Spread Option Pricing in a Copula Affine GARCH(p,q) Model

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Abstract

In this study, we construct a bivariate market model combining the copula function with the affine GARCH(p,q) process used to describe the marginal dynamics of the log price. We then provide a numerical procedure for pricing European spread option contracts. To assess the accuracy of our approach we present a comparison with the Monte Carlo simulation method.

keywords: Copula Function, Affine GARCH(p,q) process, Spread Options

1 Introduction

A spread option is a contract written on the price difference between two assets, referred to as *spread* and defined as $S_{1,t} - S_{2,t}$ for $t = 0, ..., T$. Accordingly, the payoff of a European call spread option can be written as $\max\{S_{1,T} - S_{2,T} - K; 0\}$. Although few spread options are traded on large exchanges, most of the transactions take place over-the-counter (OTC). Spread options are particularly suited for commodity markets, where their features make them remarkably appropriate. The OTC nature of the contract does not fit ordinary speculation purposes but rather allows to hedge certain risks embedded in the features of commodity markets. Notably, "processing" spreads allow the trader to reduce or gain exposure on a commodity processing routine. For instance, a "Crack Spread" option (an option whose underlying is the spread between crude oil and the petroleum product refined from it) allows a refiner to cover against unfavourable fluctuations in the prices of input or output.

The most common approach in the literature, as well as the prevailing in practice, is the Multivariate Geometric Brownian Motion (henceforth MGBM) approach. MGBM model is often used alongside Monte Carlo simulation and provides the framework for further developments. The approach relies on Pearson's ρ to collect full information about the processes' dependence structure. Many authors have provided their contribution to this approach by deriving a closed form solution to the pricing problem. Notably, [19] derives a formula to price options when the $K = 0$. [16] provides a further approximation that is currently the market standard for the contract. [5] compute an analytical approximation by reducing the price range to a bounded interval that is particularly tight for certain parameters values. Finally, [1] introduce a lower bound for the price that has been proven to be more accurate than the approximation in [16].

The vivacity of the market for spread options has recently drawn much attention. Several works have provided new theoretical insights on the topic of spread options. [4] and [27] present new methodologies for the computation of the price of spread options, while [21] provides an assessment of spread option price sensitivities through Malliavin calculus. Further enrichments to the spread option framework are introduced by [26] and [25]. The former proposes a joint model for day-ahead electricity prices in interconnected markets and observes that the model yields analytical prices. The latter develops a multi-factor model and shows how to compute the joint characteristic function of a pair of futures prices.

An alternative method to price spread options by means of Copula functions is provided in [6] and [12]. These works propose a Copula-based dependence structure and different processes for underlying assets. Previous literature exhibits two major limitations: (i) past pricing methods rely heavily on Gaussian framework, even though different processes may better reflect the underlying's behaviour, and (ii) most of the literature (although not all) imposes linear correlation as a measure of dependence.

In this study, we develop a framework for spread option pricing that relies on the application of affine GARCH(p,q) models. The class of GARCH models is introduced in [2] as a useful adaptation of the ARCH model presented in [11]. GARCH models assume that the volatility process at time t is influenced not only by previous error terms but also by its lagged values, favouring volatility clustering dynamics. Another advantage of GARCH modelling is that it does not require returns to be independent. Indeed, despite being considered Gaussian distributed conditional to past values, returns' unconditional distribution exhibits leptokurtosis.

From the seminal work in [10], several GARCH models for option pricing have been proposed in the financial literature. A major breakthrough occurred with the GARCH model with Gaussian innovations developed in [14] where the specification of the variance process yields a recursive procedure for the computation of the characteristic function of the log price. Therefore the price of a European option can be obtained by means of the Fourier Transform as in [13]. To distinguish this model from to the original GARCH, in literature the model in [14] is known as "Affine" GARCH model with Gaussian innovation. Affine Garch model with non-guassian innovations are the model proposed in [7] with Inverse Gaussian innovations and the model in [20] with Tempered Stable Innovations.

We embed the dependence structure of the processes within a Copula function. The Copula-based setup provides a flexible method of capturing the dependence between the marginals without loss of fit on other crucial features of financial data (heavier tails, non linear dependence and asymmetrical dependence). Considering an appropriate Copula family, we derive a formula for a spread option that involves the first partial derivatives of the Copula function. Numerical results of this formula are benchmarked against an adequate Monte Carlo simulation to assess the accuracy and the performance of the introduced method.

This paper is organized as follows. Sections 2 and 3 review the main results about the affine GARCH model with normal innovations and the main properties of the copula functions respectively. In Section 4 we propose a numerical method for pricing the spread options based on the evaluation of a single integral. Section 5 assesses the numerical precision of the formula proposed in Section 4 assuming that the dependence structure between assets is described by a Plackett copula and the univariate distribution of log-returns can be generated from two Heston-Nandi GARCH models. Section 6 concludes the paper.

2 Heston-Nandi GARCH model with Gaussian Innovations

In this section we review the affine GARCH model with normal innovations proposed in [14], henceforth referred to as HN-GARCH.

This specification retains the features of the GARCH introduced in [2] and overcomes the drawback related to symmetrical effects of shocks. In addition, it is able to capture simultaneously the stochastic nature of volatility and its negative correlation with spot returns, enabling quick adjustments in variance dictated by changes in market levels.

Continuous-time stochastic volatility models that have been previously introduced, see [8],[13], are difficult to implement and computationally intensive, whereas the HN-GARCH process is much easier to apply to available data, and volatility is readily observable from the history of asset prices. Besides, the GARCH process defined in [14] allows option pricing in a semi-analytical closed-form formula, leveraging to a certain extent on numerical integration. Lastly, it can be proven that, as the observation interval shrinks, the results are numerically close to the continuous-time stochastic volatility model of [13].

The model has two basic assumptions. The first pertains the specification of the GARCH that models the volatility of the spot asset log price as well as the process for the spot log price itself.

Assumption 1. *The spot asset price, S_t, follows the process below over time steps of length Δ:*

$$log(S_t) = log(S_{t-\Delta}) + r_f + \lambda h_t + \sqrt{h_t} z_t \tag{1a}$$

where:

$$h_t = \omega + \sum_{j=1}^{p} \beta_j h_{t-j\Delta} + \sum_{j=1}^{q} \alpha_j (z_{t-j\Delta} - \gamma\sqrt{h_{t-j\Delta}})^2 \tag{1b}$$

Where r_f is the continuously compounded risk-free interest rate over the time interval Δ and z_t is a standard normal noise. h_t is the conditional variance of the log returns between time $t - \Delta$ and time t and is known from the information set at time $t - \Delta$.

We can see in (1a) that the conditional variance h_t is included in the mean as a risk premium. Therefore the expected spot return is proportional to the variance by a parameter λ[1]. Successive analysis and implementation will focus on the $(p = 1, q = 1)$ specification of the HN-GARCH, deemed the most parsimonious yet effective in modeling the time series. The process therefore includes one lag of both the variance and the error term, and can be written as

$$h_t = \omega + \beta h_{t-1} + \alpha(z_{t-1} - \gamma\sqrt{h_{t-1}})^2 \tag{2}$$

The expected value of variance at period $t + 1$ is given by

$$E[h_{t+1}] = \omega + \alpha + (\beta + \alpha\gamma^2) h_t \tag{3}$$

[1]Since volatility equals $\sqrt{h_t}$, the return premium per unit of risk is proportional to the square root of h_t as well exactly as in [9].

Imposing the constraint $\beta+\alpha\gamma^2 < 1$, the first order process is mean reverting, with finite mean and variance. Further, the process will tend to a long-time unconditional variance level of

$$V_L = \frac{\omega + \alpha}{1 - \beta - \alpha\gamma^2} \tag{4}$$

The process for $h_{t+\Delta}$ is fully observable at time t as a function of the spot price

$$h_{t+\Delta} = \omega + \beta h_t + \alpha \frac{(R_t - r_f - \lambda h_t - \gamma h_t)^2}{h_t} \tag{5}$$

where R_t is the log return at time t defined as $log(S_t) - log(S_{t-\Delta})$. The α parameter determines the kurtosis of the distribution, and imposes deterministic time-varying variance if set equal to zero. γ accounts for asymmetric impact of shocks; a large negative shock increases the variance more than a large positive shock, consistently with financial theory.

The covariance between the variance process and the spot price is defined by the following equation

$$Cov(h_{t+\Delta}, log(S_t)) = -2\alpha\gamma h_t \tag{6}$$

Given the non-negativity constraint imposed on the α parameter, a positive value of γ would imply for the spot price and variance to exhibit negative correlation.

We report the following result provided in [14].

Proposition 1. *The value of a call option with one period to expiry obeys the Black-Scholes-Rubinstein formula.*

This result implies that the conditional risk neutral distribution is still normal and, as in [10], the risk neutral dynamics for the log price and for the variance can be rewritten as follows:

$$log(S_t) = log(S_{t-\Delta}) + r_f - \frac{1}{2}h_t + \sqrt{h_t}z_t^* \tag{7a}$$

$$h_t = \omega + \beta h_{t-1} + \alpha(z_{t-1}^* - \gamma^* \sqrt{h_{t-1}})^2 \tag{7b}$$

where

$$z_t^* = z_t + (\lambda + \frac{1}{2})\sqrt{h_t}$$

$$\gamma^* = \gamma + \lambda + \frac{1}{2}$$

$$\lambda^* = -\frac{1}{2}.$$

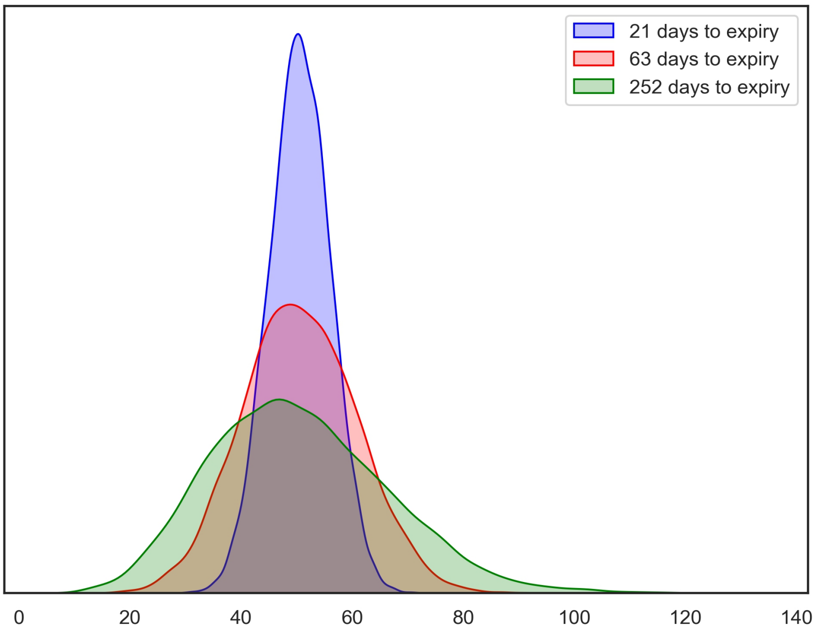

Figure 1: Simulated distributions of Heston-Nandi GARCH prices over different periods. Parameters: $S_0 = 50.52\$$, $\sigma^2 = 0.036\%$, $\alpha = 7.5e-6$, $\beta = 0.91$, $\gamma = 91.98$, $\omega = 5.4e-30$.

2.1 Heston-Nandi Closed Form Option Pricing

The main advantage of the introduction of this model is to allow closed-form option pricing. That is, compute option price based on the analytical martingale distribution of the underlying at expiry, rather than through numerical procedures such as Monte Carlo methods.

Note that the conditional generating function of the spot price at time t is computed mathematically as

$$G(u) = \mathsf{E}_{\mathsf{t}}[S_T^u] \tag{8}$$

Considering that for a general process X the moment generating function is defined as

$$MGF_X(u) = \mathsf{E}[e^{uX}] \tag{9}$$

we conclude that the conditional generating function of the spot price S_T in (8) is also the moment generating function of the logarithm of S_T.

In order to obtain the probability density function and probability distribution of the log price under the risk neutral measure $\mathbb{Q}$ at expiry, [14] obtains a log linear form for the moment generating function of the log price, that we report in the following proposition for the HN-GARCH(1,1) model.

Proposition 2. *The moment generating function of the log price has the following form*

$$MGF_{\ln(S_T)}(u) = S_t^u \exp\left(A_t + B_t h_{t+\Delta}\right) \tag{10}$$

Where S_t is the spot price of the asset, and

$$A_t = A_{t+\Delta} + ur_f + B_{t+\Delta}\omega - \frac{1}{2}\log\left(1 - 2\alpha B_{t+\Delta}\right) \tag{11a}$$

$$B_t = u(\lambda + \gamma) - \frac{1}{2}\gamma^2 + \beta B_{t+\Delta} + \frac{1}{2}\frac{(u-\gamma)^2}{1 - 2\alpha B_{t+\Delta}} \tag{11b}$$

Coefficients A_t and B_t can be derived recursively from the terminal conditions $A_T = B_T = 0$.

From (10), it is possible to obtain the characteristic function by substituting u with iu. Therefore the characteristic function of the logarithm of the spot price reads:

$$\phi_X(u) = S_t^{iu} \exp\left(A_t + B_t h_{t+\Delta}\right) \tag{12}$$

The characteristic function, under the martingale measure, can be obtained by setting $\gamma = \gamma^*$ and $\lambda = -\frac{1}{2}$.

Given the risk neutral characteristic function, $\phi_X^*(u)$ of the logarithm of the spot price, the probability density function $f_X(x)$ and the cumulative distribution function $F_X(x)$ can be obtained through *inverse Fourier transform* as follows:

$$f_X(x) = \frac{1}{\pi}\int_0^\infty \mathbb{R}\left[e^{-iux}\phi_X(u)\right] du \tag{13a}$$

$$F_X(x) = \frac{1}{2} - \frac{1}{\pi}\int_0^\infty \mathbb{R}\left[\frac{e^{-iux}\phi_X(u)}{iu}\right] du \tag{13b}$$

Where $\mathbb{R}[z] = \frac{z+\bar{z}}{2}$ represents the real part of a complex number z. Figure 2

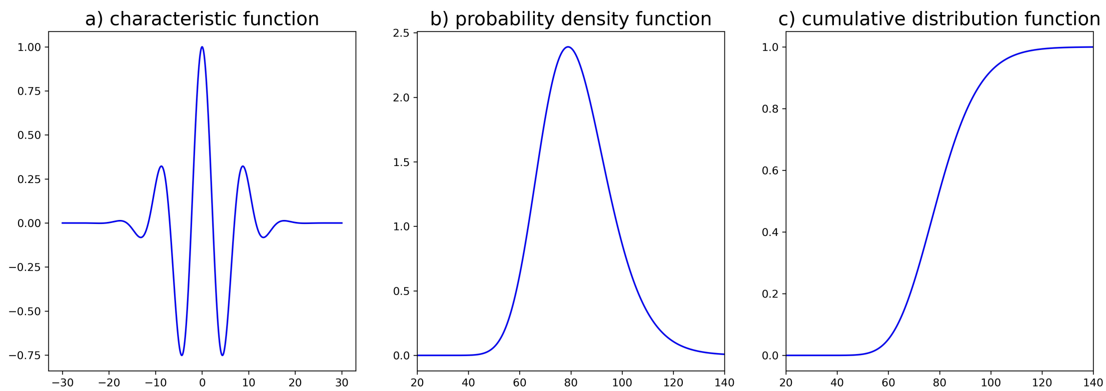

Figure 2: Characteristic function $\phi_X(u)$ and derived probability density function and distribution function over 90 days. Parameters: $S_0 = 80$, $\sigma^2 = 0.5\%$, $\alpha = 7e-5$, $\beta = 0.56$, $\gamma = 0.45$, $\omega = 1.6e-5$.

represents the characteristic function (Figure 2a) and the derived probability density (Figure 2b) and distribution function (Figure 2b). The Inverse Fourier Transform can be used to price a European call option following the approach developed first in [13].

3 Copula Function and Dependence Structure

In this section we briefly report the theoretical concept of bivariate Copula, outlining the main theorems alongside some of the principal properties. We then describe the most relevant copula families and their applications. Further, we introduce the Archimedean Copulas and give some examples of the most significant specifications. Finally, we review the notion of concordance measure and define the two most common measures: Kendall's τ and Spearman's ρ.

Definition 1. *A function $C : [0,1]^2 \to [0,1]$ is called a bivariate Copula if*

$$C(u,v) = \mathsf{Pr}(U \leq u, V \leq v) \tag{14}$$

where U, V are two independent Uniform random variables.

In order to be a valid Copula, a function must satisfy certain properties that follow from probability theory. Specifically, a Bivariate Copula must comply with the following three properties:

1. **Normalized marginals**: $C(u,1) = C(1,u) = u$. From the uniform marginals property, it holds that $\mathsf{Pr}(U \leq u, V \leq 1) = \mathsf{Pr}(U \leq u) = u$ and vice versa.

2. **Groundedness**: $C(u,v)$ is non-decreasing in each element. If at least one component is equal to zero, then $C(u,v) = 0$.

3. ***2*-increasingness**: for each $a, b, c, d \in \mathbf{I}$, with $a \leq b$ and $c \leq d$, $\mathsf{Pr}(U \in [a,b], V \in [c,d])$ must be non negative. Hence

$$0 \leq C(b,d) - C(a,d) - C(b,c) + C(a,c) \leq 1 \tag{15}$$

The above properties can be extended to any d-dimensional Copula, although for the sake of simplicity we address only the bivariate specification.

Let us now recall the fundamental theorem of Copula theory. Sklar's Theorem provides the theoretical foundations for the application of Copulas and shows how joint distributions can be set up by a number of marginal distributions and a copula function C that specifies the dependence structure among them.

Theorem 1. *A function F: $\mathbb{R}^2 \to [0,1]$ is the distribution function of some bivariate random vector (X_1, X_2) if and only if there exist a Copula $C : [0,1]^2 \to [0,1]$ and univariate distributions $F_1, F_2 : \mathbb{R} \to [0,1]$ such that*

$$C(F_1(x_1), F_2(x_2)) = F(x_2, x_2) \qquad x_1, x_2 \in \mathbb{R} \tag{16}$$

If the distribution function F_j of all components X_j with $j = 1, 2$ is continuous, then the correspondence between C and F is uniquely determined.

An interesting implication of Sklar's theorem is that, conditional on the marginal distributions F_1 and F_2 being absolutely continuous, the corresponding unique Copula distribution can be rewritten as

$$C(u,v) = F(F_1^{-1}(u), F_2^{-1}(v)) \tag{17}$$

The latter illustrates how Copulas can be extracted from multivariate joint cumulative distribution functions with continuous margins. Noting that $C(u,v)$ corresponds to the probability that X_1 and X_2 lie below the u-th and v-th quantile respectively, it appears clear how Copulas express dependence on a quantile scale.

It is most often meaningful to define how the Copula function is distributed conditional on the marginals.

Definition 2. *The conditional distribution function of the Copula $C(U,V)$ given $V = v$, which we denote $c_v(u)$, is given by:*

$$c_v(u) = \lim_{\Delta v \to 0} \frac{C(u, v+\Delta v) - C(u,v)}{\Delta v} = \frac{\partial C(u,v)}{\partial v} \tag{18}$$

The conditional distribution $c_v(u)$ is a distribution in $[0,1]$ and is Uniform if and only if the Copula corresponds to the Independence Copula Π. The conditional distribution function is of significant importance when simulating a random sample that is jointly distributed according to the Copula C.

The second order mixed partial derivative of a Copula function provides the joint density function, denoted $c(u,v)$. Formally, in the bivariate case

$$c(u,v) = \frac{\partial^2 C(u,v)}{\partial u \partial v} \tag{19}$$

Not all Copulas posses, however, a closed form density function. For instance, the Countermonotonicity Copula and Comonotonicity Copula are not absolutely continuous and do not admit the computation of the density function. Yet, most parametric specifications provide analytical densities to ease further computations.

In the following section we mention the Fréchet–Hoeffding bounds and their significant dependence structure implications, as well as the most relevant families of Copulas.

Definition 3. *Consider any Copula $C(u,v)$, then it always holds that*

$$\max\{u+v-1, 0\} \leq C(u,v) \leq \min\{u,v\} \tag{20}$$

Unlike numeric dependence measures, a Copula is a mathematical object that collects full information about the dependence structure without projecting it onto a single number. The inequality in (20) defines the so called Fréchet–Hoeffding bounds that represent the two extreme dependence structures a Copula can map. Analogously to numeric dependence measures, these bounds may be interpreted as the cases of extreme dependence.

A remarkable result of (20) is the family of Fundamental Copulas. These Copulas arise as the functions that describe perfect positive dependence, perfect negative dependence and independence. Unlike Comonotonicity and Independence Copula, the Countermonotonicity Copula does not exist for dimensions $d > 2$. It is easy to see that, in the bivariate case, the Comonotonicity Copula is the joint distribution of a random vector (U, V) where $V = T(U)$ is a strictly increasing transformation of U. Diametrically, the Countermonotonicity Copula is the Copula that collects information regarding the dependence structure of the random vector $(U, 1-U)$. Finally, it follows from Sklar's Theorem that the components (U, V) of a random vector are independent if and only if their Copula is the Independence Copula.

Copula functions can be explicitly defined or extracted from an existing joint distribution by virtue of Sklar's Theorem results. The family of Implicit Copulas stems from the latter method and includes the Gaussian Copula and the t Copula. Copulas belonging to this family reproduce the dependence structure of the existing corresponding joint distribution, although it is often the case that these Copulas cannot be extracted in a simple closed form. Table 2 reports the most common examples of Implicit Copulas.

In contrast with Implicit Copulas, explicitly defined Copulas arise from the need of a simple closed-form algebraic expression to compute joint probability. Among these Explicit Copulas, Archimedean Copulas enjoy considerable popularity in a number of practical applications in several disciplines. This collection of Copulas has in fact a convenient algebraic expression and

	bivariate definition	dependence structure
Comonotonicity Copula	$M(u, v) = \min\{u, v\}$	extreme positive dependence
Countermonotonicity Copula	$W(u, v) = \max\{u + v - 1, 0\}$	extreme negative dependence
Independence Copula	$\Pi(u, v) = uv$	independence

Table 1: Fundamental Copulas in the bivariate case and the dependence structure they represent.

	bivariate definition
Gaussian Copula	$C^G_\rho(u,v) = \mathbf{\Phi}_\rho(\Phi^{-1}(u), \Phi^{-1}(v))$
t **Copula**	$C^t_{\nu,\rho}(u,v) = \mathbf{t}_{\nu,\rho}(t_\nu^{-1}(u), t_\nu^{-1}(v))$

Table 2: Gaussian Copula and t Copula in the bivariate case. ρ is the correlation of the two random variables, $\Phi(\cdot)$ represents the standard normal distribution function and $\mathbf{\Phi}$ the joint distribution function. $\mathbf{t}_{\nu,\rho}$ indicates the multivariate t distribution with ν degrees of freedom whereas t_ν denotes the standard univariate t distribution with ν degrees of freedom.

is capable of capturing asymmetry.

Definition 4. *A Copula C_φ is an Archimedean Copula if it possesses the functional form*

$$C_\varphi(u,v) = \varphi(\varphi^{-1}(u), \varphi^{-1}(v)) \tag{21}$$

For a suitable non-increasing function $\varphi : [0,\infty) \to [0,1]$ with $\varphi(0) = 1$ and $\lim_{x\to\infty} \varphi(x) = 0$, called the "Archimedean Generator".

[17] postulates that C_φ is a valid distribution if the archimedean generator φ is d-monotone. The generator φ is said to be *d-monotone* if it has derivatives of all orders in (a,b) and if $(-1)^k f^{(k)}(x) \geq 0$ for any $x \in (a,b)$ and for $k = 0, 1, ..., d$. Each completely monotone generator function φ can be applied to define a Copula of arbitrary dimension $d \geq 2$. Table 3 shows the most common Archimedean Copula specifications with their generator function.

Although Archimedean Copulas are typically parametrized in terms of a function φ, in practice one chooses a parametric family of Laplace transforms governed by some real parameter θ, so that $\varphi = \varphi_\theta$. This allows the Copula to be completely parametrized by the single parameter θ, so that C_θ is equivalent to C_{φ_θ}. Hence, it is the parameter θ that provides the range of different dependence structures. An evident drawback of this specification, considering non-negativity of φ, is that this class of Copulas cannot model negative dependence structures.

Copula functions allow for a shift in dependence measurement, in the following section we stylize some concepts about rank correlation. Most multivariate distributions rely on a dependence structure described by matrices of pairwise linear correlation coefficients. Pearson's ρ represents in fact the most common measure of correlation within a pair of random variables in theoretical statistics as well as in financial applications. This simple

name	$\varphi_\theta(x)$	$\varphi_\theta^{-1}(y)$	$\theta \in$
Clayton Copula	$(1+x)^{-1/\theta}$	$y^{-\theta}$	$(0,\infty)$
Gumbel Copula	$\exp\{-x^{1/\theta}\}$	$(-\log(y))^\theta$	$[1,\infty)$
Ali-Mikhail-Haq Copula	$\frac{1-\theta}{e^x-\theta}$	$\log(\frac{1-\theta}{y}+\theta)$	$[0,1)$

Table 3: Archimedean Copulas with the respective generator function φ and θ parameter range.

yet fundamental measure has some drawbacks when it comes to modelling multivariate distributions. First of all, it is only able to capture pairwise correlation. When dealing with a multivariate random vector with $d > 2$ in fact, a correlation matrix of dimension $d \times d$ needs to be computed, with at least $d(d-1)/2$ correlations to be estimated. Secondly, the relationship between the random components might be non-linear and thus be distorted by Pearson's ρ. Lastly, financial applications of ρ show that this measure may result in an incorrect estimation of risk.

In order to account for the broader intuition underpinning correlation, [24] introduces the notion of *Concordance Measure*. A Concordance Measure is a mapping I from the set of bivariate distributions with continuous marginals to the interval [-1,1] satisfying the following set of conditions:

- **Symmetry**: $I(X_1, X_2) = I(X_2, X_1)$;
- **Coherence**: let (X_1, X_2) and (Y_1, Y_2) be two bivariate random vectors having Copulas $C_X(u,v)$ and $C_Y(u,v)$ respectively, and these are ordered as $C_X(u,v) \leq C_Y(u,v)$ for every $(u,v) \in [0,1]^2$. Then $I(X_1, X_2) \leq I(Y_1, Y_2)$;
- **Independence**: if X_1 and X_2 are independent, $I(X_1, X_2) = 0$. The opposite is not necessarily true;
- **Change in sign**: $I(-X_1, X_2) = -I(X_1, X_2)$. Intuitively, a change in sign inverts the ordering of one random variable, hence the concordance measure inverts itself.
- **Convergence**: if $\{(X_1^{(n)}, X_2^{(n)})\}_{n\in N}$ is a sequence of random vectors with associated sequence of joint distribution functions $\{F_n\}_{n\in N}$ and F_n converges point-wise to F, the joint distribution of a random vector (X_1, X_2), then
$$\lim_{n\to\infty} I(X_1^{(n)}, X_2^{(n)}) = I(X_1, X_2)$$

Following, we introduce the most common concordance measures, but let us first define "*concordant*" a pair of observations (x_1, x_2) and (y_1, y_2) such that $(x_1 - y_1)(x_2 - y_2) > 0$. Conversely, the same pair is said to be "*discordant*" if $(x_1 - y_1)(x_2 - y_2) < 0$.

Definition 5. ***(Kendall's** τ)* *Let (X_1, X_2) be a random vector with copula C as joint distribution function. Then Kendall's τ is defined as*

$$\tau = 4 \int_0^1 \int_0^1 C(u,v) \frac{\partial^2 C(u,v)}{\partial u \partial v} du dv - 1 = 4E[C(U,V)] - 1 \tag{22}$$

Kendall's τ is defined for a bivariate random vector (U, V) with continuous marginals, so that the joint distribution is uniquely defined by the copula C, and hence the concordance measure does not reflect any information about the single marginals. The intuition behind (22) can be interpreted as the probability of observing a concordant pair minus the probability of observing a discordant pair, formalized as follows

$$\tau = \mathsf{Pr}((X_1 - Y_1)(X_2 - Y_2) > 0) - \mathsf{Pr}((X_1 - Y_1)(X_2 - Y_2) < 0) \tag{23}$$

For *iid* samples, $(X_1^{(1)}, X_2^{(1)})...(X_1^{(n)}, X_2^{(n)})$, the empirical version of (22) is defined as

$$\begin{aligned} \hat{\tau} &= \frac{\text{n. of concordant pairs} - \text{n. of discordant pairs}}{\text{n. of total pairs}} \\ &= \frac{\sum_{1 \le i < j \le n} \text{sign}[(X_1^{(j)} - X_1^{(i)})(X_2^{(j)} - X_2^{(i)})]}{n(n-1)/2} \end{aligned} \tag{24}$$

Definition 6. ***(Spearman's** ρ)* *Given the random vector (X_1, X_2) with continuous marginal distributions F_i with $i = 1, 2$, let the vector (U, V) be the transformed vector $(U, V) = (F_1(X_1), F_2(X_2))$. Note that the vector (U, V) has joint distribution function C. Spearman's ρ is defined as the Pearson's correlation coefficient of (U, V), formally*

$$\rho_S = Corr(U, V) = Corr(F_1(X_1), F_2(X_2)) \tag{25}$$

Again, this dependence measure does not depend on the marginal laws F_1, F_2 as the influence has been removed by transforming the random variables into Uniforms. Mathematically, (25) is equivalent [see [22] for more details] to

$$\rho_s = 12 - \int_0^1 \int_0^1 C(u,v) du dv - 3 \tag{26}$$

Properties of Pearson's ρ are naturally inherited by Spearman's ρ, such as symmetry and values of 0 and 1 in case of independence and perfect positive dependence respectively. Analogously to Kendall's τ, ρ_S is increasing in the point-wise ordering of the Copulas, such that, if $C_1(u,v) \leq C_2(u,v)$ for all $(u,v) \in [0,1]^2$, then $\rho_{S,1} \leq \rho_{S,2}$. Let $\hat{\rho_S}$ denote the empirical counterpart of Spearman's ρ. $\hat{\rho_S}$ is somewhat related to the empirical version of the correlation coefficient, but levels of the random variables are substituted by ranks. Formally, consider the *iid* samples $(X_1^{(1)}, X_2^{(1)}), ..., (X_1^{(n)}, X_2^{(n)})$, the sample version can be computed as

$$\hat{\rho_S} = \frac{\sum_{i=1}^{n}(\text{rank}[X_1^{(i)}] - \frac{n+1}{2})(\text{rank}[X_2^{(i)}] - \frac{n+1}{2})}{\sqrt{\sum_{i=1}^{n}(\text{rank}[X_1^{(i)}] - \frac{n+1}{2})^2}\sqrt{\sum_{i=1}^{n}(\text{rank}[X_2^{(i)}] - \frac{n+1}{2})^2}} \tag{27}$$

where $\text{rank}[X_j^{(i)}]$ represents the rank statistic of $X_j^{(i)}$ with $j = 1, 2$.

To close this section, note that many families of Copulas can be parametrized in terms of a single parameter θ, which typically collects the information about the dependence structure between the marginals. It is often the case that one can express the parameter θ as a function of either τ, ρ_S or both.

In the next part, we introduce the Copula specification we rely on for computations.

3.1 Plackett Copula

In the continuance of the study, the Plackett Copula is used to model the dependence structure of two asset prices in order to price a spread option. It seems then natural in this section to highlight the theoretical framework of such Copula and to review its features.

It is useful to firstly illustrate the concepts of "*contingency table*" and "*odds ratio*", inasmuch as the Plackett Copula heavily leverages and extends these concepts to continuous margins. Let us suppose, for instance, that

	up	down	
up	a	b	$a+b$
down	c	d	$c+d$
	$a+c$	$b+d$	

Table 4: 2×2 contingency table

the contingency table in Table 4 shows the number of days in a year where two given stocks' closing prices are up/down relative to the opening prices. For simplicity, let **A** denote the row-stock and **B** the column-stock. It is equivalent to express a, b, c, d in Table 4 as the absolute number of days or as the percentage of the total days n, where $n = a + b + c + d$.

Intuitively, $a + b$ and $c + d$ represent the unconditional probabilities for stock **A** to close at a higher price or at a lower price respectively. The "*odds*" for this stock price to increase are $(a+b)/(c+d)$, it can be interpreted as the probability of a favourable outcome in terms of the number of unfavourable outcomes. Analogously, the odds of stock **B**'s price to increase are computed as $(a+c)/(b+d)$. The odds can also be computed on the conditional probabilities of one stock's development, given the other stock's performance. The conditional probability of Stock **A** closing with a higher price given Stock **B** result is "up" is $a/(a + c)$ whereas the opposite conditional probability will be computed as $c/(a + c)$. One could then compute the odds as the ratio of one probability to the other.

To assess the dependence between row variable and column variable one should calculate the "*cross-product ratio*" or "*odds ratio*" as

$$\theta = \frac{\dfrac{a}{a+c} \Big/ \dfrac{c}{a+c}}{\dfrac{b}{b+d} \Big/ \dfrac{d}{b+d}} = \frac{ad}{bc} \tag{28}$$

The resulting parameter θ captures the dependence structure of the stylized distribution. Note that, when $\theta > 1$ probability tends to concentrate in the main diagonal (up-up and down-down) which intuitively corresponds to positive dependence; conversely, a value of $\theta < 1$ would correspond to negative dependence. Lastly, $\theta = 1$ implies independence, suggesting that each observed entry is equal to its expected value.

[23] introduces a bivariate distribution that provides a pragmatic extension of the concept of contingency table dependence, allowing for continuous margins.

Let X and Y be continuous random variables with joint distribution H and marginal distributions F and G respectively. Let us now rewrite "*up*" and "*down*" as $X \leq x$ and $X > x$, and analogously for Y random variable. Table 4 can be rearranged as illustrated in Table 5.

The cross-product ratio is then rewritten as

$$\theta = \frac{H(x,y)[1 - F(x) - G(y) + H(x,y)]}{[F(x) - H(x,y)][G(y) - H(x,y)]} \tag{29}$$

By means of Theorem 1, if the margins are continuous then their Copula is uniquely determined, so one can substitute $H(x, y)$ with $C(F(x), G(y))$ and

	$Y \leq y$	$Y > y$	
$X \leq x$	$H(x,y)$	$F(x) - H(x,y)$	$F(x)$
$X > x$	$G(y) - H(x,y)$	$1 - F(x) - G(y) + H(x,y)$	$1 - F(x)$
	$G(y)$	$1 - G(y)$	

Table 5: 2×2 contingency table in the continuous margin domain.

$F(x), G(y)$ with u, v respectively. Accordingly, θ will become

$$\theta = \frac{C(u,v)[1 - u - v + C(u,v)]}{[u - C(u,v)][v - C(u,v)]} \tag{30}$$

Holding θ constant, solving for $C(u,v)$ yields the following

$$C(u,v) = \frac{[1 + (\theta - 1)(u+v)] \pm \sqrt{[1 + (\theta - 1)(u+v)]^2 - 4uv\theta(\theta - 1)}}{2(\theta - 1)} \tag{31}$$

Note that [18] proves that (31) with the "+" sign before the radical is never a Copula, whereas with "-" sign it always is. Furthermore, one needs to encircle the parameter θ for (31) to comply with theoretical requirements. Consequently the analytical expression of the so-called "*Plackett Copula*" is

$$C_\theta(u,v) = \frac{[1 + (\theta - 1)(u+v)] - \sqrt{[1 + (\theta - 1)(u+v)]^2 - 4uv\theta(\theta - 1)}}{2(\theta - 1)} \tag{32a}$$

for $\theta > 0$ and $\neq 1$.

$$C_\theta(u,v) = uv \tag{32b}$$

for $\theta = 1$.

The latter are 2-increasing and absolutely continuous. It can be shown by proving that

$$\frac{\partial^2 C_\theta(u,v)}{\partial u \partial v} \geq 0 \qquad u, v \in \mathbf{I}^2$$

and

$$C_\theta(u,v) = \int_0^u \int_0^v \frac{\partial^2 C_\theta(s,t)}{\partial s \partial t} dt ds$$

Moreover, Plackett Copulas form a comprehensive family of Copulas, ranging continuously from the lower Fréchet–Hoeffding bound in the limit case $\lim_{\theta \to 0} C_\theta(u,v) = W(u,v)$, to the upper Fréchet–Hoeffding bound in the limit case $\lim_{\theta \to \infty} C_\theta(u,v) = M(u,v)$. It is trivial to note that when $\theta = 1$,

$C_1(u,v) = uv = \Pi(u,v)$. Another advantage of Plackett Copula is that it possesses-closed form analytical probability density function

$$c(u,v) = \frac{\theta\,[1+(u-2uv+v)(\theta-1)]}{\{[1+(\theta-1)(u+v)]^2 - 4uv\theta(\theta-1)\}^{3/2}} \tag{33}$$

The Plackett Copula lends itself particularly well to modelling dependence structure of financial phenomena, and it is well spread in practice. As only one parameter θ captures the entire dependence dynamic, it is notably easy to estimate via Maximum Likelihood from past data. One could also estimate θ by computing the observed cross-product ratio in the dataset using some arbitrary quadrant separation.

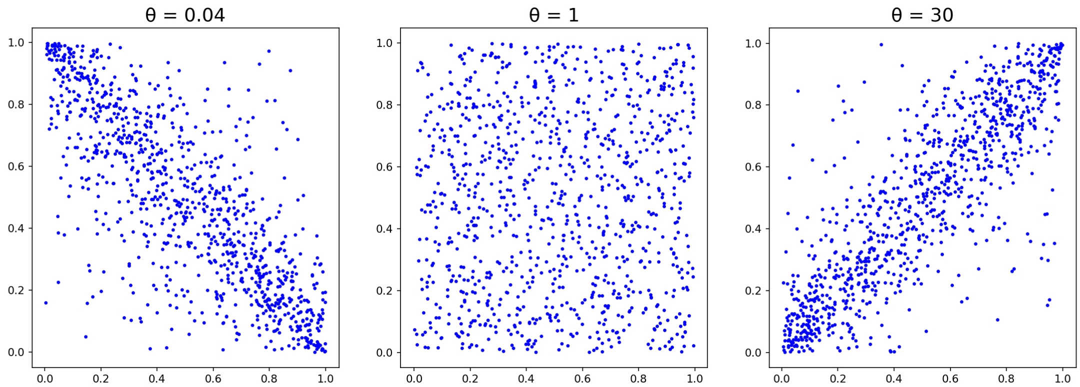

Figure 3: Scatter plot of Plackett Copula in the following cases: Negative dependence ($\theta < 1$), Independence ($\theta = 1$) and Positive dependence ($\theta > 1$).

The scatter plots in Figure 3 show the pattern in the joint distribution, when θ ranges from $\theta = 0.04$ to $\theta = 30$. The two lateral plots exhibit clear negative and positive dependence respectively, whilst the middle one represents independence. Furthermore, one can express Spearman's ρ as a function of θ as follows

$$\rho_S = \begin{cases} \dfrac{\theta+1}{\theta-1} - \dfrac{2\theta}{(\theta-1)^2}\log\theta & \text{if } \theta \neq 1 \\ 0 & \text{if } \theta = 1 \end{cases} \tag{34}$$

3.2 Plackett's θ

The relationship between the marginal distributions under the Plackett Copula is entirely captured by the non-negative θ parameter. A simple yet attractive approach to θ estimation is the observed cross-product ratio, previously

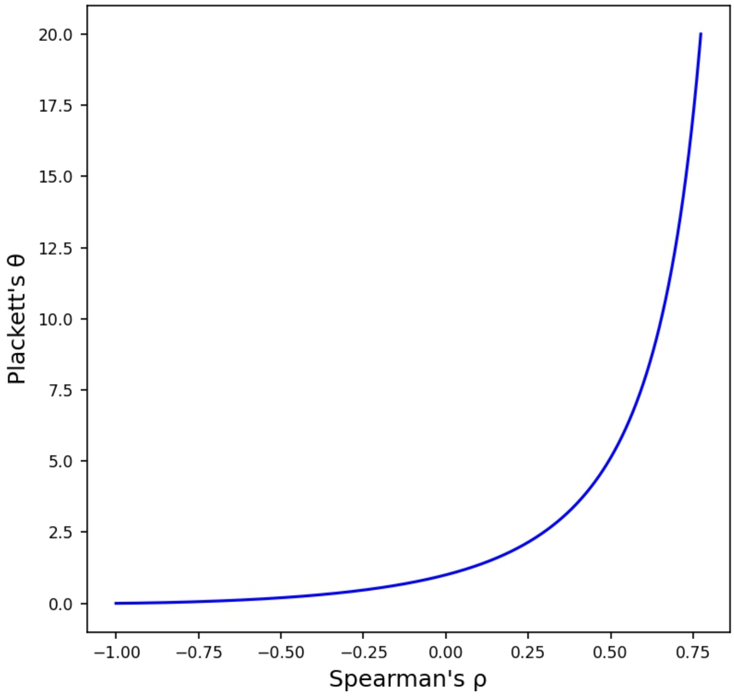

Figure 4: Relationship between Plackett's θ and Spearman's ρ.

described in (28), $\theta^* = ad/bc$ where a, b, d and c are the observed frequencies in the four quadrants determined by lines in $\mathbb{R}^2$ parallel to the axes through a point (p, q). An optimum choice for (p, q) is presented in [18] to be the sample median vector, which in turn minimizes the asymptotic variance of the estimator θ^*. Considering the following contingency table, where p is the median of variable X and q is the median for variable Y, let a, b, c and d be the relative frequencies in the quadrants.

By the definition of median, it holds that $a + c = b + d = 0.5$. And so it follows that $c = (0.5 - a)$ and $b = (0.5 - d)$. Again, from the definition of median it must hold also that $a + b = c + d = 0.5$. We equate previous results and obtain that $a = d$. Then, taking the cross-product ratio, we can substitute

$$\theta^* = \frac{ad}{bc} = \frac{ad}{(0.5 - a)(0.5 - d)} = \frac{a^2}{(0.5 - a)^2} \tag{35}$$

where a represents the observed frequency of observations in which neither of the variables exceeds its median value.

4 Spread Option Pricing with a Copula Function

In this section we propose a method for pricing a spread option based on a numerical integration of one-dimensional integral. We assume that the joint distribution of log-returns is modelled through a Copula function. For sake of brevity we discuss only the case of a European spread call option. The formula proposed in this section is easy to apply to any model where the univariate cumulative distribution function has an analytical (or semi-analytical) formula. Under a martingale measure $\mathbb{Q}$, the price of a spread call option with strike price K can be obtained as:

$$P_{t_0} = e^{-r(T-t_0)} \mathsf{E}^{\mathbb{Q}} \left[\max \left\{ S_{1,T} - S_{2,T} - K; 0 \right\} | \mathcal{F}_{t_0} \right]. \tag{36}$$

Denoting with $X_{i,t_0,T}$ with $i = 1, 2$ the log-return for each asset over the period $[t_0, T]$, the joint density function $f_{X_{1,t_0,T}, X_{2,t_0,T} | \mathcal{F}_{t_0}} (x_1, x_2)$ of log-returns given the information available at time t_0 can be written using a Copula $C(u, v)$ function as:

$$f_{X_{1,t_0,T}, X_{2,t_0,T} | \mathcal{F}_{t_0}} (x_1, x_2) = c \left(F_{X_{1,t_0,T} | \mathcal{F}_{t_0},} (x_1), F_{X_{2,t_0,T} | \mathcal{F}_{t_0},} (x_2) \right) f_{X_{1,t_0,T} | \mathcal{F}_{t_0}} (x_1) f_{X_{2,t_0,T} | \mathcal{F}_{t_0}} (x_2)$$

where the function $c(u, v)$ is the mixed second order derivative of the Copula function $C(u, v)$, i.e. $c(u, v) := \frac{\partial^2 C(u,v)}{\partial u \partial v}$; $F_{X_{i,t_0,T} | \mathcal{F}_{t_0},} (x_i)$ is for asset i the conditional univariate cumulative distribution function of the log-returns and $f_{X_{i,t_0,T} | \mathcal{F}_{t_0}} (x_i)$ is the associated conditional density function. For any twice differentiable Copula function, the formula in (36) can be obtained using the following result:

Theorem 2. *Let $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P} \right)$ be a filtered probability space. If the Copula function is twice differentiable (i.e. $C(u, v) \in \mathcal{C}^{2,2}$), the non-arbitrage price for the spread call option is:*

$$P_{t_0} = e^{-r(T-t_0)} \left[I_{t_0,1}^T - I_{t_0,2}^T - I_{t_0,3}^T \right], \tag{37}$$

where the quantities $I_{t_0,1}^T$, $I_{t_0,2}^T$ and $I_{t_0,3}^T$ are defined as:

$$I_{t_0,1}^T = \int_{d_3}^1 S_{1,t_0} \exp \left(F^{-1}_{X_{1,t_0,T} | \mathcal{F}_{t_0}} (u) \right) \partial_u C (u, d_2 (u)) \, du$$

$$I_{t_0,2}^T = \mathsf{E}^{\mathbb{Q}} \left[S_{2,T} | \mathcal{F}_{t_0} \right] - \int_0^1 S_{2,t_0} \exp \left[F^{-1}_{X_{2,t_0,T} | \mathcal{F}_{t_0}} (v) \right] \partial_v C (d_1 (v), v) \, dv$$

$$I_{t_0,3}^T = K\left[1 - \int_0^1 \partial_v C\left(d_1\left(v\right), v\right) dv\right]$$

and the terms $d_1(v)$*,* $d_2(u)$ *and* d_3 *can be determined using the following three formulas*

$$d_1\left(v\right) = F_{X_{1,t_0,T}|\mathcal{F}_{t_0}}\left(\ln\left[\frac{S_{2,t_0}\exp\left[F^{-1}_{X_{2,t_0,T}|\mathcal{F}_{t_0}}\left(v\right)\right] + K}{S_{1,t_0}}\right]\right),$$

$$d_2\left(u\right) = F_{X_{2,t_0,T}|\mathcal{F}_{t_0}}\left(\ln\left(\frac{S_{1,t_0}\exp\left(F^{-1}_{X_{1,t_0,T}|\mathcal{F}_{t_0}}\left(u\right)\right) - K}{S_{2,t_0}}\right)\right)$$

and

$$d_3 = F_{X_{1,t_0,T}|\mathcal{F}_{t_0}}\left(\ln\left(\frac{K}{S_{1,t_0}}\right)\right)$$

where the univariate conditional distributions $F_{X_{i,t_0,T}|\mathcal{F}_{t_0}}(\cdot)$ *with* $i = 1, 2$ *are determined under the martingale measure* $\mathbb{Q}$*;* $F^{-1}_{X_{i,t_0,T}}(\cdot)$ *with* $i = 1, 2$ *denote the corresponding quantile functions.*

Proof. We start from the expectation in (36) that can be rewritten as:

$$I_{t_0}^T = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \max\left\{S_{1,t_0}e^{x_{1,t_0,T}} - S_{2,t_0}e^{x_{2,t_0,T}} - K; 0\right\} dF_{X_{1,t_0,T},X_{2,t_0,T}|\mathcal{F}_{t_0}}\left(x_1, x_2\right).$$

where $F_{X_{1,t_0,T},X_{2,t_0,T}|\mathcal{F}_{t_0}}(x_1, x_2)$ is the conditional joint cumulative distribution function under the martingale measure $\mathbb{Q}$. Exploiting the representation in (16) and the twice differentiability of the Copula function $C(u, v)$ we obtain the following formula for the differential of the conditional joint distribution:

$$dF_{X_{1,t_0,T},X_{2,t_0,T}|\mathcal{F}_{t_0}}\left(x_1, x_2\right) = c\left(F_{X_{1,t_0,T}|\mathcal{F}_{t_0}}\left(x_1\right), F_{X_{2,t_0,T}|\mathcal{F}_{t_0}}\left(x_2\right)\right) \mathrm{d}F_{X_{1,t_0,T}|\mathcal{F}_{t_0},}\left(x_1\right) \mathrm{d}F_{X_{2,t_0,T}|\mathcal{F}_{t_0}}\left(x_2\right) \tag{38}$$

Function $c(\cdot, \cdot)$ is defined in (19). Substituting (38) in the integral $I_{t_0}^T$, we obtain the following result:

$$I_{t_0}^T = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \left(S_{1,t_0}e^{x_1} - S_{2,t_0}e^{x_2} - K\right)^+ c\left(F_{X_{1,t_0,T}|\mathcal{F}_{t_0}}\left(x_1\right), F_{X_{2,t_0,T}|\mathcal{F}_{t_0}}\left(x_2\right)\right) \mathrm{d}F_{X_{1,t_0,T}|\mathcal{F}_{t_0},}\left(x_1\right) \mathrm{d}F_{X_{2,t_0,T}|\mathcal{F}_{t_0}}\left(x_2\right),$$

where $(x)^+ = x\mathbb{1}_{\{x>0\}}$ and $\mathbb{1}_{\{x>0\}}$ is the indicator function that assumes value 1 if $x > 0$, 0 otherwise.

Defining $u := F_{X_{1,t_0,T}|\mathcal{F}_{t_0}}, (x_1)$ and $v := F_{X_{2,t_0,T}|\mathcal{F}_{t_0}}, (x_2)$, we obtain:

$$I_{t_0}^T = \int_0^1 \int_0^1 \left(S_1^{t_0} \exp\left[F^{-1}_{X_{1,t_0,T}|\mathcal{F}_{t_0}} (u) \right] - S_2^{t_0} \exp\left[F^{-1}_{X_{2,t_0,T}|\mathcal{F}_{t_0}} (v) \right] - K \right)^+ c(u,v)\, \mathrm{d}u \mathrm{d}v. \tag{39}$$

The integral $I_{t_0}^T$ can be decomposed into three terms:

$$I_{t_0}^T = I_{t_0,1}^T - I_{t_0,2}^T - I_{t_0,3}^T \tag{40}$$

defined as follows:

$$I_{t_0,1}^T := \int_0^1 \int_0^1 S_{1,t_0} \exp\left[F^{-1}_{X_{1,t_0,T}|\mathcal{F}_{t_0}} (u) \right] \mathbb{1}_{\left\{ S_{1,t_0} \exp\left[F^{-1}_{X_{1,t_0,T}|\mathcal{F}_{t_0}} (u) \right] > S_{2,t_0} \exp\left[F^{-1}_{X_{2,t_0,T}|\mathcal{F}_{t_0}} (v) \right] + K \right\}} c(u,v)\, \mathrm{d}u \mathrm{d}v \tag{41}$$

$$I_{t_0,2}^T := \int_0^1 \int_0^1 S_{2,t_0} \exp\left[F^{-1}_{X_{2,t_0,T}|\mathcal{F}_{t_0}} (v) \right] \mathbb{1}_{\left\{ S_{1,t_0} \exp\left[F^{-1}_{X_{1,t_0,T}|\mathcal{F}_{t_0}} (u) \right] > S_{2,t_0} \exp\left[F^{-1}_{X_{2,t_0,T}|\mathcal{F}_{t_0}} (v) \right] + K \right\}} c(u,v)\, \mathrm{d}u \mathrm{d}v \tag{42}$$

and

$$I_{t_0,3}^T := \int_0^1 \int_0^1 K \mathbb{1}_{\left\{ S_{1,t_0} \exp\left[F^{-1}_{X_{1,t_0,T}|\mathcal{F}_{t_0}} (u) \right] > S_{2,t_0} \exp\left[F^{-1}_{X_{2,t_0,T}|\mathcal{F}_{t_0}} (v) \right] + K \right\}} c(u,v)\, \mathrm{d}u \mathrm{d}v. \tag{43}$$

We introduce the following function:

$$d_1(v) = F_{X_{1,t_0,T}|\mathcal{F}_{t_0}}, \left(\ln \left[\frac{S_{2,t_0} \exp\left[F^{-1}_{X_{2,t_0,T}|\mathcal{F}_{t_0}} (v) \right] + K}{S_{1,t_0}} \right] \right), \tag{44}$$

we obtain this identity

$$\mathbb{1}_{\left\{ S_{1,t_0} \exp\left[F^{-1}_{X_{1,t_0,T}|\mathcal{F}_{t_0}} (u) \right] > S_{2,t_0} \exp\left[F^{-1}_{X_{2,t_0,T}|\mathcal{F}_{t_0}} (v) \right] + K \right\}} = \mathbb{1}_{u > d_1(v)}$$

that we use to rewrite $I_{t_0}^T(2)$ and $I_{t_0}^T(3)$ in the following way:

$$\begin{aligned} I_{t_0,2}^T &= \int_0^1 S_{2,t_0} \exp\left[F^{-1}_{X_{2,t_0,T}|\mathcal{F}_{t_0}} (v) \right] \left[\int_{d_1(v)}^1 c(u,v)\, \mathrm{d}u \right] \mathrm{d}v \\ &= \int_0^1 S_{2,t_0} e^{F^{-1}_{X_{2,t_0,T}|\mathcal{F}_{t_0}} (v)} \left[1 - \partial_v C(d_1(v), v) \right] \mathrm{d}v \\ &= \mathsf{E}^{\mathbb{Q}} \left[S_{2,T} \,|\mathcal{F}_{t_0} \right] - \int_0^1 S_{2,t_0} \exp\left[F^{-1}_{X_{2,t_0,T}|\mathcal{F}_{t_0}} (v) \right] \partial_v C(d_1(v), v)\, \mathrm{d}v \end{aligned} \tag{45}$$

$$
\begin{aligned}
I_{t_0,3}^{T} &= K\int_0^1\left[\int_0^1 \mathbb{1}_{u>d_1(v)}c\left(u,v\right)\mathrm{d}u\right]\mathrm{d}v \\
&= K\int_0^1\left[\int_{d_1(v)}^1 c\left(u,v\right)\mathrm{d}u\right]\mathrm{d}v \\
&= K\int_0^1\left[\partial_v C\left(1,v\right)-\partial_v C\left(d_1\left(v\right),v\right)\right]\mathrm{d}v \\
&= K\int_0^1\left[1-\partial_v C\left(d_1\left(v\right),v\right)\right]\mathrm{d}v \\
&= K\left[1-\int_0^1 \partial_v C\left(d_1\left(v\right),v\right)\mathrm{d}v\right] \qquad (46)
\end{aligned}
$$

Now we compute the integral $I_{t_0,1}^{T}$. The following identity holds true:

$$
\mathbb{1}_{\left\{S_{1,t_0}\exp\left[F^{-1}_{X_{1,t_0,T}\left|\mathcal{F}_{t_0}\right.}(u)\right]>S_{2,t_0}\exp\left[F^{-1}_{X_{2,t_0,T}\left|\mathcal{F}_{t_0}\right.}(v)\right]+K\right\}} = \mathbb{1}_{\{d_2(u)>v\}}\mathbb{1}_{\{1>u>d_3\}}
$$

where

$$
d_2\left(u\right):=F_{X_{2,t_0,T}\left|\mathcal{F}_{t_0}\right.}\left(\ln\left(\frac{S_{1,t_0}\exp\left(F^{-1}_{X_{1,t_0,T}\left|\mathcal{F}_{t_0}\right.}\left(u\right)\right)-K}{S_{2,t_0}}\right)\right) \qquad (47)
$$

and

$$
d_3 = F_{X_{1,t_0,T}\left|\mathcal{F}_{t_0}\right.}\left(\ln\left(\frac{K}{S_{1,t_0}}\right)\right). \qquad (48)
$$

The integral $I_{t_0}^{T}\left(1\right)$ becomes

$$
\begin{aligned}
I_{t_0}^{T}\left(1\right) &= \int_{d_3}^1 S_{1,t_0}\exp\left(F^{-1}_{X_{1,t_0,T}\left|\mathcal{F}_{t_0}\right.}\left(u\right)\right)\left[\int_0^{d_2(u)} c\left(u,v\right)\mathrm{d}v\right]\mathrm{d}u \\
&= \int_{d_3}^1 S_{1,t_0}\exp\left(F^{-1}_{X_{1,t_0,T}\left|\mathcal{F}_{t_0}\right.}\left(u\right)\right)\partial_u C\left(u,d_2\left(u\right)\right)\mathrm{d}u \qquad (49)
\end{aligned}
$$

The result is obtained considering (49), (45), (46), (47) and (48). $\square$

It is worth noting that the result in (37) can be applied to any Copula function discussed in Section 3 and requires the evaluation of one-dimensional integrals with bounded support that can be easily computed using several integral techniques available in any programming language.

5 Numerical Analysis

In this section, we conduct two numerical analyses. First, we analyze the computation efficiency and the accuracy of the result in (37). In the second analysis, we compare our formula with a numerical integration method applied for the evaluation of the double integral in (36). Moreover, we determine the confidence interval at the 95% level for the Monte-Carlo prices [see [3] for more details] to assess whether the prices obtained applying our formula belong to this interval.

We consider, as an example, the case of the Plackett Copula that has an analytical expression for partial derivatives of any order. In order to reproduce the heteroskedasticity observed in the financial time series, we require that the univariate log prices, under martingale measure, follow two HN Affine GARCH(1,1) models as described in Section 2. From this latter assumption, the conditional univariate distribution of log-returns can be determined using the Inverse Fourier Transform.

The dataset, that we use for the estimation of the parameters, is composed of the futures prices of the contract nearest to expiry for Brent crude oil and West Texas Intermediate (WTI) crude oil. Both contracts are quoted in U.S. Dollars. Three years of futures contract prices is gathered on a daily basis from March 1$^{\text{st}}$, 2017 to February 29$^{\text{th}}$, 2020. The dataset consists of 774 closing price records for Brent and 792 closing price records for WTI.

The Heston-Nandi GARCH models are estimated through Maximum Likelihood Estimation on the series of log returns of both contracts. Table 6 shows the estimated parameters of the two GARCH models and their respective standard errors. The annualized volatility of the series, computed as $\sqrt{252(\omega+\alpha)/(1-\beta-\alpha\gamma^2)}$, is 0.2972 for Brent futures returns and 0.2985 for WTI futures returns.Plackett's parameter θ is estimated using the result in (35) ($\theta^\star = 50.52$).

	ω	α	β	γ	λ	$\beta+\alpha\gamma^2$
Brent	9.124e-33	7.081e-6	0.914	96.505	-0.418	0.979
	(183.187)	(0.625)	(4.415)	(270.264)	(12.989)	
WTI	2.845e-4	7.155e-6	0.175	0.161	-0.522	0.175
	(1.774)	(2.897)	(12.587)	(2.961)	(1.322)	

Table 6: Estimates of the parameters of the GARCH models through MLE (standard errors of the estimates).

For the usage of the formula in (37), the numerical integrals in (49), (45) and (46) are computed applying the midpoint integration method that, in our case, requires a partition of the support $[0,1]$. For each natural number N, we divide the support $[0,1]$ into N sub-intervals of the same length $\Delta = \frac{1}{N}$. Thus, we approximate the integrals in (49), (45) and (46) with a Riemann summation of the following form:

$$\sum_{i=1}^{N} f\left(i-1+\frac{1}{2N}\right)\frac{1}{N} \to \int_0^1 f(x)dx \text{ a.s. } N \to +\infty,$$

where $f(x)$ is an integrable function in $[0,1]$. Table 7 analyzes the accuracy and the computation time for the formula in (37) as N varies. We observe

K	$N = 100$	$N = 500$	$N = 1000$	$N = 5000$	$N = 10000$
0.0	6.081	6.142	6.147	6.149	6.149
	(1.12)	(1.12)	(1.12)	(1.12)	(1.12)
2.5	3.964	4.014	4.017	4.019	4.019
	(1.11)	(1.11)	(1.12)	(1.12)	(1.12)
5	2.224	2.262	2.265	2.266	2.266
	(1.13)	(1.14)	(1.14)	(1.20)	(1.20)
7.5	1.073	1.1071	1.109	1.111	1.111
	(1.13)	(1.13)	(1.13)	(1.13)	(1.13)
10	0.493	0.528	0.539	0.533	0.533
	(1.12)	(1.12)	(1.12)	(1.13)	(1.13)

Table 7: Comparison of European spread call option prices with time to maturity 3 months. The prices are computed using the formula in (37) where the integrals are evaluated by means of the midpoint integration approach. We report the computational times in parenthesis and N denotes the number of sub-intervals in the partition of the support $[0,1]$.

that computational time increases slowly as N increases; all prices seem to reach a stable behaviour for $N \geq 5000$.

We describe alternative approaches for pricing a spread European Call option that we compare with our formula in (37). The first approach is based on the numerical evaluation of the double integral in (39). The partition is now associated to the set $[0,1]^2$ of the Cartesian plane. In this case, the double integral can be approximated by a double Riemann summation of the following form:

$$\sum_{i=1}^{N}\sum_{j=1}^{N} f\left(i-1+\frac{1}{2N}, j-1+\frac{1}{2N}\right)\frac{1}{N^2} \to \int_0^1\int_0^1 f(x,y)dxdy \text{ a.s. } N \to +\infty,$$

where $f(x,y)$ is a integrable function defined on the support $[0,1]^2$. The second alternative approach is based on the Monte Carlo simulation applying the following algorithm:

1. We generate two independent random numbers $\hat{u}$, $\hat{s}$ from an Uniform distribution with support $[0,1]$.

2. We determine the simulated log-return $\hat{x}_1$ for the first asset as:

$$\hat{u} = F_{X_{1,t_0,T}|\mathcal{F}_{t_0}}(x_1).$$

3. We determine a random number $\hat{v}$ that satisfies the following equation:

$$\hat{s} = c_u(v). \tag{50}$$

   $c_u(\cdot)$ is the partial derivative of the Copula function with respect to the variable u and corresponds to the conditional distribution of V given the event $U = u$. In the Plackett Copula, equation (50) has an analytical solution as explicited in [15].

4. We determine the simulated log-return $\hat{x}_2$ for the second asset solving with respect to x_2 the following equation:

$$\hat{u} = F_{X_{2,t_0,T}|\mathcal{F}_{t_0}}(x_2).$$

5. We repeat M times steps 1 - 4.

The formula in (36) is finally approximated as follows:

$$\hat{P}_{t_0} = e^{-r(T-t_0)} \frac{1}{M} \sum_{j=1}^{M} \max\left\{S_{1,t_0,T}e^{\hat{x}_{1,j}} - S_{2,t_0,T}e^{\hat{x}_{2,j}} - K; 0\right\}. \tag{51}$$

The results are reported in Table 8 where all prices, obtained by applying (37), belong to the Monte Carlo confidence interval determined with 10^5 simulations. Prices obtained with (37) coincide with those obtained using the double integration method. As expected, the main advantage of our approach is the reduction of the computational time.

K	Single	Double	$CI^-_{95\%}$	$CI^+_{95\%}$
0.0	6.149	6.149	6.127	6.182
	(1.12)	(34.66)	(51.22)	
2.5	4.019	4.019	3.982	4.031
	(1.12)	(36.54)	(101.95)	
5	2.266	2.266	2.251	2.293
	(1.20)	(34.32)	(103.62)	
7.5	1.111	1.111	1.107	1.140
	(1.13)	(34.24)	(45.96)	
10	0.533	0.532	0.526	0.551
	(1.13)	(34.07)	(44.94)	

Table 8: Comparison of 90 days European spread call option prices computed using Formula (37) (Single), double integration method (Double) and Monte Carlo 95% confidence interval. To determine the Monte Carlo upper and lower bound we simulate 10^5 realization of the spread payoff.

6 Conclusion

In this paper we proposed a new formula for pricing a spread option based on a generic twice differentiable Copula function. The accuracy of the newly introduced formula is investigated using both Monte Carlo simulations and a numerical method for approximating multiple integrals. We showed that the pricing of spread options based on a copula function is more efficient than the alternative methods considered. The main advantage refers to the fact that the pricing formula depends on a single integral that can be easily computed, at least numerically, in any standard programming language.

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