

# Modelowanie wielowymiarowe za pomocą kopuł

Przykład opcji spread

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# Agenda

## ○ Wstęp

O czym będziemy mówić?

## ○ Motywacja

Producent a ryzyko rynkowe

## ○ Spread

Popularne rodzaje spreadu, opcje na spread

## ○ Kopuły

Definicja, przykłady, intuicja

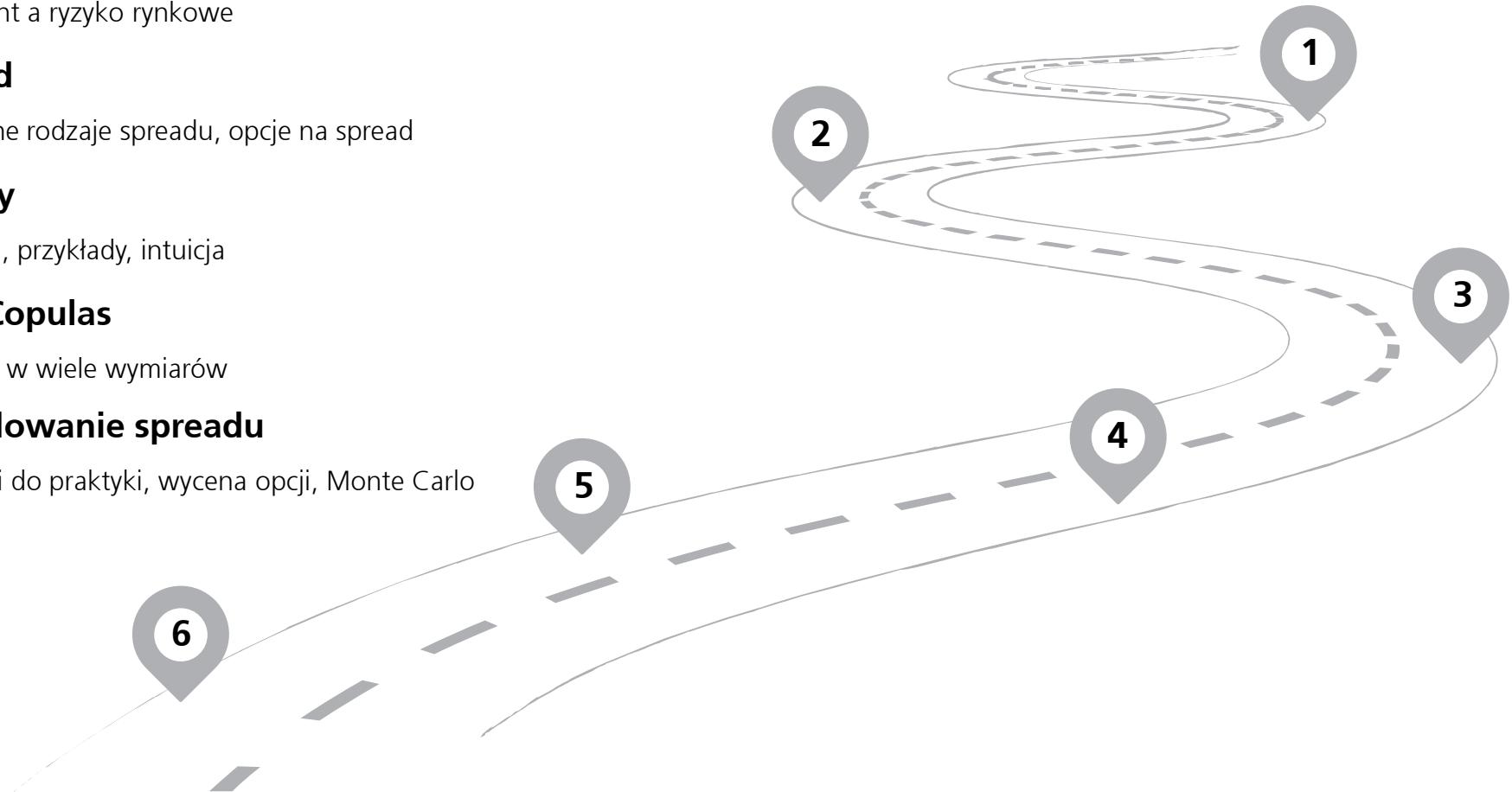
## ○ Vine Copulas

Przejście w wiele wymiarów

## ○ Modelowanie spreadu

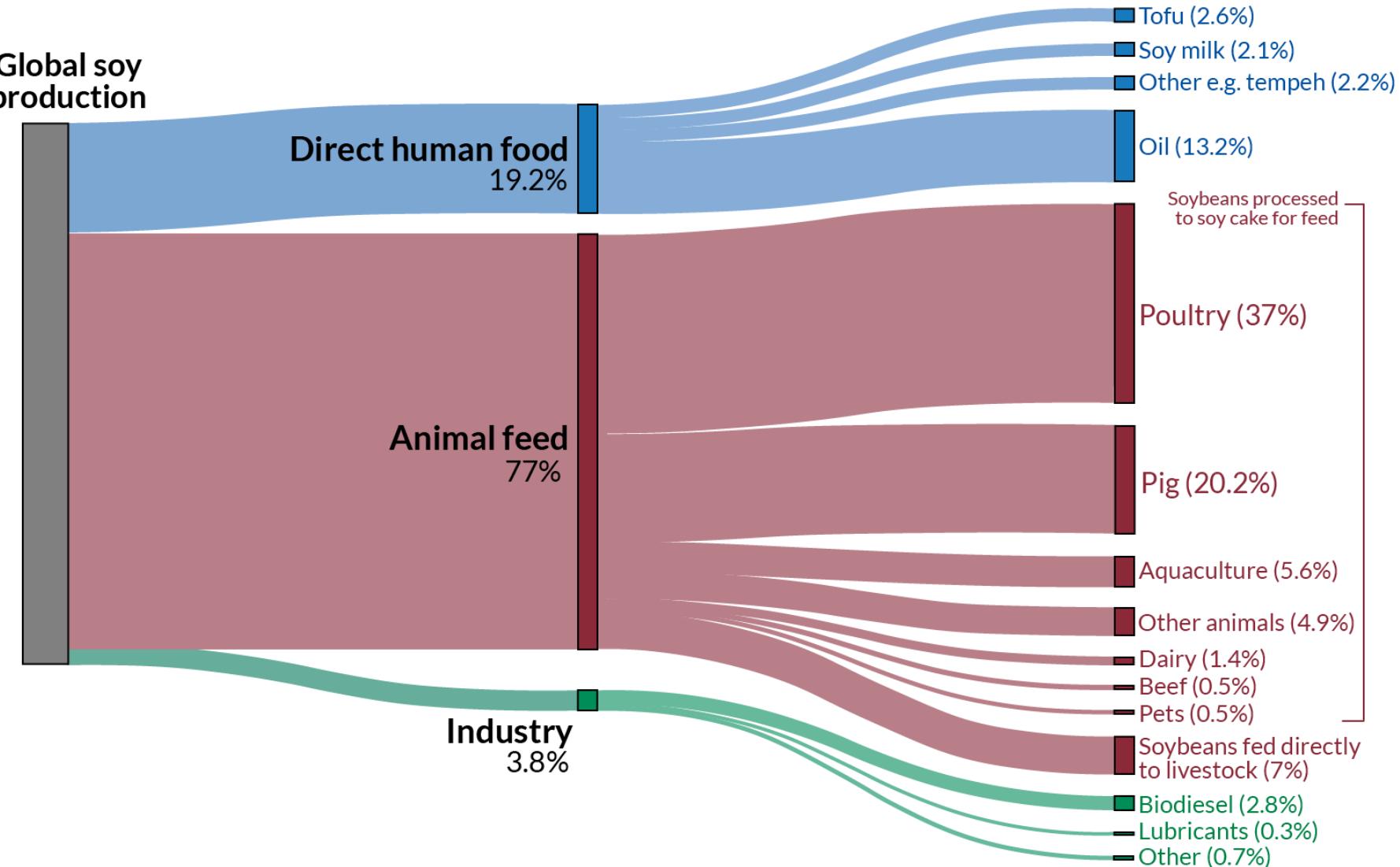
Od teorii do praktyki, wycena opcji, Monte Carlo

## ○ Q&A



# The World's Soy: is it used for Food, Fuel, or Animal Feed?

Shown is the allocation of global soy production to its end uses by weight. This is based on data from 2017 to 2019.



Data source: Food Climate Resource Network (FCRN), University of Oxford; and USDA PSD Database.

OurWorldInData.org – Research and data to make progress against the world's largest problems.

Licensed under CC-BY by the author Hannah Ritchie.

# Spread w roli aktywa

Problem przetwarzania soi



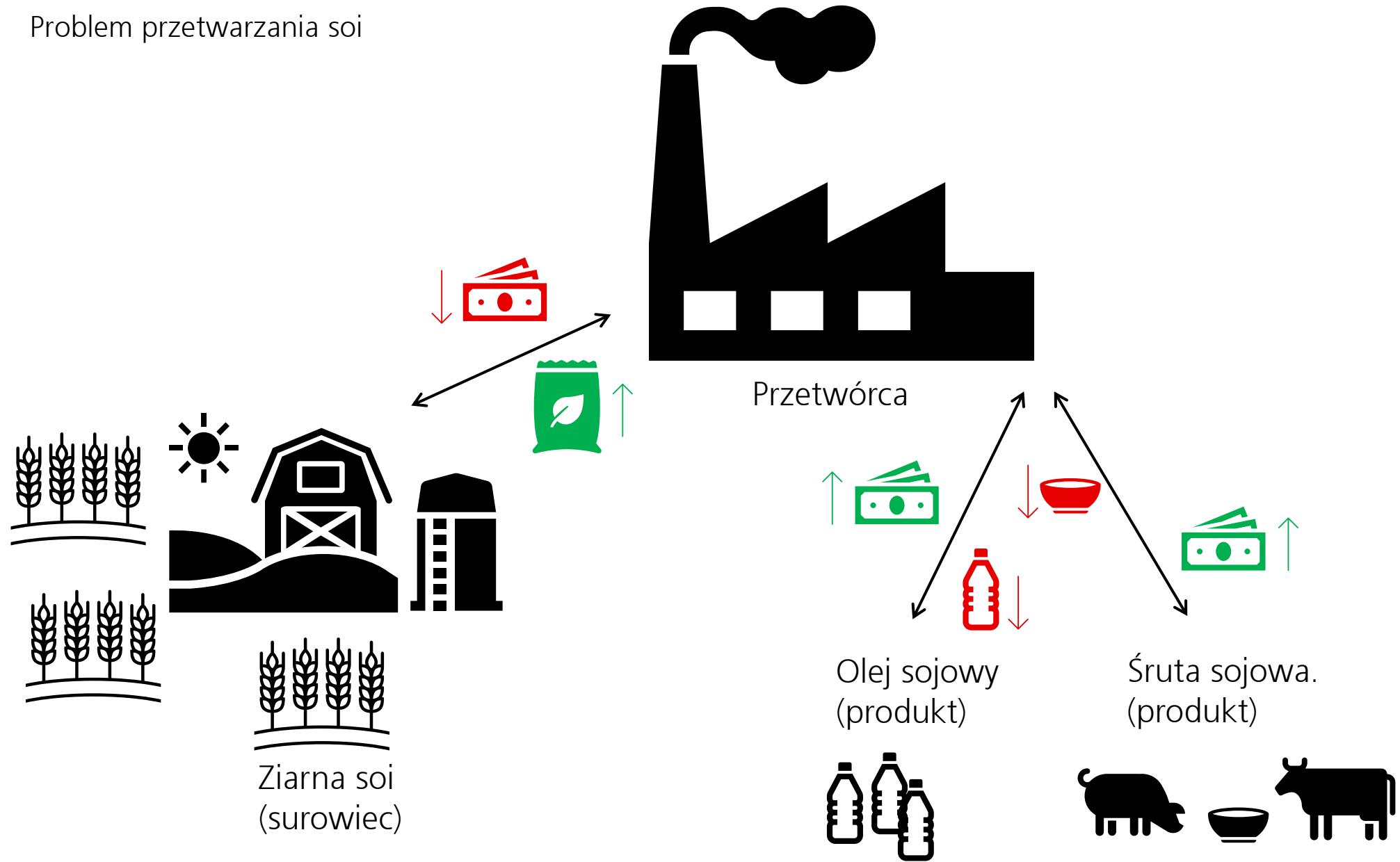
# Spread w roli aktywa

Problem przetwarzania soi



# Spread w roli aktywa

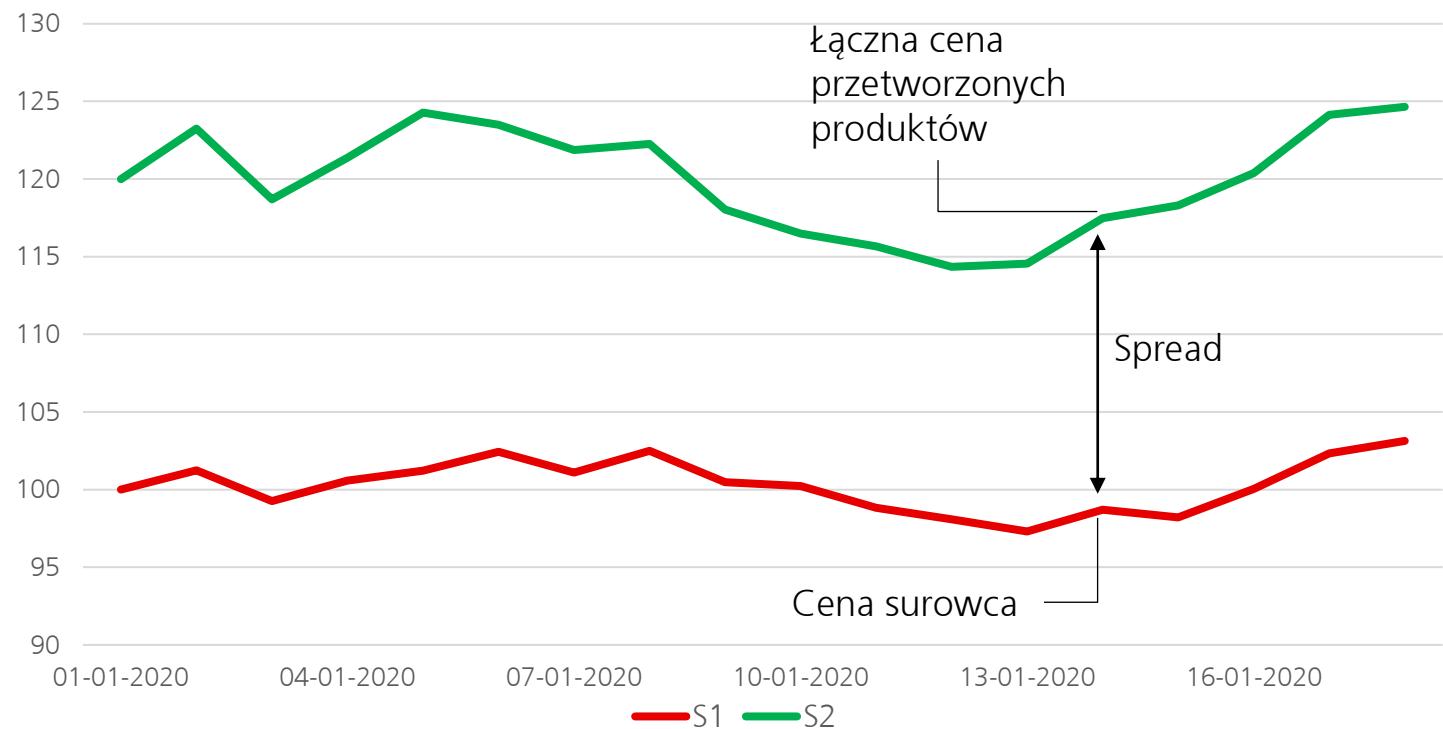
Problem przetwarzania soi



# Spread w roli aktywa

## Dwa źródła ryzyka rynkowego

Kluczowy jest spread



Source: Opracowanie własne

# Spread w roli aktywa

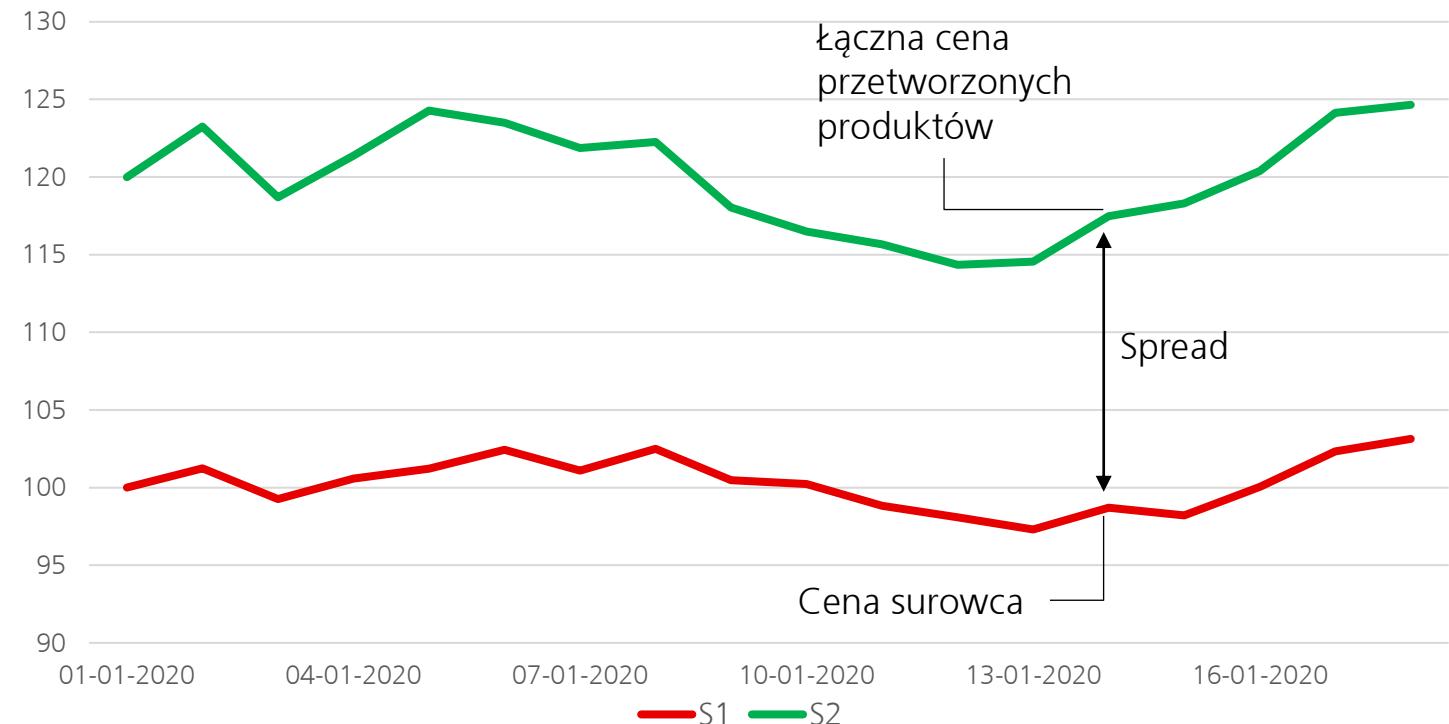
Kluczowy jest spread

A gdy się zmniejsza?

- Znika dochód
- Wstrzymujemy produkcję?
- Na jak długo?
- Jak kosztowna jest taka decyzja?

Nie lepiej się zabezpieczyć?

Dwa źródła ryzyka rynkowego



Source: Opracowanie własne

# Opcja spread

To europejska opcja waniliowa, w której instrumentem bazowym zamiast pojedynczego aktywa jest różnica cen określonych w kontrakcie aktywów.

Funkcja wypłaty opcji spread (o cenie wykonania  $K$ , na spread pomiędzy aktywami  $S_2$  i  $S_1$ ) to:

- dla opcji call:

$$h(S_1(T), S_2(T)) = ([S_1(T) - S_2(T)] - K)^+$$

- dla opcji put:

$$h(S_1(T), S_2(T)) = (K - [S_1(T) - S_2(T)])^+$$

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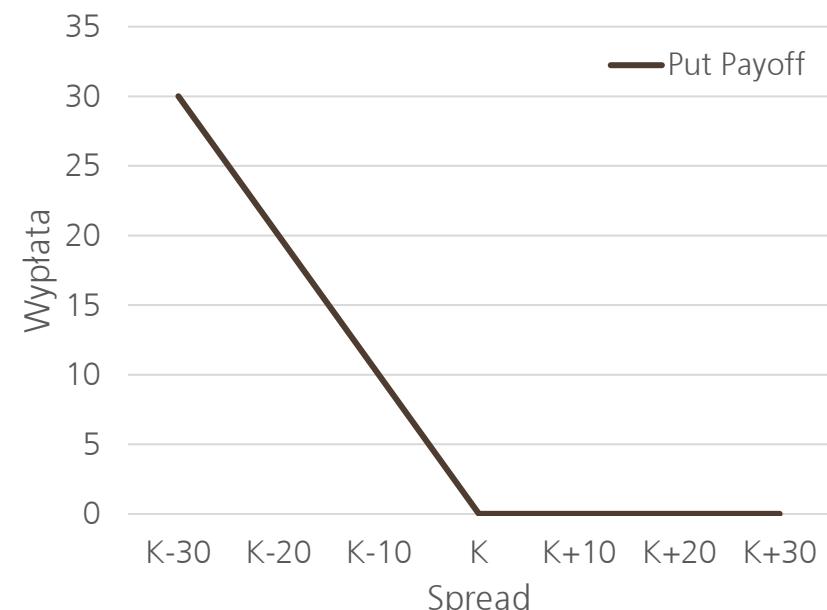
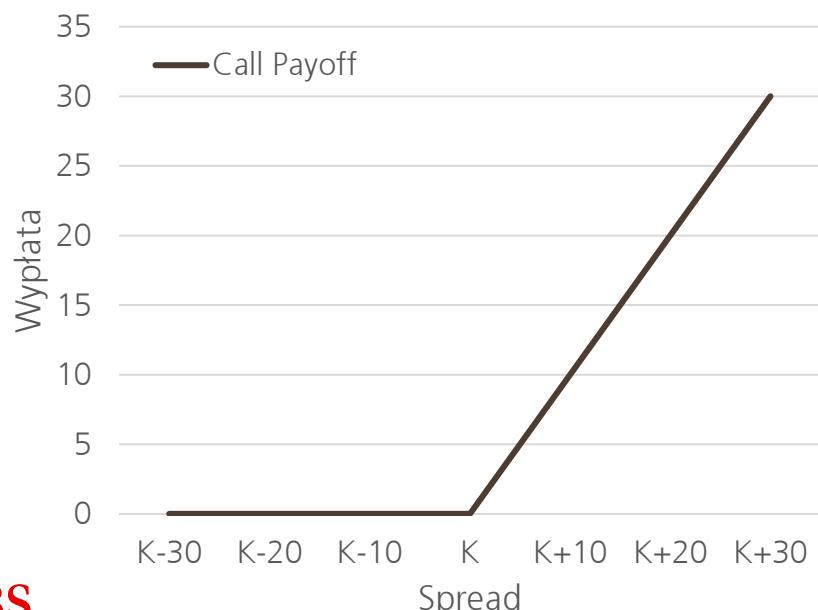
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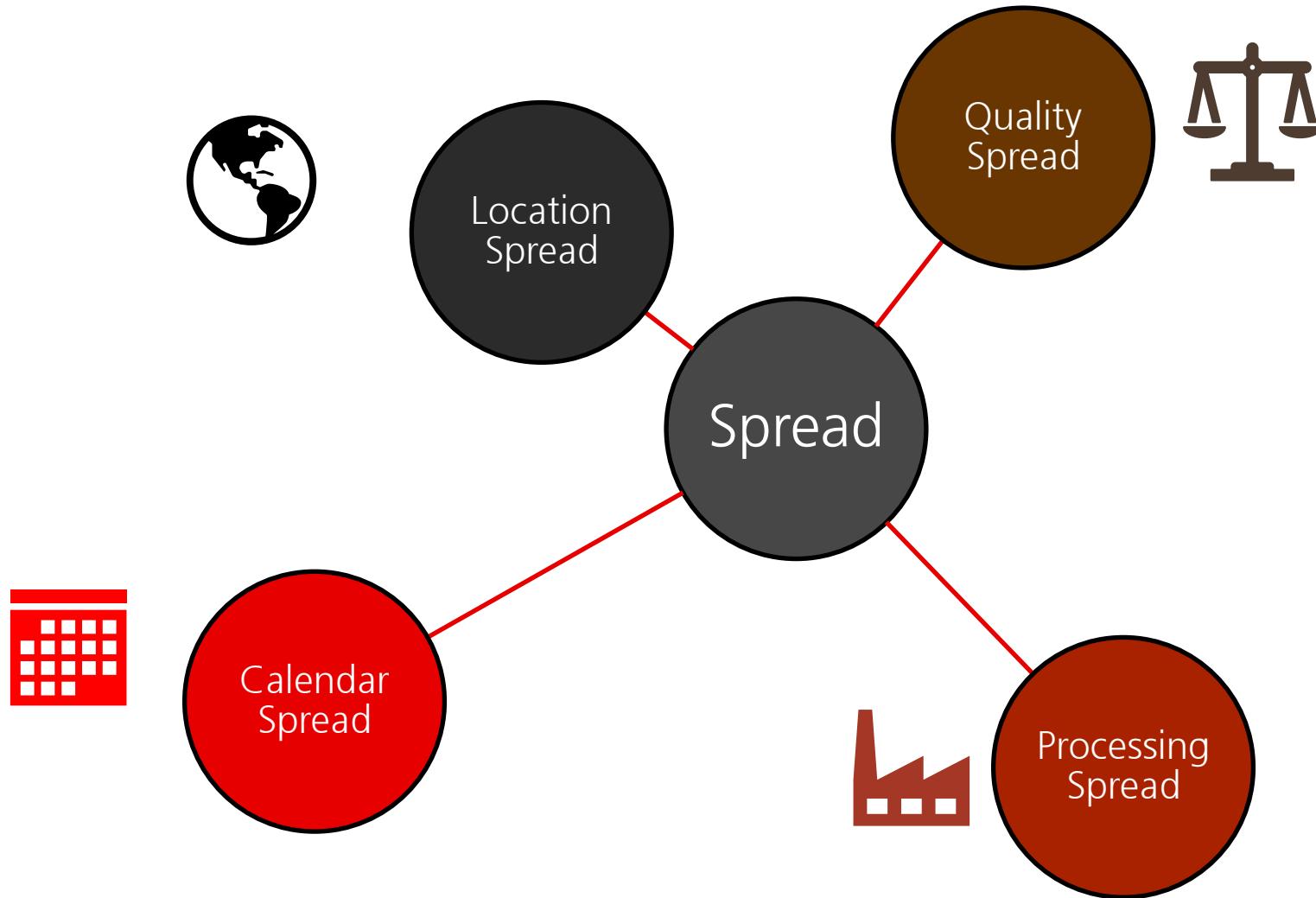
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# Galeria spreadów

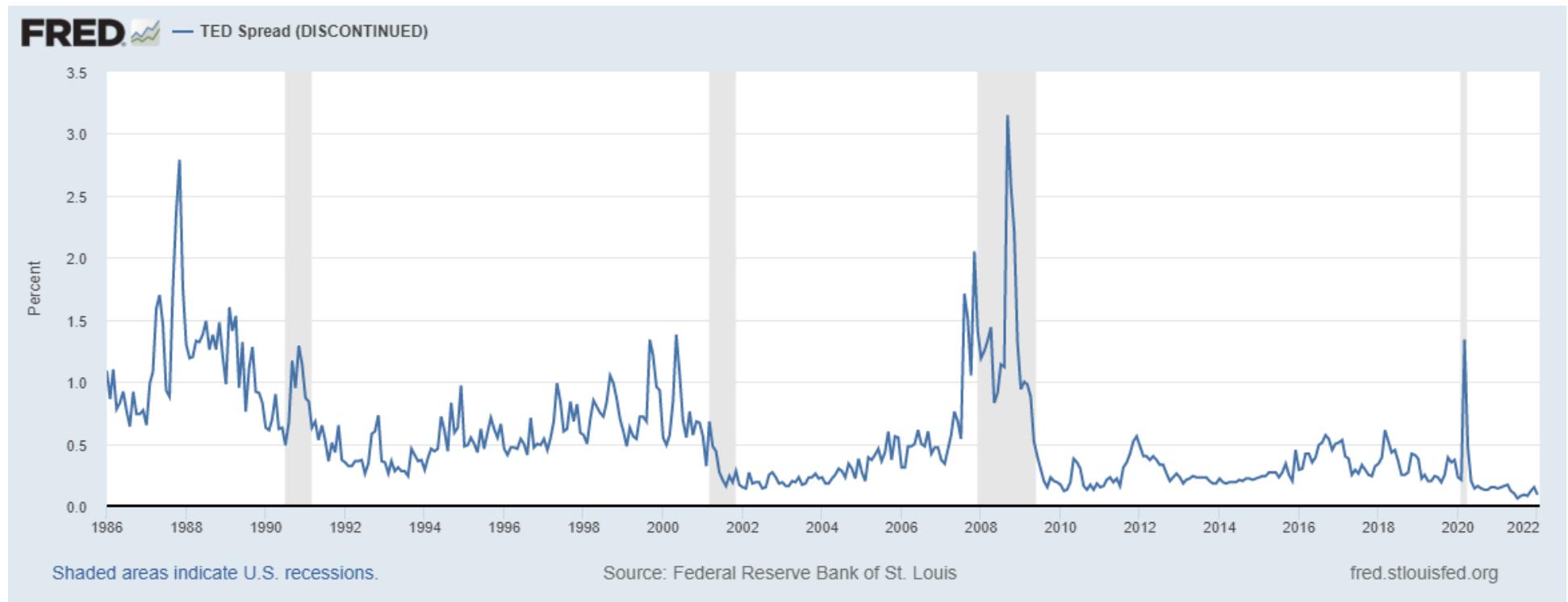


# Galeria spreadów

Przykład z rynku Fixed Income:

$$TS_t = LIB_t - TB_t$$

**TED spread** (US Libor 3M - T-Bill 3M)



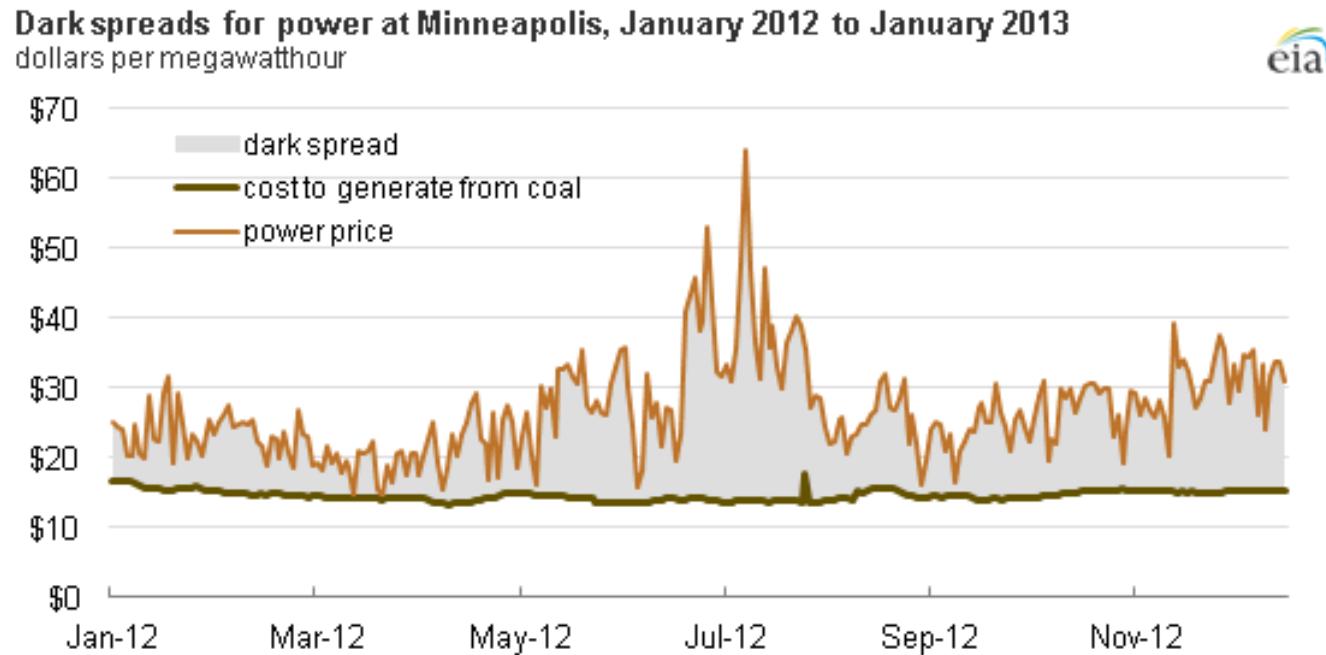
Source: <https://fred.stlouisfed.org/series/TEDRATE>

# Galeria spreadów

Przykład z rynku energii:

$$DS_t = E_t - 0.35 \cdot C_t$$

**Dark spread** (Cena energii - Koszt węgla użytego do jej produkcji)



Source: eia.gov

# Galeria spreadów

Przykład z rynku surowców:

**Crush spread** (Cena śruty sojowej + oleju sojowego - Cena soi użytej do produkcji)



Soja: \$/*buszel*

(1 buszel soi  $\approx$  60 funtów)



Śruta sojowa: \$/short ton

(1 buszel soi  $\rightarrow$  44 funty śruty)



Olej sojowy: \$/pound

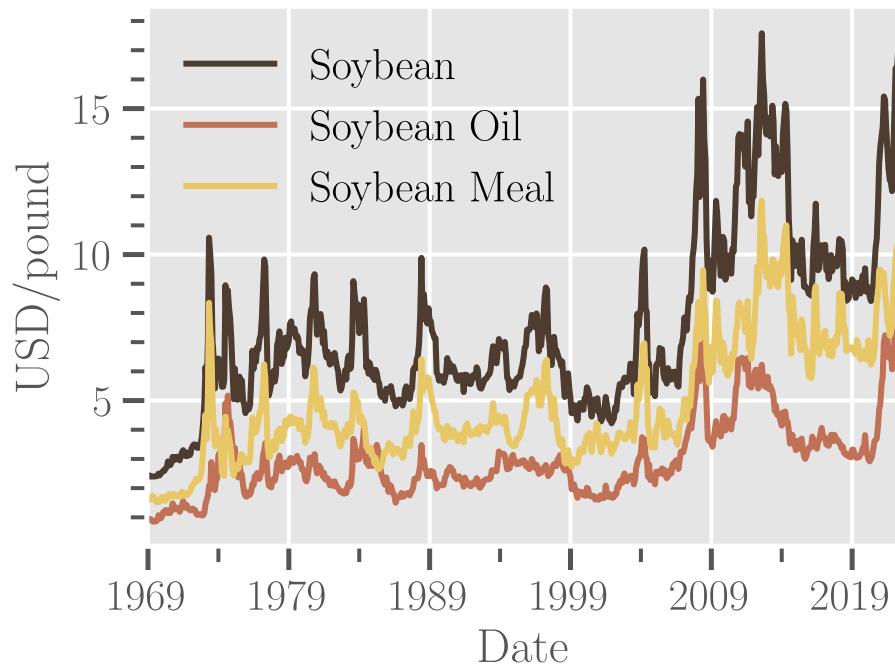
(1 buszel soi  $\rightarrow$  11 funtów oleju)

# Galeria spreadów

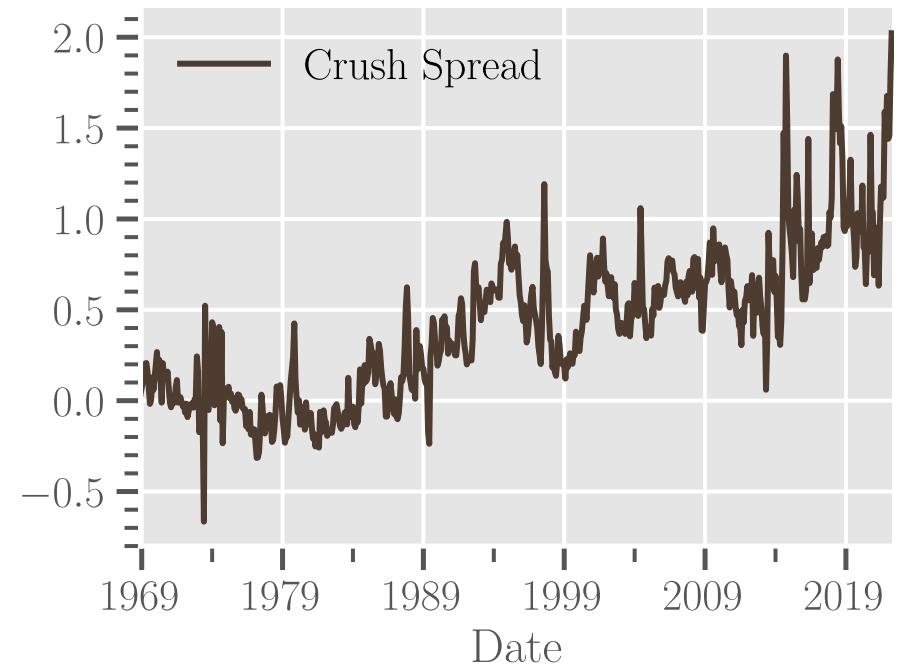
Przykład z rynku surowców:

**Crush spread** (Cena śruty sojowej + oleju sojowego - Cena soi użytej do produkcji)

$$CS_t = (0.022 \cdot SM_t + 0.11 \cdot SO_t) - S_t$$



Data source: [www.macrotrends.net](http://www.macrotrends.net)



# Zmienne zależne

## Korelacja Pearsona

### Współczynnik korelacji Pearsona

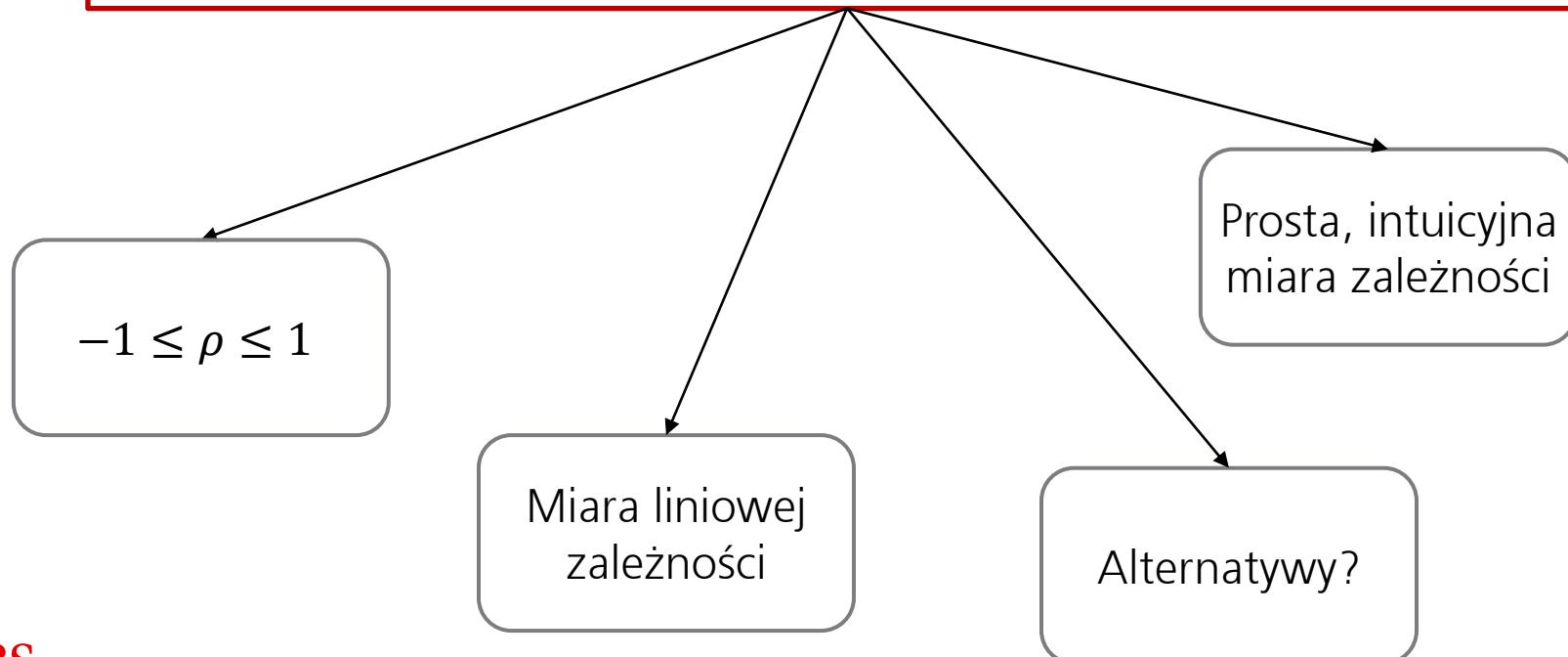
$$\rho := Cor(X, Y) = \frac{Cov(X, Y)}{\sqrt{Var(X)}\sqrt{Var(Y)}}$$

# Zmienne zależne

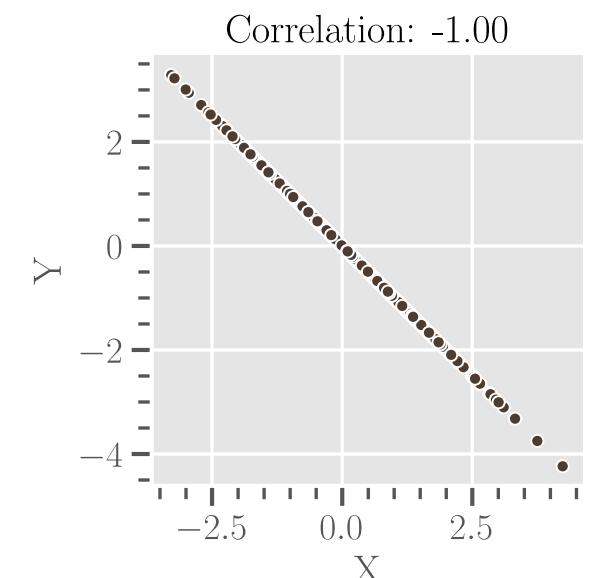
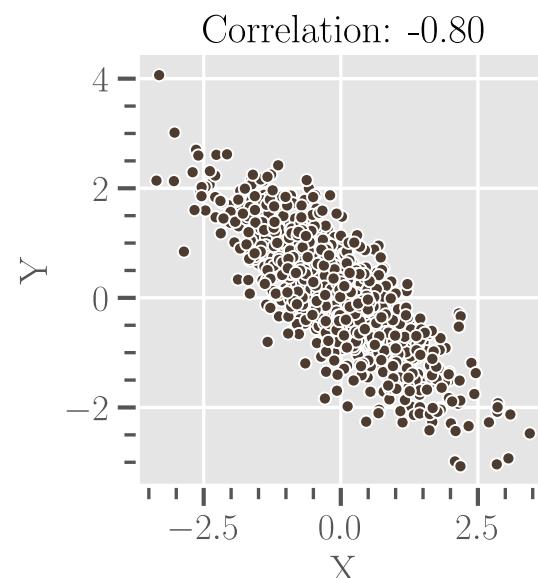
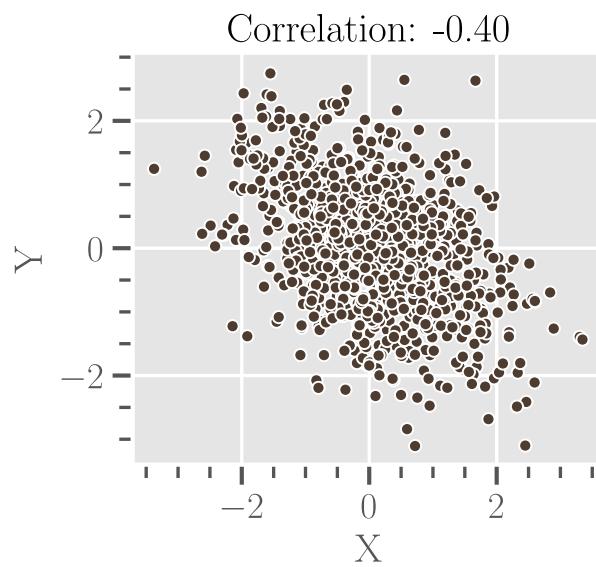
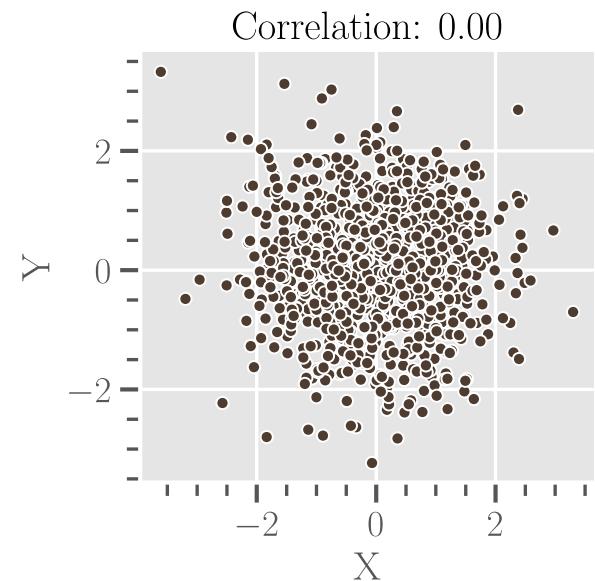
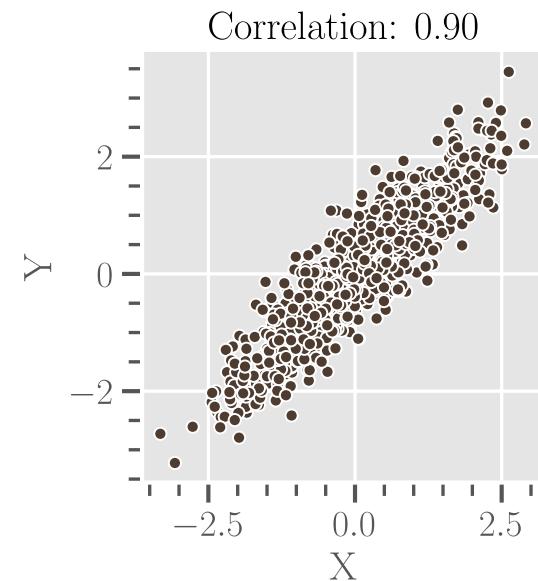
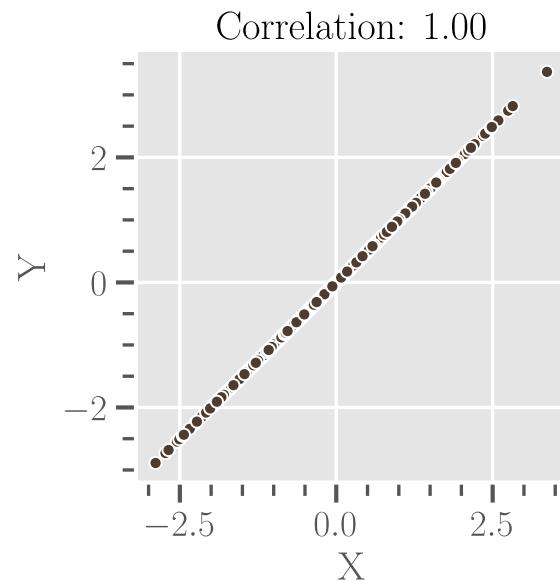
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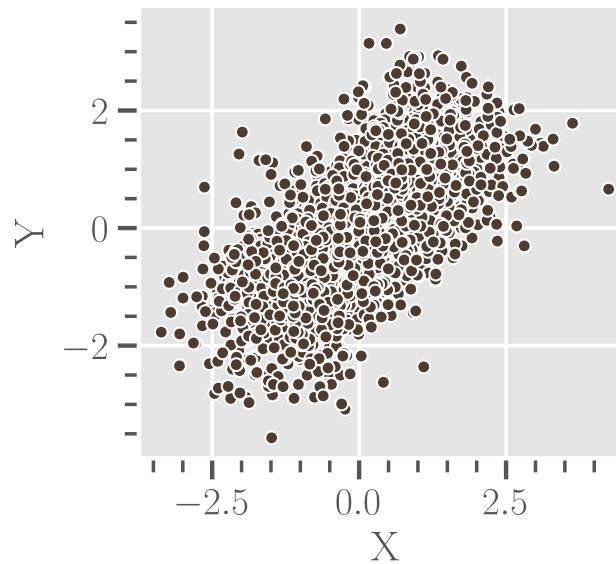
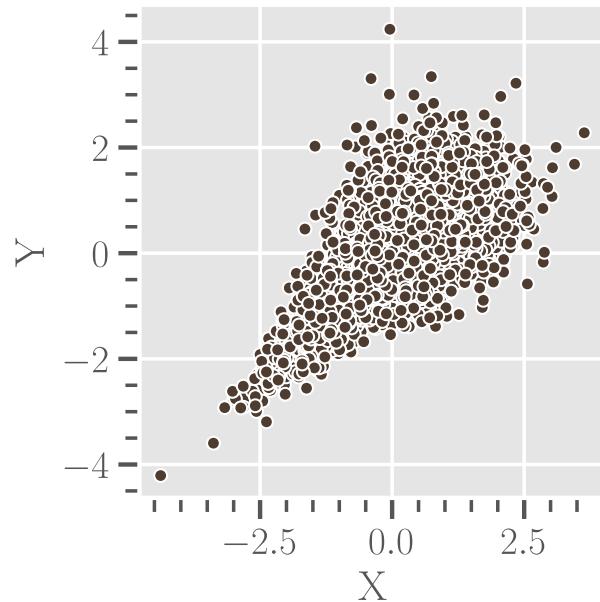
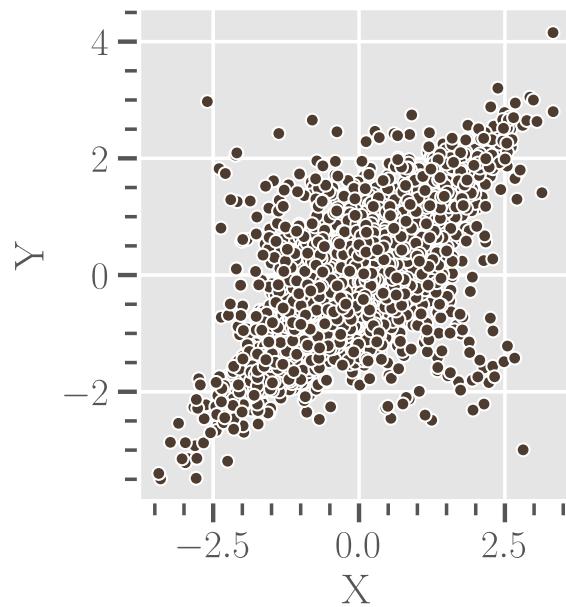
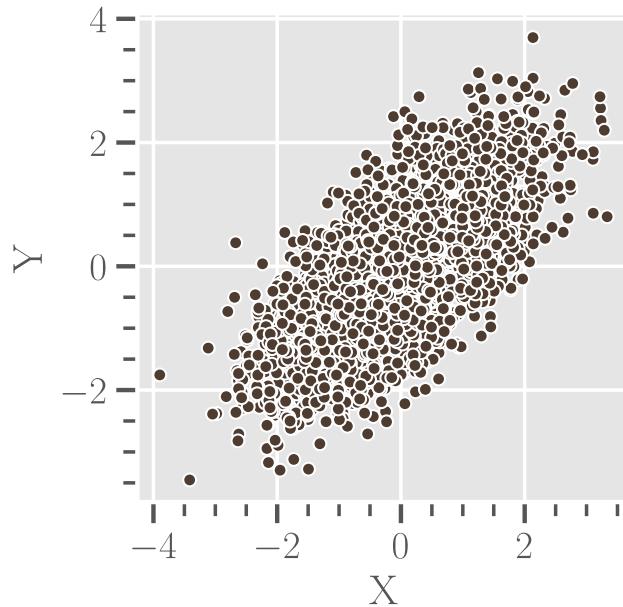
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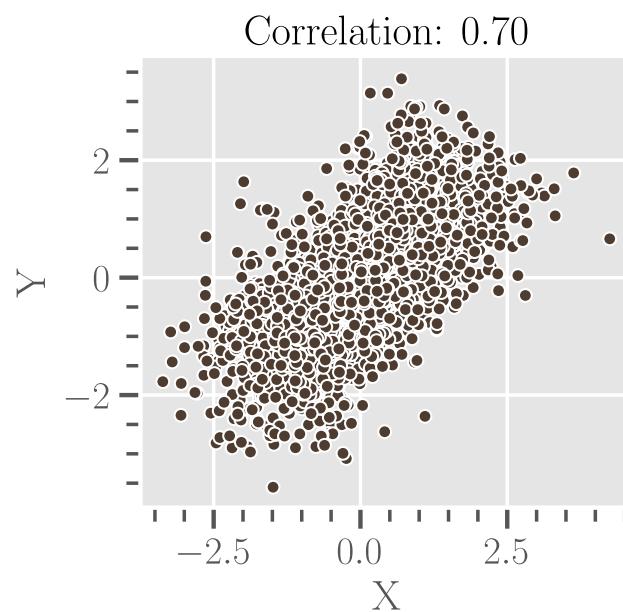
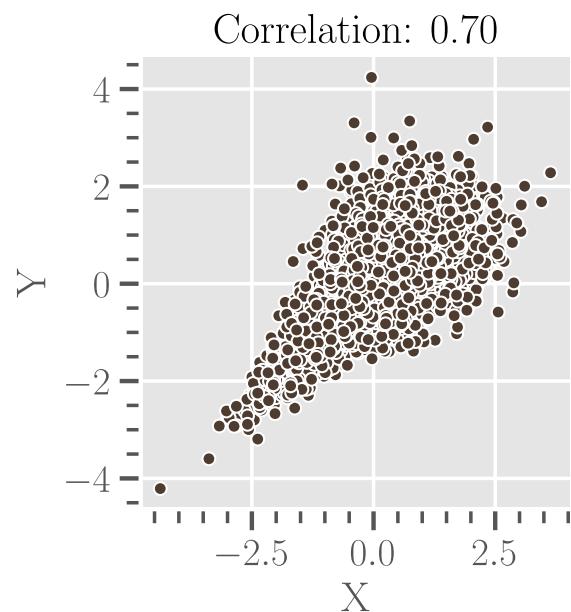
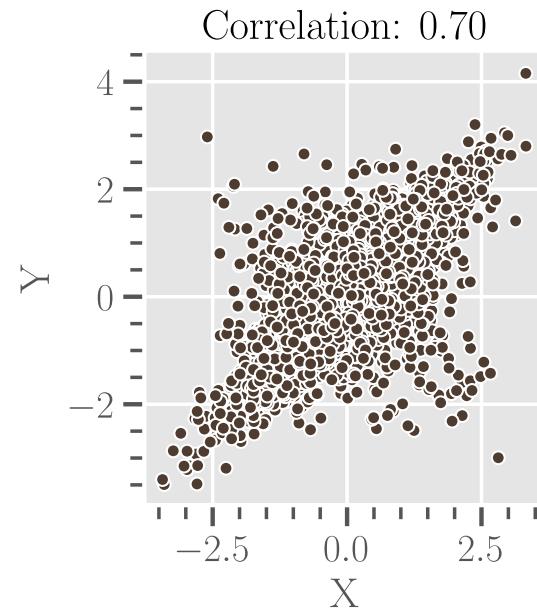
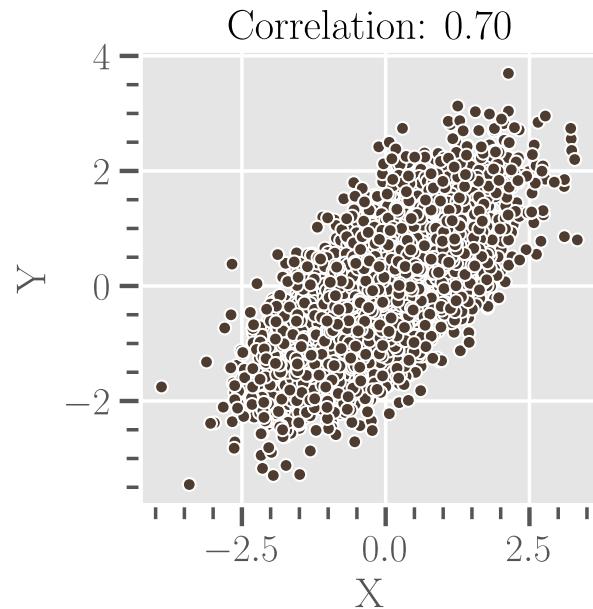
# Zmienne zależne



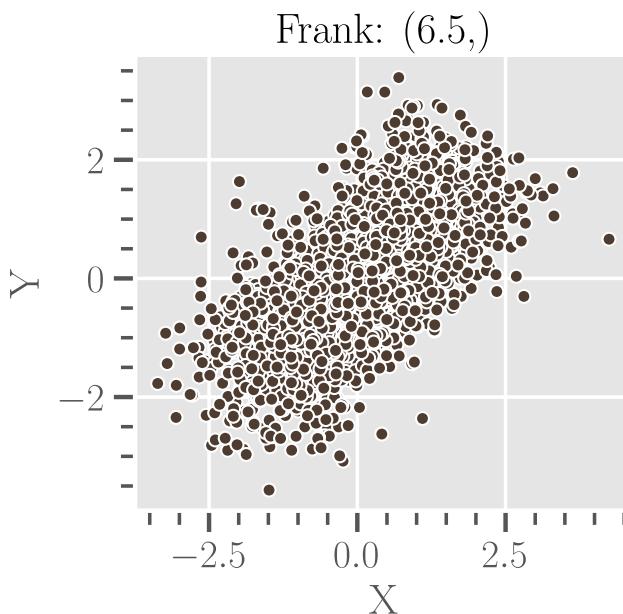
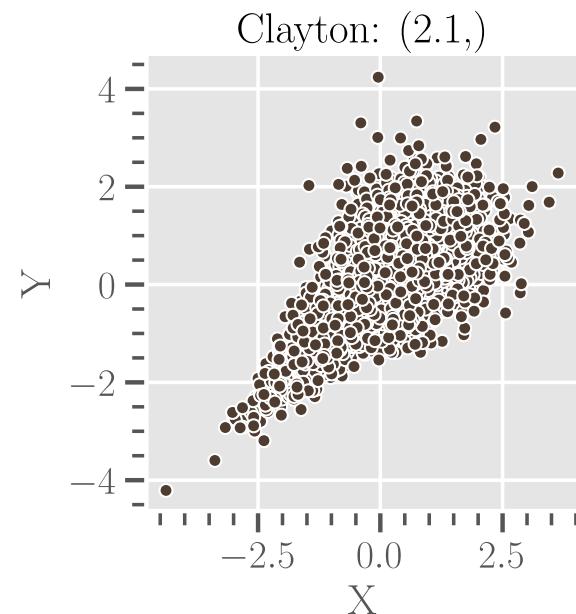
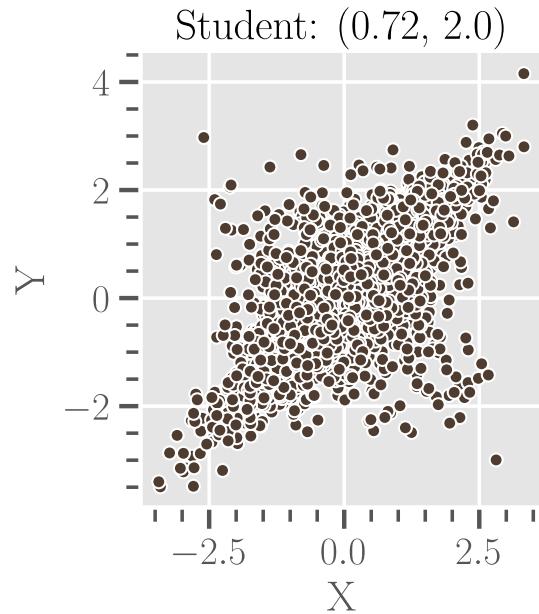
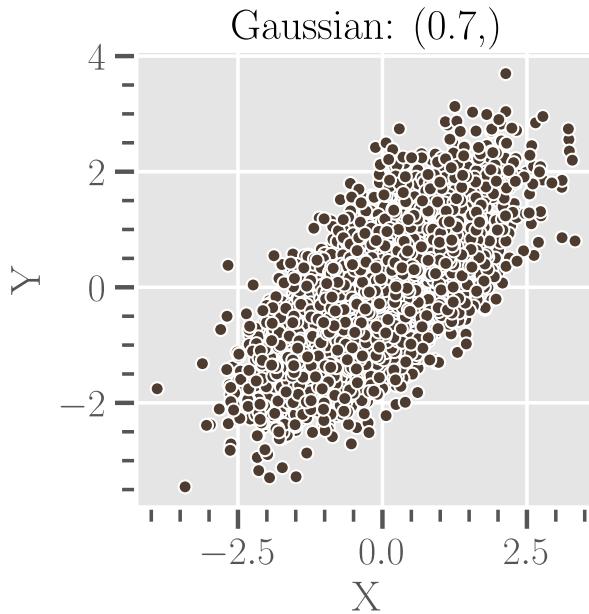
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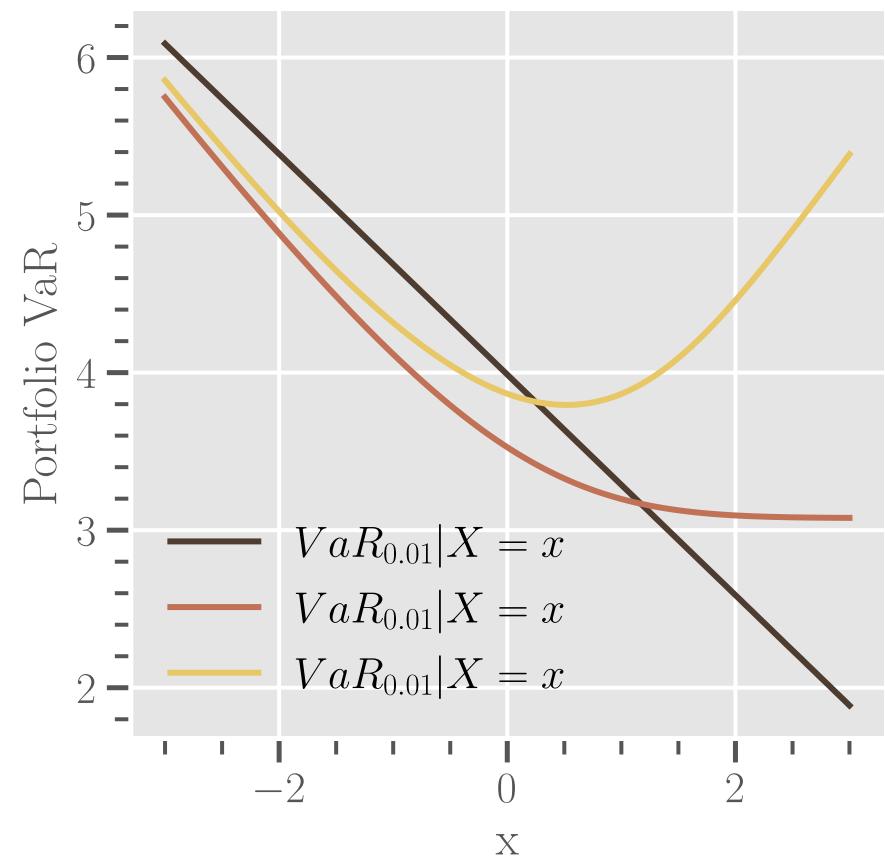
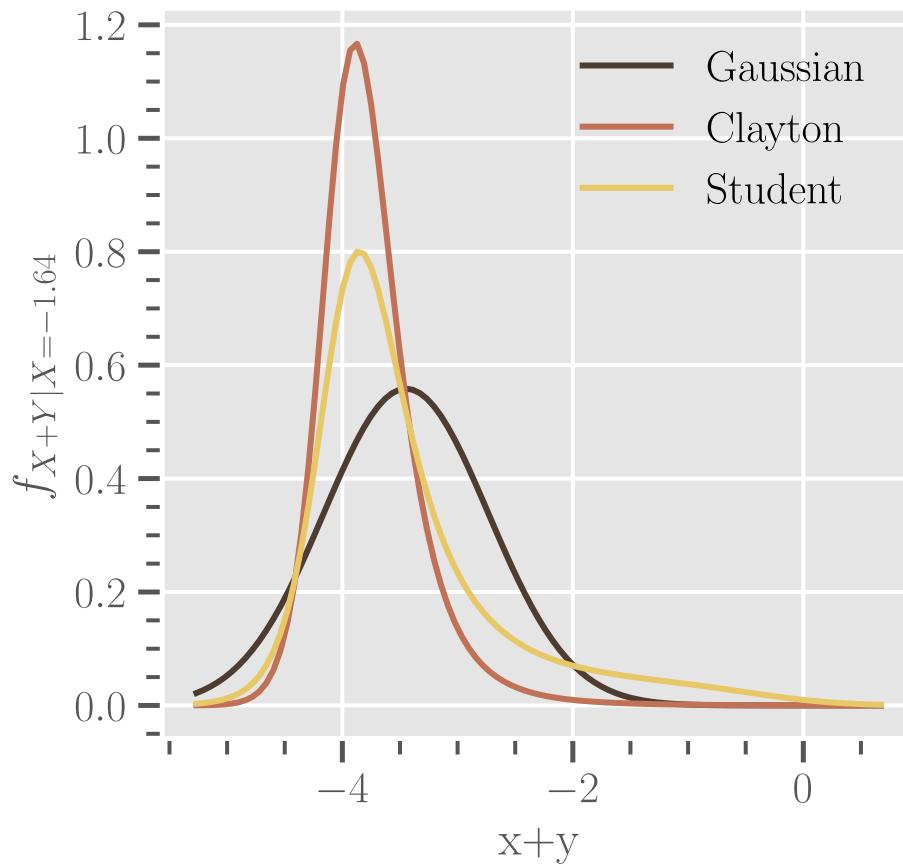
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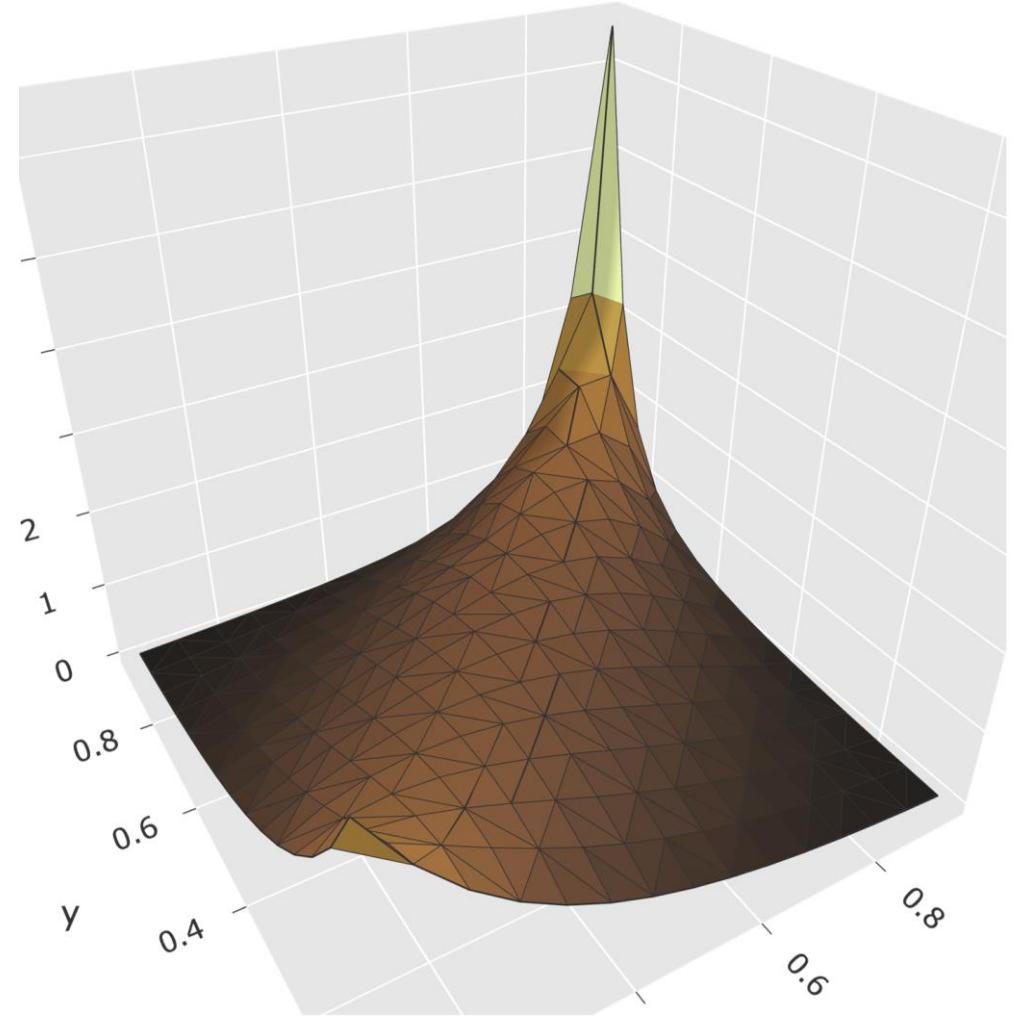
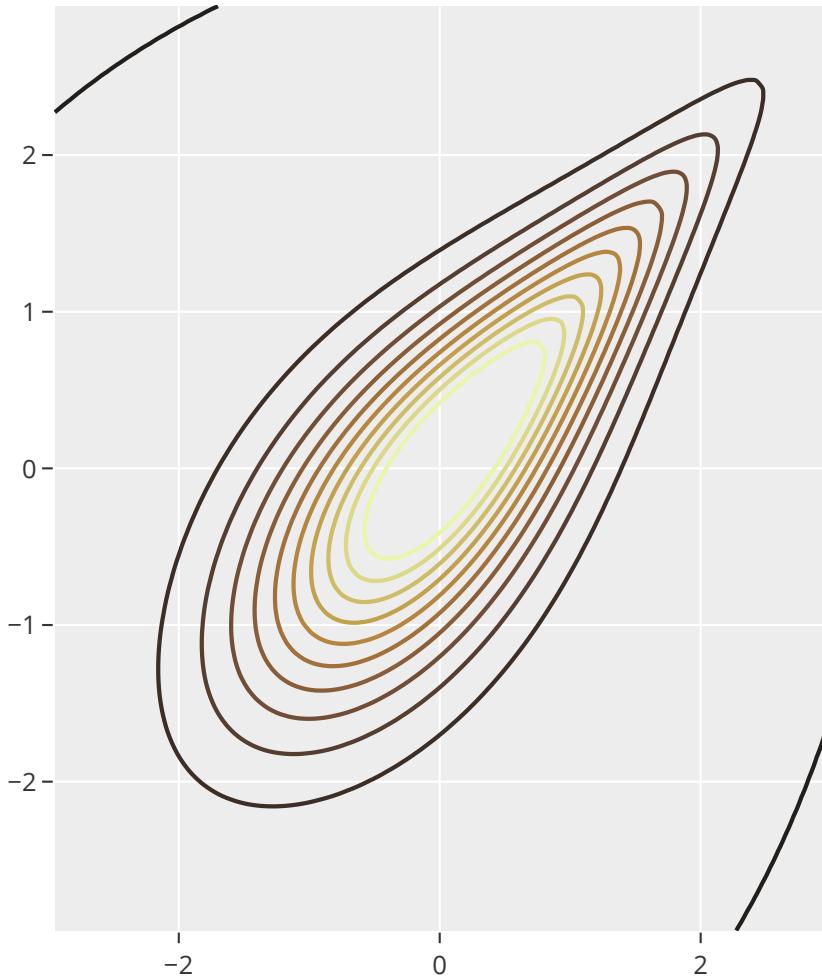


# Zmienne zależne



# Kopuły

Co to właściwie jest?



# Kopuły

Formalna definicja

## Dwuwymiarowa kopuła

To funkcja  $C: [0, 1] \times [0, 1] \rightarrow [0, 1]$ , spełniająca warunki:

- $C(u, 0) = C(0, v) = 0$
- $C(u, 1) = u, C(1, v) = v$
- Rosnąca ze względu na oba argumenty

# Kopuły

Formalna definicja

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- Rosnąca ze względu na oba argumenty

Kopuły to nic innego jak rozkłady łączne, dla zmiennych jednostajnych.

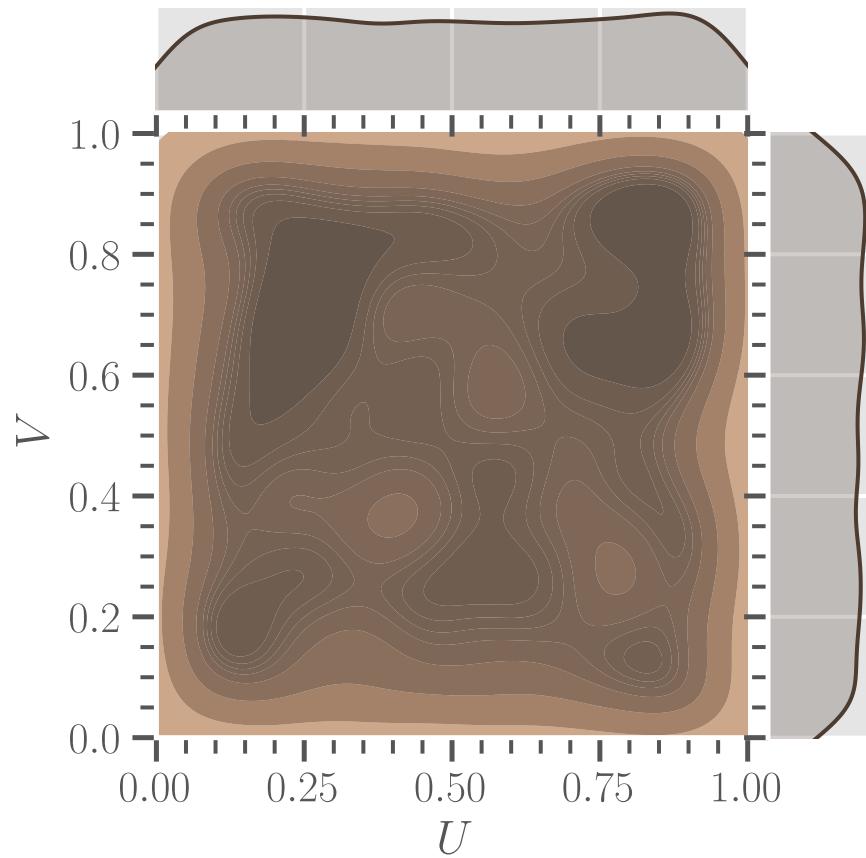
$$C(u, v) = P(U \leq u, V \leq v)$$

Można też więc rozważyć gęstość kopuły  $c(u, v)$ , czy rozkłady warunkowe  $C_{U|V}(u|v)$

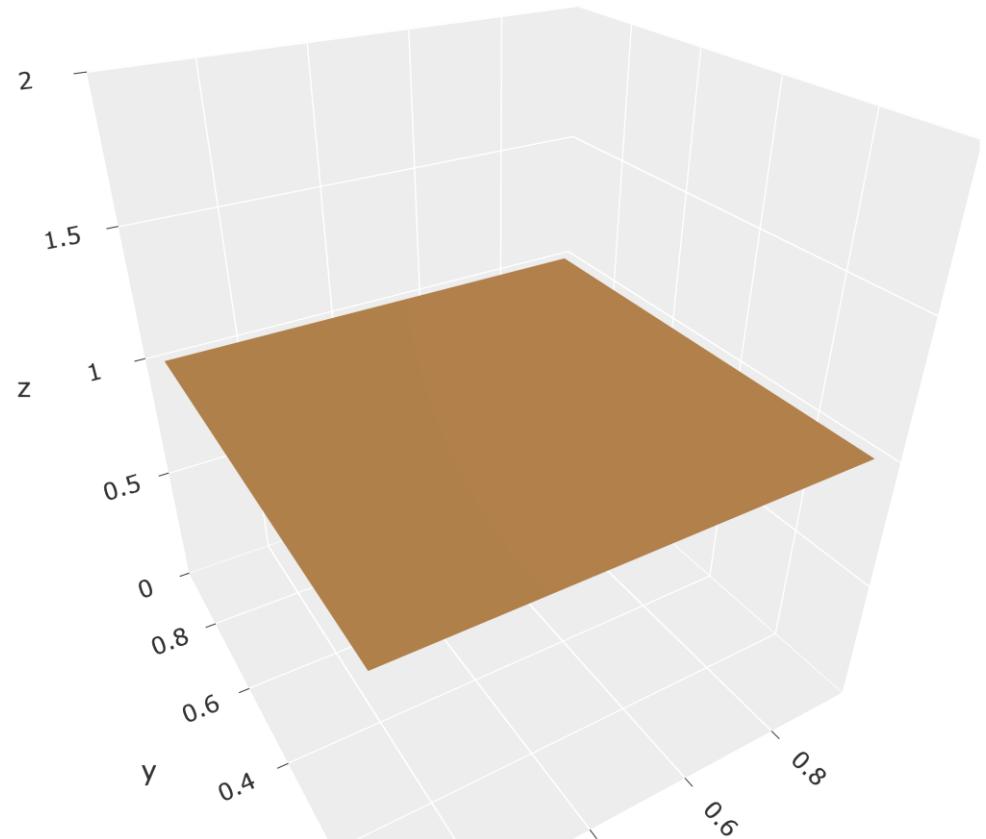
- $c(u, v) = \frac{\partial^2 C(u, v)}{\partial u \partial v}$
- $C_{U|V}(u|v) = P(U \leq u | V = v) = \frac{\partial C(u, v)}{\partial v}$

# Kopuły

Formalna definicja

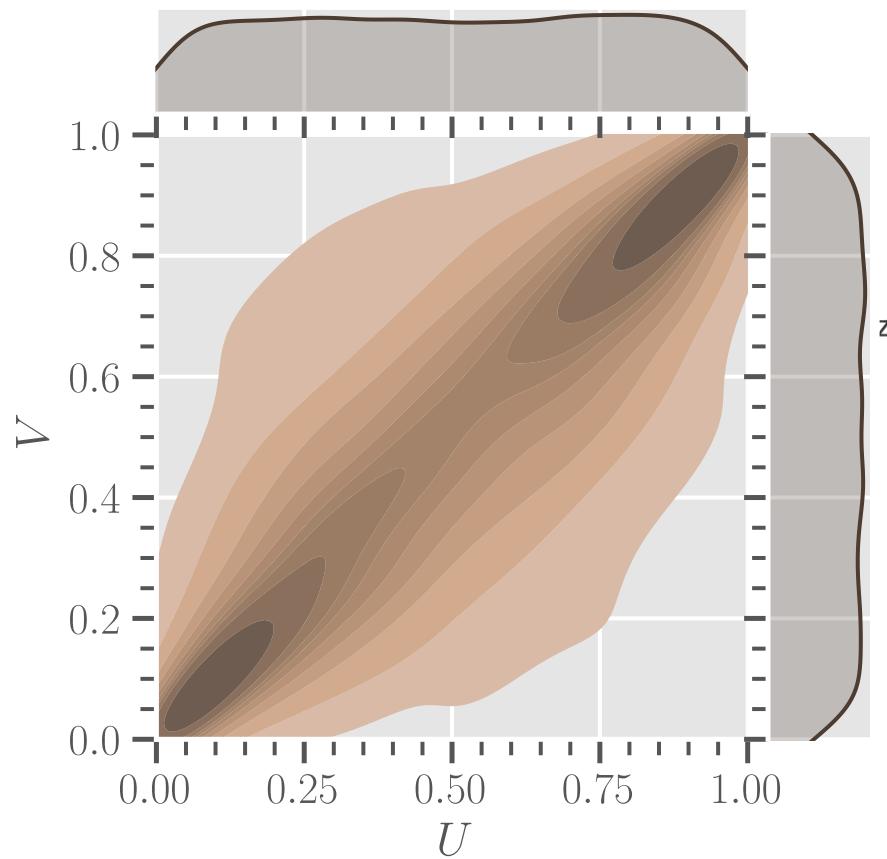


Independence copula

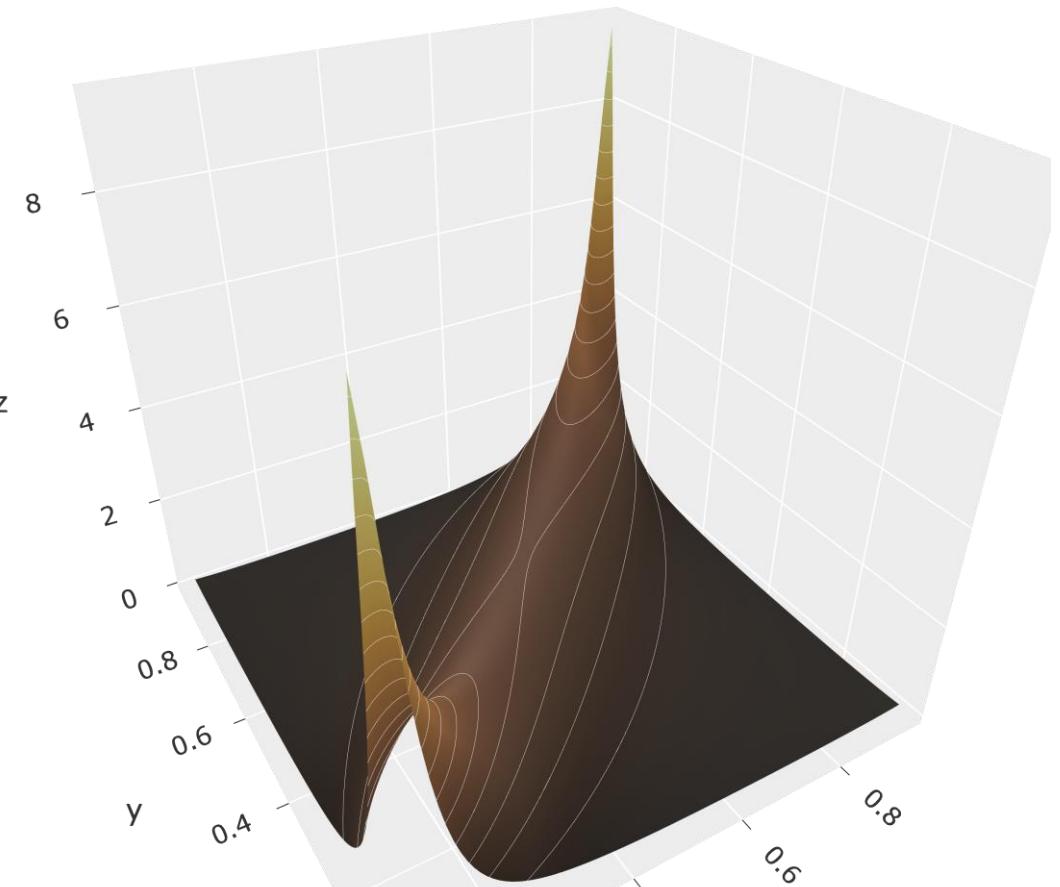


# Kopuły

Formalna definicja

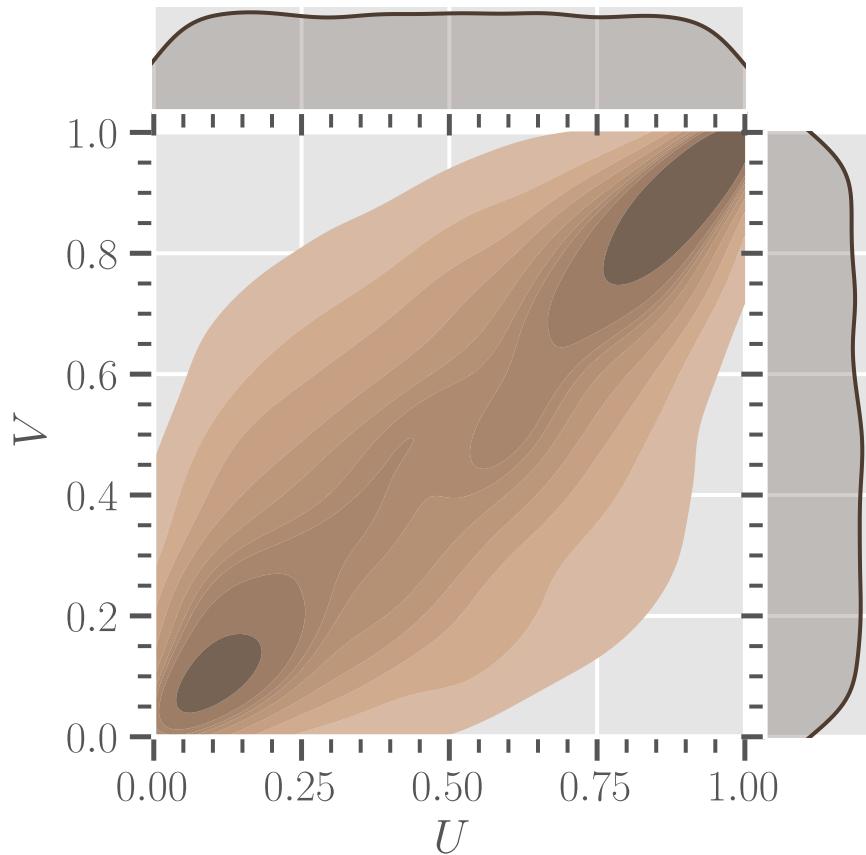


Student copula

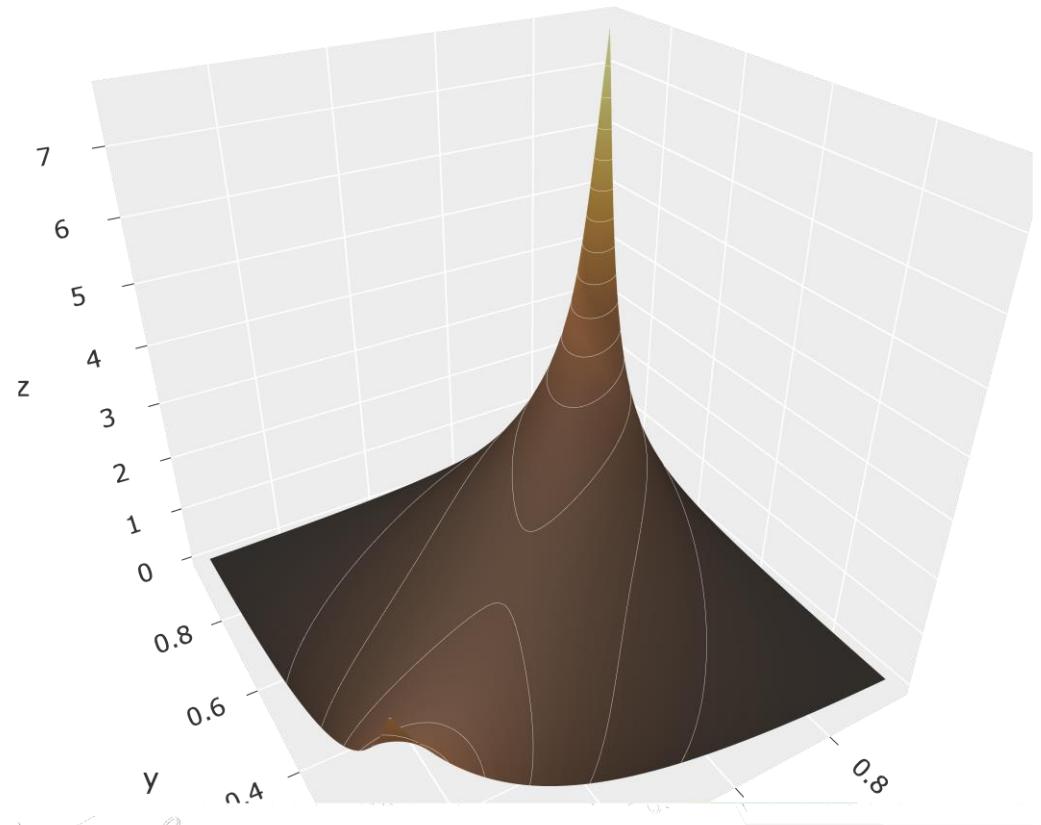


# Kopuły

Formalna definicja



Gumbel copula



# Kopuły

Formalna definicja

## Twierdzenie Sklara (1959)

Jeśli  $X$  jest  $d$ -wymiarowym wektorem losowym o rozkładzie łącznym  $F(x_1, \dots, x_d)$  i rozkładach brzegowych  $F_i(x_i)$ , gdzie  $i = 1, \dots, d$ , to dystrybuanta i gęstość tego rozkładu łącznego daje się przedstawić jako:

$$F(x_1, \dots, x_d) = C(F_1(x_1), \dots, F_d(x_d))$$

$$f(x_1, \dots, x_d) = c(F_1(x_1), \dots, F_d(x_d)) \cdot f_1(x_1) \cdot \dots \cdot f_d(x_d)$$

# Kopuły

Formalna definicja

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Jest i odwrotnie: kopuła  $C(\mathbf{u})$  i jej gęstość  $c(\mathbf{u})$  która odpowiada rozkładowi łącznemu  $F$  o rozkładach brzegowych  $F_i$ , daje się przedstawić jako:

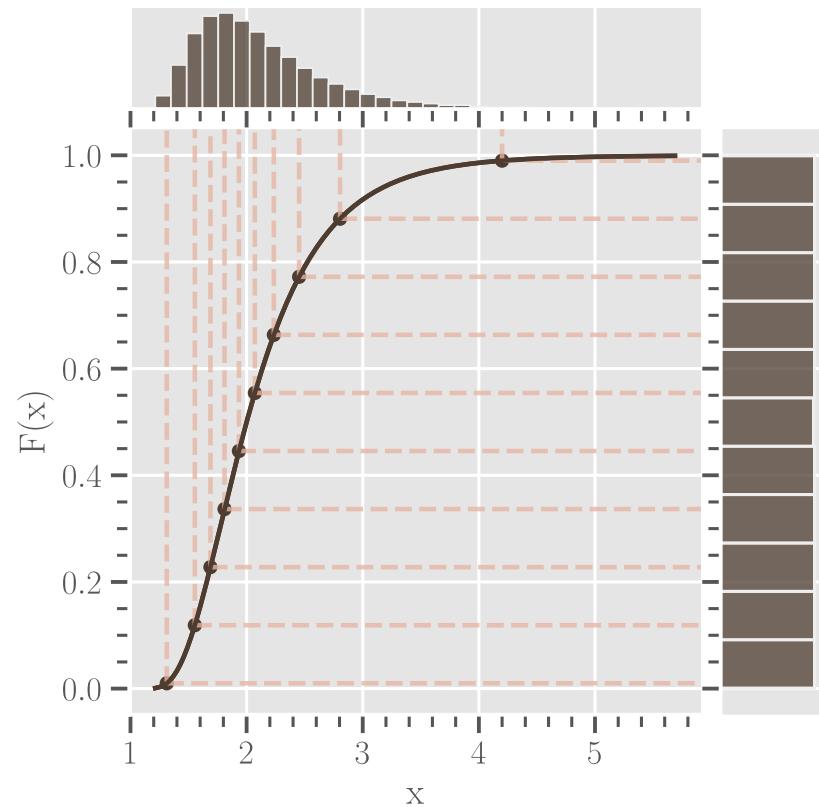
$$C(u_1, \dots, u_d) = F(F_1^{-1}(u_1), \dots, F_d^{-1}(u_d))$$

$$c(u_1, \dots, u_d) = \frac{f(F_1^{-1}(u_1), \dots, F_d^{-1}(u_d))}{f_1(F_1^{-1}(u_1)) \cdot \dots \cdot f_d(F_d^{-1}(u_d))}$$

# Estymacja

## Probability integral transform

Jeżeli  $X$  jest ciągłą zmienną losową o dystrybuancie  $F(x)$ , to zmienna losowa  $U := F(X)$  ma rozkład jednostajny na odcinku  $[0, 1]$



$$U \sim U(0, 1)$$

$$F_X(X)$$

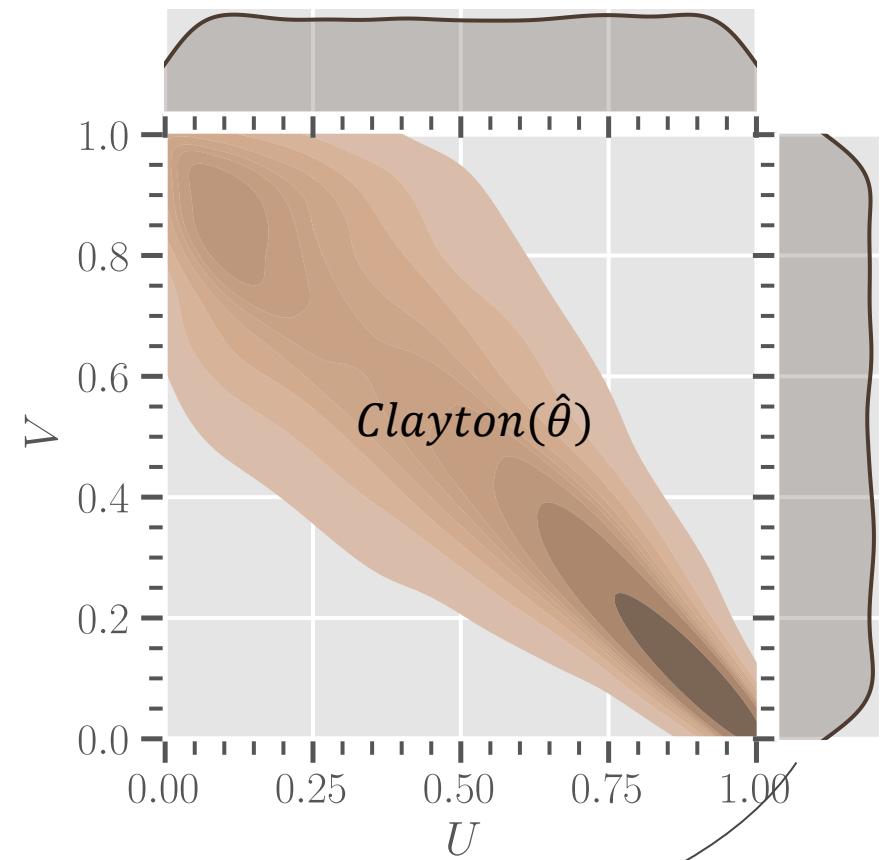
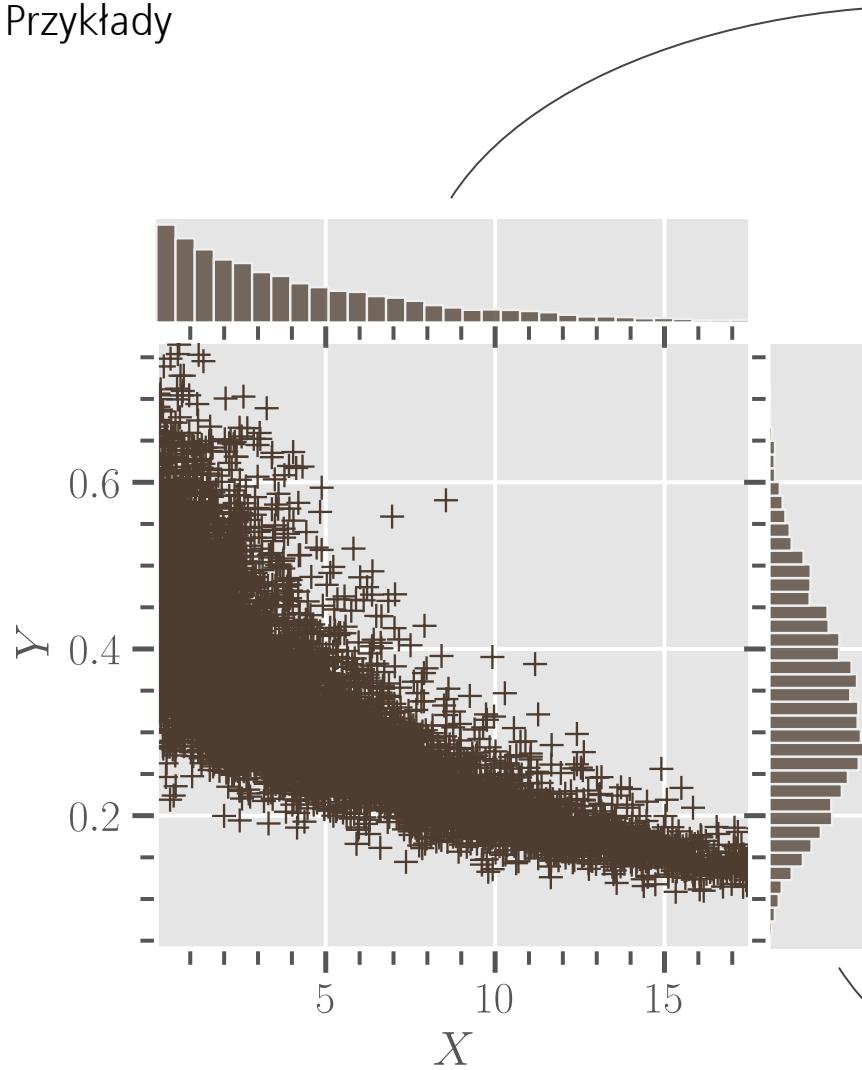
$$F_X^{-1}(U)$$

$$X \sim F$$

# Kopuły

Przykłady

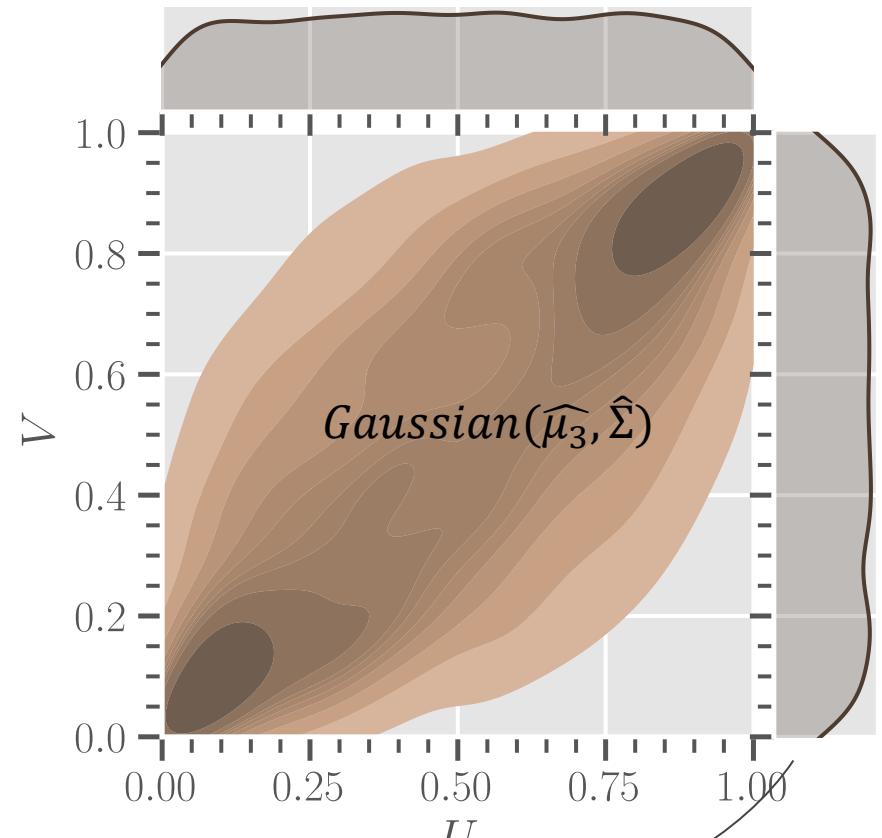
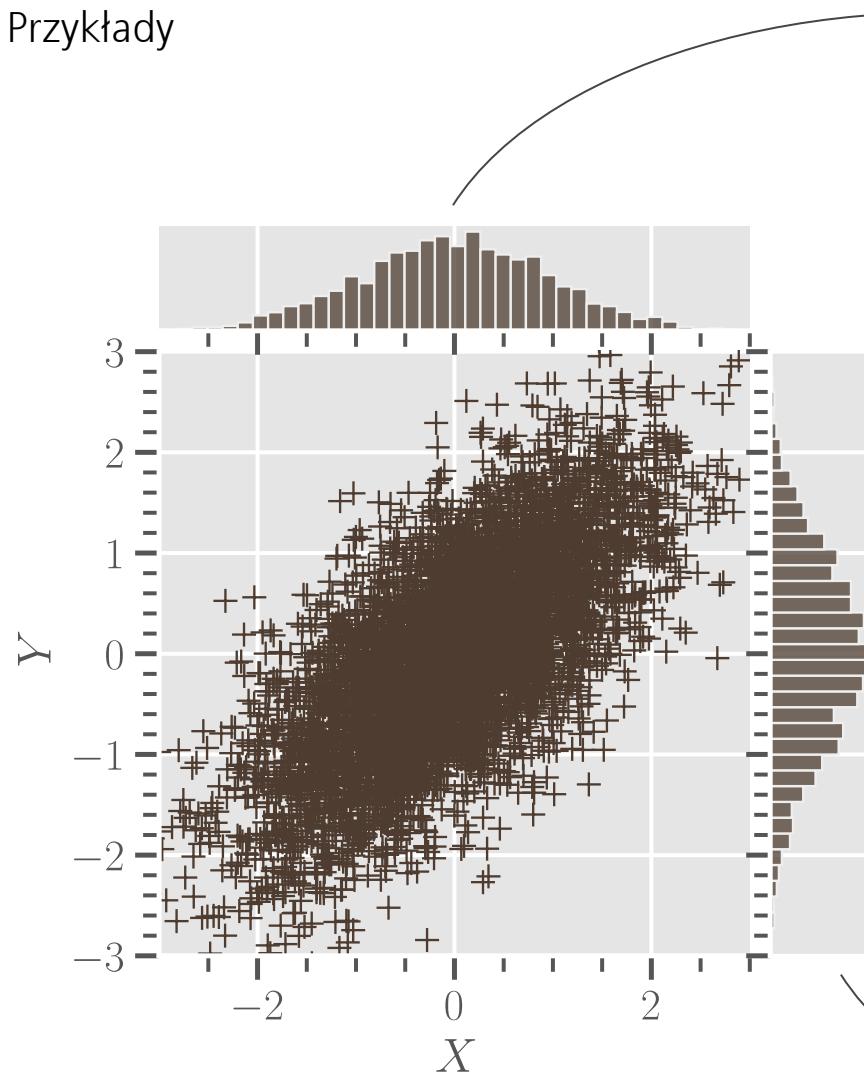
$$Exp(\hat{\lambda})_{CDF}$$



# Kopuły

Przykłady

$$N(\widehat{\mu}_1, \widehat{\sigma}_1)_{CDF}$$

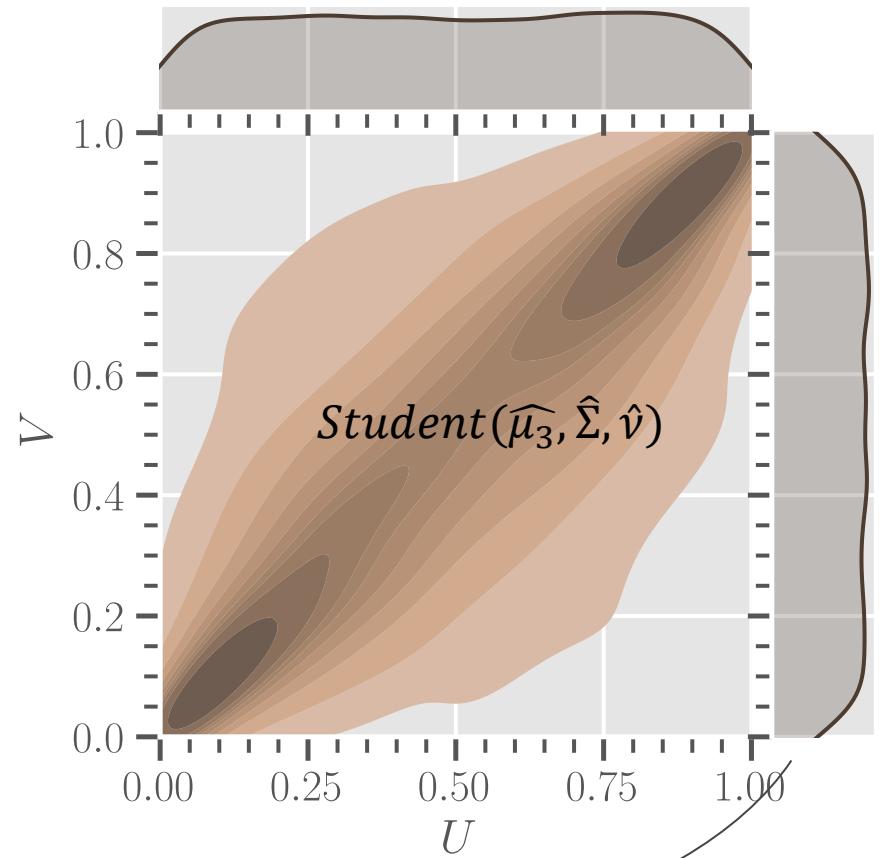
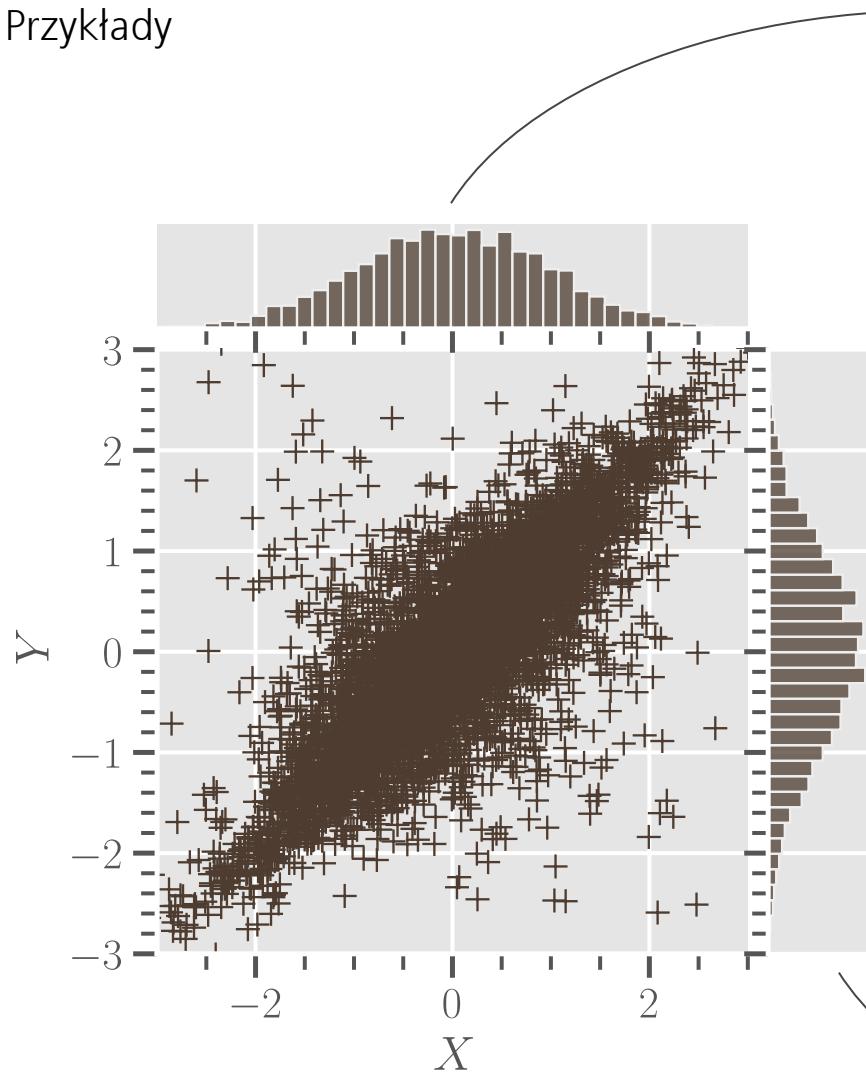


$$N(\widehat{\mu}_2, \widehat{\sigma}_2)_{CDF}$$

# Kopuły

Przykłady

$$N(\widehat{\mu}_1, \widehat{\sigma}_1)_{CDF}$$



$$N(\widehat{\mu}_2, \widehat{\sigma}_2)_{CDF}$$

# Vine Copula

Krok do większej ilości wymiarów

# Vine Copula

Krok do większej ilości wymiarów

$$f(x_1, x_2, x_3) = f_{3|12}(x_3|x_1, x_2) \cdot f_{2|1}(x_2|x_1) \cdot f_1(x_1)$$

# Vine Copula

Krok do większej ilości wymiarów

$$f(x_1, x_2, x_3) = f_{3|12}(x_3|x_1, x_2) \cdot f_{2|1}(x_2|x_1) \cdot f_1(x_1)$$

I)

$$f_{2|1}(x_2|x_1) = c_{12}(F_1(x_1), F_2(x_2)) \cdot f_2(x_2)$$

$$f_{3|12}(x_3|x_1, x_2) = c_{13;2}(F_{1|2}(x_1|x_2), F_{3|2}(x_3|x_2); x_2) \cdot f_{3|2}(x_3|x_2)$$

II)

$$f_{3|2}(x_3|x_2) = c_{23}(F_2(x_2), F_3(x_3)) \cdot f_3(x_3)$$

# Vine Copula

Krok do większej ilości wymiarów

$$f(x_1, x_2, x_3) = f_{3|12}(x_3|x_1, x_2) \cdot f_{2|1}(x_2|x_1) \cdot f_1(x_1)$$

I)

$$f_{2|1}(x_2|x_1) = c_{12}(F_1(x_1), F_2(x_2)) \cdot f_2(x_2)$$

$$f_{3|12}(x_3|x_1, x_2) = c_{13;2}(F_{1|2}(x_1|x_2), F_{3|2}(x_3|x_2); x_2) \cdot f_{3|2}(x_3|x_2)$$

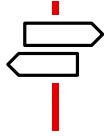
II)

$$f_{3|2}(x_3|x_2) = c_{23}(F_2(x_2), F_3(x_3)) \cdot f_3(x_3)$$

$$f(x_1, x_2, x_3) = c_{13;2} \left( F_{1|2}(x_1|x_2), F_{3|2}(x_3|x_2) \right) \cdot c_{23}(F_2(x_2), F_3(x_3)) \cdot \\ c_{12}(F_1(x_1), F_2(x_2)) \cdot f_3(x_3) \cdot f_2(x_2) \cdot f_1(x_1)$$

# Vine Copula

Krok do większej ilości wymiarów

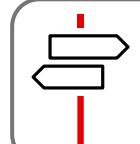


Którą Pair Copula Construction wybrać?

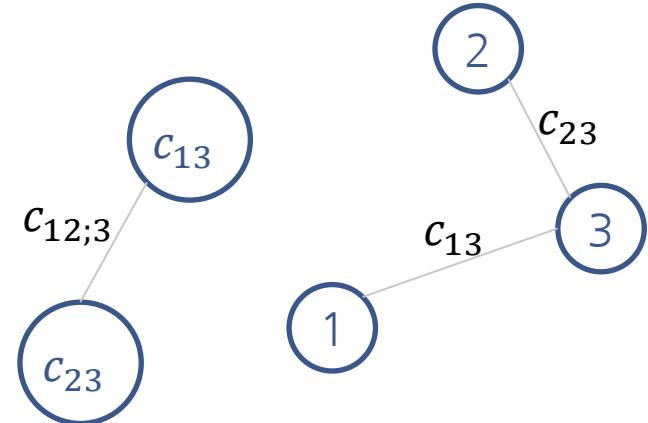
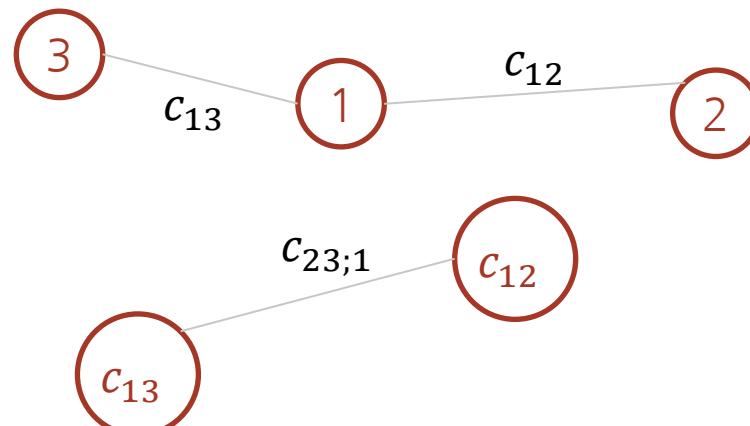
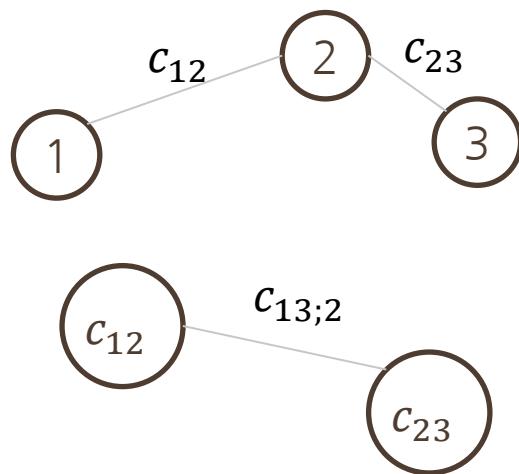
- $f(x_1, x_2, x_3) = c_{13;2}(F_{1|2}(x_1|x_2), F_{3|2}(x_3|x_2); x_2) \cdot c_{23}(F_2(x_2), F_3(x_3)) \cdot c_{12}(F_1(x_1), F_2(x_2)) \cdot f_3(x_3) \cdot f_2(x_2) \cdot f_1(x_1)$
- $f(x_1, x_2, x_3) = c_{12;3}(F_{1|3}(x_1|x_3), F_{2|1}(x_2|x_3); x_3) \cdot c_{13}(F_1(x_1), F_3(x_3)) \cdot c_{23}(F_2(x_2), F_3(x_3)) \cdot f_3(x_3) \cdot f_2(x_2) \cdot f_1(x_1)$
- $f(x_1, x_2, x_3) = c_{23;1}(F_{2|1}(x_2|x_1), F_{3|1}(x_3|x_1); x_1) \cdot c_{13}(F_1(x_1), F_3(x_3)) \cdot c_{12}(F_1(x_1), F_2(x_2)) \cdot f_3(x_3) \cdot f_2(x_2) \cdot f_1(x_1)$

# Vine Copula

Krok do większej ilości wymiarów



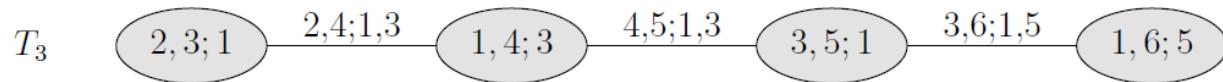
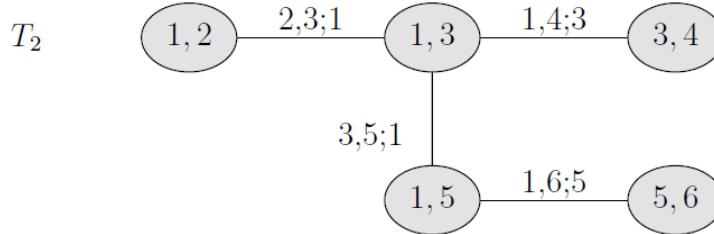
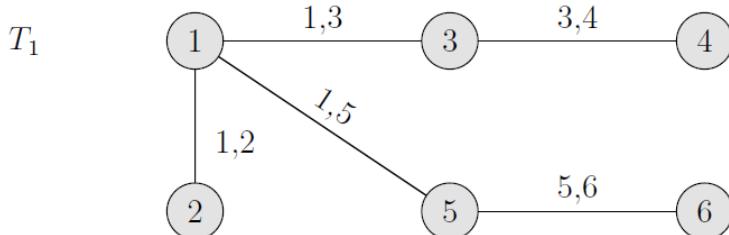
Która Pair Copula Construction wybrać?



# Vine Copula

Krok do większej ilości wymiarów

- Ilość możliwości (bardzo) szybko rośnie
- $\left(\frac{d!}{2}\right) \cdot 2^{\binom{d-2}{2}}$
- Przycinanie drzewek?
- Subklasy vine copula?



# Vine Copula

Krok do większej ilości wymiarów

**D(rawable) vine**

$$f_{1,\dots,d}(x_1, \dots, x_d) = \left[ \prod_{j=1}^{d-1} \prod_{i=1}^{d-j} c_{i,(i+j);(i+1)\dots(i+j-1)} \right] \cdot \left[ \prod_{k=1}^d f_k(x_k) \right]$$

**C(anonical) vine**

$$f_{1,\dots,d}(x_1, \dots, x_d) = \left[ \prod_{j=1}^{d-1} \prod_{i=1}^{d-j} c_{j,(j+1);1\dots j-1} \right] \cdot \left[ \prod_{k=1}^d f_k(x_k) \right]$$

# Vine Copula

Krok do większej ilości wymiarów

## D-vine

$$f(x_1, \dots, x_5) =$$

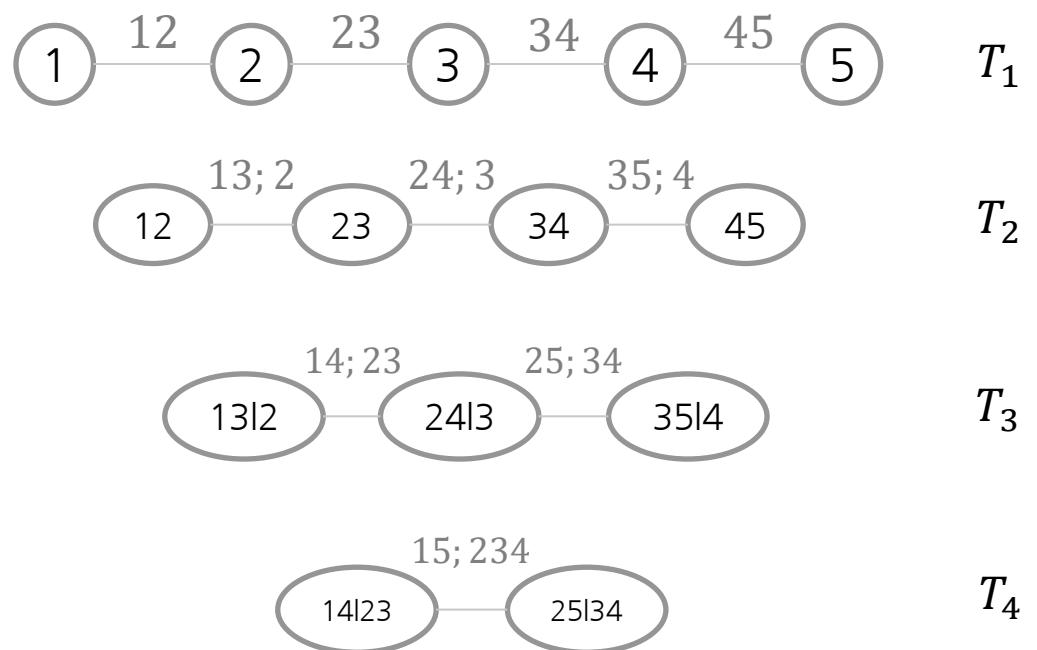
$$f_1(x_1)f_2(x_2)f_3(x_3)f_4(x_4)f_5(x_5) \cdot$$

$$\cdot c_{12} \cdot c_{23} \cdot c_{34} \cdot c_{45} \cdot$$

$$\cdot c_{13;2} \cdot c_{24;3} \cdot c_{35;4} \cdot$$

$$\cdot c_{14;23} \cdot c_{25;34} \cdot$$

$$\cdot c_{15;234}$$



# Vine Copula

Krok do większej ilości wymiarów

## D-vine

$$f(x_1, \dots, x_5) =$$

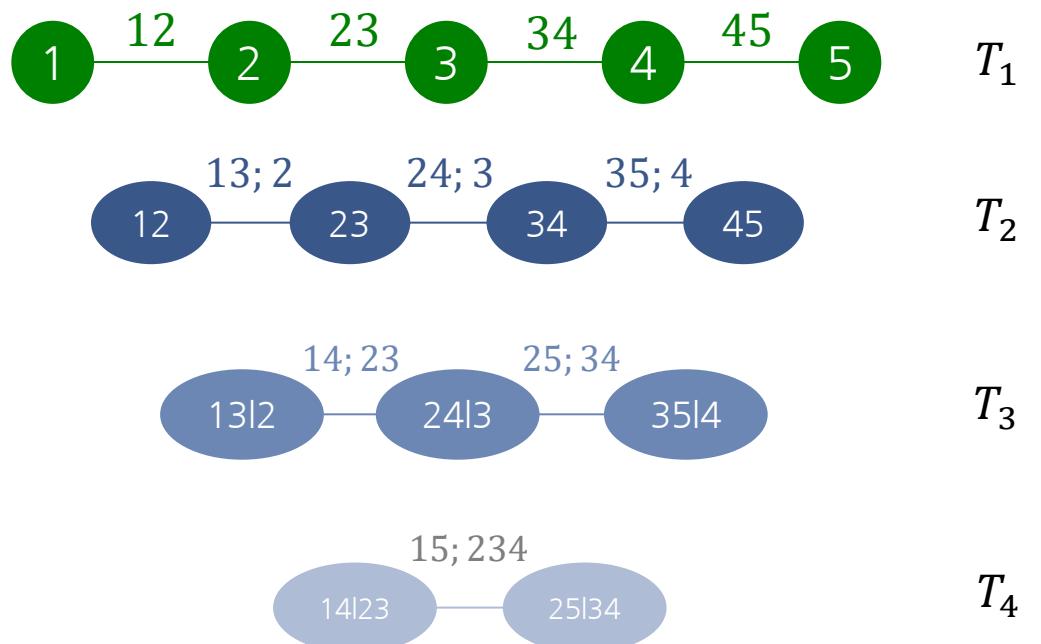
$$f_1(x_1)f_2(x_2)f_3(x_3)f_4(x_4)f_5(x_5) \cdot$$

$$\cdot c_{12} \cdot c_{23} \cdot c_{34} \cdot c_{45} \cdot$$

$$\cdot c_{13;2} \cdot c_{24;3} \cdot c_{35;4} \cdot$$

$$\cdot c_{14;23} \cdot c_{25;34} \cdot$$

$$\cdot c_{15;234}$$



# Vine Copula

Krok do większej ilości wymiarów

## C-vine

$$f(x_1, \dots, x_5) =$$

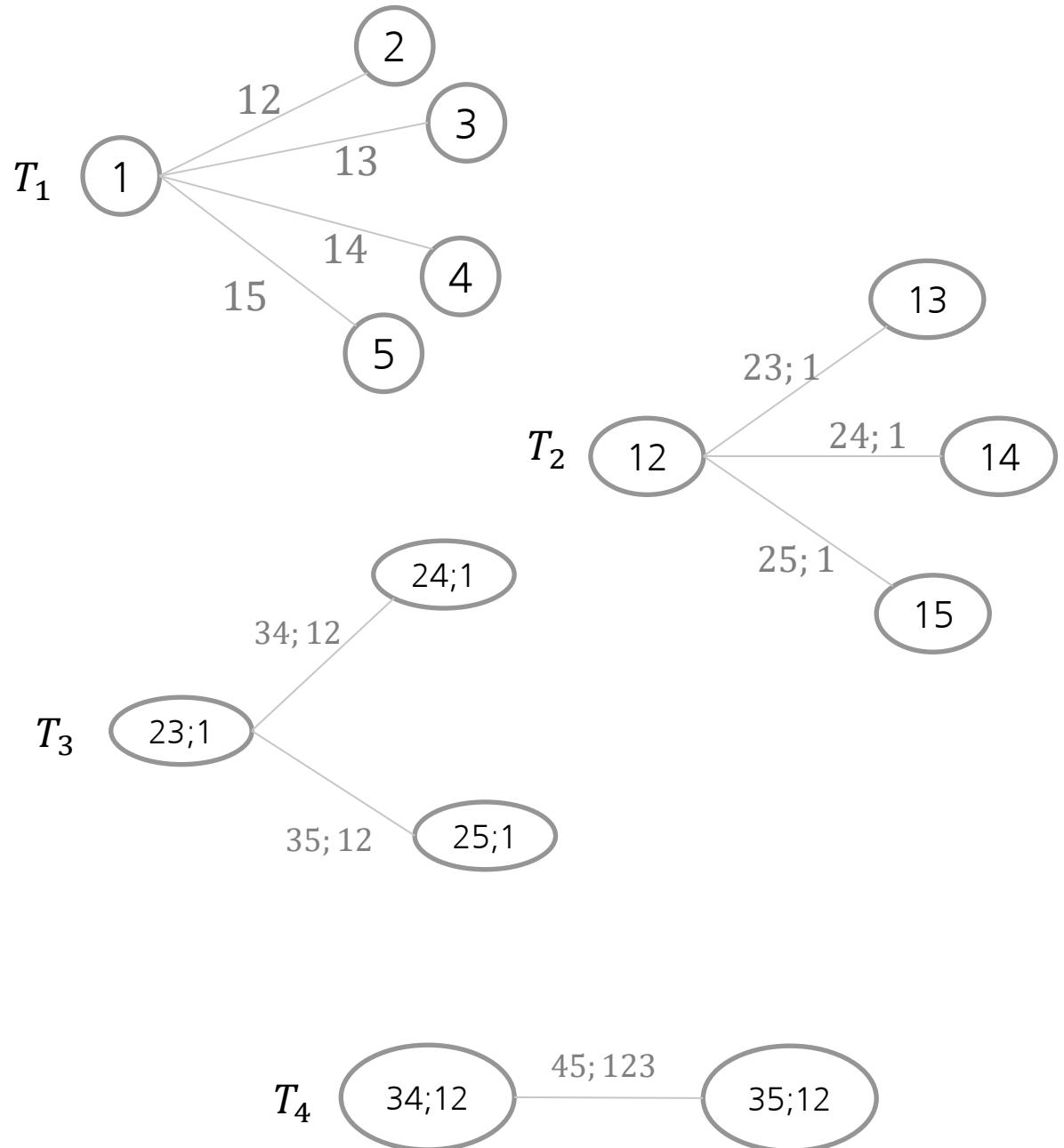
$$f_1(x_1)f_2(x_2)f_3(x_3)f_4(x_4)f_5(x_5) \cdot$$

$$\cdot c_{12} \cdot c_{13} \cdot c_{14} \cdot c_{15} \cdot$$

$$\cdot c_{23;1} \cdot c_{24;1} \cdot c_{25;1} \cdot$$

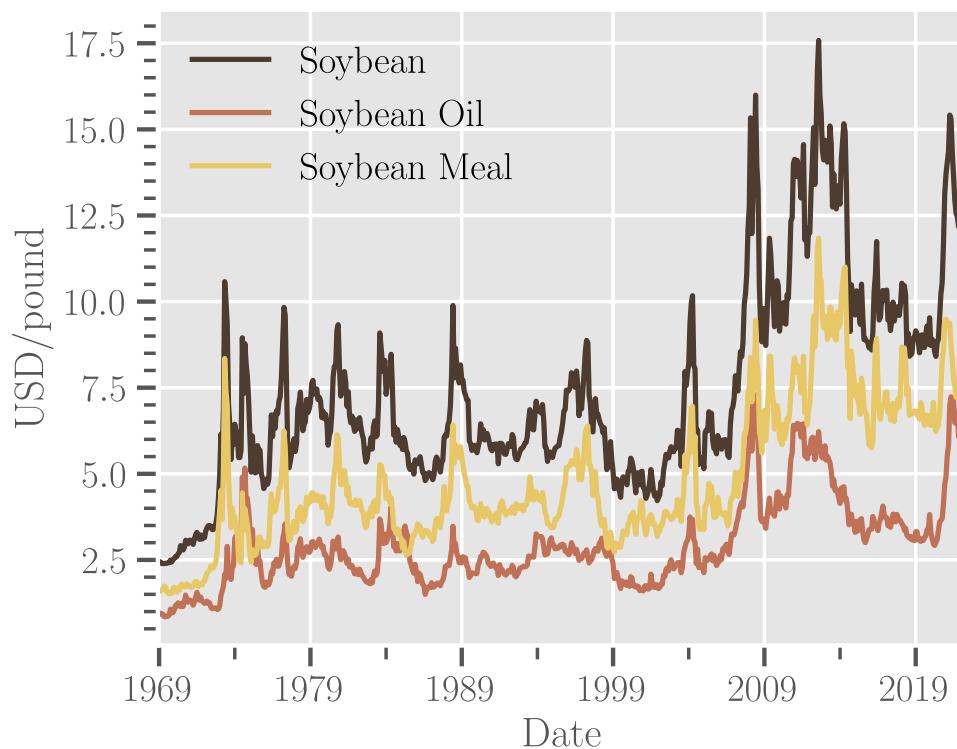
$$\cdot c_{34;12} \cdot c_{35;12} \cdot$$

$$\cdot c_{45;123}$$

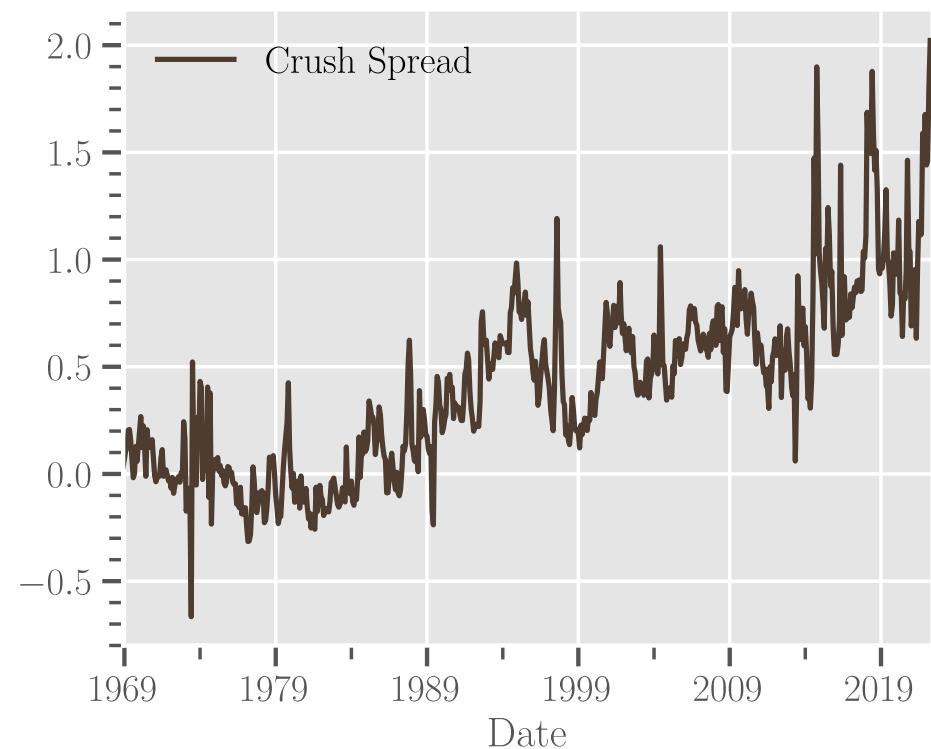


# Wycena opcji na crush spread

Dane rynkowe

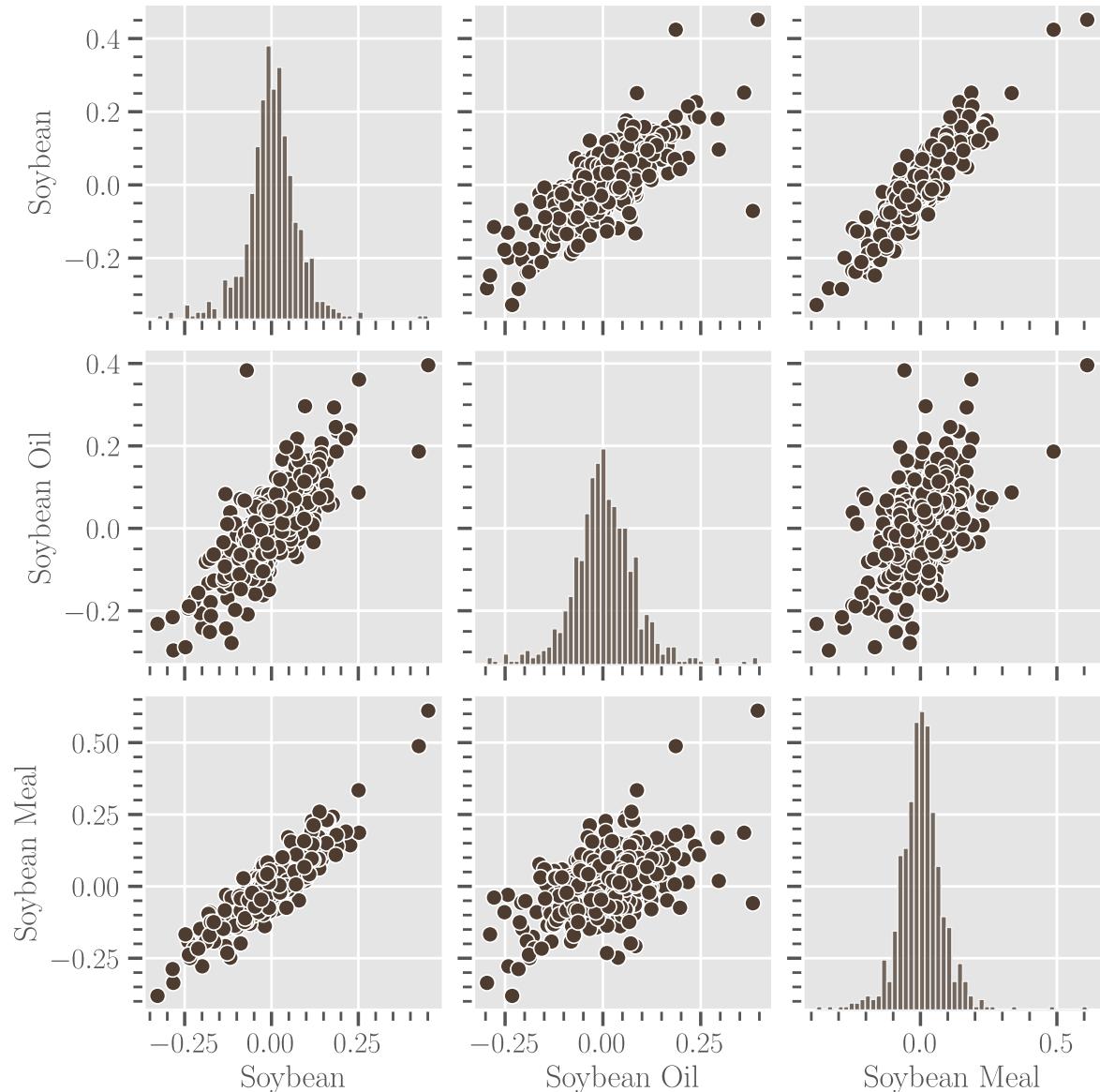


Data source: [www.macrotrends.net](http://www.macrotrends.net)



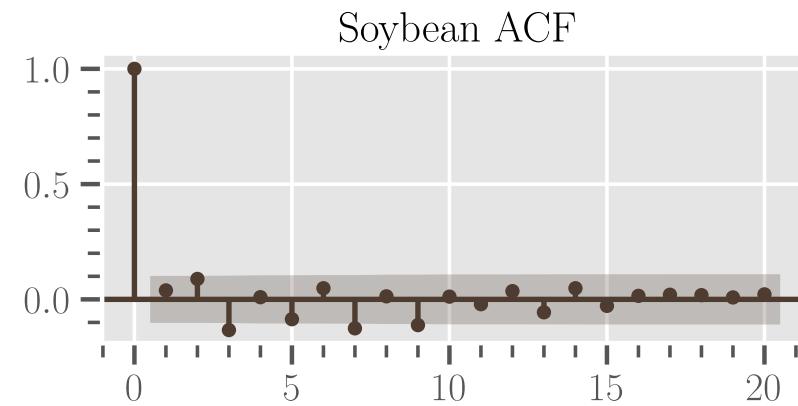
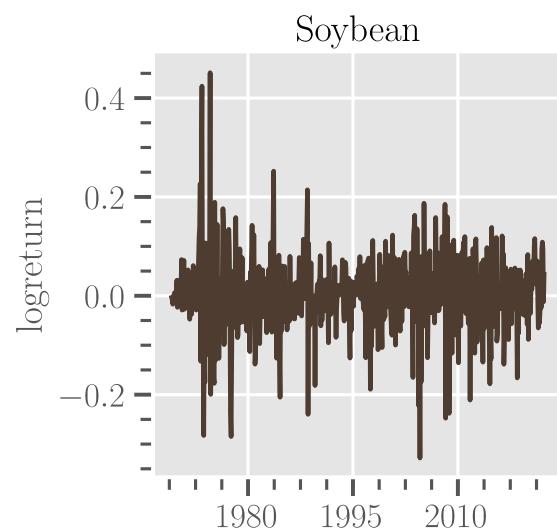
# Wycena opcji na crush spread

Zależności między zmiennymi

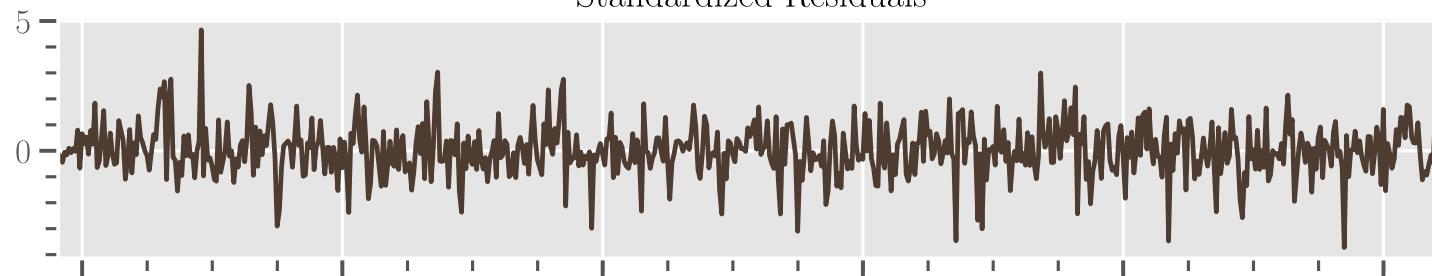


# Wycena opcji na crush spread

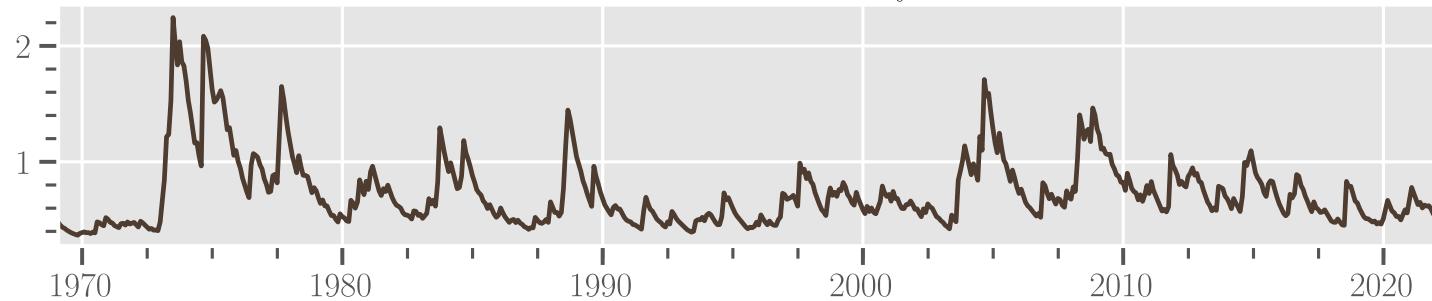
TSA



Standardized Residuals

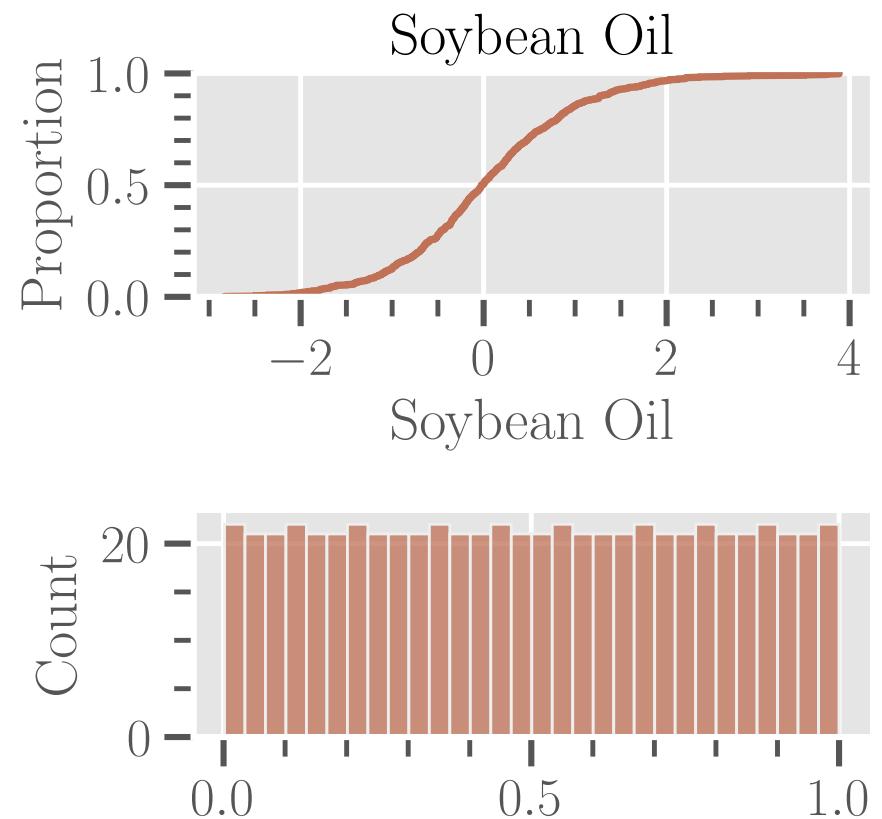
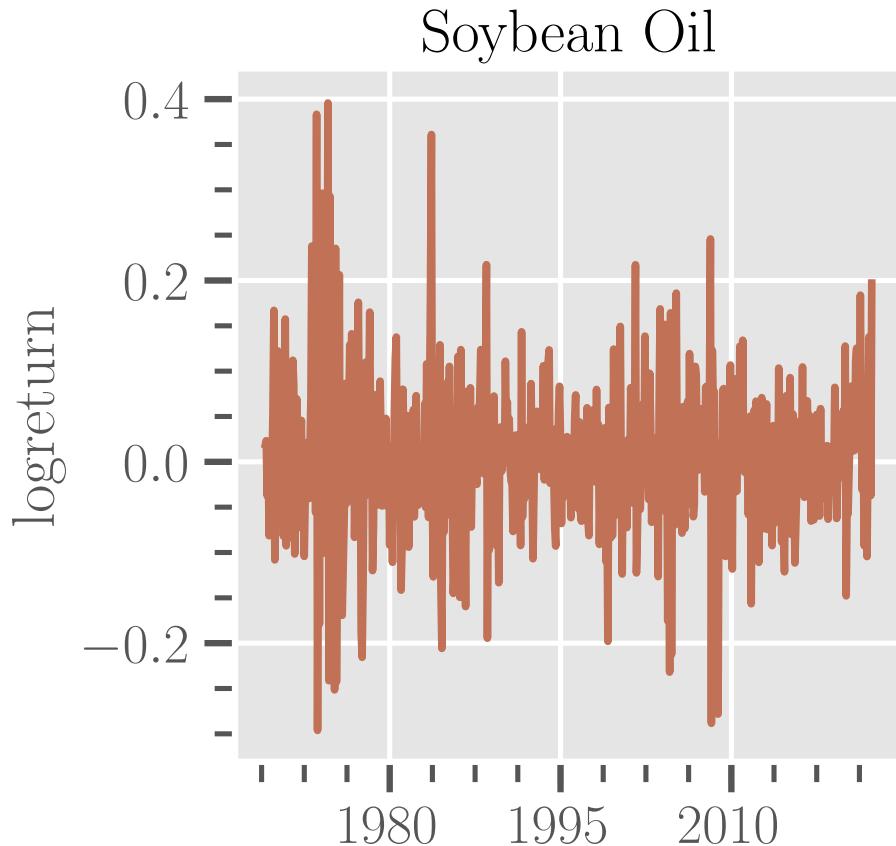


Conditional Volatility



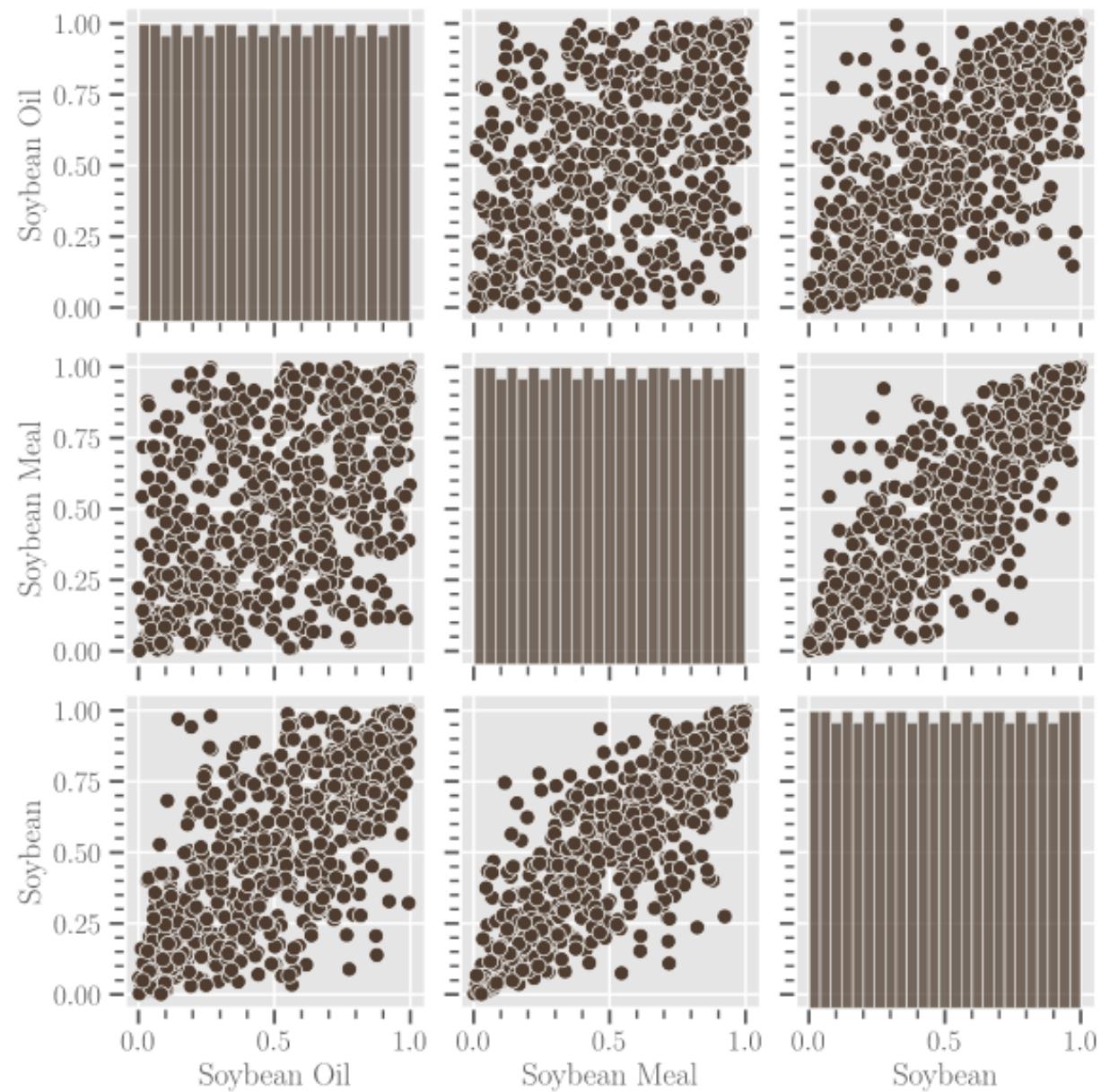
# Wycena opcji na crush spread

Probability integral transform

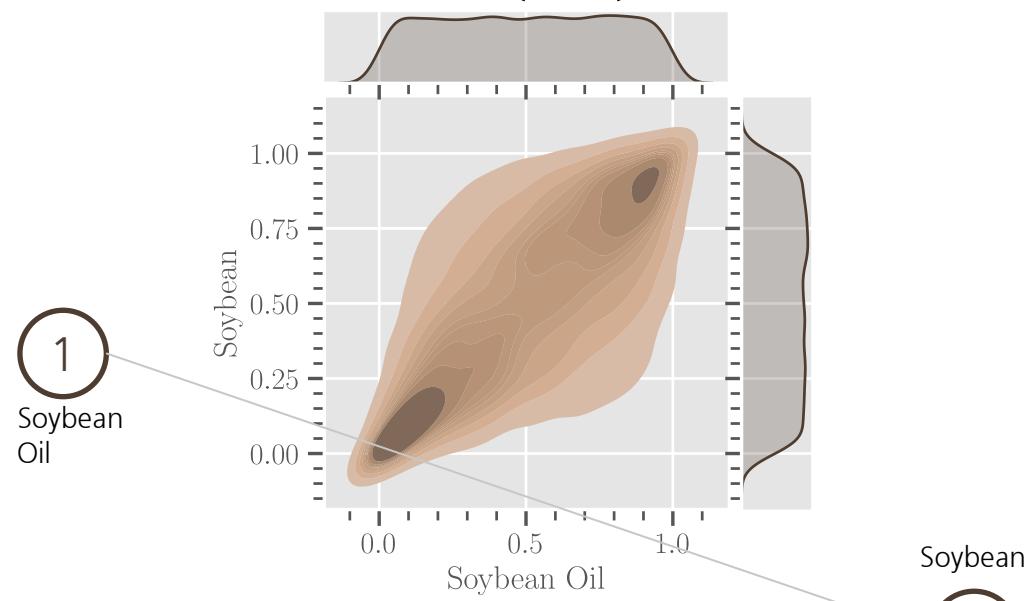


# Wycena opcji na crush spread

Estymacja modelu



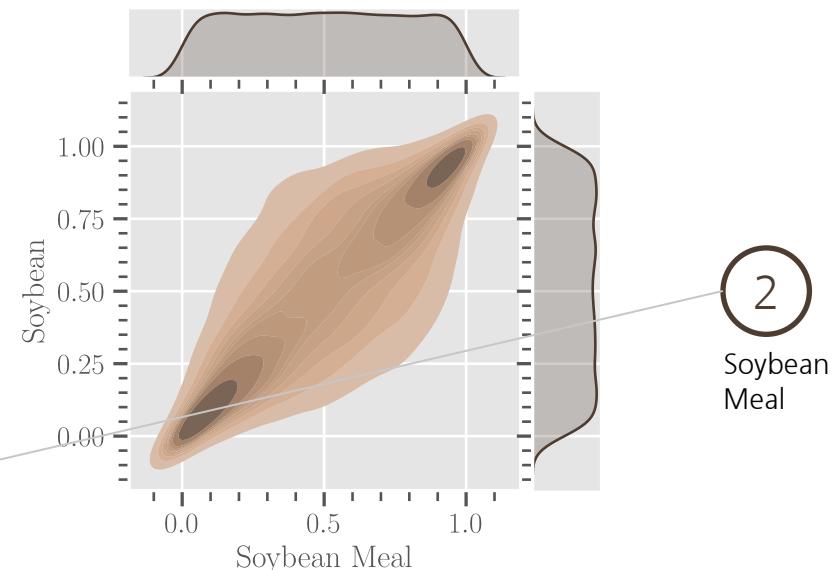
*Gumbell(2.05), 180°*



1  
Soybean  
Oil

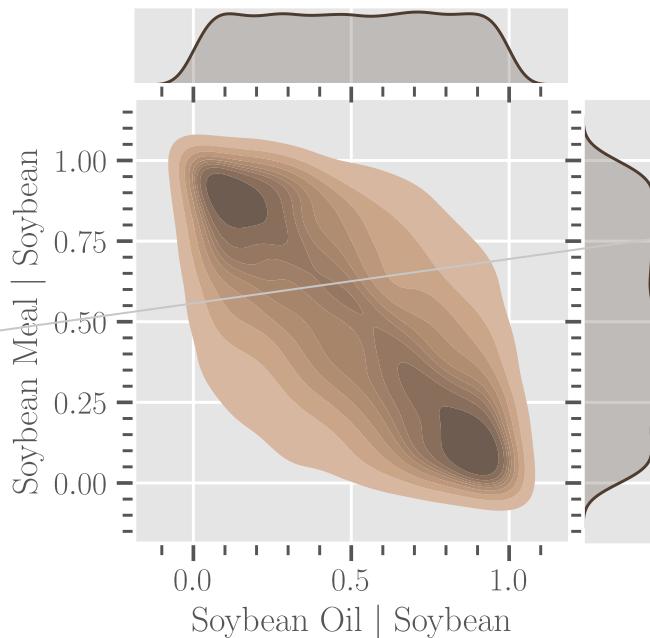
Soybean  
3

*BB7(2.01)*



2  
Soybean  
Meal

*Frank(-4.45)*

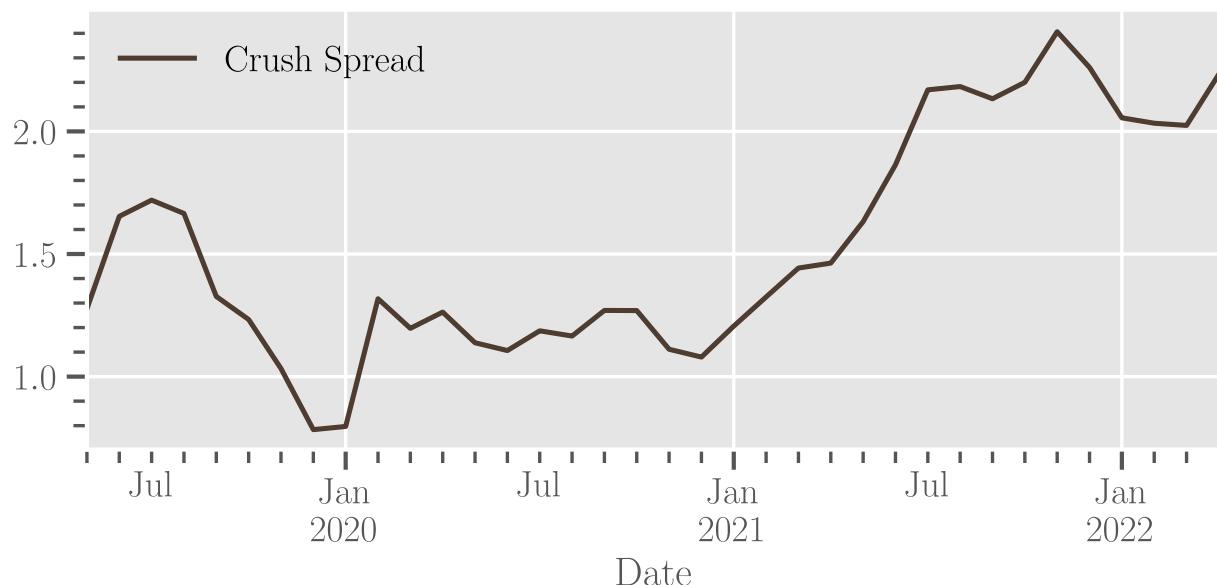
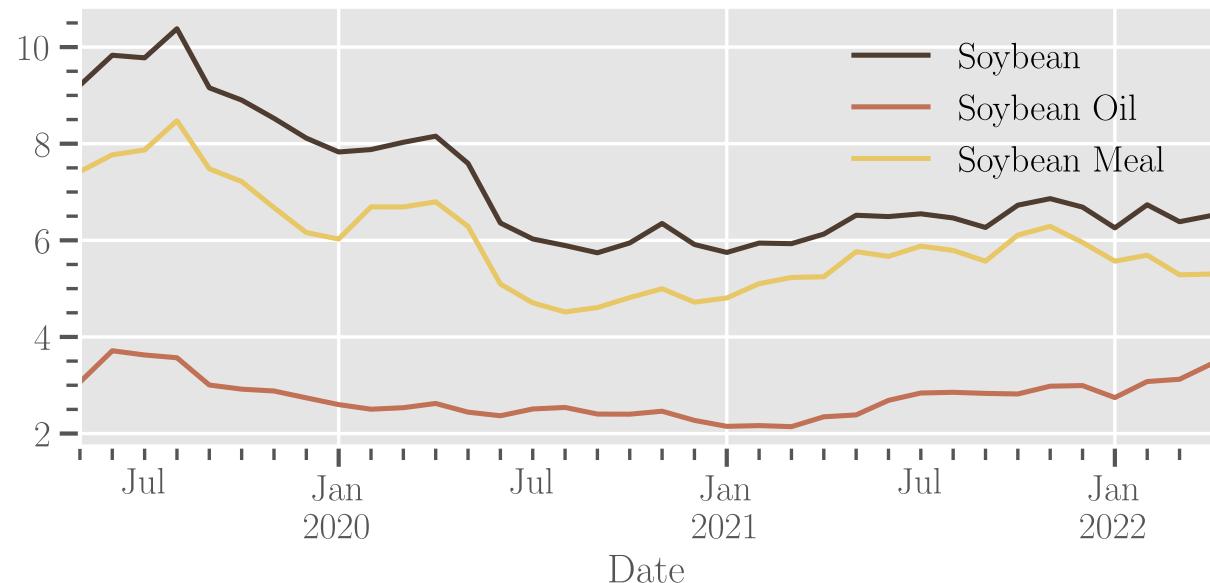


$c_{13}$

$c_{23}$

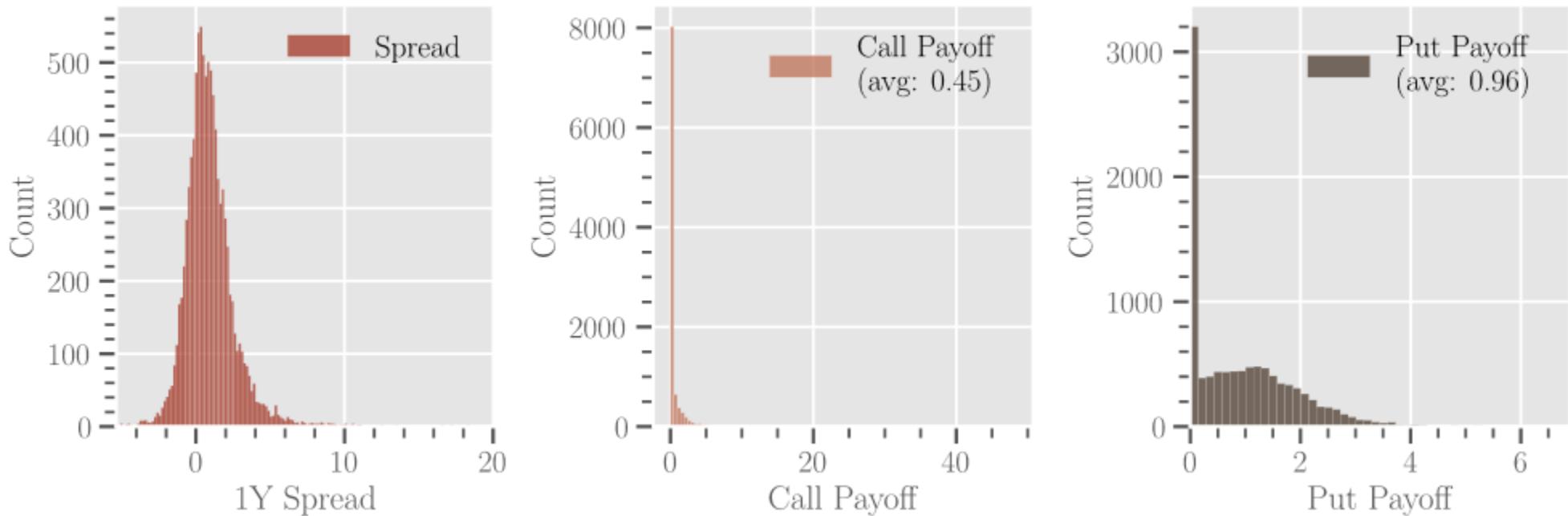
# Wycena opcji na crush spread

Pełna symulacja



# Wycena opcji na crush spread

Wysymulowany rozkład spreadu



Instrument	Cena	Std. Err.
Call option K = 1.5, T = 3Y	0.427	0.014
Put option K = 1.5, T = 3Y	0.899	0.0097

Thank you

Questions?