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This paper considers the valuation of a spread call when asset prices are log-normal. The implicit strategy of the Kirk formula is to exercise if the price of the long asset exceeds a given power function of the price of the short asset. We derive a formula for the spread call value, conditional on following this feasible, but non-optimal, exercise strategy. Numerical investigations indicate that the lower bound produced by our formula is extremely accurate. The precision is much greater than the Kirk formula. Moreover, optimizing with respect to the strategy parameters (which corresponds to the Carmona–Durrleman procedure) yields only a marginal improvement of accuracy (if any).

Keywords: Finance; Financial applications; Financial derivatives; Financial engineering; Financial mathematics; Financial modelling; Financial options

JEL Classification: C6, C63, D8, D81, G1, G12, G13

1. Introduction

This paper considers the valuation of a spread call when the asset prices are log-normal. The starting point is the observation that the implicit strategy of the Kirk formula is to exercise if the price of the long asset exceeds a given power function of the price of the short asset. We derive a formula that evaluates the spread call, conditional on following this exercise strategy. The formula consists of three terms, one for each of the two assets, and one for the strike. A standard normal cumulative probability enters into each term, and each argument is a function of the forward prices, time to exercise, volatilities, and correlation. The formula fits well into the tradition of Black–Scholes, Black76, and Margrabe.

Numerical investigations indicate that our formula is extremely accurate. The precision is much higher than for the Kirk formula. Furthermore, the accuracy of our formula is comparable to the precision of the lower bound procedure suggested by Carmona and Durrleman, which requires a two-dimensional optimization scheme.

2. Assumptions

Consider a frictionless market with no arbitrage opportunities and with a constant riskless interest rate r. Assume two assets where the prices at the future date T are

$$S_1 = F_1 \exp\left\{-\frac{1}{2}\sigma_1^2 T + \sigma_1\sqrt{T}\varepsilon_1\right\}, \tag{1}$$

$$S_2 = F_2 \exp\left\{-\frac{1}{2}\sigma_2^2 T + \sigma_2\sqrt{T}\varepsilon_2\right\}, \tag{2}$$

with respect to the equivalent martingale measure (EMM), where F_1 and F_2 are the current forward prices for delivery at the future date T, σ_1 and σ_2 are volatilities, and ε_1 and ε_2 are standard normal random variables with correlation ρ. It follows from the above that the two asset prices are log-normal, and that the expected future price for each asset (w.r.t. the EMM) coincides with the current forward price.

3. The spread call

Consider a European call option on the price spread $S_1 - S_2$ with strike $K \geq 0$ and time to exercise T. The call option pay-off at time T is

$$C(T) = (S_1 - S_2 - K)^+, \tag{3}$$

where $(\ )^+$ denotes the positive part. The call value at time 0 can be represented by (see, e.g., Cox and Ross (1976), Harrison and Kreps (1979), and Harrison and Pliska (1981))

$$C = e^{-rT} E_0[(S_1 - S_2 - K)^+], \tag{4}$$

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where the expectation is taken with respect to the EMM, and r is the riskless interest rate. It follows from the put–call parity that the value of a European put option on the price spread $S_1 - S_2$ with strike $K \geq 0$ and time to exercise T is given by $P = C - e^{-rT}(F_1 - F_2 - K)$.

With both S_1 and S_2 being log-normal, there is no known general formula for the spread call value. However, closed-form solutions are available for the following limiting cases. Firstly, if $F_2 = 0$, the call spread collapses into a standard call on S_1, and the value is given by the Black76 formula (Black 1976). Secondly, if $K = 0$, the call spread collapses into an option to exchange one asset for another. The option value in this case is given by the Margrabe formula (Margrabe 1978).

4. The Kirk formula

In the general case, however, we have to rely on either approximation formulas or extensive numerical methods. Approximation formulas allow quick calculations and facilitate analytical tractability, whereas numerical methods typically produces more accurate results. Practitioners are very focused on simple calculations and real-time solutions, hence a closed-form approximation formula is typically the preferred alternative.

Kirk (1995) suggests the following approximation to the spread call:†

$$c_K = e^{-rT}\{F_1 N(d_{K,1}) - (F_2 + K)N(d_{K,2})\}, \tag{5}$$

where $N(\ )$ denotes the standard normal cumulative probability function, and $d_{K,1}$ and $d_{K,2}$ are given by

$$d_{K,1} = \frac{\ln(F_1/(F_2+K)) + \frac{1}{2}\sigma_K^2 T}{\sigma_K\sqrt{T}}, \tag{6}$$

$$d_{K,2} = d_1 - \sigma_K\sqrt{T}, \tag{7}$$

$$\sigma_K = \sqrt{\sigma_1^2 - 2\frac{F_2}{F_2+K}\rho\sigma_1\sigma_2 + \left(\frac{F_2}{F_2+K}\right)^2\sigma_2^2}. \tag{8}$$

5. The Carmona–Durrleman procedure

Carmona and Durrleman (2003a,b) represent the future spot prices by two independent state variables and model the correlation using trigonometric functions. In particular, equations (1) and (2) above translate into

$$S_1 = F_1 \exp\left\{-\frac{1}{2}\sigma_1^2 T + (z_1 \sin\phi + z_2\cos\phi)\sigma_1\sqrt{T}\right\}, \tag{9}$$

$$S_2 = F_2 \exp\left\{-\frac{1}{2}\sigma_2^2 T + \sigma_2\sqrt{T}z_2\right\}, \tag{10}$$

where z_1 and z_2 are standard normal and independent random variables, and $\cos\phi = \rho$ where $\phi \in [0, \pi]$. The authors consider the value from exercising the spread option according to a feasible, but non-optimal, strategy conditional on the two state variables. In particular, the strategy is to exercise when

$$Y_{\theta^*} \equiv z_1 \sin\theta^* - z_2\cos\theta^* \leq d^*, \tag{11}$$

where $\theta^* \in [\pi, 2\pi]$ and d^* are found numerically by maximizing the option value.‡

The value from following this strategy, which represents a lower bound to the true spread option value, is§

$$\begin{aligned} c_{\mathrm{CD}} &= e^{-rT}E_0[(S_1 - S_2 - K)\cdot I(z_1\sin\theta^* - z_2\cos\theta^* \leq d^*)] \\ &= e^{-rT}\{F_1 N(d^* + \sigma_1\sqrt{T}\cos(\theta^*+\phi)) \\ &\quad - F_2 N(d^* + \sigma_2\sqrt{T}\cos\theta^*) - KN(d^*)\} \end{aligned} \tag{12}$$

(see appendix B).¶ Numerical investigations by Carmona and Durrleman (2003a) indicate that their lower bound optimization procedure produces very accurate estimates to the true option value.

6. A closed-form spread option formula

It can be verified (see appendix C) that the Kirk formula follows from the expectation

$$c_K = e^{-rT}E_0\left[\left(S_1 - \frac{aS_2^b}{E_0[S_2^b]}\right)\cdot I\left(S_1 \geq \frac{aS_2^b}{E_0[S_2^b]}\right)\right], \tag{13}$$

where $I(\ )$ represents the indicator function, assuming unity whenever the argument is true and zero otherwise, where $a = F_2 + K$, $b = F_2/(F_2 + K)$, and

$$E_0[S_2^b] = \exp\left\{\frac{1}{2}b(b-1)\sigma_2^2 T\right\}F_2^b.$$

Note that $0 \leq b < 1$ when $K \geq 0$. Observe from equation (13) that the implicit strategy of the Kirk formula is to exercise if and only if S_1 exceeds a scaled power function of S_2.

We want to use the above insight to obtain an alternative spread option approximation formula. Now consider the future spread call pay-off conditional on

†By the put–call parity, the Kirk approximation of a put on the price spread $S_1 - S_2$ with strike $K \geq 0$ and time to exercise T is $p_K = c_K - e^{-rT}(F_1 - F_2 - K)$.

‡$\phi \in [0, \pi]$ and $\theta^* \in [\pi, 2\pi]$ translate into $\sin\phi \geq 0$ and $\cos\theta^* \leq 0$. To motivate this, observe from equations (9) and (11) that an increase in z_1 will increase the pay-off from asset 1, and push the call more in-the-money (less out-of-the-money).

§There is a typo in equation (20) of Carmona and Durleman (2003a) as well as in equation (6.3) of Carmona and Durrleman (2003b). The trigonometric function entering the second term should read cos and not sin.

¶By the put–call parity, the Carmona–Durrleman approximation of a put on the price spread $S_1 - S_2$ with strike $K \geq 0$ and time to exercise T is $p_{\mathrm{CD}} = c_{\mathrm{CD}} - e^{-rT}(F_1 - F_2 - K)$.

following the Kirk exercise strategy. We can express the future pay-off from following this strategy as

$$c(T) = (S_1 - S_2 - K) \cdot I\left(S_1 \geq \frac{aS_2^b}{E_0[S_2^b]}\right). \tag{14}$$

The exercise strategy in equation (14) is clearly feasible. However, we know that the optimal strategy is to exercise the spread option if and only if the pay-off from exercising is positive (or zero). The exercise strategy that is specified by the indicator function in equation (14) deviates from the optimal one. Hence, the value from following this feasible, but non-optimal, strategy represents a lower bound to the true spread option value.

Proposition 6.1: *Approximate the spread call value by the following formula*:

$$\begin{aligned} c(a,b) &= e^{-rT}E_0\left[(S_1 - S_2 - K) \cdot I\left(S_1 \geq \frac{aS_2^b}{E_0[S_2^b]}\right)\right] \\ &= e^{-rT}\{F_1N(d_1) - F_2N(d_2) - KN(d_3)\}, \end{aligned} \tag{15}$$

where d_1, d_2, and d_3 are defined by

$$d_1 = \frac{\ln(F_1/a) + \left(\frac{1}{2}\sigma_1^2 - b\rho\sigma_1\sigma_2 + \frac{1}{2}b^2\sigma_2^2\right)T}{\sigma\sqrt{T}}, \tag{16}$$

$$d_2 = \frac{\ln(F_1/a) + (-\frac{1}{2}\sigma_1^2 + \rho\sigma_1\sigma_2 + \frac{1}{2}b^2\sigma_2^2 - b\sigma_2^2)T}{\sigma\sqrt{T}}, \tag{17}$$

$$d_3 = \frac{\ln(F_1/a) + (-\frac{1}{2}\sigma_1^2 + \frac{1}{2}b^2\sigma_2^2)T}{\sigma\sqrt{T}}, \tag{18}$$

$$\sigma = \sqrt{\sigma_1^2 - 2b\rho\sigma_1\sigma_2 + b^2\sigma_2^2}, \tag{19}$$

and where the constants a and b are given by

$$a = F_2 + K, \tag{20}$$

$$b = \frac{F_2}{F_2 + K}. \tag{21}$$

By the put–call parity, the put option on the price spread $S_1 - S_2$ with strike K and time to exercise T is approximated by $p = c - e^{-rT}(F_1 - F_2 - K)$.

Proof: See appendix D. □

The Black–Scholes, the Black76, and the Margrabe formulas consist of one term for each component that enters into the future option pay-off. Equations (15)–(19) conform with this tradition. This form is similar to equation (6.3) of Carmona and Durrleman (2003b). However, by comparing with our equations (15)–(19), there should be no doubt that our representation of the arguments d_1, d_2, and d_3 is more along the lines of the Black–Scholes, Black76, and Margrabe than the corresponding arguments found in Carmona and Durrleman (2003b).

In order to obtain a stricter lower bound, one could optimize the spread call value $c(a, b)$ above with respect to a and b. Optimal parameters a^* and b^* can be obtained by expanding the first-order conditions and applying the Newton–Raphson iterative procedure, using equations (20) and (21) as the initial guess. The first-order conditions, as well as the second-order partials needed for Newton–Raphson, are provided in appendix E.

Optimizing our formula with respect to a and b is in fact equivalent to the Carmona and Durrleman procedure (see appendix F). Extensive numerical investigations, however, indicate that, with our initial choice a and b, there is very little to gain from implementing such an optimization procedure. Put differently, the formula stated in equations (15)–(21) represents a very tight lower bound to the true spread option value.

7. Numerical results

In the following, we compare the accuracy of the Kirk approximation in equations (5)–(8), our formula in equations (15)–(21), and the optimal lower bound following from maximizing the formula with respect to the parameters a and b (which produces the same results as the Carmona–Durrleman optimization procedure).

To approximate the true spread option value, we apply a Monte Carlo simulation procedure using the first 100,000 pairs of numbers from a two-dimensional Halton sequence. In order to reduce the simulation error, we use our representation of the spread call as control variate.†

We adopt the numerical example in Carmona and Durrleman (2003a), where the annual riskless interest rate is $r = 0.05$ and the time horizon is $T = 1$ year. Their numerical case translates into forward prices $F_1 = e^{(0.05-0.03)\cdot 1}110 \approx 112.22$ and $F_2 = e^{(0.05-0.02)\cdot 1}100 \approx 103.05$. The annualized volatilities are $\sigma_1 = 0.10$ and $\sigma_2 = 0.15$.

We consider different combinations of strike K and correlation ρ. In the case $K = 0$, the spread option reduces to the Margrabe exchange option (Margrabe 1978). With $K > 0$, the option corresponds to a call on the price spread $S_1 - S_2$. In the case $K < 0$, the option represents a put on the opposite price spread, i.e. $S_2 - S_1$. The put values are obtained by the put–call parity.

Table 1 shows the results of the spread option valuations. The first row for each K value gives the result from the Kirk formula. The second row (in italics)

†Rewrite equation (4) as

$$E_0[e^{-rT}C(T)] = c + E_0[e^{-rT}(C(T) - c(T))],$$

where the pay-offs $C(T)$ and $c(T)$ are defined in equations (3) and (14), and c is our spread option formula in equations (15)–(21). Clearly, the two pay-offs $C(T)$ and $c(T)$ are highly correlated. Consequently, the simulation error from evaluating the expectation on the RHS is much smaller than the simulation error from evaluating the expectation on the LHS.

Table 1. Spread option value.

	ρ					
K	−1	−0.5	0	0.3	0.8	1
−20	29.6752	29.0056	28.3848	28.0709	27.7704	27.7538
	29.6561	*28.9948*	*28.3811*	*28.0701*	*27.7701*	*27.7538*
	(2.36E−6)	(1.07E−5)	(1.72E−5)	(1.11E−05)	(6.14E−6)	(8.24E−6)
	29.6561	28.9948	28.3811	28.0701	27.7701	27.7538
	29.6561	28.9948	28.3811	28.0701	27.7701	27.7538
−10	21.8787	20.9114	19.8917	19.2710	18.3816	18.2444
	21.8686	*20.9050*	*19.8889*	*19.2701*	*18.3811*	*18.2439*
	(2.24E−6)	(2.24E−6)	(7.59E−6)	(5.68E−6)	(5.16E−6)	(1.06E−5)
	21.8686	20.9050	19.8889	19.2701	18.3811	18.2439
	21.8686	20.9049	19.8888	19.2701	18.3811	18.2438
0	15.1332	13.9180	12.5237	11.5618	9.6325	8.8212
	15.1332	*13.9180*	*12.5237*	*11.5618*	*9.6325*	*8.8212*
	(0)	(0)	(0)	(0)	(0)	(0)
	15.1332	13.9180	12.5237	11.5618	9.6325	8.8212
	15.1332	13.9180	12.5237	11.5618	9.6325	8.8212
5	12.2425	10.9543	9.4431	8.3649	5.9628	4.4420
	12.2441	*10.9562*	*9.4453*	*8.3674*	*5.9670*	*4.4542*
	(1.61E−5)	(1.29E−5)	(1.21E−5)	(1.51E−5)	(1.68E−5)	(3.35E−6)
	12.2441	10.9562	9.4453	8.3674	5.9670	4.4542
	12.2441	10.9562	9.4453	8.3674	5.9670	4.4542
15	7.5376	6.2559	4.7562	3.6907	1.3545	0.0724
	7.5218	*6.2422*	*4.7445*	*3.6798*	*1.3425*	*0.0488*
	(3.25E−5)	(2.82E−5)	(2.71E−5)	(2.99E−5)	(3.95E−5)	(3.71E−5)
	7.5218	6.2422	4.7444	3.6797	1.3422	0.0488
	7.5217	6.2421	4.7443	3.6796	1.3421	0.0479
25	4.2475	3.1686	1.9923	1.2441	0.1124	0.0000
	4.2014	*3.1300*	*1.9621*	*1.2200*	*0.1041*	*0.0000*
	(2.06E−5)	(2.89E−5)	(4.72E−5)	(6.05E−5)	(7.79E−5)	(0)
	4.2014	3.1300	1.9620	1.2198	0.1039	0.0000
	4.2013	3.1298	1.9617	1.2194	0.1032	0.0000

The first row for each K value is from Kirk's formula.
The second row (in italics) is the simulation result (100,000 trials).
The third row (in parentheses) is the standard error of the simulation result.
The fourth row is from optimizing our formula w.r.t. a and b.
The fifth row is from our formula (a and b fixed).

is the spread option value obtained by Monte Carlo simulation with 100,000 trials. We use the simulation results as the benchmark for the true spread option value. The third row gives the standard error of the simulation result. The fourth row is the result from optimizing our formula with respect to a and b, which is similar to the Carmona–Durrleman optimization procedure. The last row for each K value represents the result from our formula. It is interesting to observe from table 1 that the Kirk formula violates the lower bound (provided by our formula) for all correlations when the strike is $K = 5$.

Table 2 shows the pricing error associated with the Kirk formula (first row for each K value) and our formula (second row) compared with the benchmark. Note that the Kirk formula seems to underprice the spread option when the strike is closer to zero, and to overprice the spread option when the strike is further away from zero. Our formula represents a lower bound, hence the pricing error (if any) is negative. Observe that, for all relevant cases in table 2,† our formula performs much better than the Kirk formula. In our view, practitioners looking for a pricing formula are better off using our formula than the Kirk formula when evaluating spread options.

Table 3 shows the reduced pricing error from optimizing our formula with respect to a and b compared with using our formula with parameter values for a and b given by equations (20) and (21). Recall that optimizing our formula corresponds to the Carmona–Durrleman optimization procedure. Hence, we may interpret the results in table 3 as the gain from using their numerical optimization procedure compared with our formula, which is closed form. The results in the table indicate that the improved accuracy from implementing numerical optimization is either marginal or zero. For practical purposes, the benefits of a closed-form solution are obvious. In our view, the numerical results indicate that the accuracy of our formula is comparable to the accuracy of using an optimization procedure. Consequently, practitioners should settle for our formula rather than the Carmona–Durrleman procedure when evaluating spread options.

†When $K = 0$, both formulas degenerate to the Margrabe exchange option formula, which represents the true value in this case. Hence, the pricing errors are zero in these cases.

Table 2. Pricing error.

	ρ					
K	−1	−0.5	0	0.3	0.8	1
−20	0.0191	0.0108	0.0037	0.0008	0.0003	0.0001
	0.0000	0.0000	−0.0001	0.0000	0.0000	0.0000
−10	0.0101	0.0065	0.0029	0.0010	0.0005	0.0006
	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
5	−0.0016	−0.0019	−0.0023	−0.0025	−0.0042	−0.0122
	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
15	0.0158	0.0137	0.0118	0.0109	0.0120	0.0236
	−0.0001	−0.0001	−0.0001	−0.0002	−0.0004	−0.0009
25	0.0461	0.0386	0.0301	0.0241	0.0083	0.0000
	−0.0001	−0.0002	−0.0004	−0.0006	−0.0009	0.0000

The first row for each K value is the pricing error following from the Kirk formula.
The second row is the pricing error following from our formula.

Table 3. Reduced pricing error from optimization of our formula w.r.t. a and b.

	ρ					
K	−1	−0.5	0	0.3	0.8	1
−20	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
−10	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
5	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
15	0.0001	0.0001	0.0001	0.0001	0.0002	0.0009
25	0.0001	0.0001	0.0003	0.0004	0.0007	0.0000

8. Closed-form Greeks

The Greeks are of great importance for risk management and hedging. The Greeks of our formula in equations (15)–(19), assuming a and b constant, are reported in appendix G. In this section, we focus on the option price sensitivity with respect to the two forward prices. We find that the two deltas are

$$\frac{\partial c}{\partial F_1} = \mathrm{e}^{-rT} N(d_1) + \mathrm{e}^{-rT}\{F_1 n(d_1) - F_2 n(d_2) - K n(d_3)\} \times \frac{1}{F_1 \sigma\sqrt{T}}, \tag{22}$$

$$\frac{\partial c}{\partial F_2} = -\mathrm{e}^{-rT} N(d_2) + \mathrm{e}^{-rT}\{F_1 n(d_1) - F_2 n(d_2) - K n(d_3)\} \times \frac{-b}{F_2 \sigma\sqrt{T}}, \tag{23}$$

respectively, where $n(\ )$ denotes the standard normal probability density function. In the following, we investigate the accuracy of our analytical deltas in equations (22) and (23) assuming that the exercise strategy parameters a and b are fixed (and evaluated in equations (20) and (21)).

The true deltas of equation (4) must be found by a numerical method. We obtain the benchmark deltas from the following expressions:

$$\frac{\partial C}{\partial F_1} = \mathrm{e}^{-rT} N(d_1) + \mathrm{e}^{-rT} E_0\left[\frac{S_1}{F_1}\cdot\left\{I(S_1 \geq S_2 + K) - I\left(S_1 \geq \frac{aS_2^b}{E_0[S_2^b]}\right)\right\}\right], \tag{24}$$

$$\frac{\partial C}{\partial F_2} = -\mathrm{e}^{-rT} N(d_2) - \mathrm{e}^{-rT} E_0\left[\frac{S_2}{F_2}\cdot\left\{I(S_1 \geq S_2 + K) - I\left(S_1 \geq \frac{aS_2^b}{E_0[S_2^b]}\right)\right\}\right], \tag{25}$$

where d_1 and d_2 are defined in equations (16) and (17). The first term on the right-hand side of each equation is closed form, whereas the second term is evaluated by Monte Carlo simulation using the first 100,000 pairs of numbers from a two-dimensional Halton sequence.†

We use the same input parameters as above, i.e. a riskless interest rate of $r = 0.05$ per annum, a time horizon of $T = 1$ year, forward prices $F_1 = \mathrm{e}^{(0.05-0.03)\cdot 1}\ 110 \approx 112.22$ and $F_2 = \mathrm{e}^{(0.05-0.02)\cdot 1}\ 100 \approx 103.05$, and volatilities $\sigma_1 = 0.10$ and $\sigma_2 = 0.15$. We consider the combinations of strike and correlation that translate into a non-degenerate spread call option, i.e. strikes $K \geq 0$ and correlations $|\rho| < 1$. The results are provided in tables 4 and 5. These tables indicate that the analytical deltas are very accurate. However, the difference between the true delta and the closed-form delta seem to increase with correlation as well as strike.

9. Conclusions

This paper considers the valuation of a European spread option when the asset prices are log-normal. We derive a

†It can be shown that

$$\frac{\partial C}{\partial F_1} \equiv \frac{\partial C}{\partial F_1}\{\mathrm{e}^{-rT} E_0[(S_1 - S_2 - K)^+]\} = \mathrm{e}^{-rT} E_0\left[\frac{S_1}{F_1}\cdot I(S_1 \geq S_2 + K)\right].$$

One could of course simulate the delta directly from this expression. In order to increase the precision, however, we use the Kirk exercise strategy as a control variate and apply the result

$$\mathrm{e}^{-rT} E_0\left[\frac{S_1}{F_1}\cdot I\left(S_1 \geq \frac{aS_2^b}{E_0[S_2^b]}\right)\right] = \mathrm{e}^{-rT} N(d_1)$$

(see appendix C) to obtain equation (24). We obtain equation (25) in a similar fashion using a result from appendix D.

Table 4. The spread call delta w.r.t. forward price 1.

	ρ			
K	−0.5	0	0.3	0.8
0	*0.6579*	*0.6786*	*0.7008*	*0.7936*
	(0.0000)	(0.0000)	(0.0000)	(0.0000)
	0.6579	0.6786	0.7008	0.7936
5	*0.5831*	*0.5901*	*0.5996*	*0.6520*
	(0.0001)	(0.0001)	(0.0001)	(0.0001)
	0.5830	0.5901	0.5994	0.6516
15	*0.4297*	*0.3928*	*0.3669*	*0.2623*
	(0.0001)	(0.0001)	(0.0001)	(0.0002)
	0.4199	0.3926	0.3668	0.2610
25	*0.2643*	*0.2131*	*0.1673*	*0.0360*
	(0.0001)	(0.0001)	(0.0001)	(0.0001)
	0.2640	0.2122	0.1659	0.0340

The first row for each K value (in italics) is the true delta from equation (24) using simulation (100,000 trials).
The second row (in parentheses) is the standard error of the simulation result.
The third row is the closed-form delta from equation (22).

Table 5. The spread call delta w.r.t. forward price 2.

	ρ			
K	−0.5	0	0.3	0.8
0	*−0.5814*	*−0.6175*	*−0.6510*	*−0.7708*
	(0.0000)	(0.0000)	(0.0000)	(0.0000)
	−0.5814	−0.6175	−0.6510	−0.7708
5	*−0.5018*	*−0.5233*	*−0.5433*	*−0.6201*
	(0.0001)	(0.0001)	(0.0001)	(0.0001)
	−0.5018	−0.5234	−0.5432	−0.6198
15	*−0.3401*	*−0.3280*	*−0.3130*	*−0.2345*
	(0.0001)	(0.0001)	(0.0001)	(0.0002)
	−0.3401	−0.3277	−0.3127	−0.2330
25	*−0.2000*	*−0.1660*	*−0.1330*	*−0.0297*
	(0.0001)	(0.0001)	(0.0001)	(0.0001)
	−0.1994	−0.1649	−0.1314	−0.0278

The first row for each K value (in italics) is the true delta from equation (25) using simulation (100,000 trials).
The second row (in parentheses) is the standard error of the simulation result.
The third row is the closed-form delta from equation (23).

spread option formula that consists of three terms, one for each of the two assets and one for the strike. A standard normal cumulative probability enters into each term, and each argument is a function of the forward prices, time to exercise, volatilities, and correlation. The formula fits well into the tradition of Black–Scholes, Black76, and Margrabe.

Numerical investigations indicate that our formula is extremely accurate. The precision is much better than for the Kirk formula, which is the current market standard in practice. Moreover, the precision of our formula is comparable to the lower bound procedure of Carmona and Durrleman, which requires a two-dimensional optimization scheme.

Option pricing in practice involves market prices that do not obey the assumption of constant volatility. The market prices of standard options are often translated into a so-called implicit volatility dependent on strike and time to exercise. The implicit volatility is calculated using an appropriate closed-form option pricing formula (e.g., Black76). Now, assume a market has two underlying assets, standard call and put options on each asset, as well as spread options. Our spread option formula may then be used to calculate an implicit correlation between the underlying asset prices. The modeling of a strike- and time-dependent correlation represents a challenging area for further research.

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Appendix A: Bivariate normal variables—A useful result

The standard bivariate normal density function is defined as

$$m(x, y; \rho) = \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left\{-\frac{x^2 - 2\rho xy + y^2}{2(1-\rho^2)}\right\},$$

where ρ is the correlation. The density function satisfies the identity

$$\begin{aligned}&\exp\left\{(ax + by) - \frac{1}{2}(a^2 + 2\rho ab + b^2)\right\} m(x, y; \rho)\\ &\quad = m(x - (a + \rho b), y - (\rho a + b); \rho),\end{aligned}$$

where $E[\exp\{(ax+by)-\frac{1}{2}(a^2+2\rho ab+b^2)\}]=1$. Consequently,

$$\begin{aligned}
&E\left[\exp\left\{(ax+by)-\frac{1}{2}(a^2+2\rho ab+b^2)\right\}h(x,y)\right]\\
&=\int\int\exp\left\{(ax+by)-\frac{1}{2}(a^2+2\rho ab+b^2)\right\}\\
&\quad\times h(x,y)m(x,y;\rho)\mathrm{d}y\mathrm{d}x\\
&=\int\int h(x,y)m(x-(a+\rho b),y-(\rho a+b);\rho)\mathrm{d}y\mathrm{d}x\\
&=\int\int h(x+(a+\rho b),y+(\rho a+b))m(x,y;\rho)\mathrm{d}y\mathrm{d}x\\
&=E[h(x+(a+\rho b),y+(\rho a+b))].
\end{aligned}$$

Appendix B: The Carmona–Durrleman result

For notational convenience, write

$$S_1=F_1\exp\left\{-\frac{1}{2}v_1^2+v_1(z_1\sin\phi+z_2\cos\phi)\right\},$$
$$S_2=F_2\exp\left\{-\frac{1}{2}v_2^2+v_2z_2\right\},$$

where z_1 and z_2 are independent and standard normal. Consider the expectation

$$\begin{aligned}
c_{\mathrm{CD}}&=E[(S_1-S_2-K)I(z_1\sin\theta^*-z_2\cos\theta^*\le d^*)]\\
&=E[S_1I(z_1\sin\theta^*-z_2\cos\theta^*\le d^*)]\\
&\quad-E[S_2I(z_1\sin\theta^*-z_2\cos\theta^*\le d^*)]\\
&\quad-E[KI(z_1\sin\theta^*-z_2\cos\theta^*\le d^*)].
\end{aligned}$$

Observe that, due to the identity

$$(\sin\phi)^2+(\cos\phi)^2=1,$$

both $z\equiv(z_1\sin\phi+z_2\cos\phi)$ and $Y_{\theta^*}\equiv(z_1\sin\theta^*-z_2\cos\theta^*)$ are standard normal. Evaluate the last term

$$E[KI(z_1\sin\theta^*-z_2\cos\theta^*\le d^*)]=KE[I(z\le d^*)]=KN(d^*).$$

Evaluate the second term,

$$\begin{aligned}
&E[S_2I(z_1\sin\theta^*-z_2\cos\theta^*\le d^*)]\\
&=E\left[F_2\exp\left\{-\frac{1}{2}v_2^2+v_2z_2\right\}I(z_1\sin\theta^*-z_2\cos\theta^*\le d^*)\right]\\
&=F_2E[I(z_1\sin\theta^*-(z_2+v_2)\cos\theta^*\le d^*)]\\
&=F_2E[I(z_1\sin\theta^*-z_2\cos\theta^*\le d^*+v_2\cos\theta^*)]\\
&=F_2E[I(z\le d^*+v_2\cos\theta^*)]\\
&=F_2N(d^*+v_2\cos\theta^*),
\end{aligned}$$

using the result in appendix A. Finally, evaluate the first term

$$\begin{aligned}
&E[S_1I(z_1\sin\theta^*-z_2\cos\theta^*\le d^*)]\\
&=E\left[F_1\exp\left\{-\frac{1}{2}v_1^2+v_1(z_1\sin\phi+z_2\cos\phi)\right\}\right.\\
&\quad\left.\times I(z_1\sin\theta^*-z_2\cos\theta^*\le d^*)\right]\\
&=F_1E[I(\sin\theta^*(z_1+v_1\sin\phi)-\cos\theta^*(z_2+v_1\cos\phi)\le d^*)]\\
&=F_1E[I(z_1\sin\theta^*-z_2\cos\theta^*\le d^*\\
&\quad+v_1(\cos\theta^*\cos\phi-\sin\theta^*\sin\phi))]\\
&=F_1E[I(z\le d^*+v_1\cos(\theta^*+\phi))]\\
&=F_1N(d^*+v_1\cos(\theta^*+\phi)),
\end{aligned}$$

where we use the result in appendix A and the identity

$$\cos\theta^*\cos\phi-\sin\theta^*\sin\phi=\cos(\theta^*+\phi).$$

Now collect the results, apply riskless discounting, and translate $v_1=\sigma_1\sqrt{T}$ and $v_2=\sigma_2\sqrt{T}$ to obtain the result in equation (12).

Appendix C: The implicit Kirk strategy

For notational convenience, write

$$S_1=F_1\exp\left\{-\frac{1}{2}v_1^2+v_1\varepsilon_1\right\},$$

$$S_2=F_2\exp\left\{-\frac{1}{2}v_2^2+v_2\varepsilon_2\right\},$$

$$\frac{aS_2^b}{E[S_2^b]}=a\exp\left\{-\frac{1}{2}b^2v_2^2+bv_2\varepsilon_2\right\},$$

and consider the expectation

$$\begin{aligned}
&E\left[\left(S_1-\frac{aS_2^b}{E[S_2^b]}\right)^+\right]\\
&=E\left[S_1I\left(S_1\ge\frac{aS_2^b}{E[S_2^b]}\right)\right]-E\left[aS_2^bI\left(S_1\ge\frac{aS_2^b}{E[S_2^b]}\right)\right].
\end{aligned}$$

The two terms are evaluated as follows:

$$\begin{aligned}
&E\left[S_1I\left(S_1\ge\frac{aS_2^b}{E[S_2^b]}\right)\right]\\
&=E\left[F_1\exp\left\{-\frac{1}{2}v_1^2+v_1\varepsilon_1\right\}I\left(F_1\exp\left\{-\frac{1}{2}v_1^2+v_1\varepsilon_1\right\}\right.\right.\\
&\qquad\left.\left.\ge a\exp\left\{-\frac{1}{2}b^2v_2^2+bv_2\varepsilon_2\right\}\right)\right]\\
&=F_1E\left[I\left(F_1\exp\left\{-\frac{1}{2}v_1^2+v_1(\varepsilon_1+v_1)\right\}\right.\right.\\
&\qquad\left.\left.\ge a\exp\left\{-\frac{1}{2}b^2v_2^2+bv_2(\varepsilon_2+\rho v_1)\right\}\right)\right]\\
&=F_1E\left[I\left(v_1\varepsilon_1-bv_2\varepsilon_2\ge-\ln(F_1/a)-\frac{1}{2}v_1^2+b\rho v_1v_2-\frac{1}{2}b^2v_2^2\right)\right]\\
&=F_1E\left[I\left(\varepsilon\ge-\frac{\ln(F_1/a)+\frac{1}{2}v_1^2-b\rho v_1v_2+\frac{1}{2}b^2v_2^2}{\sqrt{v_1^2-2b\rho v_1v_2+b^2v_2^2}}\right)\right]\\
&=F_1N\left(\frac{\ln(F_1/a)+\frac{1}{2}v_1^2-b\rho v_1v_2+\frac{1}{2}b^2v_2^2}{\sqrt{v_1^2-2b\rho v_1v_2+b^2v_2^2}}\right)
\end{aligned}$$

and

$$E\left[\frac{aS_2^b}{E[S_2^b]}I\left(S_1 \geq \frac{aS_2^b}{E[S_2^b]}\right)\right]$$

$$= E\left[a\exp\left\{-\frac{1}{2}b^2v_2^2 + bv_2\varepsilon_2\right\}I\left(F_1\exp\left\{-\frac{1}{2}v_1^2 + v_1\varepsilon_1\right\} \geq a\exp\left\{-\frac{1}{2}b^2v_2^2 + bv_2\varepsilon_2\right\}\right)\right]$$

$$= aE\left[\exp\left\{-\frac{1}{2}b^2v_2^2 + bv_2\varepsilon_2\right\}I\left(F_1\exp\left\{-\frac{1}{2}v_1^2 + v_1\varepsilon_1\right\} \geq a\exp\left\{-\frac{1}{2}b^2v_2^2 + bv_2\varepsilon_2\right\}\right)\right]$$

$$= aE\left[I\left(F_1\exp\left\{-\frac{1}{2}v_1^2 + v_1(\varepsilon_1 + \rho bv_2)\right\} \geq a\exp\left\{-\frac{1}{2}b^2v_2^2 + bv_2(\varepsilon_2 + bv_2)\right\}\right)\right]$$

$$= aE\left[I\left(v_1\varepsilon_1 - bv_2\varepsilon_2 \geq -\ln(F_1/a) + \frac{1}{2}v_1^2 - b\rho v_1v_2 + \frac{1}{2}b^2v_2^2\right)\right]$$

$$= aE\left[I\left(\varepsilon \geq -\frac{\ln(F_1/a) - \frac{1}{2}v_1^2 + b\rho v_1v_2 - \frac{1}{2}b^2v_2^2}{\sqrt{v_1^2 - 2b\rho v_1v_2 + b^2v_2^2}}\right)\right]$$

$$= aN\left(\frac{\ln(F_1/a) - \frac{1}{2}v_1^2 + b\rho v_1v_2 - \frac{1}{2}b^2v_2^2}{\sqrt{v_1^2 - 2b\rho v_1v_2 + b^2v_2^2}}\right).$$

Now collect the results, apply riskless discounting, choose the constant a such that

$$a = F_2 + K,$$

and translate $v_1 = \sigma_1\sqrt{T}$, $v_2 = \sigma_2\sqrt{T}$, and $\sqrt{v_1^2 - 2b\rho v_1v_2 + b^2v_2^2} = \sigma\sqrt{T}$ to obtain the Kirk formula stated in equations (5)–(8).

Appendix D: Derivation of the spread option formula

Consider the expectation

$$E\left[(S_1 - S_2 - K)I\left(S_1 \geq \frac{aS_2^b}{E[S_2^b]}\right)\right]$$

$$= E\left[S_1I\left(S_1 \geq \frac{aS_2^b}{E[S_2^b]}\right)\right] - E\left[S_2I\left(S_1 \geq \frac{aS_2^b}{E[S_2^b]}\right)\right] - E\left[KI\left(S_1 \geq \frac{aS_2^b}{E[S_2^b]}\right)\right].$$

The first term is evaluated in appendix C. The two remaining terms are evaluated as follows:

$$E\left[S_2I\left(S_1 \geq \frac{aS_2^b}{E[S_2^b]}\right)\right]$$

$$= E\left[F_2\exp\left\{-\frac{1}{2}v_2^2 + v_2\varepsilon_2\right\}I\left(F_1\exp\left\{-\frac{1}{2}v_1^2 + v_1\varepsilon_1\right\} \geq a\exp\left\{-\frac{1}{2}b^2v_2^2 + bv_2\varepsilon_2\right\}\right)\right]$$

$$= F_2E\left[I\left(F_1\exp\left\{-\frac{1}{2}v_1^2 + v_1(\varepsilon_1 + \rho v_2)\right\} \geq a\exp\left\{-\frac{1}{2}b^2v_2^2 + bv_2(\varepsilon_2 + v_2)\right\}\right)\right]$$

$$= F_2E\left[I\left(v_1\varepsilon_1 - bv_2\varepsilon_2 \geq -\ln(F_1/a) + \frac{1}{2}v_1^2 - \rho v_1v_2 - \frac{1}{2}b^2v_2^2 + bv_2^2\right)\right]$$

$$= F_2E\left[I\left(\varepsilon \geq -\frac{\ln(F_1/a) + \frac{1}{2}v_1^2 - \rho v_1v_2 - \frac{1}{2}b^2v_2^2 + bv_2^2}{\sqrt{v_1^2 - 2b\rho v_1v_2 + b^2v_2^2}}\right)\right]$$

$$= F_2N\left(\frac{\ln(F_1/a) - \frac{1}{2}v_1^2 + \rho v_1v_2 + \frac{1}{2}b^2v_2^2 - bv_2^2}{\sqrt{v_1^2 - 2b\rho v_1v_2 + b^2v_2^2}}\right),$$

$$E\left[KI\left(S_1 \geq \frac{aS_2^b}{E[S_2^b]}\right)\right]$$

$$= KE\left[I\left(F_1\exp\left\{-\frac{1}{2}v_1^2 + v_1\varepsilon_1\right\} \geq a\exp\left\{-\frac{1}{2}b^2v_2^2 + bv_2\varepsilon_2\right\}\right)\right]$$

$$= KE\left[v_1\varepsilon_1 - bv_2\varepsilon_2 \geq -\ln(F_1/a) + \frac{1}{2}v_1^2 - \frac{1}{2}b^2v_2^2\right]$$

$$= KE\left[\varepsilon \geq -\frac{\ln(F_1/a) - \frac{1}{2}v_1^2 + \frac{1}{2}b^2v_2^2}{\sqrt{v_1^2 - 2b\rho v_1v_2 + b^2v_2^2}}\right]$$

$$= KN\left(\frac{\ln(F_1/a) - \frac{1}{2}v_1^2 + \frac{1}{2}b^2v_2^2}{\sqrt{v_1^2 - 2b\rho v_1v_2 + b^2v_2^2}}\right).$$

Collect the results, and translate $v_1 = \sigma_1\sqrt{T}$, $v_2 = \sigma_2\sqrt{T}$, and

$$\sqrt{v_1^2 - 2b\rho v_1v_2 + b^2v_2^2} = \sqrt{(\sigma_1^2 - 2b\rho\sigma_1\sigma_2 + b^2\sigma_2^2)T} = \sigma\sqrt{T}$$

to obtain the result stated as equations (15)–(19).

Appendix E: Optimizing w.r.t. the exercise strategy

Define

$$
\begin{aligned}
H(a,b) &= \mathrm{e}^{-rT}\{F_1N(d_1) - F_2N(d_2) - KN(d_3)\},\\
d_1 &= \frac{\ln(F_1/a) + (\frac{1}{2}\sigma_1^2 - b\rho\sigma_1\sigma_2 + \frac{1}{2}b^2\sigma_2^2)T}{\sigma\sqrt{T}},\\
d_2 &= \frac{\ln(F_1/a) + (-\frac{1}{2}\sigma_1^2 + \rho\sigma_1\sigma_2 + \frac{1}{2}b^2\sigma_2^2 - b\sigma_2^2)T}{\sigma\sqrt{T}},\\
d_3 &= \frac{\ln(F_1/a) + (-\frac{1}{2}\sigma_1^2 + \frac{1}{2}b^2\sigma_2^2)T}{\sigma\sqrt{T}},\\
\sigma &= \sqrt{\sigma_1^2 - 2b\sigma_1\sigma_2 + b^2\sigma_2^2}.
\end{aligned}
$$

First, establish

$$
\begin{aligned}
&\frac{\partial d_1}{\partial a} = \frac{\partial d_2}{\partial a} = \frac{\partial d_3}{\partial a} = \frac{-1}{a\sigma\sqrt{T}},\\
&\frac{\partial \sigma}{\partial b} = \widehat{\rho}\sigma,\\
&\frac{\partial d_1}{\partial b} = C_1 - d_1\widehat{\rho},\\
&\frac{\partial d_2}{\partial b} = C_2 - d_2\widehat{\rho},\\
&\frac{\partial d_3}{\partial b} = C_3 - d_3\widehat{\rho},
\end{aligned}
$$

where, for notational convenience, we define

$$
\begin{aligned}
\widehat{\rho} &= \frac{b\sigma_2^2 - \rho\sigma_1\sigma_2}{\sigma^2},\\
C_1 &= \frac{-\rho\sigma_1\sigma_2T + b\sigma_2^2T}{\sigma\sqrt{T}},\\
C_2 &= \frac{b\sigma_2^2T - \sigma_2^2T}{\sigma\sqrt{T}},\\
C_3 &= \frac{b\sigma_2^2T}{\sigma\sqrt{T}}.
\end{aligned}
$$

Consequently, the first-order partials of H are

$$
\begin{aligned}
H_a &= -\mathrm{e}^{-rT}\{F_1n(d_1) - F_2n(d_2) - Kn(d_3)\}\frac{1}{a\sigma\sqrt{T}},\\
H_b &= -\mathrm{e}^{-rT}\{F_1n(d_1)d_1 - F_2n(d_2)d_2 - Kn(d_3)d_3\}\widehat{\rho}\\
&\quad + \mathrm{e}^{-rT}\{F_1n(d_1)C_1 - F_2n(d_2)C_2 - Kn(d_3)C_3\}.
\end{aligned}
$$

We now obtain the second-order partials,

$$
\begin{aligned}
&\frac{\partial^2 d_1}{\partial a^2} = \frac{\partial^2 d_2}{\partial a^2} = \frac{\partial^2 d_3}{\partial a^2} = \frac{1}{a^2\sigma\sqrt{T}},\\
&\frac{\partial^2 d_1}{\partial a\,\partial b} = \frac{\partial^2 d_2}{\partial a\,\partial b} = \frac{\partial^2 d_3}{\partial a\,\partial b} = \frac{\widehat{\rho}}{a\sigma\sqrt{T}},\\
&\frac{\partial^2 d_1}{\partial b^2} = \frac{\sigma_2^2}{\sigma} - 2C_1\widehat{\rho} + \left(3\widehat{\rho}^2 - \frac{\sigma_2^2}{\sigma^2}\right)d_1,\\
&\frac{\partial^2 d_2}{\partial b^2} = \frac{\sigma_2^2}{\sigma} - 2C_2\widehat{\rho} + \left(3\widehat{\rho}^2 - \frac{\sigma_2^2}{\sigma^2}\right)d_2,\\
&\frac{\partial^2 d_2}{\partial b^2} = \frac{\sigma_2^2}{\sigma} - 2C_3\widehat{\rho} + \left(3\widehat{\rho}^2 - \frac{\sigma_2^2}{\sigma^2}\right)d_3,
\end{aligned}
$$

and the second-order partials of H are

$$
\begin{aligned}
H_{aa} &= -\mathrm{e}^{-rT}\{F_1n(d_1)d_1 - F_2n(d_2)d_2 - Kn(d_3)d_3\}\frac{1}{a^2\sigma^2T}\\
&\quad + \mathrm{e}^{-rT}\{F_1n(d_1) - F_2n(d_2) - Kn(d_3)\}\frac{1}{a^2\sigma\sqrt{T}},\\
H_{ab} = H_{ba} &= -\mathrm{e}^{-rT}\{F_1n(d_1)d_1^2 - F_2n(d_2)d_2^2 - Kn(d_3)d_3^2\}\frac{\widehat{\rho}}{a\sigma\sqrt{T}}\\
&\quad + \mathrm{e}^{-rT}\{F_1n(d_1)C_1d_1 - F_2n(d_2)C_2d_2 - Kn(d_3)C_3d_3\}\frac{1}{a\sigma\sqrt{T}}\\
&\quad + \mathrm{e}^{-rT}\{F_1n(d_1) - F_2n(d_2) - Kn(d_3)\}\frac{\widehat{\rho}}{a\sigma\sqrt{T}},\\
H_{bb} &= -\mathrm{e}^{-rT}\{F_1n(d_1)d_1^3 - F_2n(d_2)d_2^3 - Kn(d_3)d_3^3\}\widehat{\rho}^2\\
&\quad + \mathrm{e}^{-rT}\{F_1n(d_1)C_1d_1^2 - F_2n(d_2)C_2d_2^2 - Kn(d_3)C_3d_3^2\}2\widehat{\rho}\\
&\quad - \mathrm{e}^{-rT}\{F_1n(d_1)C_1^2d_1 - F_2n(d_2)C_2^2d_2 - Kn(d_3)C_3^2d_3\}\\
&\quad + \mathrm{e}^{-rT}\{F_1n(d_1)d_1 - F_2n(d_2)d_2 - Kn(d_3)d_3\}\left(3\widehat{\rho}^2 - \frac{\sigma_2^2}{\sigma^2}\right)\\
&\quad - \mathrm{e}^{-rT}\{F_1n(d_1)C_1 - F_2n(d_2)C_2 - Kn(d_3)C_3\}2\widehat{\rho}\\
&\quad + \mathrm{e}^{-rT}\{F_1n(d_1) - F_2n(d_2) - Kn(d_3)\}\frac{\sigma_2^2}{\sigma}.
\end{aligned}
$$

We now have the necessary results to implement the Newton–Raphson iterative procedure, using $a = F_2 + K$ and $b = F_2/(F_2 + K)$ as our initial guess.

Appendix F: Optimizing our formula and the Carmona–Durrleman procedure

Compare equation (12) with equations (15)–(19), assuming that a and b are optimal. Note that it is sufficient to show that the arguments of $N(\,)$ are equal for each of the three terms. First, let $d^* = d_3$. Second, let $\sigma_2\sqrt{T}\cos\theta^* = d_2 - d_3$, which leads to $\cos\theta^* = (\rho\sigma_1 - b\sigma_2)/\sigma$. Hence, we need to show that $\sigma_1\sqrt{T}\cos(\theta^* + \phi) = d_1 - d_3$. Recall that $\cos\phi = \rho$, and that $\phi \in [0, \pi]$ and $\theta^* \in [\pi, 2\pi]$ (see footnote ‡). Obtain the result as follows:

$$
\begin{aligned}
\cos(\theta^* + \phi) &= \cos(\theta^*)\cos(\phi) - \sin(\theta^*)\sin(\phi)\\
&= \frac{\rho\sigma_1 - b\sigma_2}{\sigma}\rho - (-1)\sqrt{1 - \left(\frac{\rho\sigma_1 - b\sigma_2}{\sigma}\right)^2}\sqrt{1-\rho^2}\\
&= \frac{\rho^2\sigma_1 - b\rho\sigma_2}{\sigma} + \sqrt{\frac{\sigma_1^2 - \rho^2\sigma_1^2}{\sigma^2}}\sqrt{1-\rho^2}\\
&= \frac{\rho^2\sigma_1 - b\rho\sigma_2}{\sigma} + \frac{\sigma_1(1-\rho^2)}{\sigma}\\
&= \frac{\sigma_1 - b\rho\sigma_2}{\sigma}\\
&= \frac{1}{\sigma_1\sqrt{T}}\frac{(\sigma_1^2 - b\rho\sigma_1\sigma_2)T}{\sigma\sqrt{T}}\\
&= \frac{d_1 - d_3}{\sigma_1\sqrt{T}}.
\end{aligned}
$$

Consequently, optimizing our formula with respect to a and b is equivalent to the Carmona–Durrleman procedure.

Appendix G: The Greeks

In the following, we consider the Greeks in equations (15)–(19). Keeping a and b constant, delta$_1$ and delta$_2$ are

$$\frac{\partial c}{\partial F_1} = \mathrm{e}^{-rT}N(d_1) + \mathrm{e}^{-rT}\{F_1 n(d_1) - F_2 n(d_2) - Kn(d_3)\}\frac{1}{F_1\sigma\sqrt{T}},$$
$$\frac{\partial c}{\partial F_2} = -\mathrm{e}^{-rT}N(d_2) + \mathrm{e}^{-rT}\{F_1 n(d_1) - F_2 n(d_2) - Kn(d_3)\}\frac{-b}{F_2\sigma\sqrt{T}}.$$

Gamma$_{11}$, gamma$_{12}$, and gamma$_{22}$ are

$$\begin{aligned}\frac{\partial^2 c}{(\partial F_1)^2} &= 2\mathrm{e}^{-rT}n(d_1)\frac{1}{F_1\sigma\sqrt{T}}\\ &\quad - \mathrm{e}^{-rT}\{F_1 n(d_1) - F_2 n(d_2) - Kn(d_3)\}\frac{1}{F_1}\frac{1}{F_1\sigma\sqrt{T}}\\ &\quad - \mathrm{e}^{-rT}\{F_1 n(d_1)d_1 - F_2 n(d_2)d_2 - Kn(d_3)d_3\}\left(\frac{1}{F_1\sigma\sqrt{T}}\right)^2,\\ \frac{\partial^2 c}{\partial F_1\,\partial F_2} &= \mathrm{e}^{-rT}n(d_1)\frac{-b}{F_2\sigma\sqrt{T}} - \mathrm{e}^{-rT}n(d_2)\frac{1}{F_1\sigma\sqrt{T}}\\ &\quad - \mathrm{e}^{-rT}\{F_1 n(d_1)d_1 - F_2 n(d_2)d_2 - Kn(d_3)d_3\}\frac{-b}{F_2\sigma\sqrt{T}}\frac{1}{F_1\sigma\sqrt{T}},\\ \frac{\partial^2 c}{(\partial F_2)^2} &= -2\mathrm{e}^{-rT}n(d_2)\frac{-b}{F_2\sigma\sqrt{T}}\\ &\quad - \mathrm{e}^{-rT}\{F_1 n(d_1) - F_2 n(d_2) - Kn(d_3)\}\frac{1}{F_2}\frac{-b}{F_2\sigma\sqrt{T}}\\ &\quad - \mathrm{e}^{-rT}\{F_1 n(d_1)d_1 - F_2 n(d_2)d_2 - Kn(d_3)d_3\}\left(\frac{-b}{F_2\sigma\sqrt{T}}\right)^2.\end{aligned}$$

Vega$_1$ and vega$_2$ are

$$\begin{aligned}\frac{\partial c}{\partial \sigma_1} &= \mathrm{e}^{-rT}\left\{F_1 n(d_1)\frac{(\sigma_1 - b\rho\sigma_2)T}{\sigma\sqrt{T}} + F_2 n(d_2)\frac{(\sigma_1 - \rho\sigma_2)T}{\sigma\sqrt{T}}\right.\\ &\quad \left. + Kn(d_3)\frac{\sigma_1 T}{\sigma\sqrt{T}}\right\}\\ &\quad - \mathrm{e}^{-rT}\{F_1 n(d_1)d_1 - F_2 n(d_2)d_2 - Kn(d_3)d_3\}\frac{\sigma_1 - b\rho\sigma_2}{\sigma^2},\\ \frac{\partial c}{\partial \sigma_2} &= \mathrm{e}^{-rT}\left\{F_1 n(d_1)\frac{(b\sigma_2 - b\rho\sigma_1)T}{\sigma\sqrt{T}} - F_2 n(d_2)\frac{(\rho\sigma_1 - b\sigma_2)T}{\sigma\sqrt{T}}\right.\\ &\quad \left. - Kn(d_3)\frac{b\sigma_2 T}{\sigma\sqrt{T}}\right\}\\ &\quad - \mathrm{e}^{-rT}\{F_1 n(d_1)d_1 - F_2 n(d_2)d_2 - Kn(d_3)d_3\}\frac{b^2\sigma_2 - b\rho\sigma_1}{\sigma^2}.\end{aligned}$$

The partial w.r.t. ρ is

$$\begin{aligned}\frac{\partial c}{\partial \rho} &= \mathrm{e}^{-rT}\left\{-F_1 n(d_1)\frac{b\sigma_1\sigma_2 T}{\sigma\sqrt{T}} - F_2 n(d_2)\frac{\sigma_1\sigma_2 T}{\sigma\sqrt{T}}\right\}\\ &\quad + \mathrm{e}^{-rT}\{F_1 n(d_1)d_1 - F_2 n(d_2)d_2 - Kn(d_3)d_3\}\frac{b\sigma_1\sigma_2}{\sigma^2}.\end{aligned}$$

The theta is

$$\begin{aligned}\frac{\partial c}{\partial T} &= -rc - \mathrm{e}^{-rT}\{F_1 n(d_1) - F_2 n(d_2) - Kn(d_3)\}\frac{1}{T}\frac{\ln(F_1/aF_2^b)}{\sigma\sqrt{T}}\\ &\quad + \mathrm{e}^{-rT}\{F_1 n(d_1)d_1 - F_2 n(d_2)d_2 - Kn(d_3)d_3\}\frac{1}{2T},\end{aligned}$$

and the rho is

$$\frac{\partial c}{\partial r} = -Tc.$$

It can be verified that the Greeks satisfy the fundamental partial differential equation for contingent claims,

$$\frac{1}{2}\sigma_1^2 F_1^2\frac{\partial^2 c}{\partial F_1^2} + \rho\sigma_1 F_1\sigma_2 F_2\frac{\partial^2 c}{\partial F_1\,\partial F_2} + \frac{1}{2}\sigma_2^2 F_2^2\frac{\partial^2 c}{\partial F_2^2} - \frac{\partial c}{\partial T} = rc.$$

Moreover, it can also be verified that the Greeks satisfy

$$\frac{1}{2}\sigma_1\frac{\partial c}{\partial \sigma_1} + \frac{1}{2}\sigma_2\frac{\partial c}{\partial \sigma_2} + r\frac{\partial c}{\partial r} = T\frac{\partial c}{\partial T}.$$

In optimum, i.e. $a = a^*$ and $b = b^*$, it follows from the first-order conditions associated with a that the following identities hold:

$$\begin{aligned}F_1 n(d_1^*) - F_2 n(d_2^*) - Kn(d_3^*) &= 0,\\ bF_1 n(d_1^*) - F_2 n(d_2^*) &= 0,\\ F_1 n(d_1^*)d_1^* - F_2 n(d_2^*)d_2^* - Kn(d_3^*)d_3^* &= \sigma\sqrt{T}F_1 n(d_1^*).\end{aligned}$$

Using these identities, the Greeks in optimum simplify to the following:

$$\begin{aligned}\frac{\partial c}{\partial F_1} &= \mathrm{e}^{-rT}N(d_1^*),\\ \frac{\partial c}{\partial F_2} &= -\mathrm{e}^{-rT}N(d_2^*),\\ \frac{\partial^2 c}{(\partial F_1)^2} &= \mathrm{e}^{-rT}n(d_1^*)\frac{1}{F_1\sigma\sqrt{T}},\\ \frac{\partial^2 c}{\partial F_1\,\partial F_2} &= -\mathrm{e}^{-rT}n(d_2^*)\frac{1}{F_1\sigma\sqrt{T}},\\ \frac{\partial^2 c}{(\partial F_2)^2} &= \mathrm{e}^{-rT}n(d_2^*)\frac{b^*}{F_2\sigma\sqrt{T}},\\ \frac{\partial c}{\partial \sigma_1} &= \mathrm{e}^{-rT}F_1 n(d_1^*)\frac{(\sigma_1 - b^*\rho\sigma_2)T}{\sigma\sqrt{T}},\\ \frac{\partial c}{\partial \sigma_2} &= -\mathrm{e}^{-rT}F_2 n(d_2^*)\frac{(\rho\sigma_1 - b^*\sigma_2)T}{\sigma\sqrt{T}},\\ \frac{\partial c}{\partial \rho} &= -\mathrm{e}^{-rT}F_1 n(d_1^*)\frac{b^*\sigma_1\sigma_2 T}{\sigma\sqrt{T}},\\ \frac{\partial c}{\partial T} &= -rc + \mathrm{e}^{-rT}\sigma\sqrt{T}F_1 n(d_1^*)\frac{1}{2T},\\ \frac{\partial c}{\partial r} &= -Tc.\end{aligned}$$