NYU Tandon Bridge Summer 2023 - HW8

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Question 5

a

Proof
Expression $P(n) = n^3 + 2n$
We shall prove that P(n) is divisible by 3 for all positive integers
Using induction on

Base Case
For $n = 1, n^3 + 2n = 1 + 2 = 3$, which is divisible by 3

Inductive Assumption
We shall assume that the proposition holds true for n=k.
i.e. $k^3 + 2 = 3P$, where P is some integer.
We shall prove that the proposition is true for n=k+1
$P(k+1) = (k+1)^3 + 2 = k3 + 3k^2 + 3k + 1 + 2 = k^3 + 2n + 3k(k+1)$
We can replace $k^3 + 2$ using 3P from our inductive assumption. we get,
$P(k+1) = 3P + 3k(k+1) = 3Q$, where Q is an integer=$P + k(k+1)$■

b) Ans:
proof using induction on n

Base case For n=2, 2 can be written as product of a prime number 2.

Inductive assumption We shall assume that all integers $\geq$ 2 upto and including k can be expressed as product of prime numbers

ie $k = p1 \times p2 \times ... \times p_k$, where P1 to P_k are prime numbers.

We shall prove that K+1 can be expressed as product of prime numbers.
K+1 can either be a prime number or a composite number.
Case1: k+1 is prime. In this case, k+1 can be written a product of prime number k+1
Case2: k+1 is composite. In this case $k + 1 = a \times b$, where a and b are integers greater than 1(by definition of composite numbers).
Here a and b has to be less than or equal to k. Say a=k+1, then b has to be less than 1, which violates the above statement. This can be established for b as well. Hence a and b are both less than or equal to k.
Thus, using our inductive assumption both a and b can be written as product of prime numbers.
Since, k+1 is product of a and b, k+1 can also be written as product of prime numbers ■

Question 6

7.4.1
$\Sigma_{j=1}^{n} j^2 = n(n+1)(2n+1)/6$

a) P(3)
$\Sigma_{j=1}^{n} j^2 = 1 + 4 + 9 = 14$
$3(3+1)(6+1)/6 = 14$
Hence, P(3) is true.

b) P(k) is the assertion that
$\Sigma_{j=1}^{k} j^2 = k(k+1)(2k+1)/6$

c) P(k+1) is the assertion that
$\Sigma_{j=1}^{k+1} j^2 = (k+1)(k+2)(2k+3)/6$

d) In the base case,we must prove that P(1) is true
e) In the inductive step we must prove that $p(k) \rightarrow P(k+1)$
f) The inductive hypothesis is P(k) is true for any $k \geq 1$
g) **Proof** Using induction on n
Base case
$P(1) = 1^2 = 1 = 1 * (1+1) * 3/6 = 1$
Inductive Assumption
P(k) is true. i.e. $\Sigma_{j=1}^{k} j^2 = k(k+1)(2k+1)/6$
P(k+1) $= \Sigma_{j=1}^{k} j^2 + (k+1)^2$
Substituting for $\Sigma_{j=1}^{k} j^2$ from the assumption, we get,
$k(k+1)(2k+1)/6 + (k+1)^2$
$1/6(k+1)(2k^2 + k + 6k + 6)$
$1/6(k+1)(2k^2 + 7k + 6$
$1/6(k+1)(k+2)(2k+3)$■

b) 7.4.3 c) Prove that for $n \geq 1, \Sigma_{j=1}^{n}(1/j^2) \leq 2 - 1/n$

Proof Using induction on n
Base case $P(1) = 1/1 = 1 \leq 2 - 1 = 1$. Thus, P(1) is true.
Inductive Assumption
P(k) is true. i.e. $\Sigma_{j=1}^{k}(1/j^2) \leq 2 - 1/k$

$p(k+1) = \Sigma_{j=1}^{k+1}(1/j^2) = \Sigma_{j=1}^{k}(1/j^2) + 1/(k+1)^2$

Substituting from our inductive assumption, we get $\Sigma_{j=1}^{k+1}(1/j^2) \leq 2 - 1/k + 1/(k+1)^2)$
Using the fact $1/k(k+1) \geq 1/(k+1)^2$ we get, $\Sigma_{j=1}^{k+1}(1/j^2) \leq 2 - 1/k + 1/k(k+1) = 2 - 1/k(1 - 1/(k+1)) = 2 - 1/(k+1)$
Thus, $\Sigma_{j=1}^{k+1}(1/j^2) \leq 2 - 1/(k+1)$■

c) Prove that for any positive integer n, 4 evenly divides $3^{2n} - 1$

Proof Using induction on n
Base Case $P(1) = 3^2 - 1 = 8$. 8 is evenly divisible by 4. Hence P(1) is true.

Inductive assumption p(k) is true. i.e. $3^{2k} - 1 = 4m$, where m is an integer.
We shall prove that p(k+1) is true.
$p(k+1) = 3^{2k+2} - 1 = 3^{2k} * 9 - 1$. Substituting for 3^{2k} from the inductive assumption we get,

$p(k+1) = (4m+1) * 9 - 1 = 9 * 4m - 8 = 4 * n$, where n is an integer(9m - 2) .
Thus, P(k+1) is true. ■