下推自动机

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- 下推自动机接受的语言
  - 从终态方式到空栈方式
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- 下推自动机与文法的等价性

- 确定型下推自动机

定义

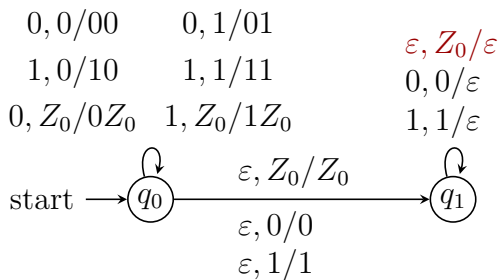PDA $P = (Q, \Sigma, \Gamma, \delta, q_0, Z_0, F)$, 以两种方式接受语言:

- P 以**终态方式**接受的语言, 记为 $\mathbf{L}(P)$, 定义为

$$\mathbf{L}(P) = \{w \mid (q_0, w, Z_0) \vdash^* (p, \varepsilon, \gamma), p \in F\}.$$

- P 以**空栈方式**接受的语言, 记为 $\mathbf{N}(P)$, 定义为

$$\mathbf{N}(P) = \{w \mid (q_0, w, Z_0) \vdash^* (p, \varepsilon, \varepsilon)\}.$$

续例2. 识别 L_{wwr} 的 PDA P, 从终态方式接受, 改为空栈方式接受.
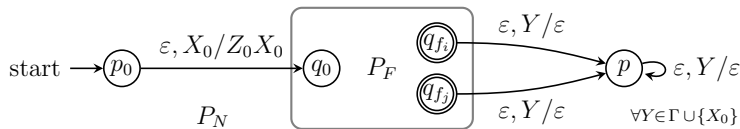用 $\delta(q_1, \varepsilon, Z_0) = \{(q_1, \varepsilon)\}$ 代替 $\delta(q_1, \varepsilon, Z_0) = \{(q_2, Z_0)\}$ 即可.

$$
\begin{array}{ll}
0, 0/00 & 0, 1/01 \\
1, 0/10 & 1, 1/11 \\
0, Z_0/0Z_0 & 1, Z_0/1Z_0
\end{array}
\qquad
\begin{array}{l}
\varepsilon, Z_0/\varepsilon \\
0, 0/\varepsilon \\
1, 1/\varepsilon
\end{array}
$$

$$
\text{start} \longrightarrow \underset{q_0}{\circlearrowright} \xrightarrow[\substack{\varepsilon, 0/0 \\ \varepsilon, 1/1}]{\varepsilon, Z_0/Z_0} \underset{q_1}{\circlearrowright}
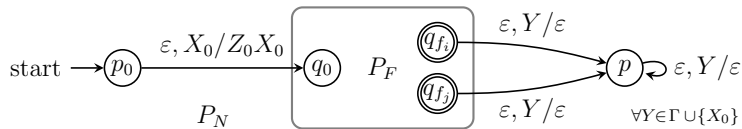$$

从终态方式到空栈方式

定理 25

如果 PDA P_F 以终态方式接受语言 L, 那么
一定存在 PDA P_N 以空栈方式接受 L.

证明: 设 $P_F = (Q, \Sigma, \Gamma, \delta_F, q_0, Z_0, F)$, 构造 PDA P_N,

$$P_N = (Q \cup \{p_0, p\},\ \Sigma,\ \Gamma \cup \{X_0\},\ \delta_N,\ p_0,\ X_0,\ \varnothing).$$

其中 δ_N 定义如下:

❶ P_N 首先将 P_F 的栈底符号压栈, 开始模拟 P_F:
$$\delta_N(p_0, \varepsilon, X_0) = \{(q_0, Z_0 X_0)\};$$

❷ P_N 模拟 P_F 的动作: $\forall q \in Q,\ \forall a \in \Sigma \cup \{\varepsilon\},\ \forall Y \in \Gamma$:
$$\delta_N(q, a, Y) \text{ 包含 } \delta_F(q, a, Y) \text{ 的全部元素};$$

❸ 从 $q_f \in F$ 开始弹出栈中符号, 即 $\forall q_f \in F, \forall Y \in \Gamma \cup \{X_0\}$:
$$\delta_N(q_f, \varepsilon, Y) \text{ 包含 } (p, \varepsilon);$$

❹ 在状态 p 时, 弹出全部栈中符号, 即 $\forall Y \in \Gamma \cup \{X_0\}$:
$$\delta_N(p, \varepsilon, Y) = \{(p, \varepsilon)\}.$$

对 $\forall w \in \Sigma^*$ 有

$$
\begin{aligned}
w \in \mathbf{L}(P_F) &\Rightarrow (q_0, w, Z_0) \vdash^*_{P_F} (q_f, \varepsilon, \gamma) \\
&\Rightarrow (q_0, w, Z_0 X_0) \vdash^*_{P_F} (q_f, \varepsilon, \gamma X_0) && \text{定理23} \\
&\Rightarrow (q_0, w, Z_0 X_0) \vdash^*_{P_N} (q_f, \varepsilon, \gamma X_0) && P_N 模拟 P_F \\
&\Rightarrow (p_0, w, X_0) \vdash_{P_N} (q_0, w, Z_0 X_0) \vdash^*_{P_N} (q_f, \varepsilon, \gamma X_0) && \delta_N 构造 p_0 \text{ 部分} \\
&\Rightarrow (p_0, w, X_0) \vdash^*_{P_N} (q_f, \varepsilon, \gamma X_0) \vdash^*_{P_N} (p, \varepsilon, \varepsilon) && \delta_N 构造 q_f 和 p \text{ 部分} \\
&\Rightarrow w \in \mathbf{N}(P_N)
\end{aligned}
$$

即 $\mathbf{L}(P_F) \subseteq \mathbf{N}(P_N)$.

对 $\forall w \in \Sigma^*$ 有

$$
\begin{aligned}
w \in \mathbf{N}(P_N) &\Rightarrow (p_0, w, X_0) \vdash_{P_N}^* (p, \varepsilon, \varepsilon) \\
&\Rightarrow (p_0, w, X_0) \vdash_{P_N} (q_0, w, Z_0 X_0) \vdash_{P_N}^* (p, \varepsilon, \varepsilon) \quad \text{第一个动作必然到 } q_0 \\
&\Rightarrow (q_0, w, Z_0 X_0) \vdash_{P_N}^* (q_f, \varepsilon, \gamma X_0) \vdash_{P_N}^* (p, \varepsilon, \varepsilon) \quad \text{必经 } q_f \in F \text{ 消耗完 } w \\
&\Rightarrow (q_0, w, Z_0) \vdash_{P_N}^* (q_f, \varepsilon, \gamma) \quad\quad\quad\quad\quad P_N \text{ 中未用过栈底的 } X_0 \\
&\Rightarrow (q_0, w, Z_0) \vdash_{P_F}^* (q_f, \varepsilon, \gamma) \quad\quad\quad\quad\quad \text{均为模拟 } P_F \\
&\Rightarrow w \in \mathbf{L}(P_F)
\end{aligned}
$$

即 $\mathbf{N}(P_N) \subseteq \mathbf{L}(P_F)$.
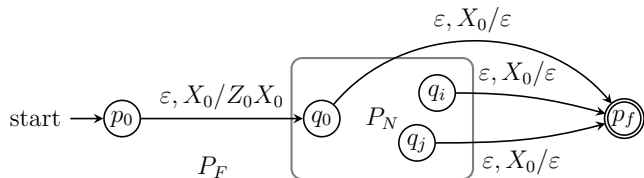
所以 $\mathbf{N}(P_N) = \mathbf{L}(P_F)$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \square$

从空栈方式到终态方式

定理 26

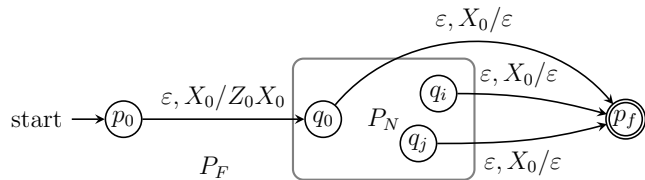如果 PDA P_N 以空栈方式接受语言 L, 那么
一定存在 PDA P_F 以终态方式接受 L.

证明: 设 $P_N = (Q, \Sigma, \Gamma, \delta_N, q_0, Z_0, \varnothing)$. 构造 PDA P_F,

$$P_F = (Q \cup \{p_0, p_f\},\ \Sigma,\ \Gamma \cup \{X_0\},\ \delta_F,\ p_0,\ X_0,\ \{p_f\})$$

其中 δ_F 定义如下:

❶ P_F 开始时, 将 P_N 栈底符号压入栈, 并开始模拟 P_N,
$$\delta_F(p_0, \varepsilon, X_0) = \{(q_0, Z_0 X_0)\};$$

❷ P_F 模拟 P_N, $\forall q \in Q$, $\forall a \in \Sigma \cup \{\varepsilon\}$, $\forall Y \in \Gamma$:
$$\delta_F(q, a, Y) = \delta_N(q, a, Y);$$

❸ 在 $\forall q \in Q$ 时, 看到 P_F 的栈底 X_0, 则转移到新终态 p_f:
$$\delta_F(q, \varepsilon, X_0) = \{(p_f, \varepsilon)\}.$$

对 $\forall w \in \Sigma^*$ 有

$$\begin{aligned}
w \in \mathbf{N}(P_N) &\Rightarrow (q_0, w, Z_0) \vdash_{P_N}^* (q, \varepsilon, \varepsilon) \\
&\Rightarrow (q_0, w, Z_0 X_0) \vdash_{P_N}^* (q, \varepsilon, X_0) & \text{定理23} \\
&\Rightarrow (q_0, w, Z_0 X_0) \vdash_{P_F}^* (q, \varepsilon, X_0) & P_F \text{ 模拟 } P_N \\
&\Rightarrow (p_0, w, X_0) \vdash_{P_F} (q_0, w, Z_0 X_0) \vdash_{P_F}^* (q, \varepsilon, X_0) & \delta_F \text{构造}, p_0 \text{ 部分} \\
&\Rightarrow (p_0, w, X_0) \vdash_{P_F}^* (q, \varepsilon, X_0) \vdash_{P_F} (p_f, \varepsilon, \varepsilon) & \delta_F \text{构造}, p_f \text{ 部分} \\
&\Rightarrow (p_0, w, X_0) \vdash_{P_F}^* (p_f, \varepsilon, \varepsilon) \\
&\Rightarrow w \in \mathbf{L}(P_F)
\end{aligned}$$

即 $\mathbf{N}(P_N) \subseteq \mathbf{L}(P_F)$.

对 $\forall w \in \Sigma^*$ 有

$$
\begin{aligned}
w \in \mathbf{L}(P_F) &\Rightarrow (p_0, w, X_0) \vdash^*_{P_F} (p_f, \varepsilon, \varepsilon) \\
&\Rightarrow (p_0, w, X_0) \vdash^*_{P_F} (q, \varepsilon, X_0) \vdash_{P_F} (p_f, \varepsilon, \varepsilon) && \text{经 } q \text{ 才可达 } p_f \\
&\Rightarrow (p_0, w, X_0) \vdash_{P_F} (q_0, w, Z_0 X_0) \vdash^*_{P_F} (q, \varepsilon, X_0) && P_F \text{ 第一个动作} \\
&\Rightarrow (q_0, w, Z_0 X_0) \vdash^*_{P_F} (q, \varepsilon, X_0) && \text{即上式} \\
&\Rightarrow (q_0, w, Z_0) \vdash^*_{P_N} (q, \varepsilon, \varepsilon) && P_N \text{ 与 } X_0 \text{ 无关} \\
&\Rightarrow w \in \mathbf{N}(P_N)
\end{aligned}
$$

即 $\mathbf{N}(P_F) \subseteq \mathbf{L}(P_N)$.
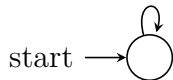
所以 $\mathbf{L}(P_F) = \mathbf{N}(P_N)$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

例 3. 接受 $L = \{w \in \{0,1\}^* \mid w$ 中字符 0 和 1 的数量相同$\}$ 的 PDA.

非确定

$$0, Z_0/0Z_0 \quad 1,0/10 \quad 0,0/00$$
$$1, Z_0/1Z_0 \quad 1,1/11 \quad 0,1/01$$
$$\varepsilon, Z_0/\varepsilon \quad 1,0/\varepsilon \quad 0,1/\varepsilon$$

start $\longrightarrow$ ◯

例 4. 接受 $L = \{0^n 1^m \mid 0 \le n \le m \le 2n\}$ 的 PDA.