

Asymmetric Lévy flight in financial ratios

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Because financial crises are characterized by dangerous rare events that occur more frequently than those predicted by models with finite variances, we investigate the underlying stochastic process generating these events. In the 1960s Mandelbrot [Mandelbrot B (1963) *J Bus* 36:394–419] and Fama [Fama EF (1965) *J Bus* 38:34–105] proposed a symmetric Lévy probability distribution function (PDF) to describe the stochastic properties of commodity changes and price changes. We find that an asymmetric Lévy PDF, \mathcal{L} , characterized by infinite variance, models several multiple credit ratios used in financial accounting to quantify a firm's financial health, such as the Altman [Altman EI (1968) *J Financ* 23:589–609] Z score and the Zmijewski [Zmijewski ME (1984) *J Accounting Res* 22:59–82] score, and models changes of individual financial ratios, ΔX_i . We thus find that Lévy PDFs describe both the static and dynamics of credit ratings. We find that for the majority of ratios, ΔX_i scales with the Lévy parameter $\alpha \approx 1$, even though only a few of the individual ratios are characterized by a PDF with power-law tails $X_i^{-1-\alpha}$ with infinite variance. We also find that α exhibits a striking stability over time. A key element in estimating credit losses is the distribution of credit rating changes, the functional form of which is unknown for alphabetical ratings. For continuous credit ratings, the Altman Z score, we find that $P(\Delta Z)$ follows a Lévy PDF with power-law exponent $\alpha \approx 1$, consistent with changes of individual financial ratios. Estimating the conditional $P(\Delta Z|Z)$ versus Z, we demonstrate how this continuous credit rating approach and its dynamics can be used to evaluate credit risk.

complex systems | econophysics | rating migrations

Most tests and tools used in statistics assume that any errors in a financial model are Gaussian distributed, and it is a common practice in economics to use a Gaussian distribution to approximate empirical data. Mandelbrot (1) and Fama (2) were among the first to notice that the logarithm of cotton price fluctuations and common stock price fluctuations have fatter tails than those produced by a Gaussian distribution, and they proposed a stable Lévy distribution to model the stochastic properties of the fluctuations. Analyzing high-frequency data, Mantegna and Stanley (3) reported that the stable Lévy distribution accurately models only a broad central region of the probability distribution function (PDF) of stock price changes, whereas Gopikrishnan et al. reported that a power law with an exponent value beyond the Lévy regime is needed to describe the tails (4, 5).

The central limit theorem (CLT) implies that the mean of a sufficiently large number of independent random variables, each with finite variance, will approximately follow a normal distribution (6). A generalization of the CLT shows that the mean of a sufficiently large number of independent random variables, each with infinite variance, approximately follows a stable Lévy distribution $L_{\alpha,\gamma}(x) = (1/\pi) \int_0^\infty dq \exp(-\gamma q^\alpha) \cos(qx)$, where $\gamma > 0$ and $0 < \alpha < 2$ (6, 7). Infinite variances are related to power-law distributions, and the general rule when combining two or more power-law variables, $x^{1+\alpha}$, is that the one with the smallest power-law exponent (the fattest power law) dominates when $x \rightarrow \infty$, which holds even if some variables are Gaussian distributed (8, 9). Because in finance one commonly deals with credit ratios defined as multiple financial ratios, if only one ratio is found to be

power-law distributed, the credit ratio itself is also power-law distributed.

In contrast to the previous literature on financial ratios, we focus not on ratios, X_i , but on dynamics of credit ratios, represented by their changes, ΔX_i . For each of eight individual ratios X_i comprising the Altman Z score (10), the Zmijewski Z_m score (11), and also the Shumway Hazard model (12), we find asymmetric Lévy \mathcal{L} PDFs in changes of financial ratios, ΔX_i , which are related to credit rating changes and thus to credit risk. We find that \mathcal{L} models several multiple credit ratios such as the Altman Z score and the Zmijewski score. $P(Z)$ follows an \mathcal{L} PDF with scale parameter $\alpha = 1.06 \pm 0.02$ and skewness parameter $\beta = 0.70 \pm 0.02$. We depart from the usual discrete alphabet credit ratings, such as Moody's (13), and choose the Z score as a proxy for the continuous credit rating (14), where the ΔZ quantifies credit rating migrations. We find that $P(\Delta Z)$ follows a Lévy PDF with a power-law exponent $\alpha \approx 1$. We demonstrate how our previous findings can be used to model credit risk.

Methods and Data

In modeling changes of financial ratios and multiple credit scores, we choose the asymmetric Lévy \mathcal{L} because, e.g., multiple credit scores Z are characterized by heavy tails in $P(Z)$, and we fit them with a scale parameter α . We model asymmetry in the PDF tails using skewness parameter β , the location (mean) of multiple credit scores using shift parameter μ , and the spread using parameter σ . For both the symmetric Lévy $L_{\alpha,\gamma}$ and its generalization, \mathcal{L} , the PDF generally cannot be written analytically. \mathcal{L} is determined by its characteristic function $\varphi(t)$: $\mathcal{L} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \varphi(t) e^{-itx} dt$, where

$$\varphi(t; \mu, c, \alpha, \beta) = \exp\{it\mu - |\sigma t|^\alpha [1 - i\beta \operatorname{sgn}(t)\Phi]\}. \quad [1]$$

In Eq. 1, $\operatorname{sgn}(t)$ is the sign of t , $\Phi = \tan(\pi\alpha/2)$, and $\beta \in [-1, 1]$ (15). When $\beta = 0$ and $\alpha = 1$, the \mathcal{L} becomes the Cauchy distribution, the analytic form of which is well-known.

For power-law distributed variables with a cumulative distribution function (CDF), $P(s > x) \sim x^{-\zeta}$, a Zipf plot of size s versus rank R asymptotically ($R \gg 1$) follows a power law with exponent ζ (16),

$$\zeta = 1/\zeta'. \quad [2]$$

If CDF is a Lévy distribution, then $\zeta' = \alpha$.

We analyze financial data for each quarter during the period 2000–2009 of 488 publicly traded manufacturing firms. The data are available at <http://www.wikinvest.com>. Our body of data includes (i) working capital to total assets (X_1), (ii) retained earnings divided by total assets (X_2), (iii) earnings before taxes and interest divided by total assets (X_3), (iv) market value of equity divided by book value of total liabilities (X_4), (v) sales divided by total assets (X_5), (vi) net income divided by total assets (X_6), (vii) total liabilities divided by total assets (X_7), and (viii) current

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otic limits for $r + s$ versus ΔZ , we fit this dependence to a hyperbolic tangent,

$$a \tanh(bx + c) + d. \quad [7]$$

We set the lower asymptotic limit ($r + s = 1$ when $\Delta Z \ll -1$) to calculate a recovery rate (20, 27) of approximately 50% after bankruptcy is declared ($r + s$ is calculated when $B_{if} \approx 0.5 \cdot 105$ in Eq. 6).

We next apply the previous approach to assess 1% risk as a specified percentile level for the portfolio value distribution (28). The lowest value that the portfolio will achieve 1% of the time is the first percentile. We then perform Monte Carlo simulations. For each simulation we generate ΔZ from $P(\Delta Z)$, and based on ΔZ we calculate B_{if} in Eq. 6 by using [7]. In Fig. 6B we show the PDF of loan values due to the increase and decrease of Z values. The PDF has a rapidly decreasing upside tail and a long downside tail, as found in empirical data on loan values with a *Baa* initial rating (20). Having this PDF one may estimate the 1% risk by calculating the B_1 value below, which there are 1% of all B values.

In our approach, stochasticity exists in credit rating migrations, and interest rate and credit rating are deterministically related (7). Our approach contradicts, e.g., the Black–Derman–Toy model (29), where the interest rate is stochastically evolved and follows a lognormal process. Now we demonstrate how we calculate the price of a bond maturing at time T , when applying the same approach to bond options (29). We subdivide a period between 0 and T on, e.g., n steps, each representing one quarter. If the option expiration date T is 3 y, then $n = 12$. In the first step, having information about the initial ranking i we apply a CDF of migration $P(\Delta Z|i)$ of Fig. 5B to determine a new ranking f' , where $\Delta Z = f' - i$. The new ranking, f' , in the previous step is also the initial ranking i' for the next step. After 12 steps we are

able to calculate the final ranking f . By performing Monte Carlo simulations on the exercise date we obtain the final credit ranking, and also the final bond value, using formulas similar to Eq. 6 and [7].

Summary and Conclusion

Recently we have witnessed rapid growth in the study of power-law tail phenomena in economics and finance (1, 2, 4, 5, 9, 30–35). We model the power-law scaling properties of credit rating changes using a multivariate Simon model, which is an extension of the Simon model used in the theory of firm growth (36). We perform 100,000 Monte Carlo time steps, and for each existing company, calculate the Z score. We set the time step to be 1 h, define a working day to be eight working hours, and a working year to be ≈ 250 working days. Hence 100,000 steps represent ≈ 50 y. We calculate the Z score after 90,000 time steps and after 92,000 steps, a timespan of ≈ 1 y. Then we calculate $P(\Delta Z)$ over the year. For $\sigma = 0.006$ in Fig. 7A, the tail is well fit by a power law with exponent ≈ 2 , as is found in the data. In Fig. 7B, using numerical simulations, we calculate that the choice for σ in the Gaussian distribution determines the spread of $P(\Delta Z)$. It is the rich get richer formalism that naturally leads to fat power-law tails in the distribution of rating changes. Geometric Brownian motion is needed to assure the spread in $P(\Delta Z)$.

Lévy PDFs were first proposed in finance to describe the commodity and price changes. We find asymmetric Lévy PDFs, \mathcal{L} , in multiple credit ratios and changes of individual financial ratios, ΔX_i , related to credit rating changes and hence credit risk. Although power-law exponents in ratios X_i are highly diversified, surprisingly, ΔX_i are all fit by the power laws of a Lévy stable regime. Existence of the Lévy PDFs in financial ratios has an important implication: It calls for the development of a statistical approach based on infinite variances (37).

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