*a)* $X_0 = 0$ *almost surely,*

*b)* $X_t - X_s$ *is independent of* $\mathcal{F}_s$ *for all* $0 \le s < t$,

*c)* $X_t - X_s \sim \mathsf{N}(0, t-s)$, $0 \le s < t$.

It follows from Karatzas and Shreve (2000, Theorem 5.12, p. 75) that this process is a Markov process. We can now extract the underlying bivariate copula family $C\langle X\rangle$.

**5.23 Example (copula of Brownian motion)**
Let $(B_t)_{t\in\mathbb{R}^+}$ be a standard Brownian motion. In the sequel, let $\Phi$ denote the distribution function of a standard normal random variable and $\varphi$ the corresponding density.

The transition probabilities for a standard Brownian motion are given by

$$\mathsf{P}(x, s; y, t) := \mathsf{P}(B_t \le y \mid B_s = x) = \Phi\left(\frac{y-x}{\sqrt{t-s}}\right), \; t > s, \; x, y \in \mathbb{R}. \tag{5.13}$$

From Lemma 2.22 we have

$$\mathsf{P}(x, s; y, t) = D_1 C^B_{s,t}\big(F_s(x), F_t(y)\big),$$

where $C^B_{s,t}$ denotes the copula of $B_s$ and $B_t$, $F_s$ and $F_t$ are the corresponding marginals. Thus, we get

$$\begin{aligned} C^B_{s,t}\big(F_s(x), F_t(y)\big) &= \int_{-\infty}^{x} D_1 C^B_{s,t}\big(F_s(z), F_t(y)\big)\, \mathrm{d}F_s(z) \\ &= \int_{-\infty}^{x} \Phi\left(\frac{y-z}{\sqrt{t-s}}\right) \mathrm{d}F_s(z) \quad \text{for } 0 < s < t. \end{aligned} \tag{5.14}$$

From the assumption $B_0 = 0$ we have $B_t - B_0 \sim \mathsf{N}(0, t)$ so that $F_t(x) = \Phi(x/\sqrt{t})$ which is equivalent to $x = \sqrt{t}\cdot\Phi^{-1}(F_t(x))$. Substitution into (5.14) yields

$$C^B_{s,t}(u, v) = \int_0^u \Phi\left(\frac{\sqrt{t}\Phi^{-1}(v) - \sqrt{s}\Phi^{-1}(w)}{\sqrt{t-s}}\right) \mathrm{d}w. \tag{5.15}$$

It is now easy to derive the corresponding partial derivatives and the copula density. Elementary calculus yields

$$D_1 C^B_{s,t}(u, v) = \Phi\left(\frac{\sqrt{t}\Phi^{-1}(v) - \sqrt{s}\Phi^{-1}(u)}{\sqrt{t-s}}\right), \qquad (u, v) \in (0,1)^2, \tag{5.16}$$

$$D_2 C^B_{s,t}(u, v) = \frac{1}{\varphi\big(\Phi^{-1}(v)\big)}\sqrt{\frac{t}{t-s}} \int_0^u \varphi\left(\frac{\sqrt{t}\Phi^{-1}(v) - \sqrt{s}\Phi^{-1}(w)}{\sqrt{t-s}}\right) \mathrm{d}w, \quad (u, v) \in (0,1)^2. \tag{5.17}$$

For the last result, an interchange of integration and differentiation is necessary which holds from the "differentiation lemma" (e. g., Bauer, 1992, Lemma 16.2, p. 102). Differentiation of (5.16) yields the density

$$c^B_{s,t}(u,v) = \sqrt{\frac{t}{t-s}} \frac{\varphi\left((\sqrt{t}\Phi^{-1}(v) - \sqrt{s}\Phi^{-1}(u))/\sqrt{t-s}\right)}{\varphi\big(\Phi^{-1}(v)\big)}, \quad (u,v) \in (0,1)^2, \tag{5.18}$$

for the Brownian copula. ■

After having derived the copula we are free to generate a new stochastic process (again of Markovian nature as the Markov structure is characterized by the bivariate copulas, see Theorem 5.44) with the same intertemporal dependence structure as Brownian motion but with different marginals. An empirical example for this is given in Figure 5.1. Every path is generated the following way:

- Choose a time grid $(t_1, \ldots, t_n)$ where $t_i > 0$, $t_i < t_{i+1}$.
- A realization $(u_1, \ldots, u_n)$ from the Brownian copula process is generated on the time grid (cf. Section B.3).
- For the path of standard Brownian motion, set $b_n := \sqrt{t_n}\Phi^{-1}(u_n)$ such that $b_n$ can be interpreted as a realization of a normal random variable with $\mu = 0$ and $\sigma^2 = t_n$.
- For the path of the generalized Brownian motion with scaled $t_\nu$-marginals, set $d_n := \sqrt{t_n \frac{\nu-2}{\nu}} \cdot F_\nu^{-1}(u_n)$ where $F_\nu^{-1}$ is the quantile function of a $t_\nu$-distributed random variable (cf. Section A.2, p. 94) and take $\nu = 3$. The $d_n$ can then be interpreted as realizations from a rescaled $t_3$ distributed random variable $D_n$ such that $\mathsf{E}(D_n) = 0$, $\mathsf{Var}(D_n) = t_n$.

Note that it is not possible to distinguish the standard Brownian motion from the generalized one only considering first and second moments of the marginals and the dependence structure. This is a widely unrecognized danger in practice.

**5.24 Remark**

a) It is now straightforward to see that the copula of any strictly monotone increasing transformation of a Brownian motion is exactly the Brownian copula (see Theorem 2.15).

For example, a *geometric Brownian motion* $X = (X_t)_{t \in T}$ satisfies the stochastic integral equation

$$X_t = X_0 + \mu \int_0^t X_s \,\mathrm{d}s + \nu \int_0^t X_s \,\mathrm{d}B_s \tag{5.19}$$

whose solution is given by

$$X_t = X_0 \cdot \exp\left((\mu - \frac{1}{2}\nu^2)t + \nu B_t\right) \tag{5.20}$$

(Karatzas and Shreve, 2000, pp. 349) where $\mu \in \mathbb{R}$, $\nu > 0$, $X_0 > 0$. Thus, $X_t$ is just a strictly increasing transformation of $B_t$ by $f(z) := X_0 \cdot \exp\bigl((\mu - \frac{1}{2}\nu^2)t + \nu z\bigr)$. The copula of geometric Brownian motion is therefore given by equation (5.15).

However, this is only a special case of a related question: How can we derive the family of copulas for general transformations of diffusions or—more general—local (semi-)martingales? We will deal with this interesting point in Section 5.5.

b) As Brownian motion is also a Gaussian process[2], $Z := (B_{t_1}, \dots, B_{t_n})$ has a multivariate normal-distribution. Therefore, the $n$-dimensional copula must be the $n$-dimensional Gaussian copula which is derived from the multivariate normal distribution.

■

[2] A Gaussian process is a process whose finite-dimensional distributions are all multivariate normal distributions.

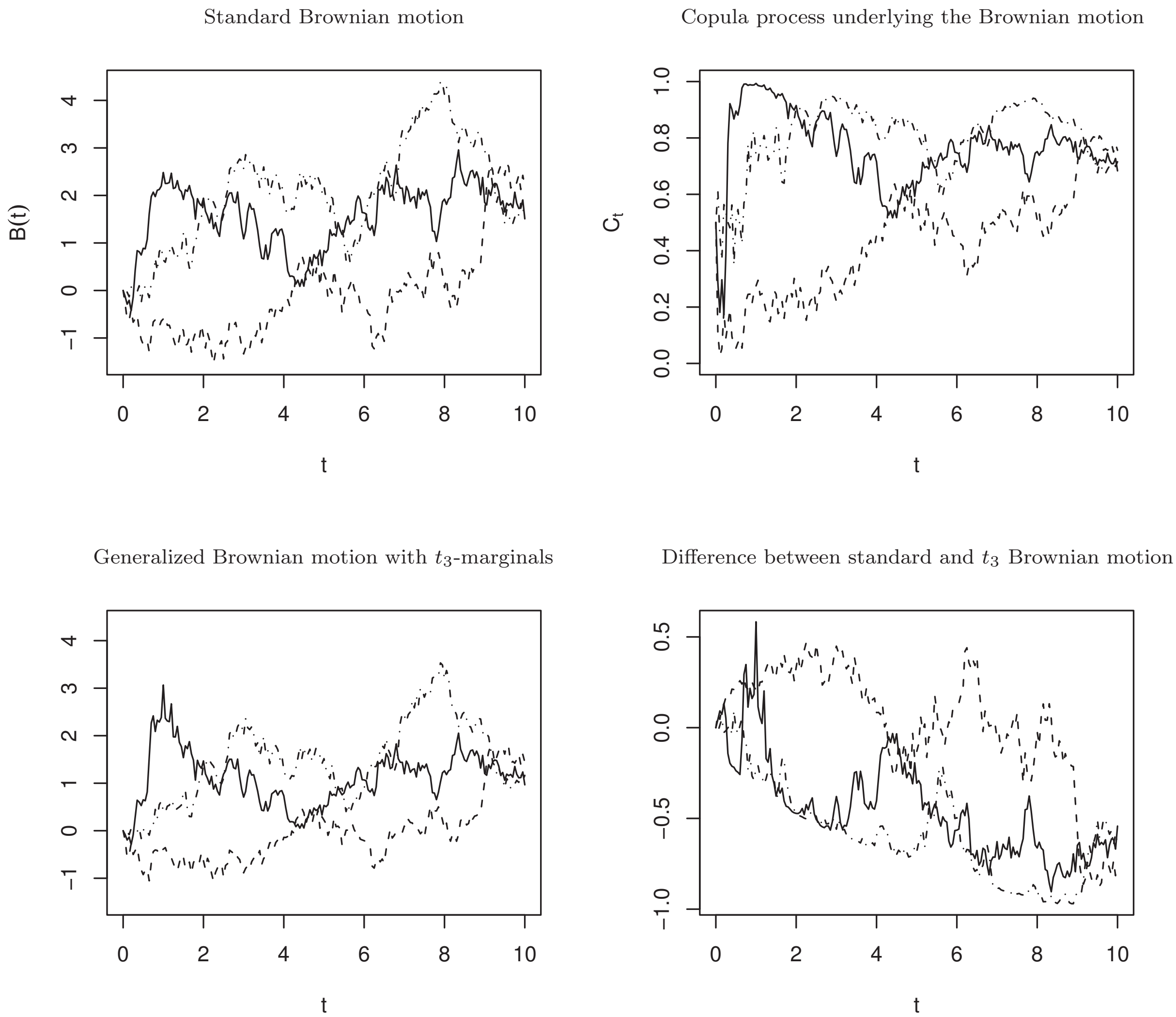


FIGURE 5.1. Standard Brownian motion and underlying copula process (upper row); "generalized Brownian motion" with scaled marginals such that $\mathsf{E}(X_t) = 0$, $\mathsf{Var}(X_t) = t$ as in the Brownian case.