On quantitative versions of Xu's Lemma

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1 Introduction

In this note we gather several quantitative versions of a useful lemma due to Hong-Kun Xu,

Lemma 1.1. [14] *Let* $(a_n) \subset (0,1)$ *and* $(r_n), (v_n)$ *be real sequences such that*

$$(i) \sum a_n = \infty; \qquad (ii) \limsup r_n \leqslant 0; \qquad (iii) \sum v_n < \infty.$$

Let (s_n) *be a non-negative real sequence satisfying* $s_{n+1} \leqslant (1-a_n)s_n + a_n r_n + v_n$*, for all* $n \in \mathbb{N}$*. Then* $\lim s_n = 0$*.*

Notation 1.2.

(i) Throughout this paper $\lceil x \rceil$ *is defined as* $\max\{0, \lceil x \rceil\}$ *with the usual definition of* $\lceil \cdot \rceil$ *in the latter.*

(ii) Consider a function φ *on tuples of variables* $\bar{x}$*,* $\bar{y}$*. If we wish to consider the variables* $\bar{x}$ *as parameters we write* $\varphi[\bar{x}](\bar{y})$*. For simplicity of notation we may then even omit the parameters and simply write* $\varphi(\bar{y})$*.*

Quantitative versions of this results and related considerations appeared in

2007 [9, Lemma 9]: Considers $r_n \equiv 0$ but general summable (v_n). Relies on a rate of divergence for $\sum a_n = \infty$. This yields a exponential rate of convergence when $a_n = 1/(n+1)$.

2011 [5, Lemma 3.1] (and end of paragraph preceding it): Obtains a simpler (quadratic) bound than that of [9] for the particular case $a_n = 1/(n+1)$.

2012 [7, Lemmas 5.2 and 5.3]: Here $v_n \equiv 0$, but considers an error term, concluding an interval of stability. Considers rate of divergence for the sum in Lemma 5.2. In Lemma 5.3 assumes $a_n < 1$ and relies on a rate of convergence for $(\prod_{i=0}^{n}(1-a_i))$ together with a lower bound witnessing that the product is always positive. In Corollary 5.3, the data in Lemma 5.3 is computed for the instance $a_n = 1/(n+1)$.

2015 [8, Lemma 2.4]: Considers general (v_n) such that $\sum v_n < \infty$. The result assumes $a_n < 1$, uses a rate of convergence for $(\prod_{i=0}^{n}(1-a_i))$ together with a lower bound witnessing that the product is always positive.

2017 [13, Lemma 3]: Through the use of a sharp trick obtain linear rates of convergence for the particular instance of Xu's lemma when $a_n = 2/(n+2)$. Recently, Cheval, Kohlenbach, and Leuştean [1] applied [13, Lemma 3] to compute linear rates for the Tikhonov-Mann iteration and the modified Halpern iteration in W-hyperbolic spaces for a particular choice of parameters. In [11], studying the alternating Halpern-Mann [3] in general UCW-hyperbolic spaces, using again the argument of [13, Lemma 3] linear rates asymptotic regularity in W-hyperbolic spaces and quadratic rates of T- and U-asymptotic regularity in CAT(0) spaces were obtained for a particular choice of parameters.

2020 [6, Lemma 3.5 (Lemma 10 in preprint)]: This considers a rate towards zero for $\prod_{i=m}^{n}(1-a_i)$ and doesn't need to assume that $a_n < 1$ nor work with a positive lower bound on the product. Only uses a '*metastable*' version of $\limsup r_n \leqslant 0$.

2021 [10, Propositions 3.4 and 3.5]: Considers general (r_n) and (v_n) under the assumptions. In Proposition 3.4, the rate of convergence for (s_n) is computed with a rate of divergence for $\sum a_n = \infty$. In Proposition 3.5, the rate is also computed under the assumption that $a_n < 1$, with a rate of convergence towards zero for $(\prod_{i=0}^{n}(1-a_i))$ together with a lower bound witnessing that the product is always positive.

2021 [12, Lemmas 12 and 13]: Lemma 12 is the conclusion of [10, Proposition 3.4] (which there appears split between several lemmas). Lemma 13 uses a rate towards zero for $\prod_{i=m}^{n}(1-a_i)$ and doesn't need to assume that $a_n < 1$ nor work with a positive lower bound on the product. It additionally discusses the general result (i.e. with both general sequences (r_n) and (v_n) under the conditions) also allowing for a sequence of error terms (cf. Lemmas 14 and 16). Note that some hypothesis were added to the statement of these technical lemmas to make them tailored for the context there, but are in fact unnecessary in their proof. Namely the monotonicity assumption on the functions and the factor $(1-a_i)$ in the error term.

2 Quantitative notions

Definition 2.1. *Let (a_n) be a sequence of real numbers.*

(i) A rate of convergence *for $\lim a_n = 0$ is a function $\gamma : (0, +\infty) \to \mathbb{N}$ such that*

$$\forall \varepsilon > 0\, \forall n \geqslant \gamma(\varepsilon)\ (|a_n| \leqslant \varepsilon)\,.$$

(*ii*) *A* rate of divergence *for* $\lim a_n = +\infty$ *is a function* $\gamma : \mathbb{N} \to \mathbb{N}$ *such that*

$$\forall k \in \mathbb{N}\, \forall n \geqslant \gamma(k)\, (a_n \geqslant k)\,.$$

(*iii*) *A* Cauchy rate *for* (a_n) *is a function* $\gamma : (0, +\infty) \to \mathbb{N}$ *such that*

$$\forall \varepsilon > 0\, \forall i, j \geqslant \gamma(\varepsilon)\, (|a_i - a_j| \leqslant \varepsilon)\,.$$

(*iv*) *For* (b_n) *a sequence of nonnegative real numbers and with* (a_n) *defined as the partial sums of* $\sum_{i=0}^{\infty} b_i$*. In the situation where* (a_n) *is convergent, a* Cauchy rate *for* (a_n) *is a function* $\gamma : (0, +\infty) \to \mathbb{N}$ *such that*

$$\forall \varepsilon > 0\, \forall n \in \mathbb{N} \left(\sum_{i=\gamma(\varepsilon)+1}^{n} b_i \leqslant \varepsilon \right).$$

(*v*) *When* $\limsup a_n \leqslant 0$*, we call* $\gamma : (0, +\infty) \to \mathbb{N}$ *a* lim sup*-rate if*

$$\forall \varepsilon > 0 \ \forall n \geqslant \gamma(\varepsilon)\, (a_n \leqslant \varepsilon)\,.$$

3 Quantitative Versions

Xu's lemma has received several quantitative formulations. The next four lemmas are essentially from [10] and [12].

Lemma 3.1. *Consider* $(s_n) \subset [0, \mathrm{D}]$*, for some upper bound* $\mathrm{D} \in \mathbb{N}\backslash\{0\}$*, and sequences* $(a_n) \subseteq [0, 1]$*,* $(r_n) \subseteq \mathbb{R}$*, and* $(v_n) \subseteq [0, \infty)$*. Under the conditions,*

(*a*) $\sum a_n = \infty$ *with rate of divergence* A*;*

(*b*) $\limsup r_n \leqslant 0$ *with* lim sup*-rate* R*;*

(*c*) $\sum v_n < \infty$ *with a Cauchy rate* V*;*

(*d*) $\forall n \in \mathbb{N}\, (s_{n+1} \leqslant (1 - a_n)s_n + a_n r_n + v_n)$*;*

we have that $\lim s_n = 0$ *with rate of convergence*

$$\Theta(\varepsilon) := \theta[\mathrm{A}, \mathrm{R}, \mathrm{V}, \mathrm{D}](\varepsilon) := A\left(K + \left\lceil \ln\left(\frac{3D}{\varepsilon}\right)\right\rceil\right) + 1,$$

where $K := \max\{\mathrm{R}(\frac{\varepsilon}{3}), \mathrm{V}(\frac{\varepsilon}{3}) + 1\}$.

Proof. Let $\varepsilon > 0$ be arbitrarily given. We may assume that $\varepsilon < \mathrm{D}$ otherwise the results trivially holds.
<u>**Claim:**</u> For all $m \in \mathbb{N}$,

$$s_{K+m+1} \leqslant \left(\prod_{i=K}^{K+m} (1 - a_i)\right) s_K + \left(1 - \prod_{i=K}^{K+m} (1 - a_i)\right) \frac{\varepsilon}{3} + \sum_{i=K}^{K+m} v_i.$$

Proof of Claim: The proof is by induction on $m \in \mathbb{N}$. The base case $m = 0$ follows from (d) together with the definition of K and hypothesis (b). For the induction step, we have

$$\begin{aligned}
s_{K+m+2} &\overset{\text{(d)}}{\leqslant} (1-a_{K+m+1})s_{K+m+1} + a_{K+m+1}r_{K+m+1} + v_{K+m+1} \\
&\overset{\text{IH}}{\leqslant} (1-a_{K+m+1})\left[\left(\prod_{i=K}^{K+m}(1-a_i)\right)s_K + \left(1-\prod_{i=K}^{K+m}(1-a_i)\right)\frac{\varepsilon}{3} + \sum_{i=K}^{K+m} v_i\right] \\
&\quad + a_{K+m+1}r_{K+m+1} + v_{K+m+1} \\
&\overset{\text{(b)}}{\leqslant} \left(\prod_{i=K}^{K+m+1}(1-a_i)\right)s_K + \left((1-a_{K+m+1}) - \prod_{i=K}^{K+m+1}(1-a_i)\right)\frac{\varepsilon}{3} \\
&\quad + a_{K+m+1}\frac{\varepsilon}{3} + \sum_{i=K}^{K+m+1} v_i \\
&= \left(\prod_{i=K}^{K+m+1}(1-a_i)\right)s_K + \left(1-\prod_{i=K}^{K+m+1}(1-a_i)\right)\frac{\varepsilon}{3} + \sum_{i=K}^{K+m+1} v_i.
\end{aligned}$$

This concludes the induction and the proof of the claim. ■

As $K \geqslant \mathrm{V}(\varepsilon/3) + 1$, we have $\sum_{i=K}^{n} v_i \leqslant \varepsilon/3$, for all $n \in \mathbb{N}$ and so

$$\forall m \in \mathbb{N}\left(s_{K+m+1} \leqslant \mathrm{D}\cdot\left(\prod_{i=K}^{K+m}(1-a_i)\right) + \frac{2\varepsilon}{3}\right). \tag{1}$$

Since $(a_n) \subseteq [0,1]$, we get $A(L+1) \geqslant L$ for all $L \in \mathbb{N}$. Indeed,

$$L + 1 \leqslant \sum_{i=0}^{\mathrm{A}(L+1)} a_i \leqslant \sum_{i=0}^{\mathrm{A}(L+1)} 1 = \mathrm{A}(L+1) + 1.$$

Consider $M := \mathrm{A}(K + \lceil \ln(3\mathrm{D}/\varepsilon)\rceil) - K$, and observe that $\varepsilon < \mathrm{D}$ entails $3\mathrm{D}/\varepsilon \geqslant 3$ and so

$$\mathrm{A}(K + \lceil \ln(3\mathrm{D}/\varepsilon)\rceil) \geqslant K + \lceil \ln(3\mathrm{D}/\varepsilon)\rceil - 1 \geqslant K.$$

Thus M is a natural number. For any $m \geqslant M$,

$$\sum_{i=0}^{K+m} a_i \geqslant \sum_{i=0}^{K+M} a_i \geqslant K + \ln\left(\frac{3\mathrm{D}}{\varepsilon}\right) \geqslant \sum_{i=0}^{K-1} a_i + \ln\left(\frac{3\mathrm{D}}{\varepsilon}\right),$$

which gives $\sum_{i=K}^{K+m} a_i \geqslant \ln(3\mathrm{D}/\varepsilon)$.
Since for $x \geqslant 0$, we have $1 - x \leqslant \exp(-x)$, we derive for $m \geqslant M$

$$\mathrm{D}\cdot\left(\prod_{i=K}^{K+m}(1-a_i)\right) \leqslant \mathrm{D}\cdot\exp\left(-\sum_{i=K}^{K+m}\right) \leqslant \mathrm{D}\cdot\frac{\varepsilon}{3\mathrm{D}} = \frac{\varepsilon}{3}.$$

This together with (1) concludes

$$\forall n \geqslant K + M + 1 =: \Theta\left(s_n \leqslant \varepsilon\right). \qquad \square$$

In Lemma 3.1, the fact that (s_n) is bounded follows trivially from the other assumptions. This translates into the easy fact that it is possible to compute a bound D from the remaining data. Namely, a possible value for D is $\lceil \max\{s_0, \mathcal{R}\} + \mathcal{V} \rceil$, where $\mathcal{R} := \max_{n \leqslant \mathrm{R}(1)}\{1, r_n\}$ and $\mathcal{V} := 1 + \sum_{i=0}^{\mathrm{V}(1)} v_i$ are bounds on the sequences (r_n) and $(\sum_{i=0}^{n} v_i)$, respectively.

Lemma 3.2. *Under the conditions of Lemma 3.1*

(1) *If $r_n \equiv 0$, then the function Θ is simplified to*

$$\Theta(\varepsilon) := \widehat{\theta}[\mathrm{A}, \mathrm{V}, \mathrm{D}](\varepsilon) := \mathrm{A}\left(\mathrm{V}\left(\frac{\varepsilon}{2}\right) + \left\lceil \ln\left(\frac{2\mathrm{D}}{\varepsilon}\right)\right\rceil + 1\right) + 1.$$

(2) *If $v_n \equiv 0$, then the function Θ is simplified to*

$$\Theta(\varepsilon) := \check{\theta}[\mathrm{A}, \mathrm{R}, \mathrm{D}](\varepsilon) := \mathrm{A}\left(\mathrm{R}\left(\frac{\varepsilon}{2}\right) + \left\lceil \ln\left(\frac{2\mathrm{D}}{\varepsilon}\right)\right\rceil\right) + 1.$$

Proof. Follow the argument in the claim of the previous proof towards concluding (1) with '$+\varepsilon/2$' instead of '$+2\varepsilon/3$'. Observe that from $\varepsilon < \mathrm{D}$ we still get $\lceil \ln(2\mathrm{D}/\varepsilon)\rceil \geqslant 1$ and $M \in \mathbb{N}$. ☐

Instead of considering $\sum a_n = \infty$, one can work with the following equivalent condition

$$\forall m \in \mathbb{N} \left(\prod_{i=m}^{\infty}(1 - a_i) = 0\right).$$

Hence, it makes sense to also consider a corresponding quantitative assumption:

$$\begin{gathered} \mathrm{A}' : \mathbb{N} \times (0, +\infty) \to \mathbb{N} \text{ is a monotone function satisfying} \\ \forall \varepsilon > 0\, \forall m \in \mathbb{N} \left(\prod_{i=m}^{\mathrm{A}'(m,\varepsilon)}(1 - a_i) \leqslant \varepsilon\right), \end{gathered} \tag{$Q\Pi$}$$

i.e. $\mathrm{A}'(m, \cdot)$ is a rate of convergence towards zero for the sequence $(\prod_{i=m}^{n}(1 - a_i))$.

Next we state a quantitative version of Lemma 1.1 which relies on the condition ($Q\Pi$) (see also [8, Lemma 2.4] and [10]).

Lemma 3.3. *Consider $(s_n) \subseteq [0, \mathrm{D}]$ for some upper bound $\mathrm{D} \in \mathbb{N}\backslash\{0\}$, and sequences $(a_n) \subseteq [0, 1]$, $(r_n) \subseteq \mathbb{R}$, and $(v_n) \subseteq [0, \infty)$. Under the conditions,*

(*a*) A' *satisfies* ($Q\Pi$)*;*

(*b*) $\limsup r_n \leqslant 0$ *with* $\limsup$*-rate* R*;*

(*c*) $\sum v_n < \infty$ *with a Cauchy rate* V*;*

(*d*) $\forall n \in \mathbb{N}\, (s_{n+1} \leqslant (1 - a_n)s_n + a_n r_n + vn)$*;*

we have that $\lim s_n = 0$ *with rate of convergence*

$$\Theta' := \theta'[\mathrm{A}', \mathrm{R}, \mathrm{V}, \mathrm{D}](\varepsilon) := \mathrm{A}'\left(K, \frac{\varepsilon}{3\mathrm{D}}\right) + 1, \quad \textit{with } K \textit{ as in Lemma 3.1.}$$

Proof. The proof is the same as that of Lemma 3.1 up until the conclusion of

$$\forall m \in \mathbb{N}\left(s_{K+m+1} \leqslant \mathrm{D} \cdot \left(\prod_{i=K}^{K+m}(1-a_i)\right) + \frac{2\varepsilon}{3}\right). \tag{1}$$

Note that by the assumption of A', whenever $\delta \in (0,1)$ we have $\mathrm{A}'(L,\delta) \geqslant L$ for all $L \in \mathbb{N}$. Indeed, the contrary would give the contradiction

$$1 = \prod_{i=L}^{\mathrm{A}'(L,\delta)}(1-a_i) \leqslant \delta < 1.$$

Consider $M := \mathrm{A}'\left(K, \frac{\varepsilon}{3\mathrm{D}}\right) - K$, which is a natural number since $\varepsilon/3\mathrm{D} < 1$. Then, for any $m \geqslant M$

$$\mathrm{D} \cdot \left(\prod_{i=K}^{K+m}(1-a_i)\right) \leqslant \mathrm{D} \cdot \left(\prod_{i=K}^{K+M}(1-a_i)\right) \leqslant \frac{\varepsilon}{3}.$$

This together with (1) entails the result. □

Lemma 3.4. *Under the conditions of Lemma 3.3*

(1) *If* $r_n \equiv 0$*, then the function* Θ' *is simplified to*

$$\Theta'(\varepsilon) := \hat{\theta}'[\mathrm{A}', \mathrm{V}, \mathrm{D}](\varepsilon) := \mathrm{A}'\left(\mathrm{V}\left(\frac{\varepsilon}{2}\right) + 1, \frac{\varepsilon}{2\mathrm{D}}\right) + 1.$$

(2) *If* $v_n \equiv 0$*, then the function* Θ' *is simplified to*

$$\Theta'(\varepsilon) := \check{\theta}'[\mathrm{A}, \mathrm{R}, \mathrm{D}](\varepsilon) := \mathrm{A}'\left(\mathrm{R}\left(\frac{\varepsilon}{2}\right), \frac{\varepsilon}{3\mathrm{D}}\right) + 1.$$

Proof. Similar to Lemma 3.2 □

We will also require particular instances of Lemma 1.1 that allow for a error term. The next two results are trivial variants of [7, Lemmas 5.2 and 5.3].

Lemma 3.5. *Consider* $(s_n) \subseteq [0, \mathrm{D}]$ *for some upper bound* $D \in \mathbb{N}\backslash\{0\}$*, and sequences* $(a_n) \subseteq [0,1]$*,* $(r_n) \subseteq \mathbb{R}$*. Assume that* $\sum a_n = \infty$ *with a rate of divergence* A*. Let* $\varepsilon > 0$*,* $K, P \in \mathbb{N}$ *be given. If for all* $n \in [K, P]$

$$(i)\ s_{n+1} \leqslant (1-a_n)s_n + a_n r_n + \mathcal{E} \qquad (ii)\ r_n \leqslant \frac{\varepsilon}{3} \qquad (iii)\ \mathcal{E} \leqslant \frac{\varepsilon}{3(P+1)},$$

then $\forall n \in [\Sigma, P]\,(s_n \leqslant \varepsilon)$*, where*

$$\Sigma := \sigma[\mathrm{A}, \mathrm{D}](\varepsilon, K) := \mathrm{A}\left(K + \left\lceil \ln\left(\frac{3\mathrm{D}}{\varepsilon}\right)\right\rceil\right) + 1.$$

Proof. First note that if $\Sigma > P$, then the result is trivial. Hence we may assume that $\Sigma \leqslant P$ (which in particular implies that $K \leqslant P$). Next, we may assume that $\varepsilon < D$, otherwise the result also trivially holds. Now, by induction one easily sees that for all $m \leqslant P - K$,

$$s_{K+m+1} \leqslant \left(\prod_{i=K}^{K+m}(1-a_i)\right)s_K + \left(1 - \prod_{i=K}^{K+m}(1-a_i)\right)\frac{\varepsilon}{3} + (m+1)\mathcal{E}.$$

Hence, for all $m \leqslant P - K$,

$$s_{K+m+1} \leqslant \left(\prod_{i=K}^{K+m}(1-a_i)\right)s_K + \frac{\varepsilon}{3} + (P+1)\mathcal{E} \leqslant \mathrm{D}\left(\prod_{i=K}^{K+m}(1-a_i)\right) + \frac{2\varepsilon}{3}. \quad (2)$$

As in Lemma 3.1, by the assumption $\varepsilon < \mathrm{D}$, we have $\lceil \ln(3D/\varepsilon)\rceil \geqslant 1$. Thus $\mathrm{A}(K + \lceil \ln(3D/\varepsilon)\rceil) \geqslant K$ and can consider $M := \mathrm{A}(K + \lceil \ln(3D/\varepsilon)\rceil) - K \in \mathbb{N}$. For all $m \geqslant M$, we have

$$\sum_{i=0}^{K+m} a_i \geqslant \sum_{i=0}^{K+M} a_i \geqslant K + \ln\left(\frac{3D}{\varepsilon}\right) \geqslant \sum_{i=0}^{K-1} a_i + \ln\left(\frac{3D}{\varepsilon}\right).$$

which entails $\sum_{i=K}^{K+m} a_i \geqslant \ln(3D/\varepsilon)$. Since for all $x \geqslant 0$, $1 - x \leqslant \exp(-x)$, we derive for all $m \geqslant M$,

$$\mathrm{D}\left(\prod_{i=K}^{K+m}(1-a_i)\right) \leqslant \mathrm{D}\cdot\exp\left(-\sum_{i=K}^{K+m} a_i\right) \leqslant \mathrm{D}\frac{\varepsilon}{3D} = \frac{\varepsilon}{3}.$$

Together with (2), we have thus concluded that for $m \in [M, P-K]$,

$$s_{K+m+1} \leqslant \mathrm{D}\left(\prod_{i=K}^{K+m}(1-a_i)\right) + \frac{2\varepsilon}{3} \leqslant \varepsilon,$$

which entails the result. $\square$

As before, instead of a rate of divergence A, we can stablish an analogue of the previous result now with a function A′ satisfying ($Q\Pi$).

Lemma 3.6. *Consider* $(s_n) \subseteq [0, \mathrm{D}]$ *for some upper bound* $D \in \mathbb{N}\backslash\{0\}$, *and sequences* $(a_n) \subseteq [0,1]$, $(r_n) \subseteq \mathbb{R}$. *Assume that* $\sum a_n = \infty$ *with a function* A *satisfying* ($Q\Pi$). *Let* $\varepsilon > 0$, $K, P \in \mathbb{N}$ *be given. If for all* $n \in [K, P]$

$$(i)\ s_{n+1} \leqslant (1-a_n)s_n + a_n r_n + \mathcal{E} \qquad (ii)\ r_n \leqslant \frac{\varepsilon}{3} \qquad (iii)\ \mathcal{E} \leqslant \frac{\varepsilon}{3(P+1)},$$

then $\forall n \in [\Sigma', P]\,(s_n \leqslant \varepsilon)$, *where*

$$\Sigma' := \sigma'[\mathrm{A}', \mathrm{D}](\varepsilon, K) := \mathrm{A}'\left(K, \frac{\varepsilon}{3\mathrm{D}}\right) + 1.$$

Proof. Following the proof of Lemma 3.5, we have for all $m \leqslant P - K$,

$$s_{K+m+1} \leqslant \mathrm{D}\left(\prod_{i=K}^{K+m}(1-a_i)\right) + \frac{2\varepsilon}{3}. \tag{2}$$

Again from the assumption on ε, we may consider $M := \mathrm{A}'(K, \varepsilon/3D) - K \in \mathbb{N}$. For all $m \geqslant M$,

$$\mathrm{D}\left(\prod_{i=K}^{K+m}(1-a_i)\right) \leqslant \mathrm{D}\left(\prod_{i=K}^{K+M}(1-a_i)\right) = \mathrm{D}\left(\prod_{i=K}^{\mathrm{A}'(K,\varepsilon/3D)}(1-a_i)\right) \leqslant \frac{\varepsilon}{3},$$

which, as before, entails the result. $\square$

In [13, Lemma 3], Sabach and Shtern proved an interesting particular version of Xu's lemma. While studying the sequential averaging method (a generalization of the Halpern iteration), they obtained linear rates of asymptotic regularity for a particular choice for the parameter sequence (a_n).

Lemma 3.7. *Let $L > 0$, $J \geqslant N \geqslant 2$, and $\gamma \in (0,1]$. Assume that $a_n = \frac{N}{\gamma(n+J)}$ and $c_n \leqslant L$ for all $n \in \mathbb{N}$. Consider a sequence of nonnegative real numbers (s_n) satisfying the following: $s_0 \leqslant L$ and, for all $n \in \mathbb{N}$,*

$$s_{n+1} \leqslant (1-\gamma a_{n+1})s_n + (a_n - a_{n+1})c_n.$$

Then

$$s_n \leqslant \frac{JL}{\gamma(n+J)} \quad \textit{for all } n \in \mathbb{N}.$$

Proof. The proof is by a simple induction on n. For the base case, we trivially have that $s_0 \leqslant \frac{L}{\gamma}$, since $\gamma \in (0,1]$. For the induction step, assume that the inequality holds for n. Then, we have

$$\begin{aligned}
s_{n+1} &\leqslant (1-\gamma a_{n+1})s_n + (a_n - a_{n+1})L \\
&\leqslant \left(1 - \frac{N}{n+1+J}\right)\frac{JL}{\gamma(n+J)} + \left(\frac{N}{\gamma(n+J)} - \frac{N}{\gamma(n+1+J)}\right)L \\
&\overset{\text{(IH)}}{=} \frac{(n+1+J-N)JL}{\gamma(n+1+J)(n+J)} + \frac{NL}{\gamma(n+J)(n+1+J)} \\
&\overset{(J \geqslant N)}{\leqslant} \frac{(n+1+J-N)JL}{\gamma(n+1+J)(n+J)} + \frac{JL}{\gamma(n+J)(n+1+J)} \\
&\overset{(N \geqslant 2)}{=} \frac{(n+J+2-N)JL}{\gamma(n+J)(n+1+J)} \leqslant \frac{JL}{\gamma(n+1+J)}.
\end{aligned}$$

$\square$

Recently, Cheval, Kohlenbach, and Leuştean [1] applied [13, Lemma 3] to compute linear rates for the Tikhonov-Mann iteration and the modified Halpern iteration in W-hyperbolic spaces. In a similar way, this lemma was used in [11] to obtain linear rates of asymptotic regularity in W-hyperbolic spaces, and quadratic rates of T- and U-asymptotic regularity for the alternating Halpern-Mann [3] iteration in CAT(0) spaces. Furthermore, this result entails that any convergence result centred largely in an application of Xu's lemma should allow for a particular case where linear rates of convergence are available (as e.g. the recent note [2], where Sabbach-Shtern's lemma is applied to obtain linear rates for different Halpern-type iterations).

References

[1] H. Cheval, U. Kohlenbach, and L. Leuştean. On modified Halpern and Tikhonov-Mann iterations. *Journal of Optimization Theory and Applications* 197, 233–251, 2023.

[2] H. Cheval, and L. Leuştean. Linear rates of asymptotic regularity for Halpern-type iterations. arXiv:2303.05406, 2023.

[3] B. Dinis and P. Pinto. Strong convergence for the alternating Halpern-Mann iteration in CAT(0) spaces. To appear in: *SIAM Journal on Optimization*. arxiv.2112.14525, 2023.

[4] F. Ferreira, L. Leuştean, and P. Pinto. On the removal of weak compactness arguments in proof mining. *Advances in Mathematics*, 354:106728, 2019.

[5] U. Kohlenbach. On quantitative versions of theorems due to F.E. Browder and R. Wittmann. *Advances in Mathematics*, 226(3): 2764–2795, (2011).

[6] U. Kohlenbach. Quantitative analysis of a Halpern-type Proximal Point Algorithm for accretive operators in Banach spaces. *Journal of Nonlinear and Convex Analysis*, 21(9): 2125–2138, 2020.

[7] U. Kohlenbach and L. Leuştean. Effective metastability of Halpern iterates in CAT(0) spaces. *Advances in Mathematics*, 231(5):2526–2556, 2012.

[8] D. Körnlein. Quantitative results for Halpern iterations of nonexpansive mappings. *Journal of Mathematical Analysis and Applications*, 428(2):1161–1172, 2015.

[9] L. Leuştean. Rates of Asymptotic Regularity for Halpern Iterations of Nonexpansive Mappings. *Journal of Universal Computer Science*, 13(11): 1680–1691, 2007.

[10] L. Leuştean and P. Pinto. Quantitative results on a Halpern-type proximal point algorithm. *Computational Optimization and Applications*, 79(1):101–125, 2021.

[11] L. Leuştean, and P. Pinto. Rates of asymptotic regularity for the alternating Halpern-Mann iteration. To appear in: *Optimization Letters*. arXiv.2206.02226, 2023.

[12] P. Pinto. A rate of metastability for the Halpern type Proximal Point Algorithm. *Numerical Functional Analysis and Optimization*, 42(3): 320-343, 2021.

[13] S. Sabach, and S. Shtern. A first order method for solving convex bilevel optimization problems. *SIAM Journal on Optimization*, 27(2): 640–660, 2017.

[14] H.-K. Xu. Iterative algorithms for nonlinear operators. *Journal of the London Mathematical Society*, 66(1):240–256, 2002.