Afternoon

Context-Free Grammars

- Formal definition
- Construction
- Parse tree
- Simplification

English Grammar

$\langle$sentence$\rangle \rightarrow \langle$noun_phrase$\rangle$ $\langle$predicate$\rangle$

$\langle$noun_phrase$\rangle \rightarrow \langle$article$\rangle$ $\langle$noun$\rangle$

$\langle$predicate$\rangle \rightarrow \langle$verb$\rangle$

$\langle$article$\rangle \rightarrow \langle$a$\rangle$ | $\langle$an$\rangle$ | $\langle$the$\rangle$

$\langle$noun$\rangle \rightarrow \langle$boy$\rangle$ | $\langle$dog$\rangle$

$\langle$verb$\rangle \rightarrow \langle$runs$\rangle$ | $\langle$walks$\rangle$

a boy runs   a dog walks

Palindrome Language

$$L=\{ w \mid w\in\{0,1\}^* \text{ and } w = w^R \}$$

- recursive definition
  - *basis* ε, 0, 1 are palindromes.
  - *induction* If w is a palindrome, so is $0w0$ and $1w1$.

Palindrome Language

$$L = \{ w \mid w \in \{0,1\}^* \text{ and } w = w^R \}$$

- definition with grammars or rules

1. ε is a palindrome.
2. 0 is a palindrome.
3. 1 is a palindrome.
4. If w is a palindrome, so is $0w0$.
5. If w is a palindrome, so is $1w1$.

Palindrome Language

$$L=\{ w \mid w \in \{0,1\}^* \text{ and } w = w^R \}$$

1. ε is a P.
2. 0 is a P.
3. 1 is a P.
4. If w is a P, so is $0w0$.
5. If w is a P, so is $1w1$.

1. $P \rightarrow \varepsilon$
2. $P \rightarrow 0$
3. $P \rightarrow 1$
4. $P \rightarrow 0P0$
5. $P \rightarrow 1P1$

Context-Free Grammar

A grammar $G=(V, T, S, P)$ is said to be context-free if all productions in P have the form

$$A \to \alpha \text{ , where } A \in V,\ \alpha \in (V \cup T)^*$$

CFG for Palindrome Language

$$L = \{ w \mid w \in \{0,1\}^* \text{ and } w = w^R \}$$

CFG for palindromes on {0,1}

$R = (\{S\}, \{0,1\}, S, P)$, P is defined as follow

$$S \to \varepsilon,\ S \to 0,\ S \to 1,\ S \to 0S0,\ S \to 1S1$$

Compact notation

$$S \to \varepsilon \mid 0 \mid 1 \mid 0S0 \mid 1S1$$

Example 1 CFG for

$L=\{ 0^n1^n \mid n \geq 0 \}$

$R = (\{S\}, \{0,1\}, P, S)$, P is defined as follow

$S \rightarrow \varepsilon \mid 0S1$

ε

0 1

0 0 1 1

0 0 0 1 1 1

0 0 0 0 1 1 1 1

Example 2 CFG for

$L = \{ 0^n 1^m \mid n \neq m \}$

$R = (\{S, A, B, C\}, \{0,1\}, P, S)$

$S \to AC \mid CB, \quad C \to 0C1 \mid \varepsilon$

$A \to A0 \mid 0, \quad B \to 1B \mid 1$

$$n \neq m \Rightarrow \begin{cases} n > m \Rightarrow n = (n - m) + m \\ n < m \Rightarrow m = n + (m - n) \end{cases}$$

Example 3 CFG for

$L = \{ w \in \{0,1\}^* \mid w$ contains same number of 0's and 1's $\}$

$R = (\{S\}, \{0,1\}, P, S)$, P is defined as follow

$S \rightarrow \varepsilon \mid 0S1 \mid 1S0 \mid SS$

Example 4 CFG for

$L = \{ w \in \{0,1\}^* \mid n_0(w) = n_1(w) \text{ and } n_0(v) \geq n_1(v)$

where v is any prefix of w $\}$

$R = (\{S\}, \{0,1\}, P, S)$, P is defined as follow

$S \rightarrow \varepsilon \mid 0S1 \mid SS$

Example 5 CFG for

$$L=\{a^{2n}b^m \mid n \geq 0, m \geq 0\}$$

$R = (\{S, A, B\}, \{a,b\}, P, S)$, P is defined as follow

$S \rightarrow AB$, $A \rightarrow \varepsilon \mid aaA$, $B \rightarrow \varepsilon \mid Bb$

Derivations

$L = \{a^{2n}b^m \mid n \geq 0, m \geq 0\}$

$S \to AB$, $A \to \varepsilon | aaA$, $B \to \varepsilon | Bb$

for $w = aabb$:

$$S \Rightarrow AB \Rightarrow aaAB \Rightarrow aaABb \Rightarrow aaBb \Rightarrow aaBbb \Rightarrow aabb$$

$S \to AB$

$A \to aaA$

$B \to Bb$

$A \to \varepsilon$

$B \to Bb$

$B \to \varepsilon$

Context-Free Language

Let G=(V, T, S, P) be context-free, then

$$L(G) = \{w \mid w \in T^* \text{ and } S \overset{*}{\Rightarrow} w\}$$

Left Most Derivations

$L=\{a^{2n}b^m \mid n \geq 0, m \geq 0\}$

$S \rightarrow AB$, $A \rightarrow \varepsilon | aaA$, $B \rightarrow \varepsilon | Bb$

for $w = aabb$:

$S \Rightarrow AB \Rightarrow aaAB \Rightarrow aaABb \Rightarrow aaBb \Rightarrow aaBbb \Rightarrow aabb$

Left most :

$S \Rightarrow AB \Rightarrow aaAB \Rightarrow aaB \Rightarrow aaBb \Rightarrow aaBbb \Rightarrow aabb$

Right most :

$S \Rightarrow AB \Rightarrow ABb \Rightarrow ABbb \Rightarrow Abb \Rightarrow aaAbb \Rightarrow aabb$

Parse Tree

Let $G = (V, T, S, P)$ be a CFG. A tree is a parse tree for G if :

1. Each interior node is labeled by a variable in V

2. Each leaf is labeled by a symbol in $T \cup \{\varepsilon\}$. Any ε-labeled leaf is the only child of its parent.

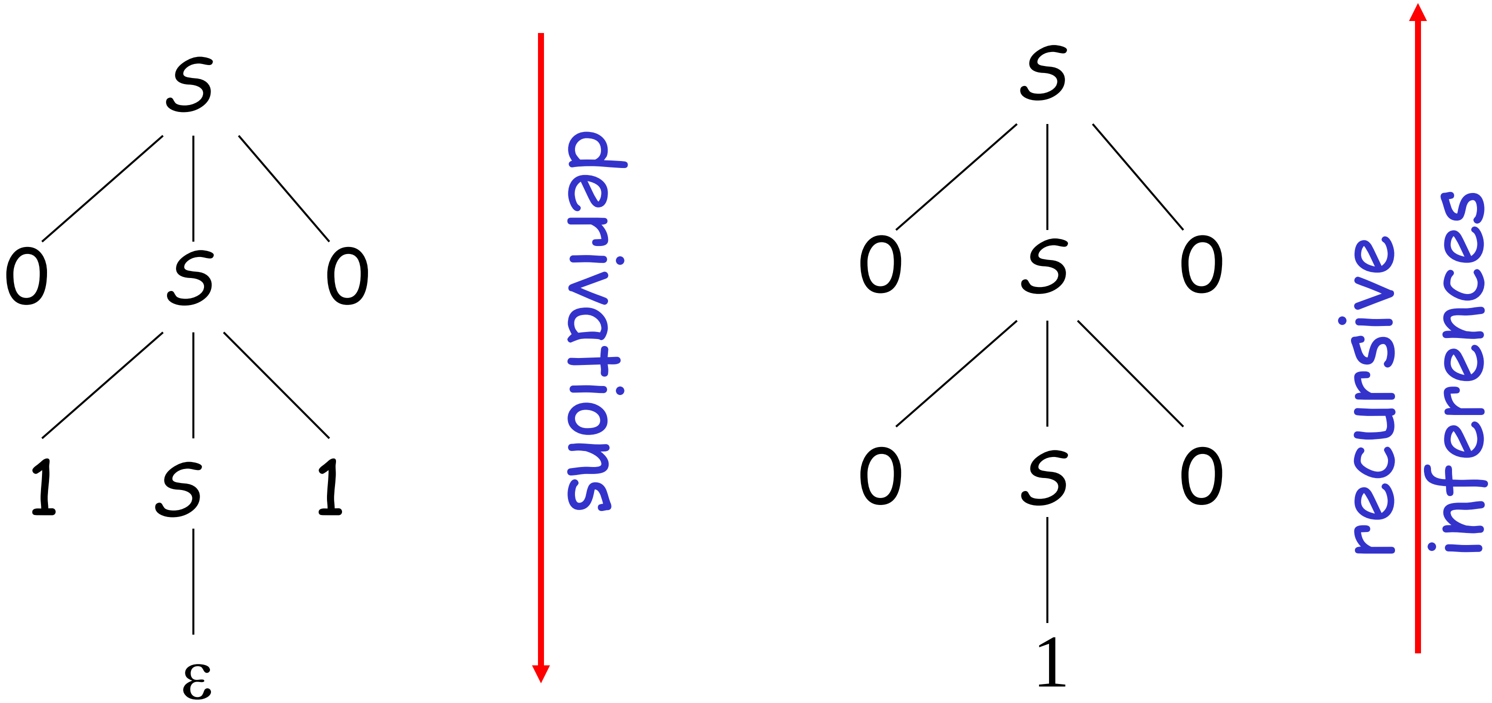
3. If an interior node is labeled A, and its children (from left to right) labeled $x_1, x_2, \ldots, x_k$,

Then $A \rightarrow x_1, x_2, \ldots, x_k \in P$.

Example 6

$L = \{\, w \mid w \in \{0,1\}^* \text{ and } w = w^R \,\}$

$S \rightarrow \varepsilon \mid 0 \mid 1 \mid 0S0 \mid 1S1$



w=0110 w=00100

Ambiguity

$G = (\{E, I\}, \{a, b, (,), +, *\}, E, P)$

$E \rightarrow I \mid E + E \mid E * E \mid (E), \quad I \rightarrow a \mid b$

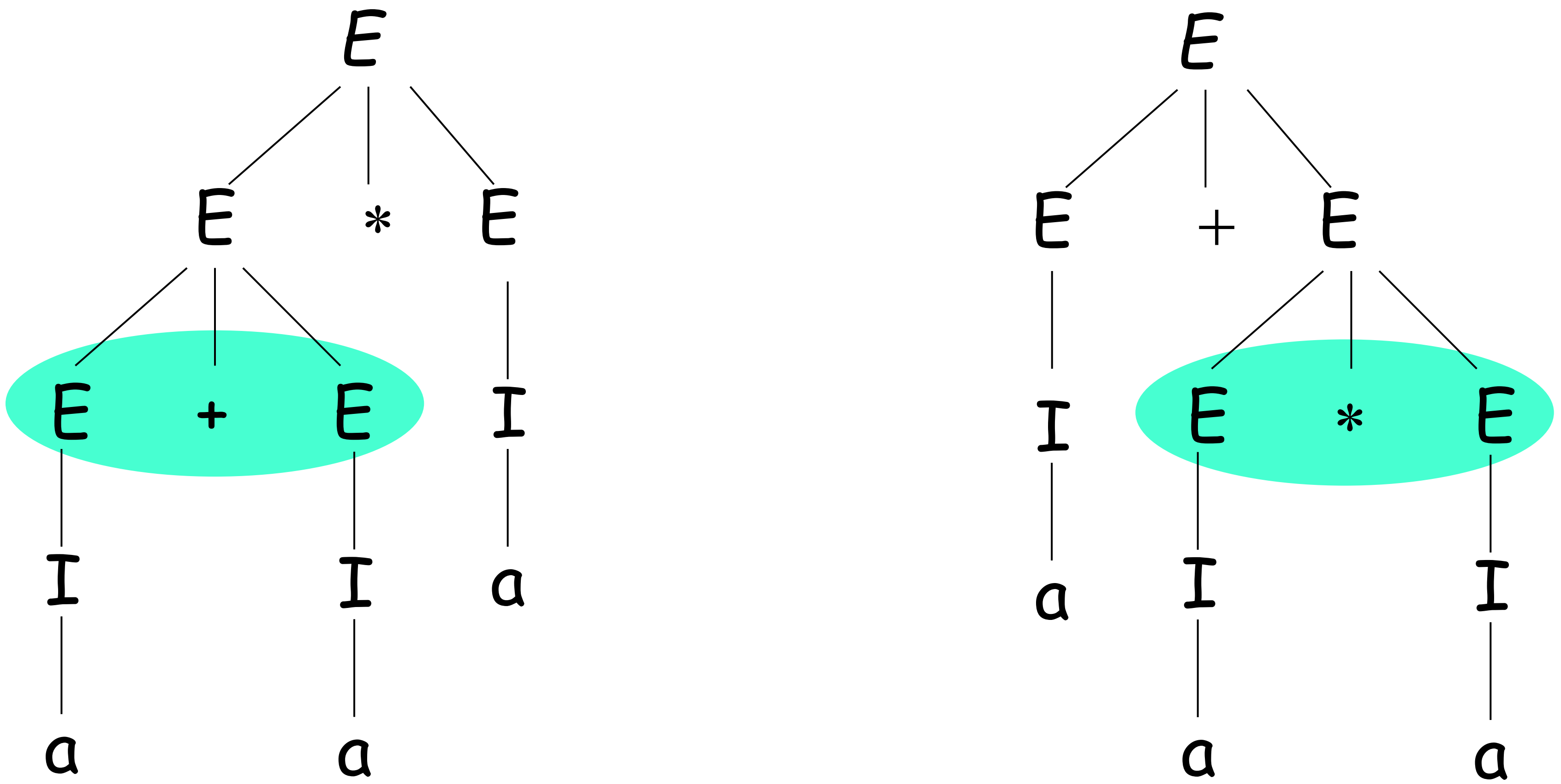
Derivation for $w = a + a * a$:

$$E \Rightarrow E*E \Rightarrow E+E*E \Rightarrow I+E*E \Rightarrow a+E*E \overset{*}{\Rightarrow} a+a*a$$

$$E \Rightarrow E+E \Rightarrow I+E \Rightarrow a+E \Rightarrow a+E*E \overset{*}{\Rightarrow} a+a*a$$

Ambiguity

parse-tree for $w = a + a * a$:



Removing Ambiguity

$E \to I \mid E+E \mid E*E \mid (E)$, $I \to a \mid b$

$E \to T|E+T$, $T \to F|T*F$, $F \to I|(E)$, $I \to a|b|Ia|Ib$

Left most derivation for $w = a + a * a$:

$$E \Rightarrow E+T \Rightarrow T+T \Rightarrow F+T \Rightarrow I+T \Rightarrow a+T \Rightarrow a+T*F$$

$$\Rightarrow a+F*F \Rightarrow a+I*F \Rightarrow a+a*F \Rightarrow a+a*I \Rightarrow a+a*a$$

$$E \Rightarrow T \Rightarrow T*T \Rightarrow (E)*T \Rightarrow (E+T)*T \overset{*}{\Rightarrow} (a+a)*a$$

Inherent Ambiguity

- What is inherent ambiguity

A CFL L is said to be *inherently ambiguous* if all grammars that generate it is ambiguous.

Example 7

Let $L=\{\, w \mid w \in \{0,1\}^* \text{ and } n_0(w)=n_1(w) \,\}$

L is not inherently ambiguous ,because there is an unambiguous CFG :

$S \to \varepsilon \mid 0S1 \mid 1S0 \mid 0S11S0 \mid 1S00S1$

$S \to \varepsilon \mid 0S1 \mid 1S0 \mid SS$

ambiguity

Example 8

$L=\{a^n b^n c^m d^m \mid n\geq 1, m\geq 1\} \cup \{a^n b^m c^m d^n \mid n\geq 1, m\geq 1\}$

The CFG for L is :

$S\rightarrow AB \mid C$, $A\rightarrow aAb \mid ab$, $B\rightarrow cBd \mid cd$

$C\rightarrow aCd \mid aDd$, $D\rightarrow bDc \mid bc$

Let w= $abcd$, there are two left most derivations

$S\Rightarrow AB \Rightarrow abB \Rightarrow abcd$

$S\Rightarrow C \Rightarrow aDd \Rightarrow abcd$

Simplification of CFG

Why & what :

$S \to A \mid B$, $A \to 1CA \mid 1DE \mid \varepsilon$, $B \to 1CB \mid 1DF$,

$C \to 1CC \mid 1DG \mid 0G$, $D \to 1CD \mid 1DH \mid 0H$,

$E \to 0A$, $F \to 0B$, $G \to \phi$, $H \to 1$

- ε-productions
- unit productions
- useless symbols and productions

ε-productions

Variable A is said to be nullable if $A \overset{*}{\Rightarrow} \varepsilon$.

Let G=(V,T,P,S) is a CFG

If $A \rightarrow \varepsilon \in P$, then A is nullable.

If $A \rightarrow A_1 A_2 \ldots A_k \in P$, and $A_i \rightarrow \varepsilon \in P$ for i=1, …,k

then A is nullable.

Example 9 ε-production

$G: S \to AB,\ A \to aAA \mid \varepsilon,\ B \to bBB \mid \varepsilon$

$$\left.\begin{array}{l} A \to \varepsilon \Rightarrow A \text{ is nullable.} \\ B \to \varepsilon \Rightarrow B \text{ is nullable.} \end{array}\right\} \quad S \to AB \Rightarrow S \text{ is nullable.}$$

Example 10 unit productions

$G: S \to A|B|0S1,\ A \to 0A|0,\ B \to 1B|1$

$S \to 0A|0|1B|1|0S1$

$A \to 0A|0$

$B \to 1B|1$

Useless productions

For a grammar $G=(V,T,P,S)$, a symbol X is

usefull, if there is a derivation for some $w \in T^*$

$$S \overset{*}{\Rightarrow} \alpha X \beta \overset{*}{\Rightarrow} w$$

generating, if $\alpha X \beta \overset{*}{\Rightarrow} w$ for some $w \in T^*$

reachable, if $S \overset{*}{\Rightarrow} \alpha X \beta$ for $\{\alpha, \beta\} \subseteq (V \cup T)^*$

Example 11 Useless productions

$G: S \to AB \mid a,\ A \to b.$

useless

non-genrating

$S \Rightarrow a$ or $S \Rightarrow AB \Rightarrow bB \Rightarrow ?$

non-reachable

$G: S \to a,\ A \to b.$

$G: S \to a$

Example 12 Simplify CFG

$S \to A \mid B,\ \ A \to 1CA \mid 1DE \mid \varepsilon,\ \ B \to 1CB \mid 1DF,$

$C \to 1CC \mid 1DG \mid 0G,\ D \to 1CD \mid 1DH \mid 0H,$

$E \to 0A,\ F \to 0B,\ G \to \phi,\ H \to 1$

- eliminating ε-productions : $A \to \varepsilon$

$S \to A \mid B,\ \ A \to 1CA \mid 1C \mid 1DE,\ \ B \to 1CB \mid 1DF,$

$C \to 1CC \mid 1DG \mid 0G,\ D \to 1CD \mid 1DH \mid 0H,$

$E \to 0A \mid 0,\ F \to 0B,\ G \to \phi,\ H \to 1$

Example 12 Simplify CFG

$S \to A \mid B$, $A \to 1CA \mid 1C \mid 1DE$, $B \to 1CB \mid 1DF$,

$C \to 1CC \mid 1DG \mid 0G$, $D \to 1CD \mid 1DH \mid 0H$,

$E \to 0A \mid 0$, $F \to 0B$, $G \to \phi$, $H \to 1$

- eliminating unit productions : $S \to A$, $S \to B$

  $S \to 1CA \mid 1C \mid 1DE \mid 1CB \mid 1DF$,

  $A \to 1CA \mid 1C \mid 1DE$, $B \to 1CB \mid 1DF$,

  $C \to 1CC \mid 1DG \mid 0G$, $D \to 1CD \mid 1DH \mid 0H$,

  $E \to 0A \mid 0$, $F \to 0B$, $G \to \phi$, $H \to 1$

Example 12 Simplify CFG

$S \to 1CA \mid 1C \mid 1DE \mid 1CB \mid 1DF$,

$A \to 1CA \mid 1C \mid 1DE$, $B \to 1CB \mid 1DF$,

$C \to 1CC \mid 1DG \mid 0G$, $D \to 1CD \mid 1DH \mid 0H$,

$E \to 0A \mid 0$, $F \to 0B$, $G \to \phi$, $H \to 1$

- eliminating useless productions

$S \to 1DE$, $A \to 1DE$, $D \to 1DH \mid 0H$, $E \to 0A \mid 0$, $H \to 1$

Chomsky Normal Form

All productions are one of following two forms :

1. $A \to BC$, $A, B, C \in V$

2. $A \to a$, $a \in T$

Example 13

Convert following CFG into CNF

$S \rightarrow ABa$, $A \rightarrow aab$, $B \rightarrow Ac$

Greibach Normal Form/GNF

All productions are shown as following form :

$$A \to ax \ , \ \text{where } a \in T \ , \ x \in V^*$$

Example 14

Convert following grammar to GNF

$$S \to AB\ ,\ \ A \to aA \mid bB \mid b\ ,\ \ B \to b$$

Example 15

Convert following grammar to GNF

$S \rightarrow 01S1 | 00$

Discussion

- eliminating ε-productions : $\varepsilon \in L$?

- Chomsky normal form

  $A \to a \mid BC$ *advantage ?*

- Greibach normal form

  $A \to a\alpha$ *advantage ?*

Good good study

day day up !

How To Be More Impressive

Unknown

Suppose we want to publish something that is as simple as

$$1 + 1 = 2 \tag{1}$$

This is not very impressive. If we want our article to be accepted by IEEE reviewers, we have to more abstract. So, we could complicate the left hand side of the expression by using

$$\ln(e) = 1 \quad \text{and} \quad \sin^2 x + \cos^2 x = 1$$

and the right hand side can be stated as

$$2 = \sum_{n=0}^{\infty} \frac{1}{2^n}.$$

Therefore, Equation (1) can be expressed more scientifically as:

$$\ln(e) + (\sin^2 x + \cos^2 x) = \sum_{n=0}^{\infty} \frac{1}{2^n} \tag{2}$$

which is far more impressive. However, we should not stop here. The expression can be further complicated by using

$$e = \lim_{z \to \infty} \left(1 + \frac{1}{z}\right)^z \quad \text{and} \quad 1 = \cosh(y)\sqrt{1 - \tanh^2 y}.$$

Equation (2) may therefore be written as

$$\ln\left[\lim_{z \to \infty} \left(1 + \frac{1}{z}\right)^z\right] + (\sin^2 x + \cos^2 x) = \sum_{n=0}^{\infty} \frac{\cosh(y)\sqrt{1 - \tanh^2 y}}{2^n} \tag{3}$$

Note: Other methods of a similar nature could also be used to enhance our prestige, once we grasp the underlying principles.