PROPOSITIONAL LOGIC :-

Proposition : A declarative sentence that is either true or false, but not both.

Examples :- Dispur is the capital of Assam.
Mumbai is the capital of India
$10 + 1 = 11$

What time is it ?
Read this carefully. } Not Propositions.

$x + 1 = 2$
$x + y = 3$ } Can be turned into propositions after assigning values to variables.

Compound Proposition : New propositions formed from existing propositions using logical operators.

Propositional Variables : Variables that denote propositions.
Ex :- p : Romeo loves Juliet.

DEFINITION 1 :- Let P be a proposition. The negation of P, denoted $\sim p$, is the statement "It is not the case that p".

Q// p : "Michael's PC runs Linux"

⇓

$\sim p$: "It is not the case that ....."

or

"Michael's PC does not run Linux"

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Q// p : "Vandana's phone has at least 32 GB of memory."

$\sim p$: "Vandana's phone does not have at least 32 GB of memory"

$\boxed{\text{OR}}$

"Vandana's phone has less than 32 GB of memory."

DEFINITION 2 :- Let P & Q be propositions. The conjunction of P & Q, denoted $P \wedge Q$, is the proposition "p and q". The conjunction $p \wedge q$ is true when both p and q are true, false otherwise.

Q// p : "Ram's PC has more than 16 GB free space"
q : "The processor in Ram's PC runs faster than 1 GHz"

$p \wedge q$: "Ram's PC has more than 16 GB free space, and its processor runs faster than 1 GHz."

DEFINITION 3 :- Let p & q be propositions. The disjunction of p and q, denoted $p \vee q$, is the proposition "p or q". The disjunction $p \vee q$ is false when both p & q are false, true otherwise.

Q// Take p & q from last example.

$p \vee q$: "Ram's PC has more than 16 GB free space, or its processor runs faster than 1 GHz."

DEFINITION 4 : Let p & q be propositions. The exclusive or of p and q, denoted $p \oplus q$, is the proposition when exactly one of p and q is true, false otherwise.

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Note :- Disjunction is "inclusive or" — the usage of "or" as in the English language.

"Students who have taken calculus or CS can take this class." — students who have taken any one or both the subjects can take the class.

"exclusive or" — "Students who have taken calculus or CS, but not both, can take this class." — imagine the usage of "either/or" in English language.

SUMMARY :-

p	q	$\sim p$	$p \wedge q$	$p \vee q$	$p \oplus q$
F	F	T	F	F	F
F	T	T	F	T	T
T	F	F	F	T	T
T	T	F	T	T	F

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<u>Conditional Statements</u> :- Let p and q be propositions. The conditional statement, also called implication, $p \to q$ is the proposition "if p, then q". The conditional statement "$p \to q$" is true when p is true and q is false, and true otherwise.

In the conditional statement $p \to q$, p is called the hypothesis /antecedent/ premise, and q is called the conclusion /consequence.

"If I am elected, then I will lower taxes."

↓

If politician is not elected, then there is no question of him breaking his promise — his promise is true.

If he is elected and lowers the tax, then he has fulfilled his promise — his promise is true.

But if he is elected and doesn't lower the tax, then he has broken his promise — his promise is false.

From the above examples, we can build the intuition that in $p \to q$, when p is true and q is false, $p \to q$ is false, and is true in all other cases.

"cause/effect" relation in English → "If it rains, I will not go out."

In mathematical logic, "cause/effect" not necessary.

"If Juan has a phone, then $2+3=5$" (True, because conclusion is true - premise doesn't matter.)

"If Juan has a phone, then $2+3=6$" (Will be true if Juan has a phone, altough conclusion is false.)

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Other ways to express the conditional statement $p \rightarrow q$.

"if p, then q" "p implies q"

"if p, q" "p only if q"

"p is sufficient for q" "q whenever p"

"q if p" "q is necessary for p"

"q when p" "q follows from p"

"q unless $\sim p$"

"a necessary condition for p is q"

"a sufficient condition for q is p"

Q// p : "Maria learns discrete maths"

q : "Maria will find a good job"

$p \rightarrow q$: "If Maria learns discrete maths, then she will find a good job."

OR

"Maria will find a good job when she learns discrete maths".

OR

"For Maria to get a good job, it is sufficient for her to learn discrete maths."

OR

"Maria will find a good job unless she does not learn discrete maths."

For a given proposition $p \to q$,

$q \to p$ is its converse,

$\sim q \to \sim p$ is its contrapositive,

$\sim p \to \sim q$ is its inverse.

Two compound propositions are "equivalent" if they have the same truth value.

The conditional proposition and its contrapositive are equivalent, and the inverse and converse of a proposition are equivalent.

Q// "The home team wins whenever it is raining".

$\downarrow$

"q whenever p"

$\downarrow$

"if p, then q"

$\downarrow$

"If it is raining, then the home team wins."

Contrapositive : "If the home team does not win, then it is not raining."

Converse : "If the home team wins, then it is raining".

Inverse : "If ~~the~~ it is not raining, then the home team does not win".

DEFINITION :- Let p and q be propositions. The biconditional proposition $p \leftrightarrow q$, also called bi-implication, is the proposition "p if and only if q". The biconditional statement $p \leftrightarrow q$ is true when both p & q have the same truth value, false otherwise.

"p is necessary and sufficient for q"
"if p, then q, and conversely."
"p iff q"

$p \leftrightarrow q$ is true when both $p \to q$ and $q \to p$ are true. In other words, $p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$.

Q// p : "You can take the flight"
q : "You buy a ticket".
$p \leftrightarrow q$: "You can take the flight if and only if you buy a ticket."

Note :- Biconditions are implicit in English language. For example, "If you finish your meal, you can have dessert." actually means, "You can have dessert if and only if you finish your meal."
In mathematical logic, biconditions have to be explicity denoted for precision.

PRECEDENCE OF LOGICAL OPERATIONS :-

$\sim$	(NOT)	↑ HIGHEST
$\wedge$	(AND)	
$\vee$	(OR)	
$\rightarrow$	(COND)	
$\leftrightarrow$	(BICOND)	↓ LOWEST

Q// Translate to logical expression :

(i) "You can access the Internet from campus only if you are a CS student or you are not a freshman".

p : "You can access the Internet from campus"
q : "You are a CS student"
r : "You are a freshman"

$$\boxed{p \rightarrow (q \vee \sim r)}$$

(ii) "You cannot ride the roller coaster if you are under 4 feet tall unless you are older than 16 years old."

q : "You can ride the roller coaster"
r : "You are under 4 feet"
s : "You are older than 16 years"

$$\boxed{r \wedge \sim s \longrightarrow \sim q}$$

(iii) "The automated reply cannot be sent when the file system is full."

p : "The automated reply can be sent"
q : "The file system is full".

$$\boxed{q \rightarrow \sim p}$$

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(iv) "Passing the final exam is a necessary condition for passing the course."

p : "I pass the final exam"
q : "I pass the course"

$$\boxed{q \rightarrow p}$$

(v) "God says you should forgive, but he forgave neither Adam nor Eve, and he punishes their descendants to this very day"

F : "God says you should forgive"
A : "God forgave Adam"
E : "God forgave Eve"
P : "God punish their descendants to this very day"

$$\boxed{F \wedge \sim(A \vee E) \wedge P}$$

OR

$$\boxed{F \wedge \sim A \wedge \sim E \wedge P}$$

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Q// Determine whether these system specifications are consistent :

"The diagnostic message is stored in a buffer or it is retransmitted"

"The diagnostic message is not stored in the buffer."

"If the diagnostic message is stored in the buffer, then it is retransmitted."

p : "The diagnostic message is stored in the buffer."

q : "The diagnostic message is retransmitted"

The specifications can be rewritten as,

$$p \vee q, \; \sim p, \; p \rightarrow q.$$

These are all true when p is false & q is true.
Hence, the specifications are consistent.

Q// Do the previous system specifications remain consistent if the specification "The diagnostic message is not retransmitted" is added ?

The specifications are : $p \vee q, \sim p, p \rightarrow q, \sim q.$

No combination of truth value to p & q can make the four specifications true. Consequently, the four specifications are inconsistent.

DEFINITION : A compound proposition that is always true, no matter what the truth values of the propositions that occur in it, is called a tautology. A compound proposition that is always false is called a contradiction. A compound proposition that is neither a tautology nor a contradiction is called a contingency.

Example of tautology : $p \vee \sim p$
Example of contradiction : $p \wedge \sim p$

Q// Show that $(p \wedge q) \rightarrow (p \vee q)$ is a tautology.

DEFINITION : Compound propositions that have the same truth values in all possible cases are called logically equivalent. The compound propositions p and q are called logically equivalent if $p \leftrightarrow q$ is a tautology. The notation $p \equiv q$ denotes that p & q are logically equivalent.

Q// Show that $\sim(p \vee q)$ and $\sim p \wedge \sim q$ are logically equivalent.

Q// Show that $p \rightarrow q$ and $\sim p \vee q$ are logically equivalent.

Logical Equivalences :

$p \wedge T \equiv p$
$p \vee F \equiv p$ } Identity laws

$p \wedge F \equiv F$
$p \vee T \equiv T$ } Domination laws

$p \vee p \equiv p$
$p \wedge p \equiv p$ } Idempotent laws

$\sim(\sim p) \equiv p$ } Double negation law

$p \vee q \equiv q \vee p$
$p \wedge q \equiv q \wedge p$ } Commutative laws

$(p \vee q) \vee r \equiv p \vee (q \vee r)$
$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$ } Associative laws

$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$ } Distributive laws

$\sim(p \wedge q) \equiv \sim p \vee \sim q$
$\sim(p \vee q) \equiv \sim p \wedge \sim q$ } De Morgan's laws

$p \vee (p \wedge q) \equiv p$
$p \wedge (p \vee q) \equiv p$ } Absorption laws

$p \vee \sim p \equiv T$
$p \wedge \sim p \equiv F$ } Negation laws

Logical Equivalences (involving Conditional Statements) :-

$p \rightarrow q \equiv \sim p \vee q.$

$p \rightarrow q \equiv \sim q \rightarrow \sim p$

$p \vee q \equiv \sim p \rightarrow q$

$p \wedge q \equiv \sim (p \rightarrow \sim q)$

$\sim (p \rightarrow q) \equiv p \wedge \sim q$

$(p \rightarrow q) \wedge (p \rightarrow r) \equiv p \rightarrow (q \wedge r).$

$(p \rightarrow r) \wedge (q \rightarrow r) \equiv (p \vee q) \rightarrow r$

$(p \rightarrow q) \vee (p \rightarrow r) \equiv p \rightarrow (q \vee r)$

$(p \rightarrow r) \vee (q \rightarrow r) \equiv (p \wedge q) \rightarrow r$

Logical Equivalences (involving Biconditionals) :-

$p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$

$p \leftrightarrow q \equiv \sim p \leftrightarrow \sim q$

$p \leftrightarrow q \equiv (p \wedge q) \vee (\sim p \wedge \sim q).$

$\sim (p \leftrightarrow q) \equiv p \leftrightarrow \sim q.$

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Q// Show that $\sim(p \to q)$ and $p \wedge \sim q$ are logically equivalent.

$$\begin{aligned} \sim(p \to q) &\equiv \sim(\sim p \vee q) \\ &\equiv \sim(\sim p) \wedge \sim q \\ &\equiv p \wedge \sim q. \end{aligned}$$

Q// S.t. $\sim(p \vee (\sim p \wedge q))$ and $\sim p \wedge \sim q$ are logically equivalent.

$$\begin{aligned} \sim(p \vee (\sim p \wedge q)) &\equiv \sim p \wedge \sim(\sim p \wedge q) \\ &\equiv \sim p \wedge (\sim(\sim p) \vee \sim q) \\ &\equiv \sim p \wedge (p \vee \sim q) \\ &\equiv (\sim p \wedge p) \vee (\sim p \wedge \sim q) \\ &\equiv F \vee (\sim p \wedge \sim q) \\ &\equiv \sim p \wedge \sim q. \end{aligned}$$

Q// S.t. $(p \wedge q) \longrightarrow (p \vee q)$ is a tautology.

$$\begin{aligned} (p \wedge q) \to (p \vee q) &\equiv \sim(p \wedge q) \vee (p \vee q) \\ &\equiv (\sim p \vee \sim q) \vee (p \vee q) \\ &\equiv (\sim p \vee p) \vee (\sim q \vee q) \\ &\equiv T \vee T \\ &\equiv T \end{aligned}$$

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Propositional Satisfiability : A compound proposition is satisfiable if there is an assignment of truth values to its variables that makes it true. When no such assignments exists, i.e., when the compound proposition is false for all assignments of truth values to its variables, the compound proposition is unsatisfiable.

An assignment that makes a compound proposition true, i.e., satisfies the compound proposition, is called the solution to the particular satisfiability problem.

Q// Determine whether each of the compound propositions is satisfiable :

(i) $(p \vee \sim q) \wedge (q \vee \sim r) \wedge (r \vee \sim p)$

(ii) $(p \vee q \vee r) \wedge (\sim p \vee \sim q \vee \sim r)$

(iii) $(p \vee \sim q) \wedge (q \vee \sim r) \wedge (r \vee \sim p) \wedge (p \vee q \vee r) \wedge (\sim p \vee \sim q \vee \sim r)$

Sol^n :-

(i) All variables must have same truth value.

(ii) At least 2 variables must have complimenting truth value

(iii) Unsatisfiable.