, obtained by taking $b_n = 0$.

Proposition 2.7. *Let $(s_n), (c_n) \subseteq [0, +\infty)$ and $(a_n) \subseteq [0,1]$ satisfy, for all $n \in \mathbb{N}$,*

$$s_{n+1} \leq (1-a_n)s_n + c_n.$$

Assume that $L \in \mathbb{N}^*$ is an upper bound on (s_n), $\sum\limits_{n=0}^{\infty} a_n$ diverges with rate of divergence θ, and $\sum\limits_{n=0}^{\infty} c_n$ converges with Cauchy modulus χ.

Then $\lim\limits_{n\to\infty} s_n = 0$ with rate of convergence Σ defined by

$$\Sigma(k) = \theta\big(\chi(2k+1) + 1 + \lceil \ln(2L(k+1)) \rceil\big) + 1. \tag{8}$$

Proof. See [7, Lemma 2.9(1)]. □

Sabach and Shtern [23, Lemma 3] proved a particularly interesting version of Xu's lemma that allowed them, for a clever choice for the sequence (a_n), to obtain linear rates of asymptotic regularity for the sequential averaging method

(SAM), a generalization of the Halpern iteration. Recently, Cheval, Kohlenbach, and the first author [6] applied [23, Lemma 3] to compute linear rates for the Tikhonov-Mann iteration and the modified Halpern iteration in W-hyperbolic spaces.

In this paper we shall use a version of Sabach and Shtern's lemma to obtain linear rates of asymptotic regularity for the alternating Halpern-Mann iteration, too. We give the proof, for completeness.

Lemma 2.8. *Let $L > 0$, $J \geq N \geq 2$, and $\gamma \in (0,1]$. Assume that $a_n = \frac{N}{\gamma(n+J)}$ and $c_n \leq L$ for all $n \in \mathbb{N}$. Consider a sequence of nonnegative real numbers (s_n) satisfying the following: $s_0 \leq L$ and, for all $n \in \mathbb{N}$,*

$$s_{n+1} \leq (1 - \gamma a_{n+1})s_n + (a_n - a_{n+1})c_n.$$

Then

$$s_n \leq \frac{JL}{\gamma(n+J)} \quad \textit{for all } n \in \mathbb{N}.$$

Proof. We show the result by induction on n.

$n = 0$: We trivially have that $s_0 \leq \frac{L}{\gamma}$, as $\gamma \in (0,1]$.

$n \Rightarrow n+1$: We get that

$$\begin{aligned}
s_{n+1} &\leq (1 - \gamma a_{n+1})s_n + (a_n - a_{n+1})L \\
&\leq \left(1 - \frac{N}{n+1+J}\right)\frac{JL}{\gamma(n+J)} + \left(\frac{N}{\gamma(n+J)} - \frac{N}{\gamma(n+1+J)}\right)L \\
&\quad \text{by the induction hypothesis} \\
&= \frac{(n+1+J-N)JL}{\gamma(n+1+J)(n+J)} + \frac{NL}{\gamma(n+J)(n+1+J)} \\
&\leq \frac{(n+1+J-N)JL}{\gamma(n+1+J)(n+J)} + \frac{JL}{\gamma(n+J)(n+1+J)} \quad \text{as } J \geq N \\
&= \frac{(n+J+2-N)JL}{\gamma(n+J)(n+1+J)} \leq \frac{JL}{\gamma(n+1+J)} \quad \text{as } N \geq 2.
\end{aligned}$$

□

3 Main results

The main results of the paper provide effective rates of (T- and U-)asymptotic regularity of the alternating Halpern-Mann iteration.

Consider throughout that X is a W-hyperbolic space, $C \subseteq X$ is a convex subset, and $T, U : C \to C$ are nonexpansive mappings. We assume that T and U have common fixed points, hence the set $Fix(T) \cap Fix(U)$ is nonempty, where $Fix(T)$ (resp. $Fix(U)$ is the set of fixed points of T (resp. U).

The sequences (x_n), (y_n) are defined by (2), $p \in Fix(T) \cap Fix(U)$, M_p is given by (5), and $K \in \mathbb{N}^*$ is such that $K \geq M_p$.

We shall use the following quantitative hypotheses on the parameter sequences (α_n), (β_n) from (2):

(Q1) $\lim_{n\to\infty} \alpha_n = 0$ with rate of convergence σ_1;

(Q2) $\sum_{n=0}^{\infty} \alpha_n$ diverges with rate of divergence σ_2;

(Q3) $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n|$ converges with Cauchy modulus σ_3;

(Q4) $\sum_{n=0}^{\infty} |\beta_{n+1} - \beta_n|$ converges with Cauchy modulus σ_4;

(Q5) $\Lambda \in \mathbb{N}$ satisfies $\Lambda \geq 2$ and $\frac{1}{\Lambda} \leq \beta_n \leq 1 - \frac{1}{\Lambda}$ for all $n \in N$.

These hypotheses were used in [7] for the quantitative study of the Halpern-Mann iteration in CAT(0) spaces.

3.1 Asymptotic regularity in W-hyperbolic spaces

The first asymptotic regularity results are essentially reformulations of [7, Proposition 3.3(i),(ii)] obtained by using $\frac{1}{k+1}$ instead of ε.

Proposition 3.1. *Assume that (Q2), (Q3), and (Q4) hold. Define*

$$\chi : \mathbb{N} \to \mathbb{N}, \quad \chi(k) = \max\{\sigma_3(4K(k+1) - 1), \sigma_4(4K(k+1) - 1)\}. \tag{9}$$

The following hold:

(i) (x_n) is asymptotically regular with rate Γ_1 defined by

$$\Gamma_1(k) = \sigma_2\left(\chi(2k+1) + 2 + \lceil \ln(4K(k+1)) \rceil\right) + 1. \tag{10}$$

(ii) (y_n) is asymptotically regular with rate Γ_2 defined by

$$\Gamma_2(k) = \max\left\{\Gamma_1(2k+1), \sigma_3(4K(k+1) - 1) + 1\right\}. \tag{11}$$

Proof. (i) As (7) holds, we can apply Proposition 2.7 with $s_n = d(x_{n+1}, x_n)$, $a_n = \alpha_{n+1}$, and $c_n = 2K(|\alpha_{n+1} - \alpha_n| + |\beta_{n+1} - \beta_n|)$. By Lemma 2.4.(ii), $L = 2K$ is an upper bound for (s_n). Furthermore, one can easily see that χ defined by (9) is a Cauchy modulus for $\sum_{n=0}^{\infty} c_n$ and that $\theta(n) = \sigma_2(n+1)$ is a rate of divergence for $\sum_{n=0}^{\infty} a_n$.

(ii) By (6), we get that for all $n \in \mathbb{N}$,

$$d(y_{n+1}, y_n) \leq d(x_{n+1}, x_n) + 2K|\alpha_{n+1} - \alpha_n|.$$

The fact that Γ_2 is a rate of asymptotic regularity of (y_n) follows easily from (i) and (Q3).

□

Remark 3.2. *The rates Γ_1 and Γ_2 depend on X, C, T, U, x_0, u only via K, an integer upper bound of M_p defined by (5). In particular, if C is bounded, then one can take M_p to be the diameter d_C of C and $K = \lceil d_C \rceil$.*

Remark 3.3. *Without loss of generality we can assume that σ_3 is increasing (by taking instead of σ_3 the increasing mapping $\sigma_3^M(k) = \max\{\sigma_3(i) \mid i \leq k\}$). In this case,*

$$\Gamma_2(k) = \Gamma_1(2k+1). \tag{12}$$

Proof. For all $k \in \mathbb{N}$, we have that $\sigma_2(k+1) \geq k$, as $\alpha_k \leq 1$. It follows that

$$\begin{aligned}\Gamma_1(2k+1) &> \chi(4k+3) + 2 \geq \sigma_3(16K(k+1)-1) + 2 \\ &> \sigma_3(4K(k+1)-1) + 1.\end{aligned}$$

□

By Remark 2.2(ii), for $T = id_C$, Γ_1 (resp. Γ_2) is a rate of asymptotic regularity of the Tikhonov-Mann (resp. modified Halpern) iteration. One can easily see that Γ_1 is slightly better than the rate obtained in [5, Theorem 4.1(i)]. Furthermore, in the case that σ_3 is increasing, Γ_2 given by (12) has a similar form with the rate computed in [6, Proposition 4.4(i)].

3.1.1 Linear rates

Let us consider the following parameter sequences:

$$\alpha_n = \tfrac{2}{n+2} \text{ and } \beta_n = \beta \in (0,1).$$

As pointed out in [7], one can apply Proposition 3.1 to get exponential rates of asymptotic regularity. The next result shows that we can obtain, as an application of Lemma 2.8, linear rates of asymptotic regularity.

Proposition 3.4. *For all $n \in \mathbb{N}$,*

$$d(x_n, x_{n+1}) \leq \frac{4K}{n+2} \quad \textit{and} \quad d(y_n, y_{n+1}) \leq \frac{4K}{n+3}. \tag{13}$$

Thus, (x_n) is asymptotically regular with rate $\Sigma_1(k) = 4K(k+1) - 2$ and (y_n) is asymptotically regular with rate $\Sigma_2(k) = 4K(k+1) - 3$.

Proof. Applying (7), we get that for all $n \in \mathbb{N}$,

$$d(x_{n+2}, x_{n+1}) \leq \left(1 - \frac{2}{n+3}\right) d(x_n, x_{n+1}) + \left(\frac{2}{n+2} - \frac{2}{n+3}\right) 2K. \tag{14}$$

One can easily see that we can apply Lemma 2.8 with $s_n = d(x_n, x_{n+1})$, $L = 2K$, $N = J = 2$, $\gamma = 1$, $a_n = \alpha_n = \frac{2}{n+2}$, and $c_n = 2K$ to get that for all $n \in \mathbb{N}$,

$$d(x_n, x_{n+1}) \leq \frac{4K}{n+2}.$$

Furthermore, by (6), we have that for all $n \in \mathbb{N}$,

$$d(y_n, y_{n+1}) \leq \frac{4K}{n+3}.$$

The conclusion follows immediately. □

3.2 T- and U-asymptotic regularity in UCW-hyperbolic spaces

Rates of T- and U-asymptotic regularity for (x_n), (y_n) were also computed in [7] in the setting of CAT(0) spaces. We now show that these results can be generalized to UCW-hyperbolic spaces.

Proposition 3.5. *Let (X, η) be a UCW-hyperbolic space. Assume that (Q1) and (Q5) hold and that (x_n) is asymptotically regular with rate Δ.*

The following are satisfied:

(i) (y_n) is U-asymptotically regular with rate Γ_3 defined by

$$\Gamma_3(k) = \max\{\Delta(2P(k+1)-1), \sigma_1(2PK(k+1)-1)\},$$

where $P = \left\lceil \dfrac{\Lambda^2}{\eta\left(K, \frac{1}{K(k+1)}\right)} \right\rceil.$

(ii) $\lim\limits_{n\to\infty} d(x_n, y_n) = 0$ with rate of convergence

$$\Omega(k) = \max\{\Delta(2k+1), \Gamma_3(2k+1)\}. \tag{15}$$

(iii) (y_n) is T-asymptotically regular with rate

$$\Gamma_4(k) = \max\{\Omega(2k+1), \sigma_1(4K(k+1)-1)\}. \tag{16}$$

(iv) (x_n) is U-asymptotically regular with rate

$$\Gamma_5(k) = \max\{\Omega(4k+3), \Gamma_3(2k+1)\}. \tag{17}$$

(v) (x_n) is T-asymptotically regular with rate

$$\Gamma_6(k) = \max\{\Omega(4k+3), \Gamma_4(2k+1)\}. \tag{18}$$

Proof. (i) Let $k \in \mathbb{N}$ and $n \geq \Gamma_3(k)$. Assume that $d(Uy_n, y_n) > \frac{1}{k+1}$. Since $d(y_n, p) \leq K$, by Lemma 2.4(i), and $d(Uy_n, y_n) \leq d(Uy_n, p) + d(y_n, p) \leq 2d(y_n, p)$, we get that

$$\frac{1}{2(k+1)} < d(y_n, p) \leq K. \tag{19}$$

Furthermore, $d(Uy_n, p) \leq d(y_n, p) \leq K$, $d(Uy_n, y_n) > \frac{1}{K(k+1)} d(y_n, p)$, and $\frac{1}{K(k+1)} < 2$. It follows that we can apply Lemma 2.1 with $x = Uy_n$, $y = y_n$, $a = p$, $r = d(y_n, p)$, $\varepsilon = \frac{1}{K(k+1)}$, $\lambda = \beta_n$, and $s = K$ to conclude that

$$
\begin{aligned}
d(x_{n+1},p) &= d((1-\beta_n)Uy_n + \beta_n y_n, p) \\
&\le \left(1 - 2\beta_n(1-\beta_n)\eta\left(K, \frac{1}{K(k+1)}\right)\right) d(y_n,p) \\
&= d(y_n,p) - 2d(y_n,p)\beta_n(1-\beta_n)\eta\left(K, \frac{1}{K(k+1)}\right) \\
&\le d(y_n,p) - 2d(y_n,p)\frac{1}{\Lambda^2}\eta\left(K, \frac{1}{K(k+1)}\right) \\
&\quad \text{ as, by (Q5), } \beta_n, 1-\beta_n \ge \frac{1}{\Lambda} \\
&< d(y_n,p) - \frac{1}{(k+1)\Lambda^2}\eta\left(K, \frac{1}{K(k+1)}\right) \quad \text{ by (19).}
\end{aligned}
$$

Since, by (W1), $d(y_n,p) \le (1-\alpha_n)d(Tx_n,p) + \alpha_n d(u,p) \le d(x_n,p) + \alpha_n K$, we get that

$$
d(x_{n+1},p) < d(x_n,p) + \alpha_n K - \frac{1}{(k+1)\Lambda^2}\eta\left(K, \frac{1}{K(k+1)}\right).
$$

It follows that

$$
\begin{aligned}
\frac{1}{P(k+1)} &\le \frac{1}{(k+1)\Lambda^2}\eta\left(K, \frac{1}{K(k+1)}\right) \\
&< d(x_n,p) - d(x_{n+1},p) + \alpha_n K \le d(x_{n+1},x_n) + \alpha_n K \\
&\le \frac{1}{P(k+1)}, \text{ as } n \ge \Gamma_3(k).
\end{aligned}
$$

We have obtained a contradiction.

(ii)-(v) are obtained easily from the following inequalities

$$
\begin{aligned}
d(x_n,y_n) &\le d(x_{n+1},x_n) + d(x_{n+1},y_n) \\
&\overset{(W7)}{=} d(x_{n+1},x_n) + (1-\beta_n)d(Uy_n,y_n) \\
&\le d(x_{n+1},x_n) + d(Uy_n,y_n), \\
d(Ty_n,y_n) &\le d(Ty_n,Tx_n) + d(Tx_n,y_n) \overset{(W7)}{=} d(Tx_n,Ty_n) + \alpha_n d(Tx_n,u) \\
&\le d(x_n,y_n) + 2K\alpha_n, \\
d(Ux_n,x_n) &\le d(Ux_n,Uy_n) + d(Uy_n,y_n) + d(y_n,x_n) \\
&\le 2d(x_n,y_n) + d(Uy_n,y_n), \\
d(Tx_n,x_n) &\le d(Tx_n,Ty_n) + d(Ty_n,y_n) + d(y_n,x_n) \\
&\le 2d(x_n,y_n) + d(Ty_n,y_n).
\end{aligned}
$$

□

Corollary 3.6. *Assume that (X,η) is a UCW-hyperbolic space and (Q1)-(Q6) hold. Then (x_n) and (y_n) are U- and T-asymptotically regular with rates obtained from the ones in Proposition 3.5 by replacing Δ with Γ_1.*

Proof. By Proposition 3.1(i), under the hypotheses (Q2), (Q3), (Q4), (x_n) is asymptotically regular with rate Γ_1. □

The following observation is inspired by [19, Remark 15]; see also [13, Theorem 3.4] for a similar remark in the context of uniformly convex normed spaces.

Remark 3.7. *Assume that*

(*) $\eta(r,\varepsilon) = \varepsilon \cdot \tilde{\eta}(r,\varepsilon)$ *for some* $\tilde{\eta}$ *that increases with* ε *(for a fixed* r*).*

Then Proposition 3.5(i) holds with

$$\widetilde{\Gamma_3}(k) = \max\{\Delta(2\tilde{P}(k+1)-1), \sigma_1(2\tilde{P}K(k+1)-1)\}.$$

where $\tilde{P} = \left\lceil \frac{\Lambda^2}{\tilde{\eta}\left(K, \frac{1}{K(k+1)}\right)} \right\rceil$.

Proof. Let $k \in \mathbb{N}$ and $n \geq \widetilde{\Gamma_3}(k)$. Assume that $d(Uy_n, y_n) > \frac{1}{k+1}$. Follow the proof of Proposition 3.5.(i), but replace $\varepsilon = \frac{1}{K(k+1)}$ with $\varepsilon = \frac{1}{d(y_n,p)(k+1)}$. Then we get that

$$\begin{aligned} d(x_{n+1},p) &\leq d(y_n,p) - \frac{2}{(k+1)\Lambda^2}\tilde{\eta}\left(K, \frac{1}{d(y_n,p)(k+1)}\right) \\ &< d(y_n,p) - \frac{1}{(k+1)\Lambda^2}\tilde{\eta}\left(K, \frac{1}{d(y_n,p)(k+1)}\right). \end{aligned}$$

As $d(y_n,p) \leq K$ and $\tilde{\eta}$ is increasing with ε, we have that $\tilde{\eta}\left(K, \frac{1}{d(y_n,p)(k+1)}\right) \geq \tilde{\eta}\left(K, \frac{1}{K(k+1)}\right)$. It follows that

$$d(x_{n+1},p) < d(y_n,p) - \frac{1}{(k+1)\Lambda^2}\tilde{\eta}\left(K, \frac{1}{K(k+1)}\right).$$

Continue as in the proof of Proposition 3.5(i) with $\tilde{P}, \tilde{\eta}$ instead of P, η. □

3.2.1 Quadratic rates in CAT(0) spaces

For the remainder of this section, consider X to be a CAT(0) space. As CAT(0) spaces are UCW-hyperbolic spaces with modulus $\eta(r,\varepsilon) = \frac{\varepsilon^2}{8}$ that obviously satisfies (*) from Remark 3.7 with $\tilde{\eta}(r,\varepsilon) = \frac{\varepsilon}{8}$, we get the following.

Proposition 3.8. *Assume that (Q1) and (Q5) hold and that* (x_n) *is asymptotically regular with rate* Δ*.*

Then (y_n) *is* U*-asymptotically regular with rate* Γ_0 *defined by*

$$\Gamma_0(k) = \max\{\Delta(2P_0(k+1)-1), \sigma_1(2P_0K(k+1)-1)\},$$

where $P_0 = 8K\Lambda^2(k+1)$.

Furthermore, Proposition 3.5 holds with Γ_3 replaced by Γ_0. As in Corollary 3.6, if, moreover, (Q2)-(Q4) hold, then one can take $\Delta = \Gamma_1$, with Γ_1 defined by (10). One can easily verify that our results, when particularized to CAT(0) spaces, actually recover the rates of (T- and U-)asymptotic regularity from [7].

For the particular parameter sequences considered in Subsubsection 3.1.1, we get quadratic rates of T- and U-asymptotic regularity.

Proposition 3.9. *Let* $\alpha_n = \frac{2}{n+2}$ *and* $\beta_n = \beta \in (0,1)$. *Define*

$$\Lambda = \left\lceil \max\left\{\frac{1}{\beta}, \frac{1}{1-\beta}\right\}\right\rceil. \tag{20}$$

The following hold:

(i) $\Sigma_3(k) = 2^6 K^2 \Lambda^2 (k+1)^2 - 2$ *is a rate of U-asymptotic regularity of* (y_n).

(ii) $\lim\limits_{n\to\infty} d(x_n, y_n) = 0$ *with rate of convergence* $\Theta(k) = 2^8 K^2 \Lambda^2 (k+1)^2 - 2$.

(iii) $\Sigma_4(k) = 2^{10} K^2 \Lambda^2 (k+1)^2 - 2$ *is a rate of T-asymptotic regularity of* (y_n).

(iv) $\Sigma_5(k) = 2^{12} K^2 \Lambda^2 (k+1)^2 - 2$ *is a rate of T- and U-asymptotic regularity of* (x_n).

Proof. Obviously, $\sigma_1(k) = 2k$ is a rate of convergence for $\left(\frac{2}{n+2}\right)$. By Proposition 3.4, $\Sigma_1(k) = 4K(k+1) - 2$ is a rate of asymptotic regularity of (x_n). Furthermore, $(Q5)$ holds with Λ defined by (20). To obtain (i) apply Proposition 3.8 with $\Delta = \Sigma_1$. For (ii)-(iv) we use Proposition 3.5(ii)-(v) with $\Delta = \Sigma_1$ and $\Gamma_3 = \Sigma_3$. □

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