

On the convergence of algorithms with Tikhonov regularization terms *

Bruno Dinis^a and Pedro Pinto^b

^a Departamento de Matemática, Faculdade de Ciências da Universidade de Lisboa,
Campo Grande, Edifício C6, 1749-016 Lisboa, Portugal
E-mail: bmdinis@fc.ul.pt

^b Department of Mathematics, Technische Universität Darmstadt,
Schlossgartenstrasse 7, 64289 Darmstadt, Germany
E-mail: pinto@mathematik.tu-darmstadt.de

Abstract

We consider the strongly convergent modified versions of the Krasnosel'skiĭ-Mann, the forward-backward and the Douglas-Rachford algorithms with Tikhonov regularization terms, introduced by Radu Boț, Ernö Csetnek and Dennis Meier. We obtain quantitative information for these modified iterations, namely rates of asymptotic regularity and metastability. Furthermore, our arguments avoid the use of sequential weak compactness and use only a weak form of the projection argument.

1 Introduction

In nonlinear analysis one is often confronted with the need to find zeros of a sum of monotone operators. Two well-known iterative splitting methods to approximate such zeros are the forward-backward and the Douglas-Rachford algorithms (see [1]). The former weakly converges to a zero of the sum of a multi-valued maximal monotone operator with a single-valued cocoercive operator, while the latter weakly converges to a zero of the sum of two maximal monotone operators. These convergence results stem from the weak convergence of the Krasnosel'skiĭ-Mann (KM) iteration to a fixed point of a nonexpansive map T in a Hilbert space H :

$$x_{n+1} = x_n + \lambda_n(T(x_n) - x_n), \quad (\text{KM})$$

with $x_0 \in H$ a starting point and $(\lambda_n) \subset [0, 1]$ a sequence of real numbers. One way to obtain strong convergence is to impose stronger conditions on the operators, such as strong monotonicity or strong convexity. However, the application to certain cases may be unfeasible, as the stronger conditions may prove to be too restrictive. The proximal-Tikhonov algorithm (see [11]) was introduced as an alternative way to obtain, under mild conditions, strong convergence to a zero of a maximal monotone operator A . This algorithm is based on the proximal point algorithm [13, 16] and, in a first step, switches from the operator A to the operator $A + \mu_n \text{Id}$, where (μ_n) is a sequence of nonnegative real numbers. As the original proximal point algorithm, in general, is only weakly convergent, the added term $\mu_n \text{Id}$ is crucial in guaranteeing the strong convergence of the iteration. Motivated by this method, in [3] Radu Boț, Ernö Csetnek and Dennis Meier considered the (KM) iteration with Tikhonov regularization terms as follows:

$$x_{n+1} = \beta_n x_n + \lambda_n(T(\beta_n x_n) - \beta_n x_n), \quad (\text{T-KM})$$

where $x_0 \in H$ is the starting point, $(\lambda_n), (\beta_n)$ are sequences of positive numbers and $T : H \rightarrow H$ is a nonexpansive mapping. Boț *et al.* obtained a strong convergence result for (T-KM) and for similar modifications of the forward-backward and the Douglas-Rachford algorithms.

In this paper we obtain rates of asymptotic regularity and metastability for these modified iterations through a quantitative analysis of the proofs in [3]. As such we are able to obtain quantitative versions of the convergence results by Boț *et al.* Similarly to [6, 5, 4, 15], our analysis bypasses the use of sequential weak compactness and only relies on a weak form of the metric projection argument (as explained in [8, 6]). Our results are guided by proof theoretical techniques (in the context of the proof mining program [7, 9, 10]), however no special knowledge of mathematical logic is required to read this paper. A summary of all the extracted information is given in the Appendix.

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2 Preliminaries

Consider H a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$. We start by recalling some notions and properties concerning operators on Hilbert spaces. An operator $A : H \rightrightarrows H$ is said to be *monotone* if and only if whenever (x, y) and (x', y') are elements of the graph of A , it holds that $\langle x - x', y - y' \rangle \geq 0$. A monotone operator A is said to be *maximal monotone* if the graph of A is not properly contained in the graph of any other monotone operator on H . We denote by $\text{zer}(A)$ the set of all zeros of A . Let $A : H \rightrightarrows H$ be a maximal monotone operator and $\gamma > 0$. The resolvent function $J_{\gamma A}$ is the single-valued function defined by $J_{\gamma A} := (\text{Id} + \gamma A)^{-1}$, and the reflected resolvent function is the function $R_{\gamma A} := 2J_{\gamma A} - \text{Id}$.

A mapping $T : H \rightarrow H$ is called *nonexpansive* if $\|T(x) - T(y)\| \leq \|x - y\|$, for all $x, y \in H$. The set of fixed points of the mapping T will be denoted by $\text{Fix } T$. If T is nonexpansive, then $\text{Fix } T$ is a closed and convex subset of H . We denote by $\text{proj}_{\text{Fix } T}$ the metric projection onto $\text{Fix } T$.

For $\alpha \in (0, 1]$, a function $T : H \rightarrow H$ is called α -averaged¹ if there exists a nonexpansive operator $T' : H \rightarrow H$ such that $T = (1-\alpha)\text{Id} + \alpha T'$. The α -averaged operators are always nonexpansive. The $\frac{1}{2}$ -averaged operators are also called firmly nonexpansive operators. The resolvent function $J_{\gamma A}$ is firmly nonexpansive and the reflected resolvent function $R_{\gamma A}$ is nonexpansive.

For $\delta > 0$, a function $B : H \rightarrow H$ is said to be δ -cocoercive if for all $x, y \in H$ it holds that $\langle x - y, B(x) - B(y) \rangle \geq \delta \|B(x) - B(y)\|^2$.

For a comprehensive introduction to convex analysis and the theory of monotone operators in Hilbert spaces we refer to [1].

The main purpose of this paper is to extract quantitative information from the proof of the following result.

Theorem 2.1. ([3, Theorem 3]) *Let $(\beta_n), (\lambda_n) \subset (0, 1]$ be real sequences satisfying:*

- (i) $\lim \beta_n = 1$,
- (ii) $\sum_{n \geq 0} (1 - \beta_n) = \infty$,
- (iii) $\sum_{n \geq 1} |\beta_n - \beta_{n-1}| < \infty$,
- (iv) $\liminf \lambda_n > 0$,
- (v) $\sum_{n \geq 1} |\lambda_n - \lambda_{n-1}| < \infty$.

Consider the iterative scheme **(T-KM)** with an arbitrary starting point $x_0 \in H$ and a nonexpansive mapping $T : H \rightarrow H$ such that $\text{Fix } T \neq \emptyset$. Then (x_n) converges strongly to $\text{proj}_{\text{Fix } T}(0)$.

Our main result (Theorem 3.6) gives a bound on the *metastability* property of the sequence (x_n) , i.e.

$$\forall k \in \mathbb{N} \forall f : \mathbb{N} \rightarrow \mathbb{N} \exists n \in \mathbb{N} \forall i, j \in [n, f(n)] (\|x_i - x_j\| \leq \frac{1}{k+1}),$$

which is (non-effectively) equivalent to the Cauchy property for (x_n) .

We now present some useful quantitative lemmas which require the notion of monotone functional for two particular cases. This relies on the strong majorizability relation from [2].

Definition 2.2. For functions $f, g : \mathbb{N} \rightarrow \mathbb{N}$ we define

$$g \leq^* f := \forall n, m \in \mathbb{N} (m \leq n \rightarrow (g(m) \leq f(n) \wedge f(m) \leq f(n))).$$

A function $f : \mathbb{N} \rightarrow \mathbb{N}$ is said to be *monotone* if $f \leq^* f$. We say that a functional $\varphi : \mathbb{N} \times \mathbb{N}^\mathbb{N} \rightarrow \mathbb{N}$ is *monotone* if for all $m, n \in \mathbb{N}$ and all $f, g : \mathbb{N} \rightarrow \mathbb{N}$,

$$(m \leq n \wedge g \leq^* f) \rightarrow (\varphi(m, g) \leq \varphi(n, f)).$$

Remark 2.3. For $f : \mathbb{N} \rightarrow \mathbb{N}$, the notion of being monotone corresponds to saying that f is a nondecreasing function, i.e. $\forall n \in \mathbb{N} (f(n) \leq f(n+1))$.

In what follows we restrict our metastability results to monotone functions in $\mathbb{N}^\mathbb{N}$. However, there is no real restriction in doing so, as for $f : \mathbb{N} \rightarrow \mathbb{N}$, one has $f \leq^* f^{\text{maj}}$, where f^{maj} is the monotone function defined by $f^{\text{maj}}(n) := \max\{f(i) : i \leq n\}$. In this way, we avoid constantly having to switch from f to f^{maj} , and simplify the notation.

Notation 2.4. Consider a function φ on tuples of variables \bar{x}, \bar{y} . If we wish to consider the variables \bar{x} as parameters we write $\varphi[\bar{x}](\bar{y})$. For simplicity of notation we may then even omit the parameters and simply write $\varphi(\bar{y})$.

Lemma 2.5 ([12]). *Let (s_n) be a bounded sequence of non-negative real numbers, with $d \in \mathbb{N} \setminus \{0\}$ an upper bound for (s_n) , such that for any $n \in \mathbb{N}$*

$$s_{n+1} \leq (1 - \alpha_n)s_n + \alpha_n r_n + \gamma_n,$$

¹The standard definition asks for $\alpha \in (0, 1)$. With this extension, 1-averaged is just another way of saying nonexpansive.

where $(\alpha_n) \subset [0, 1]$, (r_n) and $(\gamma_n) \subset [0, +\infty)$ are given sequences of real numbers.

Assume that there exist functions $A, R, G : \mathbb{N} \rightarrow \mathbb{N}$ such that, for all $k \in \mathbb{N}$

$$(i) \quad \sum_{i=1}^{A(k)} \alpha_i \geq k; \quad (ii) \quad \forall n \geq R(k) \left(r_n \leq \frac{1}{k+1} \right);$$

$$(iii) \quad \forall n \in \mathbb{N} \left(\sum_{i=G(k)+1}^{G(k)+n} \gamma_i \leq \frac{1}{k+1} \right).$$

Then

$$\forall k \in \mathbb{N} \forall n \geq \theta(k) (s_n \leq \frac{1}{k+1}),$$

with $\theta(k) := \theta[A, R, G, d](k) := A(M - 1 + \lceil \ln(3d(k+1)) \rceil) + 1$, where $M := \max\{R(3k+2), G(3k+2) + 1\}$.

The next result is an easy adaptation of [15, Lemma 14] for the case where $(\gamma_n) \equiv 0$.

Lemma 2.6. Let (s_n) be a bounded sequence of non-negative real numbers and $d \in \mathbb{N}$ a positive upper bound on (s_n) . Consider sequences of real numbers $(\alpha_n) \subset (0, 1)$, (r_n) and (v_n) and assume the existence of a monotone function A such that $\sum_{i=1}^{A(k)} \alpha_i \geq k$, for all $k \in \mathbb{N}$. For natural numbers k, n and q assume that for all $i \in [n, q]$

$$v_i \leq \frac{1}{3(k+1)(q+1)} \text{ and } r_i \leq \frac{1}{3(k+1)}.$$

Assume also that $s_{i+1} \leq (1 - \alpha_i)(s_i + v_i) + \alpha_i r_i$, for all $i \in \mathbb{N}$. Then

$$\forall i \in [\sigma(k, n), q] (s_i \leq \frac{1}{k+1}),$$

with $\sigma(k, n) := \sigma[A, d](k, n) := A(n + \lceil \ln(3d(k+1)) \rceil) + 1$.

Let $T : H \rightarrow H$ be a nonexpansive mapping. The next result corresponds to a quantitative version of a weaker form of the projection argument of $0 \in H$ onto $\text{Fix } T$. Below $B_N(p) := \{x \in H : \|x - p\| \leq N\}$, where $p \in H$ and $N \in \mathbb{N}$.

Lemma 2.7 ([15]). Let $T : H \rightarrow H$ be a nonexpansive mapping and $N \in \mathbb{N} \setminus \{0\}$ be such that $N \geq 2\|p\|$ for some point $p \in \text{Fix } T$. For any $k \in \mathbb{N}$ and monotone function $f : \mathbb{N} \rightarrow \mathbb{N}$, there are $n \leq 24N(\check{f}(r)(0) + 1)^2$ and $x \in B_N(p)$ such that $\|T(x) - x\| \leq \frac{1}{f(n)+1}$ and

$$\forall y \in B_N(p) (\|T(y) - y\| \leq \frac{1}{n+1} \rightarrow \langle x, x - y \rangle \leq \frac{1}{k+1}),$$

with $r := 4N^4(k+1)^2$ and $\check{f}(m) := \max\{f(24N(m+1)^2), 24N(m+1)^2\}$.

3 Quantitative results

In this section we give a quantitative analysis of Theorem 2.1 as well as some of its corollaries. We start by stating the relevant quantitative conditions.

Given positive real sequences $(\beta_n), (\lambda_n)$, a constant $\ell \in \mathbb{N} \setminus \{0\}$ and functions $b, D, B, L : \mathbb{N} \rightarrow \mathbb{N}$ and $h : \mathbb{N} \rightarrow \mathbb{N} \setminus \{0\}$ we consider the following conditions:

$$(Q_1) \quad \forall n \in \mathbb{N} (\beta_n \geq \frac{1}{h(n)})$$

$$(Q_2) \quad \forall k \in \mathbb{N} \forall n \geq b(k) (|1 - \beta_n| \leq \frac{1}{k+1}),$$

$$(Q_3) \quad \forall k \in \mathbb{N} \left(\sum_{i=1}^{D(k)} (1 - \beta_i) \geq k \right),$$

$$(Q_4) \quad \forall k \in \mathbb{N} \forall n \in \mathbb{N} \left(\sum_{i=B(k)+1}^{B(k)+n} |\beta_i - \beta_{i-1}| \leq \frac{1}{k+1} \right),$$

$$(Q_5) \quad \forall n \in \mathbb{N} (\lambda_n \geq \frac{1}{\ell}),$$

$$(Q_6) \quad \forall k \in \mathbb{N} \forall n \in \mathbb{N} \left(\sum_{i=L(k)+1}^{L(k)+n} |\lambda_i - \lambda_{i-1}| \leq \frac{1}{k+1} \right).$$

The first condition states that the function h witnesses the fact that $\beta_n > 0$. The conditions $(Q_2) - (Q_6)$ correspond to quantitative strengthenings of the conditions $(i) - (v)$ in Theorem 2.1. Indeed, b is a rate of convergence for $\beta_n \rightarrow 1$; the function D is a rate of divergence for $(\sum(1 - \beta_i))$; the functions B and L are rates for the Cauchy property of the convergent series $\sum |\beta_i - \beta_{i-1}|$ and $\sum |\lambda_i - \lambda_{i-1}|$, respectively, and the number ℓ is used to express the fact that (λ_n) is bounded away from 0.

Example 3.1. Consider the sequences defined by $\lambda_n = \beta_n = 1 - \frac{1}{n+2}$, for all $n \in \mathbb{N}$. Then the conditions $(Q_1) - (Q_6)$ are satisfied with $h(n) := 2$, $b = B = L = \text{Id}$, $D(k) := \lceil e^{k+2} \rceil$ and $\ell = 2$.

In the following lemmas (x_n) always denotes a sequence generated by (T-KM).

Lemma 3.2. Let $T : H \rightarrow H$ be a nonexpansive mapping and $N \in \mathbb{N} \setminus \{0\}$ be such that $N \geq \max\{\|x_0 - p\|, \|p\|\}$, for some $p \in \text{Fix } T$. Then, $\|x_n - p\| \leq N$, $\|x_n\| \leq 2N$ and $\|T(x_n)\| \leq 2N$, for all $n \in \mathbb{N}$.

Proof. Let $p \in \text{Fix } T$ be such that $N \geq \max\{\|x_0 - p\|, \|p\|\}$. Since T is nonexpansive, we have

$$\begin{aligned} \|x_{n+1} - p\| &\leq (1 - \lambda_n) \|\beta_n x_n - p\| + \lambda_n \|T(\beta_n x_n) - T(p)\| \\ &\leq \|\beta_n x_n - p\| = \|\beta_n(x_n - p) + (\beta_n - 1)p\| \\ &\leq \beta_n \|x_n - p\| + (1 - \beta_n) \|p\|. \end{aligned}$$

By induction on n one easily shows that $\|x_n - p\| \leq N$. Since T is nonexpansive, $\|T(x_n) - p\| \leq N$ and the result follows from the fact that $N \geq \|p\|$. \square

We say that a sequence (x_n) is *asymptotically regular* if $\lim \|x_{n+1} - x_n\| = 0$. Given a mapping $T : H \rightarrow H$, we say that a sequence (x_n) is *asymptotically regular with respect to T* if $\lim \|T(x_n) - x_n\| = 0$. The next lemma gives a rate of asymptotic regularity and a rate of asymptotic regularity with respect to T for the sequence (x_n) .

Lemma 3.3. Let $N \in \mathbb{N} \setminus \{0\}$ be such that $N \geq \max\{\|x_0 - p\|, \|p\|\}$, for some $p \in \text{Fix}(T)$. Consider $\ell \in \mathbb{N} \setminus \{0\}$ and monotone functions $b, D, B, L : \mathbb{N} \rightarrow \mathbb{N}$ satisfying conditions $(Q_2) - (Q_6)$. Then

- (i) $\forall k \in \mathbb{N} \forall n \geq \nu_1(k) (\|x_{n+1} - x_n\| \leq \frac{1}{k+1})$,
- (ii) $\forall k \in \mathbb{N} \forall n \geq \nu_2(k) (\|T(x_n) - x_n\| \leq \frac{1}{k+1})$,

where

$$\nu_1(k) := \nu_1[N, D, B, L](k) := \theta[D, \mathbf{0}, G, 2N](k), \text{ and}$$

$$\nu_2(k) := \nu_2[N, \ell, b, D, B, L](k) := \max\{b(4N\ell(k+1)-1), \nu_1(2\ell(k+1)-1)\},$$

with

$$G(k) := G[N, B, L](k) := \max\{B(4N(k+1)-1), L(10N(k+1)-1)\},$$

θ as in Lemma 2.5,

$\mathbf{0}$ the zero function.

Proof. For all $n \in \mathbb{N}$ we have

$$\begin{aligned} \|T(\beta_n x_n)\| &\leq \|T(\beta_n x_n) - T(\beta_n p)\| + \|T(\beta_n p) - T(p)\| + \|T(p)\| \\ &\leq \|x_n - p\| + (1 - \beta_n) \|p\| + \|p\| \leq 3N. \end{aligned} \tag{1}$$

Using (1) we derive, for $n \geq 1$,

$$\begin{aligned} \|x_{n+1} - x_n\| &\leq \|(1 - \lambda_n)(\beta_n x_n - \beta_{n-1} x_{n-1}) + (\lambda_{n-1} - \lambda_n) \beta_{n-1} x_{n-1}\| \\ &\quad + \|\lambda_n(T(\beta_n x_n) - T(\beta_{n-1} x_{n-1})) + (\lambda_n - \lambda_{n-1}) T(\beta_{n-1} x_{n-1})\| \\ &\leq \|\beta_n x_n - \beta_{n-1} x_{n-1}\| + |\lambda_n - \lambda_{n-1}| (\|\beta_{n-1} x_{n-1}\| + \|T(\beta_{n-1} x_{n-1})\|) \\ &\leq \|\beta_n x_n - \beta_{n-1} x_{n-1}\| + 5N|\lambda_n - \lambda_{n-1}| \\ &= \|\beta_n(x_n - x_{n-1}) + (\beta_n - \beta_{n-1}) x_{n-1}\| + 5N|\lambda_n - \lambda_{n-1}| \\ &\leq \beta_n \|x_n - x_{n-1}\| + 2N|\beta_n - \beta_{n-1}| + 5N|\lambda_n - \lambda_{n-1}|. \end{aligned}$$

Observe that the function G is a Cauchy rate for $(\sum \gamma_n)$, where (γ_n) is given by $\gamma_n := 2N|\beta_n - \beta_{n-1}| + 5N|\lambda_n - \lambda_{n-1}|$. Indeed, for all n, k we have

$$\begin{aligned} \sum_{i=G(k)+1}^{G(k)+n} \gamma_i &= 2N \sum_{i=G(k)+1}^{G(k)+n} |\beta_i - \beta_{i-1}| + 5N \sum_{i=G(k)+1}^{G(k)+n} |\lambda_i - \lambda_{i-1}| \\ &\leq 2N \sum_{i=B(4N(k+1)-1)+1}^{G(k)+n} |\beta_i - \beta_{i-1}| + 5N \sum_{i=L(10N(k+1)-1)+1}^{G(k)+n} |\lambda_i - \lambda_{i-1}| \\ &\leq \frac{2N}{4N(k+1)} + \frac{5N}{10N(k+1)} = \frac{1}{k+1}. \end{aligned}$$

Applying Lemma 2.5 with $d = 2N$, and for all $n \geq 1$, $s_n = \|x_n - x_{n-1}\|$, $r_n = 0$, $\gamma_n = 2N|\beta_n - \beta_{n-1}| + 5N|\lambda_n - \lambda_{n-1}|$ and $\alpha_n = 1 - \beta_n$ we conclude (i). As for (ii), observe that for all $n \in \mathbb{N}$

$$\begin{aligned} \|x_n - T(x_n)\| &\leq \|x_{n+1} - x_n\| + \|(1 - \lambda_n)(\beta_n x_n - T(x_n)) + \lambda_n(T(\beta_n x_n) - T(x_n))\| \\ &\leq \|x_{n+1} - x_n\| + (1 - \lambda_n) \|\beta_n x_n - T(x_n)\| + \lambda_n \|\beta_n x_n - x_n\| \\ &\leq \|x_{n+1} - x_n\| + (1 - \lambda_n) \|\beta_n x_n - \beta_n T(x_n)\| \\ &\quad + (1 - \lambda_n) \|\beta_n T(x_n) - T(x_n)\| + \lambda_n(1 - \beta_n) \|x_n\| \\ &\leq \|x_{n+1} - x_n\| + (1 - \lambda_n) \|x_n - T(x_n)\| \\ &\quad + (1 - \lambda_n)(1 - \beta_n) \|T(x_n)\| + \lambda_n(1 - \beta_n) \|x_n\|. \end{aligned}$$

Hence, for $n \geq \nu_2(k)$

$$\begin{aligned} \|x_n - T(x_n)\| &\leq \frac{1}{\lambda_n} (\|x_{n+1} - x_n\| + (1 - \lambda_n)(1 - \beta_n) \|T(x_n)\| + \lambda_n(1 - \beta_n) \|x_n\|) \\ &\leq \frac{1}{\lambda_n} (\|x_{n+1} - x_n\| + 2N(1 - \lambda_n)(1 - \beta_n) + 2N\lambda_n(1 - \beta_n)) \\ &\leq \ell (\|x_{n+1} - x_n\| + 2N(1 - \beta_n)) \leq \frac{1}{k+1}, \end{aligned}$$

which shows (ii). \square

Example 3.4. Let (x_n) be generated by (T-KM) with (λ_n) and (β_n) as in Example 3.1. Consider the exponential functions defined, for all $k \in \mathbb{N}$, by

- (i) $\tilde{\nu}_1(k) := 6N(k+1)\lceil e^{30N(k+1)+2} \rceil + 1$
- (ii) $\tilde{\nu}_2(k) := 24N(k+1)\lceil e^{120N(k+1)+2} \rceil + 1$.

It follows from Example 3.1 and Lemma 3.3 that $\tilde{\nu}_1$ is a rate of asymptotic regularity and $\tilde{\nu}_2$ is a rate of asymptotic regularity with respect to T .

The next result provides quantitative information on the fact that, with $\tilde{x} = \text{proj}_{\text{Fix } T}(0)$, one has $\limsup \langle \tilde{x}, \tilde{x} - x_n \rangle \leq 0$. Notice that, unlike the original proof, the quantitative version below does not require sequential weak compactness – the elimination of this principle is justified in [6].

Lemma 3.5. Let $N \in \mathbb{N} \setminus \{0\}$ be such that $N \geq \max\{\|x_0 - p\|, \|p\|\}$, for some $p \in \text{Fix}(T)$. Consider $\ell \in \mathbb{N} \setminus \{0\}$ and monotone functions $b, D, B, L : \mathbb{N} \rightarrow \mathbb{N}$ satisfying conditions $(Q_2) - (Q_6)$. For any $k \in \mathbb{N}$ and monotone function $f : \mathbb{N} \rightarrow \mathbb{N}$, there are $n \leq \psi(k, f)$ and $x \in B_{2N}(p)$ such that

$$\|T(x) - x\| \leq \frac{1}{f(n)+1} \wedge \forall m \geq n (\langle x, x - x_m \rangle \leq \frac{1}{k+1}),$$

where

$$\psi(k, f) := \psi[N, \ell, b, D, B, L](k, f) := \nu_2(48N(\check{g}^{(R)}(0) + 1)^2),$$

with

$$\begin{aligned} R &:= 64N^4(k+1)^2, \\ g(m) &:= f(\nu_2(m)), \\ \check{g}(m) &:= \max\{g(48N(m+1)^2), 48N(m+1)^2\}, \\ \text{and } \nu_2 \text{ as in Lemma 3.3.} \end{aligned}$$

Proof. Given $k \in \mathbb{N}$ and monotone $f : \mathbb{N} \rightarrow \mathbb{N}$, applying Lemma 2.7 to k and the function g one obtains $n_0 \leq 48N(\check{g}^{(R)}(0) + 1)^2$ and $x \in B_{2N}(p)$ such that $\|T(x) - x\| \leq \frac{1}{g(n_0)+1}$ and for all $y \in B_{2N}(p)$

$$\|T(y) - y\| \leq \frac{1}{n_0 + 1} \rightarrow \langle x, x - y \rangle \leq \frac{1}{k + 1}. \quad (2)$$

Define $n := \nu_2(n_0)$. By monotonicity, $n \leq \psi(k, f)$ and by the definition of g we conclude that $\|T(x) - x\| \leq \frac{1}{f(n)+1}$. By Lemma 3.3 (ii) we have $\|T(x_m) - x_m\| \leq \frac{1}{n_0+1}$, for all $m \geq n$. The result follows from (2) since $(x_n) \subset B_{2N}(p)$. \square

We are now able to prove our main result.

Theorem 3.6. *Given $(\beta_n), (\lambda_n) \subset (0, 1]$, $x_0 \in H$ and a nonexpansive mapping T , let (x_n) be generated by (T-KM). Assume that there exist $\ell \in \mathbb{N} \setminus \{0\}$ and monotone functions $b, D, B, L : \mathbb{N} \rightarrow \mathbb{N}$ and $h : \mathbb{N} \rightarrow \mathbb{N} \setminus \{0\}$ such that the conditions $(Q_1) - (Q_6)$ hold. Assume that there exists $N \in \mathbb{N} \setminus \{0\}$ such that $N \geq \max\{\|x_0 - p\|, \|p\|\}$, for some $p \in \text{Fix } T$. Then for all $k \in \mathbb{N}$ and monotone function $f : \mathbb{N} \rightarrow \mathbb{N}$*

$$\exists n \leq \mu(k, f) \forall i, j \in [n, f(n)] (\|x_i - x_j\| \leq \frac{1}{k + 1}),$$

where

$$\mu(k, f) := \mu[N, \ell, b, h, D, B, L](k, f) := \sigma(\tilde{k}, \max\{\psi(12(\tilde{k} + 1) - 1, \tilde{f}), \tilde{n}\}),$$

with

$$\begin{aligned} \tilde{k} &:= 4(k + 1)^2 - 1, \\ \tilde{n} &:= b(54N^2(\tilde{k} + 1) - 1), \\ \bar{f}(m) &:= \bar{f}[k, N, b, f, D](m) := f(\sigma(\tilde{k}, \max\{m, \tilde{n}\})), \\ \tilde{f}(m) &:= \tilde{f}[k, N, f, h](m) := 3(10N + 1)(\tilde{k} + 1)(\bar{f}(m) + 1)h(\bar{f}(m)) - 1, \\ \psi &\text{ as in Lemma 3.5 and } \sigma := \sigma[D, 9N^2] \text{ as in Lemma 2.6.} \end{aligned}$$

Proof. Let $k \in \mathbb{N}$ and $f : \mathbb{N} \rightarrow \mathbb{N}$ monotone be given. By Lemma 3.5 there exist $n_0 \leq \psi(12(\tilde{k} + 1) - 1, \tilde{f})$ and $x \in B_{2N}(p)$ such that $\|T(x) - x\| \leq \frac{1}{\tilde{f}(n_0) + 1}$ and

$$\langle x, x - x_m \rangle \leq \frac{1}{12(\tilde{k} + 1)}, \text{ for all } m \geq n_0. \quad (3)$$

We have, for all $n \in \mathbb{N}$

$$\begin{aligned} \|x_{n+1} - x\| &\leq (1 - \lambda_n) \|\beta_n x_n - x\| + \lambda_n \|T(\beta_n x_n) - T(x)\| + \lambda_n \|T(x) - x\| \\ &\leq \|\beta_n x_n - x\| + \|T(x) - x\|. \end{aligned}$$

Let $w_n := \|T(x) - x\| (2 \|\beta_n x_n - x\| + \|T(x) - x\|)$. Using (Q_1) , for all $n \in \mathbb{N}$

$$\begin{aligned} \|x_{n+1} - x\|^2 &\leq (\|\beta_n x_n - x\| + \|T(x) - x\|)^2 \\ &= \|\beta_n x_n - x\|^2 + w_n \\ &= \|\beta_n(x_n - x) + (\beta_n - 1)x\|^2 + w_n \\ &= \beta_n^2 \|x_n - x\|^2 + 2\beta_n(1 - \beta_n) \langle x, x - x_n \rangle + (1 - \beta_n)^2 \|x\|^2 + w_n \\ &\leq \beta_n \|x_n - x\|^2 + (1 - \beta_n)(2\beta_n \langle x, x - x_n \rangle + (1 - \beta_n) \|x\|^2) + w_n \\ &\leq \beta_n (\|x_n - x\|^2 + \frac{w_n}{\beta_n}) + (1 - \beta_n)(2\beta_n \langle x, x - x_n \rangle + (1 - \beta_n) \|x\|^2) \\ &\leq \beta_n (\|x_n - x\|^2 + h(n)w_n) + (1 - \beta_n)(2\beta_n \langle x, x - x_n \rangle + (1 - \beta_n) \|x\|^2) \end{aligned}$$

We will apply Lemma 2.6 with $s_n := \|x_n - x\|^2$, $v_n := h(n)w_n$, $\alpha_n := 1 - \beta_n$, $r_n := 2\beta_n \langle x, x - x_n \rangle + (1 - \beta_n) \|x\|^2$, A is instantiated with the function D and $d := 9N^2$. Let us see that the conditions of the lemma are satisfied. By (Q_3) , it is clear that A satisfies the required condition. We have that

$$s_n = \|x_n - x\|^2 \leq (\|x_n - p\| + \|x - p\|)^2 \leq (N + 2N)^2 = d.$$

With $n := \max\{n_0, \tilde{n}\}$ by (3) and (Q_2) , for all $m \geq n$,

$$r_m = 2\beta_m \langle x, x - x_m \rangle + (1 - \beta_m) \|x\|^2 \leq \frac{2}{12(\tilde{k}+1)} + \frac{(\|x - p\| + \|p\|)^2}{54N^2(\tilde{k}+1)} \leq \frac{1}{3(\tilde{k}+1)}.$$

Observe that $2\|\beta_n x_n - x\| + \|T(x) - x\| \leq 2(\|x_n\| + \|x - p\| + \|p\|) + 1 \leq 10N + 1$. For $m \leq \bar{f}(n_0) = f(\sigma(\tilde{k}, n))$, using the fact that the function h is monotone we have

$$\begin{aligned} v_m &= h(m) \|T(x) - x\| (2\|\beta_m x_m - x\| + \|T(x) - x\|) \leq \frac{h(\bar{f}(n_0))(10N+1)}{\bar{f}(n_0)+1} \\ &= \frac{h(\bar{f}(n_0))(10N+1)}{3(10N+1)(\tilde{k}+1)(\bar{f}(n_0)+1)h(\bar{f}(n_0))} = \frac{1}{3(\tilde{k}+1)(f(\sigma(\tilde{k}, n))+1)} \end{aligned}$$

This shows that $v_m \leq \frac{1}{3(\tilde{k}+1)(f(\sigma(\tilde{k}, n))+1)}$ and $r_m \leq \frac{1}{3(\tilde{k}+1)}$, for all $m \in [n, f(\sigma(\tilde{k}, n))]$. Hence, by Lemma 2.6

$$\forall m \in [\sigma(\tilde{k}, n), f(\sigma(\tilde{k}, n))] \left(\|x_m - x\|^2 \leq \frac{1}{\tilde{k}+1} \right). \quad (4)$$

We conclude that for $i, j \in [\sigma(\tilde{k}, n), f(\sigma(\tilde{k}, n))]$

$$\|x_i - x_j\| \leq \|x_i - x\| + \|x_j - x\| \leq \frac{1}{2(k+1)} + \frac{1}{2(k+1)} = \frac{1}{k+1},$$

which entails the result since $\sigma(\tilde{k}, n) \leq \mu(k, f)$. \square

Remark 3.7. The proof of the metastability result above does not require the projection argument neither sequential weak compactness. Nevertheless, in the presence of the projection point $\text{proj}_{\text{Fix } T}(0)$, following the arguments culminating in (4) one can say that Theorem 3.6 corresponds to a quantitative version of Theorem 2.1.

For the following corollaries we assume that there exist $\ell \in \mathbb{N} \setminus \{0\}$ and monotone functions $b, D, B, L : \mathbb{N} \rightarrow \mathbb{N}$ and $h : \mathbb{N} \rightarrow \mathbb{N} \setminus \{0\}$ such that the conditions $(Q_1) - (Q_6)$ hold.

Corollary 3.8. Let $\alpha \in (0, 1]$ and $T : H \rightarrow H$ be α -averaged. Given $(\beta_n) \subset (0, 1]$, $(\lambda_n) \subset (0, \frac{1}{\alpha}]$ and $x_0 \in H$, consider (x_n) generated by (T-KM). Let $a \in \mathbb{N} \setminus \{0\}$ be such that $\alpha \geq \frac{1}{a}$. Assume that there exists $N \in \mathbb{N} \setminus \{0\}$ such that $N \geq \max\{\|x_0 - p\|, \|p\|\}$, for some $p \in \text{Fix } T$. Then for every $k \in \mathbb{N}$ and monotone function $f : \mathbb{N} \rightarrow \mathbb{N}$

$$\exists n \leq \mu_1(k, f) \forall i, j \in [n, f(n)] (\|x_i - x_j\| \leq \frac{1}{k+1}),$$

where $\mu_1(k, f) := \mu_1[a, N, \ell, b, h, D, B, L](k, f) := \mu[N, a\ell, b, h, D, B, L](k, f)$, with μ as in Theorem 3.6.

Proof. Since T is α -averaged, there exists T' nonexpansive and such that $T = (1 - \alpha)\text{Id} + \alpha T'$. We have that

$$\begin{aligned} x_{n+1} &= \beta_n x_n + \lambda_n ((1 - \alpha)\text{Id} + \alpha T')(\beta_n x_n) - \beta_n x_n \\ &= \beta_n x_n + \alpha \lambda_n (T'(\beta_n x_n) - \beta_n x_n). \end{aligned}$$

Hence (x_n) is generated by (T-KM), with starting point x_0 , using the sequences $(\beta_n), (\alpha \lambda_n) \subset (0, 1]$ and T' . Since $\text{Fix } T = \text{Fix } T'$, by Theorem 3.6, we just need to check that the conditions (Q_5) and (Q_6) still hold using the sequence $(\alpha \lambda_n)$ instead of the sequence (λ_n) . The condition (Q_5) holds with $\ell := a\ell$, since for all $n \in \mathbb{N}$ we have $\alpha \lambda_n \geq \frac{\alpha}{\ell} \geq \frac{1}{a\ell}$. Since $\alpha \in (0, 1]$, for all $k, n \in \mathbb{N}$ we have

$$\sum_{i=L(k)+1}^{L(k)+n} |\alpha \lambda_i - \alpha \lambda_{i-1}| \leq \sum_{i=L(k)+1}^{L(k)+n} |\lambda_i - \lambda_{i-1}| \leq \frac{1}{k+1}.$$

Hence (Q_6) is also satisfied for the sequence $(\alpha \lambda_n)$ with the function L . \square

Corollary 3.9. Let $T_1 : H \rightrightarrows H$ be maximal monotone and $T_2 : H \rightarrow H$ be δ -cocoercive, for some $\delta > 0$. Let $\gamma \in (0, 2\delta]$. Given $(\beta_n) \subset (0, 1]$, $(\lambda_n) \subset (0, \frac{4\delta-\gamma}{2\delta}]$ and $x_0 \in H$, consider (x_n) generated, for all $n \in \mathbb{N}$, by

$$x_{n+1} = (1 - \lambda_n) \beta_n x_n + \lambda_n J_{\gamma T_1}(\beta_n x_n - \gamma T_2(\beta_n x_n)) \quad (\text{T-FB})$$

Assume that there exists $N \in \mathbb{N} \setminus \{0\}$ such that $N \geq \max\{\|x_0 - p\|, \|p\|\}$, for some $p \in \text{zer}(T_1 + T_2)$. Then for all $k \in \mathbb{N}$ and monotone function $f : \mathbb{N} \rightarrow \mathbb{N}$

$$\exists n \leq \mu_2(k, f) \forall i, j \in [n, f(n)] (\|x_i - x_j\| \leq \frac{1}{k+1}),$$

where $\mu_2(k, f) := \mu_2[N, \ell, b, h, D, B, L](k, f) := \mu[N, 2\ell, b, h, D, B, L](k, f)$, with μ as in Theorem 3.6.

Proof. It is straightforward to see that the iteration (x_n) is generated by **(T-KM)** with $T := J_{\gamma T_1} \circ (\text{Id} - \gamma T_2)$. The resolvent function $J_{\gamma T_1}$ is firmly nonexpansive, i.e. $\frac{1}{2}$ -averaged. Furthermore, we have $\frac{\gamma}{2\delta} \in (0, 1]$ and, using [1, Proposition 4.33] it follows that $(\text{Id} - \gamma T_2)$ is $\frac{\gamma}{2\delta}$ -averaged. If $\gamma < 2\delta$, then using [14, Theorem 3(b)] it follows that T is $\frac{2\delta}{4\delta-\gamma}$ -averaged. If $\gamma = 2\delta$, then $\text{Id} - \gamma T_2$ is nonexpansive, so T is nonexpansive. In other words, T is $\frac{2\delta}{4\delta-\gamma}$ -averaged since $\frac{2\delta}{4\delta-\gamma} = 1$. Since $\text{Fix } T = \text{zer}(T_1 + T_2)$ [1, Proposition 25.1(iv)] we have $N \geq \max\{\|x_0 - p\|, \|p\|\}$, for some $p \in \text{Fix } T$. Noting that $\frac{2\delta}{4\delta-\gamma} \in [\frac{1}{2}, 1]$, we may apply Corollary 3.8 with $a = 2$ and the result follows. \square

Remark 3.10. If in Corollary 3.9 the resolvent function $J_{\gamma T_1}$ is replaced by an arbitrary firmly nonexpansive mapping T_1 , and p is some point in $\text{Fix}(T_1 \circ (\text{Id} - \gamma T_2))$, then the result holds also for $\gamma = 0$, by Corollary 3.8. In such case, the sequence (λ_n) is allowed to vary in the interval $(0, 2]$.

Corollary 3.11. Let $T_1, T_2 : H \rightrightarrows H$ be two maximal monotone operators and $\gamma > 0$. Given $(\beta_n) \subset (0, 1]$, $(\lambda_n) \subset (0, 2]$ and $x_0 \in H$, consider (x_n) generated, for all $n \in \mathbb{N}$, by

$$\begin{cases} y_n = J_{\gamma T_2}(\beta_n x_n) \\ z_n = J_{\gamma T_1}(2y_n - \beta_n x_n) \\ x_{n+1} = \beta_n x_n + \lambda_n(z_n - y_n) \end{cases} \quad (\text{T-DR})$$

Assume that there exists $N \in \mathbb{N} \setminus \{0\}$ such that $N \geq \max\{\|x_0 - p\|, \|p\|\}$, for some $p \in \text{Fix}(R_{\gamma T_1} \circ R_{\gamma T_2})$. Then, for all $k \in \mathbb{N}$ and monotone function $f : \mathbb{N} \rightarrow \mathbb{N}$

- (i) $\exists n \leq \mu_3(k, f) \forall i, j \in [n, f(n)] (\|x_i - x_j\| \leq \frac{1}{k+1})$,
- (ii) $\exists n \leq \mu_4(k, f) \forall i, j \in [n, f(n)] (\|y_i - y_j\| \leq \frac{1}{k+1})$,
- (iii) $\exists n \leq \mu_5(k, f) \forall i, j \in [n, f(n)] (\|z_i - z_j\| \leq \frac{1}{k+1})$,

where

$$\begin{aligned} \mu_3(k, f) &:= \mu_3[N, \ell, b, h, D, B, L](k, f) := \mu[N, 2\ell, b, h, D, B, L](k, f), \\ \mu_4(k, f) &:= \max\{\mu_3(2k+1, g_1), b(8N(k+1)-1)\}, \\ \mu_5(k, f) &:= \max\{\mu_4(3k+2, g_2), \nu_1(6\ell(k+1)-1), b(12\ell N(k+1)-1)\}, \end{aligned}$$

with

$$\begin{aligned} g_1(m) &:= f(\max\{m, b(8N(k+1)-1)\}), \\ g_2(m) &:= f(\max\{m, \nu_1(6\ell(k+1)-1), b(12\ell N(k+1)-1)\}), \\ \text{and } \mu &\text{ as in Theorem 3.6.} \end{aligned}$$

Proof. It is easy to see that the iteration (x_n) is generated by **(T-KM)** using the sequences (β_n) and $(\frac{\lambda_n}{2})$, and with $T := R_{\gamma T_1} \circ R_{\gamma T_2}$, which is nonexpansive. Hence, applying Theorem 3.6 we obtain (i).

By (i), there exists $n_0 \leq \mu_3(2k+1, g_1)$ such that $\|x_i - x_j\| \leq \frac{1}{2(k+1)}$, for all $i, j \in [n_0, g_1(n_0)]$. Define $n := \max\{n_0, b(8N(k+1)-1)\} (\leq \mu_4(k, f))$. Since $[n, f(n)] \subseteq [n_0, g_1(n_0)]$, we have for $i, j \in [n, f(n)]$

$$\begin{aligned} \|y_i - y_j\| &\leq \|\beta_i x_i - \beta_j x_j\| \leq \|x_i - x_j\| + |\beta_i - \beta_j| \|x_j\| \\ &\leq \frac{1}{2(k+1)} + \left(\frac{1}{8N(k+1)} + \frac{1}{8N(k+1)} \right) 2N = \frac{1}{k+1}, \end{aligned}$$

which shows (ii).

Define $n_1 := \max\{\nu_1(6\ell(k+1)-1), b(12\ell N(k+1)-1)\}$, where ν_1 is as in Lemma 3.3. By the definition of x_{n+1} , for $i \geq n_1$

$$\begin{aligned} \|z_i - y_i\| &\leq \ell \|x_{i+1} - \beta_i x_i\| \leq \ell(\|x_{i+1} - x_i\| + 2N(1 - \beta_i)) \\ &\leq \frac{\ell}{6\ell(k+1)} + \frac{2\ell N}{12\ell N(k+1)} = \frac{1}{3(k+1)} \end{aligned}$$

By (ii), there exists $n_2 \leq \mu_4(3k+2, g_2)$ such that $\|y_i - y_j\| \leq \frac{1}{3(k+1)}$, for all $i, j \in [n_2, g_2(n_2)]$. Define $n := \max\{n_1, n_2\} (\leq \mu_5(k, f))$. Since $[n, f(n)] \subseteq [n_2, g_2(n_2)]$, we have for $i, j \in [n, f(n)]$

$$\|z_i - z_j\| \leq \|z_i - y_i\| + \|y_i - y_j\| + \|y_j - z_j\| \leq \frac{1}{k+1},$$

which concludes the result. \square

Appendix

Figure 1: Constants used.

Constants	Definition
$M := \max\{R(3k + 2), G(3k + 2) + 1\}$	Lemma 2.5
$r := 4N^4(k + 1)^2$	Lemma 2.7
$R := 64N^4(k + 1)^2$	Lemma 3.5
$\tilde{k} := 4(k + 1)^2 - 1$	Theorem 3.6
$\tilde{n} := b(54N^2(\tilde{k} + 1) - 1)$	Theorem 3.6

Figure 2: Functions used.

Function	Parameters	Definition	
θ	$A, R, G, d; k$	$A(M - 1 + \lceil \ln(3d(k + 1)) \rceil) + 1$	Lemma 2.5
σ	$A, d; k, n$	$A(n + \lceil \ln(3d(k + 1)) \rceil) + 1$	Lemma 2.6
\check{f}	$N, f; m$	$\max\{f(24N(m + 1)^2), 24N(m + 1)^2\}$	Lemma 2.7
ν_1	$N, D, B, L; k$	$\theta[D, \mathbf{0}, G, 2N](k)$	Lemma 3.3
ν_2	$N, \ell, b, D, B, L; k$	$\max\{b(4N\ell(k + 1) - 1), \nu_1(2\ell(k + 1) - 1)\}$	Lemma 3.3
G	$N, B, L; k$	$\max\{B(4N(k + 1) - 1), L(10N(k + 1) - 1)\}$	Lemma 3.3
ψ	$N, \ell, b, D, B, L; k, f$	$\nu_2(48N(\check{g}^{(R)}(0) + 1)^2)$	Lemma 3.5
g	$N, \ell, b, D, B, L, f; m$	$f(\nu_2(m))$	Lemma 3.5
\check{g}	$N, \ell, b, D, B, L, f; m$	$\max\{g(48N(m + 1)^2), 48N(m + 1)^2\}$	Lemma 3.5
\bar{f}	$N, b, k, f, D; m$	$f(\sigma[D, 9N^2](\tilde{k}, \max\{m, \tilde{n}\}))$	Theorem 3.6
\tilde{f}	$N, b, k, f, h, D; m$	$3(10N + 1)(\tilde{k} + 1)(\bar{f}(m) + 1)h(\bar{f}(m)) - 1$	Theorem 3.6
μ	$N, \ell, b, h, D, B, L; k, f$	$\sigma[D, 9N^2](\tilde{k}, \max\{\psi(12(\tilde{k} + 1) - 1, f), \tilde{n}\})$	Theorem 3.6
μ_1	$a, N, \ell, b, h, D, B, L; k, f$	$\mu[N, a\ell, b, h, D, B, L](k, f)$	Corollary 3.8
μ_2	$N, \ell, b, h, D, B, L; k, f$	$\mu[N, 2\ell, b, h, D, B, L](k, f)$	Corollary 3.9
μ_3	$N, \ell, b, h, D, B, L; k, f$	$\mu[N, 2\ell, b, h, D, B, L](k, f)$	Corollary 3.11
μ_4	$N, \ell, b, h, D, B, L; k, f$	$\max\{\mu_3(2k + 1, g_1), b(8N(k + 1) - 1)\}$	Corollary 3.11
μ_5	$N, \ell, b, h, D, B, L; k, f$	$\max\{\mu_4(3k + 2, g_2), \nu_1(6\ell(k + 1) - 1), b(12\ell N(k + 1) - 1)\}$	Corollary 3.11
g_1	$N, b, k, f; m$	$f(\max\{m, b(8N(k + 1) - 1)\})$	Corollary 3.11
g_2	$N, \ell, b, k, f, D, B, L; m$	$f(\max\{m, \nu_1(6\ell(k + 1) - 1), b(12\ell N(k + 1) - 1)\})$	Corollary 3.11

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