Generating Tempered Stable Random Variates from Mixture Representation

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Abstract

The paper presents a new method of random number generation for tempered stable distribution. This method is easy to implement, faster than other available approaches when tempering is moderate and more accurate than the benchmark. All the results are given as parametric formulas that may be directly used by practitioners.

JEL Codes: C15, C46, C63.
Keywords: heavy tails, random number generation, tempered stable distribution.

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1 Definition and motivation

This paper addresses the problem of generating random numbers from tempered stable (henceforth TS) distribution. A convenient parametric way to define it is the following.

Definition 1 (Tempered stable distribution) *Random variable X has tempered stable distribution $TS_\alpha(\beta,\delta,\mu,\theta)$ if its characteristic function takes the form $\Phi_X(u)=e^{\psi_X(u)+i(\mu-\mu_X)u}$ where*

$$\psi_X(u)=\begin{cases}-\frac{1}{2\cos\frac{\pi\alpha}{2}}\delta^\alpha[(1+\beta)(\theta-iu)^\alpha+(1-\beta)(\theta+iu)^\alpha-2\theta^\alpha] & \alpha\neq 1,\\ \frac{1}{\pi}\delta[(1+\beta)(\theta-iu)\ln(\theta-iu)+(1-\beta)(\theta+iu)\ln(\theta+iu)-2\theta\ln\theta] & \alpha=1,\end{cases}\tag{1}$$

and the centring term is $\mu_X=\alpha(\cos\frac{\pi\alpha}{2})^{-1}\beta\delta^\alpha\theta^{\alpha-1}$ for $\alpha\neq 1$ and $\mu_X=-\frac{2}{\pi}\beta\delta(\ln\theta+1)$ for $\alpha=1$. The admissible parameter values are $\alpha\in(0,2),\beta\in[-1,1],\delta,\theta>0,\mu\in\mathbb{R}$.

The formulation above is derived as probably most applicable special case of general definition introduced by Rosiński (2007). Tempered stable distributions constitute a family of possibly skewed, leptokurtic densities with all moments finite. As macroeconomic distributions frequently display moderate skewness and excess kurtosis, TS distribution might be remarkably useful in modelling macroeconomic uncertainty via Monte Carlo simulation. This paper provides easy to implement randomization method that enables such experiments. While tempered stable distributions utilized in finance are endowed with more parameters, in case of macroeconomic data there is no clear evidence that the reactions of economic agents to upsurges and downturns of indexes are asymmetric. Hence more parsimonious parametrization should be sufficient.

TS distributions evolved from the concept of α-stable distributions, which arise from Central Limit Theorem as limit densities for i.i.d. jumps with heavy power-law tails. Basic properties of α-stable distributions may be found in Samorodnitsky and Taqqu (2000). The idea behind TS density is to alter α-stable distribution so that resulting density had lighter tails. In order to obtain the desired effect, spectral α-stable measure (expressed in polar coordinates) is multiplied by a weighting function which dampens probabilities of generating numbers with large modulus. This approach is known as tempering. The densities obtained this way may retain desirable properties of α-stable distributions, display better fit to the actual data and have higher order moments finite. Chakrabarty and Meerschaert (2011) demonstrate that any random walk with power-law jumps may be approximated with tempered stable density. Hence these distributions provide a universal model of accumulated jump. The density considered here arises when spectral measure of univariate α-stable distribution is weighted with exponent function $e^{-\theta|x|}$. Therefore TS is also known as exponentially tempered stable distribution.

Tempered stable distribution inherits parameters α, β and δ of the α-stable distribution being tempered. The tempering does not affect their qualitative properties: $\alpha\in(0,2)$ stands for departure from normality (if $\alpha=2$ the underlying α-stable distribution is Gaussian), $\beta\in[-1,1]$ governs skewness (if $\beta=0$ then both α-stable and TS distribution are symmetric), $\delta>0$ displays scale–like behaviour. Additional parameter $\theta>0$ measures how far the resulting distribution is from the underlying α-stable density. While $\theta\approx 0$ indicates it is almost exactly α-stable, $\theta\gg 0$ signals a significant departure from the underlying distribution. Parameter $\mu\in\mathbb{R}$ stands for location.

TS distributions constitute densities of Smoothly Truncated Lévy Flights (STLF) stochastic process introduced by Koponen (1995). Boyarchenko and Levendorskii (2000) extended this initial concept, proposing Koponen–Boyarchenko–Levendorskii (KoBoL) process. Finally, Rosiński (2007) defined a general family of tempered stable Lévy processes. The class of infinitely divisible distributions that corresponds to his approach is closed under convolution, self–decomposable, has natural extension to higher dimensions, may display skewness, arbitrary gravity of tails and have all moments finite (CLT may apply).

TS distribution is somehow similar to the well established Carr–Geman–Madan–Yor (CGMY) distribution introduced in Carr et al. (2002). While both distributions are special cases of Classical Tempered Stable (CTS) distribution that represents the increments of KoBoL with single stability index α, CGMY results from asymmetric tempering of symmetric α-stable measure while TS stems from uniform tempering of arbitrary α-stable distribution. This difference translates to diverse tail behaviour.

The issue of random number generation for TS distribution has not been solved yet in a satisfactory way. Methods that are fast and easy to implement are only available for certain parameter values. Four algorithms – rejection proposed in Brix (1999), generalized Kanter method developed by Devroye (2009), Laplace transform inversion implemented in Ridout (2009) and approximate exponential rejection by Baeumer and Meerschaert (2010) – were dedicated to generating random draws from exponentially tempered stable distributions. However, the first is valid only for $\alpha < 1$, the latter requires that $\beta = 1$ while the remaining two are applicable if both conditions hold. In the general case only generic methods – shot–noise representation of Cohen and Rosiński (2007), compound Poisson approximation algorithms derived by Kawai and Masuda (2011) or rejection–squeeze technique developed in Devroye (1981) – remain viable. As it is reasonable to expect the first two approaches would be either imprecise or slow, rejection–squeeze algorithm remains the only promising option. No results for any of these methods have been reported for $\alpha \geq 1$ and $|\beta| \neq 1$.

Intuition suggests that TS distributions particularly relevant in modelling macroeconomic data display moderate departure from Gaussian and mild skewness, which translates to $\alpha \geq 1$ and $|\beta| \neq 1$. However, as numerical experiments in Palmer et al. (2008) and Kawai and Masuda (2011) consider only $\beta = 1$ (the former also assumes $\alpha < 1$), there is no literature treating this case. The aim of this text is to bridge this gap.

The problem investigated in the remaining part of this paper is formulation of random number generator valid for all admissible values of α and β. The proposed algorithm is easy to implement and much faster than the approach presented in Devroye (1981) for moderately tempered distributions. The new method relies on mixture representation of TS random variables that is parallel to decomposition property of α-stable random variates.

This paper is structured as follows. Mixture representation for tempered stable random variables along with a new randomization algorithm is introduced in section two. Section three compares its performance with two other viable procedures. The final section concludes.

2 Generating random variates with mixture representation

In the following section it is demonstrated that every TS random variable might be expressed as weighted average of two independent TS random variates with $\beta = 1$. The similar result for α-stable distributions may be found[1] in Samorodnitsky and Taqqu (2000). A random number generation algorithm is also proposed that relies on this new representation. It is valid for all parameter values and may be written in just one line of code, provided that random numbers generation for TS distribution with $\beta, \delta = 1$ is readily available. Hence it is particularly easy to implement.

Using Definition 1 it is possible to prove the following representation that may be directly employed to generate random numbers for TS distribution.

[1]See Property 1.2.13. This result is attributed to Zolotarev, but no direct reference has been traced.

Proposition 1 (Mixture representation) *Let Y^+, Y^- be independent, set $Y^\pm \sim TS_\alpha(1,1,0,\theta^\pm)$, set $v^\pm = \delta(1\pm\beta)^{1/\alpha}\,2^{-1/\alpha}$, $\theta^\pm = \theta v^\pm$, then $X = v^+Y^+ - v^-Y^- + \mu \sim TS_\alpha(\beta,\delta,\mu,\theta)$.*

Proof: It is enough to show that $\ln\Phi_X(u) = \ln\Phi_{Y^+}(v^+u) + \ln\Phi_{Y^-}(-v^-u) + i\mu u$. In case of $\alpha \neq 1$ it holds that

$$\ln\Phi_{v^\pm}(\pm v^\pm u) = -\frac{\delta^\alpha}{2\cos\frac{\pi\alpha}{2}}(1\pm\beta)[(\theta\mp iu)^\alpha - \theta^\alpha] \pm i\frac{\delta^\alpha}{2\cos\frac{\pi\alpha}{2}}\alpha(1\pm\beta)\theta^{\alpha-1}u,$$

while $\alpha = 1$ yields

$$\ln\Phi_{Y^\pm}(\pm v^\pm u) = \frac{1}{\pi}\delta[(1\pm\beta)(\theta\mp iu)\ln(\theta\mp iu) - (1\pm\beta)\theta\ln\theta] \pm i\frac{1}{\pi}\delta(1\pm\beta)(\ln\theta+1)u,$$

so in both cases $\ln\Phi_{Y^+}(v^+u) + \ln\Phi_{Y^-}(-v^-u) = \psi_X(u) - i\mu_X u$. $\square$

This result provides foundation for Algorithm 1. It may be extended to CGMY by altering $\theta^\pm$.

Although a number of different algorithms might be used to generate TS random variates $Y^\pm$ endowed with $\beta = 1$, solely the procedure proposed by Baeumer and Meerschaert (2010) will be utilized. Out of all the methods investigated in Kawai and Masuda (2011), this algorithm performed best in terms of accuracy and computation time. Assume $S_\alpha(1,1,0)$ stands for α-stable distribution with unit skewness β and scale δ and naught location μ, defined as in Samorodnitsky and Taqqu (2000). Equate c to sufficiently low percentile of this distribution. Baeumer and Meerschaert algorithm is the following.

Algorithm 0 (Baeumer & Meerschaert, 2010)
Step 0. Determine constant c.
Step 1. Generate $U \sim U(0,1)$, $V \sim S_\alpha(1,1,0)$.
Step 2. If $U \le e^{-\theta(V+c)}$, return $Y = V - \alpha\theta^{\alpha-1}/\cos\frac{\pi\alpha}{2}$ for $\alpha \neq 1$
or $Y = V + 2(\ln\theta+1)/\pi$ for $\alpha = 1$, otherwise go to Step 1.
Algorithm 0 returns pseudo–random number Y drawn from $TS_\alpha(1,1,0,\theta)$.

Note that the scope of this algorithm is limited as it is viable only if $\beta = 1$.

If $\alpha \ge 1$ constant c in Algorithm 0 depicts truncation threshold of α-stable distribution supported on the entire real line. Hence the resulting procedure is approximate. As demonstrated in Brix (1999), for $\alpha < 1$ and $c = 0$ this rejection becomes exact. Random draws from $S_\alpha(1,1,0)$ may be generated with *rdnsta* procedure[2] by McCulloch, based on Chambers et al. (1976).

The following procedure stems directly from Proposition 1.

Algorithm 1 (Mixture representation)
Step 0. Set $v^\pm = \delta(1\pm\beta)^{1/\alpha}\,2^{-1/\alpha}$, $\theta^\pm = \theta v^\pm$.
Step 1. Generate independent $Y^+ \sim TS_\alpha(1,1,0,\theta^+)$, $Y^- \sim TS_\alpha(1,1,0,\theta^-)$.
Step 2. Return $X = v^+Y^+ - v^-Y^- + \mu$.
Algorithm 1 returns pseudo–random number X obtained for $TS_\alpha(\beta,\delta,\mu,\theta)$.

Unlike most available methods, Algorithm 1 is viable for all parameter values, including $\alpha \ge 1$ and $|\beta| \neq 1$. Note that its output is endowed with arbitrary values of both β and δ.

[2] Available at: http://www.econ.ohio-state.edu/jhm/programs/RNDSSTA.

3 Comparison with other algorithms

This section contains the description of two additional randomization algorithms valid for TS distribution with $\alpha \geq 1$ and $|\beta| \neq 1$. Algorithm 2 is a benchmark where pseudo–random draws are generated by an inverse of piecewise linear cdf approximation obtained via Fast Fourier Transform (FFT). Algorithm 3 relies on rejection-squeeze technique proposed in Devroye (1981). The results obtained for the algorithm proposed in previous section are compared with the outcomes of these two procedures.

Perhaps the easiest way to obtain alternative (benchmark) random draws from TS distribution is to invert approximated cdf. Let $F(x)$ be the cdf of $TS_\alpha(\beta, \delta, 0, \theta)$ with essential support $[a, b]$ and pdf $f(x)$. Assume N is large integer, for $k = 0, \ldots, N-1$ denote $x_k = a + hk$ with $h = (b-a)/N$. The numerical inversion of linearly approximated cdf may be implemented as follows.

Algorithm 2 (Cdf inversion)
Step 0. Evaluate $F(x_0), \ldots, F(x_{N-1})$.
Step 1. Generate $U \sim U(0,1)$. Find n such that $F(x_n) \leq U < F(x_{n+1})$.
Step 2. Return $X = x_n + h(U - F(x_n))(F(x_{n+1}) - F(x_n))^{-1} + \mu$.
Algorithm 2 returns pseudo–random number X obtained for $TS_\alpha(\beta, \delta, \mu, \theta)$.

In Algorithm 2 cdf is approximated by a piecewise linear function.

To implement Algorithm 2 pointwise values of $F(x_k)$ need to be first evaluated. If pdf proxies are initially obtained by FFT, the sought quantities may be found from $F(x_{k+1}) = F(x_k) + hf(x_k)$ under boundary condition $F(x_0) = 0$. In Appendix B the modified version of FFT algorithm from Mittnik et al. (1999) is presented that may be conveniently utilized to obtain the values of $f(x_k)$. This version evaluates pdf over asymmetric interval $[a, b]$ while the original algorithm requires $a = -b$.

The final algorithm considered relies on the rejection–squeeze technique. Given

$$d_1 = \frac{1}{2\pi} \int_{\mathbb{R}} |\Phi_X(u)|\, du, \;\; d_2 = \frac{1}{2\pi} \int_{\mathbb{R}} |\Phi_X^{(2)}(u)|\, du,$$

the result formulated in Devroye (1981) states that pdf $f(x)$ of $TS_\alpha(\beta, \delta, 0, \theta)$ fulfils

$$\forall\, x \in \mathbb{R} : f(x) \leq \min\{d_1, d_2/x^2\}. \tag{2}$$

This inequality was originally utilized to derive the following rejection–squeeze algorithm.

Algorithm 3 (Devroye, 1981)
Step 0. Evaluate d_1 and d_2.
Step 1. Generate independent $U \sim U(0,1)$, $V, W \sim U(-1,1)$.
Set $Y = \sqrt{d_2/d_1} \cdot V/W$. If $|V| < |W|$, then go to Step 3.
Step 2. If $U < f(Y)Y^2/d_2$, then return $X = Y + \mu$. Otherwise, go to Step 1.
Step 3. If $U < f(Y)/d_1$, then return $X = Y + \mu$. Otherwise, go to Step 1.
Algorithm 3 returns pseudo–random number X drawn from $TS_\alpha(\beta, \delta, \mu, \theta)$.

The expected number of times Step 1 is executed to generate one random number is $4\sqrt{d_1 d_2}$.

In order to run Algorithm 3 some preliminary work is required. First of all, the formula for second order derivative of $\Phi_X(u)$ has to be found. Define $C_{\alpha,\delta} = \alpha\delta^\alpha(\cos\frac{\pi\alpha}{2})^{-1}/2$, then for $\alpha \neq 1$

this derivative is

$$\Phi_X^{(2)}(u) = -C_{\alpha,\delta}(C_{\alpha,\delta}[(1+\beta)(\theta - iu)^{\alpha-1} - (1-\beta)(\theta + iu)^{\alpha-1} - 2\beta\theta^{\alpha-1}]^2 +$$
$$+(1-\alpha)[(1+\beta)(\theta - iu)^{\alpha-2} + (1-\beta)(\theta + iu)^{\alpha-2}]) \cdot \Phi_X(u),$$

while for $\alpha = 1$ it holds that

$$\Phi_X^{(2)}(u) = -\frac{\delta}{\pi}(\frac{\delta}{\pi}[(1+\beta)\ln(\theta - iu) - (1-\beta)\ln(\theta + iu) - 2\beta\ln\theta]^2 + 2\frac{\theta + i\beta u}{\theta^2 + u^2}) \cdot \Phi_X(u).$$

Secondly, the integrals have to be determined. As analytic results for d_1 and d_2 seem difficult to compute, it is probably necessary to approximate both quantities numerically. Finally, the pdf of TS distribution needs to be evaluated in arbitrary points of its domain. If the pdf is first approximated in the points equally spaced over its essential support (as in Algorithm 2), this last step may be done via cubic spline interpolation performed on each subinterval.

Note that in case of $\alpha \geq 1$ there are no known formulas for pdf of TS distribution. Furthermore, it is no longer possible to follow the route taken in Kawai and Masuda (2011) and elicit the pdf from the relation, binding densities of TS distribution and α-stable distribution being tempered. The reason is that for $|\beta| \neq 1$ this identity does not hold. Therefore in Algorithm 2 and 3 Fourier inversion of characteristic functions is utilized. As Devroye procedure is exact and the underlying pdf approximation involves (in addition) cubic splines, it is reasonable to conjecture that Algorithm 3 would be more precise than Algorithm 2.

In order to compare the quality of random numbers generated with Algorithm 1 and by the remaining two approaches the following exercise was undertaken. Each procedure was run with eight different sets of parameters to obtain 100 samples of 10^6 pseudo–random numbers. All distributions were standardised (with $\mu = 0$ and δ implying unit variance) and thus parametrized with just α, β and θ. Both mean computation time and first five mean sample moments about the origin were recorded in each case. The results are depicted in Tables 1–2 and might be readily compared with theoretical values, obtained with the formulas from Appendix A. Figures in brackets denote unbiased estimates of standard deviation. As computation time does not vary much across replications, the corresponding estimates of standard deviation were not reported. Emphasized numbers indicate either the smallest mean execution time (in seconds), or the mean sample moment most similar to the relevant theoretic value.

In the numerical experiment conducted above the main criterion used to evaluate competing procedures was precision of mean sample moments. There are two reasons for it. First of all, in order to compute minimum distance measure, such as Kolmogorov–Smirnov metric[3], theoretical results for cdf are required. As formulas for cdf of TS distribution are not known, it would need to be approximated numerically. This approach favours randomization methods that rely on the same approximation. Second of all, sample moments are easier to interpret and allow for more intuitive assessment of the results thus obtained.

Implementation details were as follows. In all FFT procedures $N = 2^{13}$ was utilized as powers of 2 are computationally most efficient. In Algorithm 2 constant c was set to the bottom 0.1 percentile of the sample of 10^6 random draws coming from the relevant α-stable distribution. Results of all the calculations presented in this work were performed in MATLAB©8.0.0.783 (R2012b) on a PC with Inter®Core™i5-3550S CPU (3.00 GHz, 4.0 GB RAM) under 64-bit Windows®7 Enterprise operating system. The code is available upon request.

[3]Extensive review of possible approaches may be found in Basu et al. (2011).

Table 1: Theoretic vs. mean sample moments about the origin for standardised TS distribution. Emphasized numbers indicate either the smallest (mean) execution time, or the (mean) sample moment most similar to corresponding theoretic value.

Parameters	Method	Mean sample moments (1)	(2)	(3)	(4)	(5)	Mean time (sec)
$\alpha = 1.80$, $\beta = 0.50$, $\theta = 1.00$	Theoretic	0.00000	1.00000	0.10000	3.24000	1.26400	NA
	Mixture	0.00080 (0.00100)	0.99738 (0.00154)	0.10690 (0.00431)	3.21406 (0.01465)	1.33312 (0.07686)	5.261
	Inversion	0.00178 (0.00095)	**0.99997** (0.00147)	0.10486 (0.00404)	**3.24023** (0.01411)	1.28262 (0.07089)	**1.393**
	Devroye	**-0.00014** (0.00097)	0.99987 (0.00146)	**0.09954** (0.00502)	3.23877 (0.01525)	**1.26140** (0.08893)	6.539
$\alpha = 1.80$, $\beta = 0.50$, $\theta = 0.30$	Theoretic	0.00000	1.00000	0.33333	5.66666	13.11111	NA
	Mixture	**-0.00005** (0.00099)	0.99977 (0.00151)	**0.33287** (0.01359)	**5.67531** (0.23317)	**13.28997** (4.12589)	**1.348**
	Inversion	0.00422 (0.00103)	1.00020 (0.00156)	0.34674 (0.01758)	5.67578 (0.22502)	13.43504 (4.19854)	1.498
	Devroye	-0.00009 (0.0.00105)	**0.99980** (0.00147)	0.33408 (0.01489)	5.68166 (0.23422)	13.59037 (4.65564)	6.679
$\alpha = 1.80$, $\beta = 0.25$, $\theta = 1.00$	Theoretic	0.00000	1.00000	0.05000	3.24000	0.63200	NA
	Mixture	0.00051 (0.00099)	0.99825 (0.00151)	0.05384 (0.00415)	3.22427 (0.01683)	0.66312 (0.07109)	4.962
	Inversion	0.00178 (0.00103)	1.00020 (0.00156)	0.05546 (0.00399)	3.24285 (0.01637)	0.65987 (0.06877)	**1.379**
	Devroye	**-0.00009** (0.00105)	**1.00013** (0.00147)	**0.05073** (0.00500)	**3.240565** (0.01378)	**0.64543** (0.08087)	6.527
$\alpha = 1.80$, $\beta = 0.25$, $\theta = 0.30$	Theoretic	0.00000	1.00000	0.16666	5.66666	6.55555	NA
	Mixture	-0.00006 (0.00097)	**1.00006** (0.00228)	0.16556 (0.01603)	5.63382 (0.23989)	**6.41802** (3.68532)	**1.35008**
	Inversion	0.00434 (0.00091)	1.00038 (0.00209)	0.17761 (0.01661)	**5.67187** (0.22877)	6.16371 (3.87348)	1.4952
	Devroye	**-0.00001** (0.00095)	1.00015 (0.00221)	**0.16697** (0.01598)	5.67940 (0.23182)	6.76973 (5.66080)	6.63823

Table 2: Theoretic vs. mean sample moments about the origin for standardised TS distribution. Emphasized numbers indicate either the smallest (mean) execution time, or the (mean) sample moment most similar to corresponding theoretic value.

Parameters	Method	Mean sample moments (1)	(2)	(3)	(4)	(5)	Mean time (sec)
$\alpha = 1.30$, $\beta = 0.50$, $\theta = 1.00$	Theoretic	0.00000	1.00000	0.35000	4.19000	5.10650	NA
	Mixture	0.00033	0.99941	0.35195	4.18689	5.12810	4.030
		(0.00114)	(0.00171)	(0.00712)	(0.03108)	(0.20955)	
	Inversion	0.00209	**0.99997**	0.35597	4.19493	5.15209	**1.315**
		(0.00104)	(0.00150)	(0.00665)	(0.03169)	(0.25313)	
	Devroye	**-0.00006**	0.99993	**0.34955**	**4.18741**	**5.10322**	6.526
		(0.00110)	(0.00181)	(0.00692)	(0.03058)	(0.21146)	
$\alpha = 1.30$, $\beta = 0.50$, $\theta = 0.30$	Theoretic	0.00000	1.00000	1.16666	16.22222	71.16666	NA
	Mixture	**0.00003**	0.99983	1.16422	16.13521	70.29359	**1.122**
		(0.00111)	(0.00384)	(0.04026)	(0.64611)	(14.97359)	
	Inversion	0.00525	**0.99990**	1.18133	**16.23628**	70.57834	1.264
		(0.00093)	(0.004073)	(0.03903)	(0.70732)	(14.28063)	
	Devroye	-0.00011	0.99980	**1.16696**	16.15821	**70.73032**	6.622
		(0.00102)	(0.00378)	(0.04682)	(0.61055)	(14.97679)	
$\alpha = 1.30$, $\beta = 0.25$, $\theta = 1.00$	Theoretic	0.00000	1.00000	0.17500	4.19000	2.55325	NA
	Mixture	0.00028	0.99940	0.17587	4.18425	**2.55211**	3.764
		(0.00115)	(0.00178)	(0.00689)	(0.03140)	(0.20573)	
	Inversion	0.00209	1.00026	0.18108	**4.19234**	2.58993	**1.318**
		(0.00091)	(0.00171)	(0.00620)	(0.02998)	(0.20864)	
	Devroye	**0.00003**	**1.00025**	**0.17531**	4.19667	2.55781	6.471
		(0.00095)	(0.00199)	(0.00676)	(0.03453)	(0.25243)	
$\alpha = 1.30$, $\beta = 0.25$, $\theta = 0.30$	Theoretic	0.00000	1.00000	0.58333	16.22222	35.58333	NA
	Mixture	-0.00012	0.99986	0.57840	**16.19642**	**35.63813**	**1.12853**
		(0.00096)	(0.00333)	(0.03923)	(0.77978)	(21.42623)	
	Inversion	0.00541	0.99947	0.59935	16.12214	34.88052	1.26118
		(0.00097)	(0.00404)	(0.04024)	(0.67324)	(15.59258)	
	Devroye	**0.00000**	**0.99991**	**0.58520**	16.287548	36.70306	6.42880
		(0.00102)	(0.00378)	(0.04286)	(16.85838)	(14.99957)	

In the undertaken exercise rejection–squeeze method produced most precise sample moments 23 out of 40 times. Mixture representation delivered most accurate sample moments in 10 cases while cdf inversion ranked first for 7 different sets of parameters. Therefore for this instance Algorithm 3 is clearly best in terms of quality of pseudo–random numbers generated. Algorithms 1 and 2 seem to be comparable, although the previous is slightly more precise. Devroye procedure always requires most computation time. While for $\theta = 1$ cdf inversion is the fastest out of three algorithms, in case of $\theta = 0.3$ it performs marginally worse than the mixture representation in terms of execution speed. This difference would become more evident for smaller θ, when Baeumer and Meerschaert algorithm accepts candidate draws more often. Note that if $X \sim TS_\alpha(\beta, \delta, \mu, \theta)$, $a, b \in \mathbb{R}$ with $a \neq 0$ and $Y = aX + b$ then $Y \sim TS_\alpha((\operatorname{sgn} a)\beta, |a|\delta, a\mu + b, \theta/|a|)$. Hence rescaling the data generated from TS distribution by a sufficiently large factor guarantees that the resulting parameter θ is small. All procedures are less reliable in capturing higher order moments. In case of Algorithms 2 and 3 the reason is that in order to perform numerical approximation of pdf its support needs to be constrained. Hence pseudo–random numbers above (or below) certain values will not be generated. In the case of mixture representation the culprit is auxiliary Algorithm 0 where the left tail of the distribution is trimmed. Therefore extreme values are returned less often.

Out of the procedures whose precision was investigated in this section, Algorithm 2 has not been used yet to generate random numbers from TS distribution. Results for random number generation from Algorithm 3 in case of TS density with $\alpha \geq 1$ and $|\beta| \neq 1$ have not been reported in the literature.

4 Conclusions

The proposed algorithm is valid for $\alpha \in (0, 2)$ and $\beta \in [-1, 1]$. It may be written in just one line of code given that Baeumer and Meerschaert (2010) procedure is already implemented. It is more accurate than the benchmark and much faster than the method proposed in Devroye (1981) for moderately tempered distributions (sufficiently small θ). Although it is less precise than the procedure based on rejection–squeeze technique, the quality of random numbers thus obtained should be fully sufficient for most practical applications. It is also worth noting that the performance of the approximate cdf inversion is surprisingly good.

A Cumulants and moments

After Stuart and Ord (1994) define cumulants of integer order p as

$$\kappa_p = \frac{1}{i^p}\Big(\frac{d^p}{du^p}\ln \Phi_X(u)\Big)\Big|_{u=0}.$$

The moments of tempered α-stable random variates are all finite, which implies existence of all the cumulants, but do not take any convenient form. These cumulants are highly tractable.

When $\alpha \neq 1$ the cumulants may be elicited from Terdik and Woyczyński (2006). For $\alpha = 1$ these formulas are no longer valid and the cumulants need to be found directly. Combining both sets of results yields

Corollary 1 (Cumulants) *If random variable $X \sim TS_\alpha(\beta, \delta, \mu, \theta)$ then its cumulants fulfil*

$$\kappa_p = \begin{cases} \mu & \text{if } p = 1, \\ \frac{2\delta}{\pi}\theta^{1-p}(p-2)!(I_p + \beta I_{p+1}) & \text{if } p \neq 1,\ \alpha = 1, \\ \alpha\left[\prod_{j=1}^{p-1}(j-\alpha)\right](\cos\frac{\pi\alpha}{2})^{-1}\delta^\alpha\theta^{\alpha-p}(I_p + \beta I_{p+1}) & \text{if } p \neq 1,\ \alpha \neq 1, \end{cases} \tag{3}$$

where $I_p = 2^{-1}(1+(-1)^p)$.

Proof: To obtain cumulants when $\alpha \neq 1$ integrate eq. (14) in Terdik and Woyczyński (2006) with respect to Rosiński measure

$$R(dx) = C_\alpha(\theta\delta)^\alpha[(1+\beta)\delta(x-1/\theta)+(1-\beta)\delta(x+1/\theta)]\,dx,$$

where $\delta(x \pm 1/\theta)$ stands for Dirac's delta and constant C_α is equal to

$$C_\alpha = \begin{cases} \alpha(1-\alpha)[2\cos(\frac{\pi\alpha}{2})\Gamma(2-\alpha)]^{-1} & \alpha \neq 1 \\ \frac{1}{\pi} & \alpha = 1. \end{cases}$$

Definition 1 relies on different parametrization than the one used in Rosiński (2007), it implies that $\kappa_1 = \mu$ while all higher order cumulants remain intact. In case of $\alpha = 1$ it holds that

$$\frac{d}{du}\psi_X(u) = i\frac{2\delta}{\pi}\Big[\sum_{j=1}^{+\infty}(I_{j+1}+\beta I_j)\theta^{-j}(j-1)!\frac{(iu)^j}{j!} - \beta(1+\ln\theta)\Big],$$

so $\kappa_1 = (\mu-\mu_X) - i\psi_X^{'}(0) = \mu$ and $\kappa_p = (-i)^p\psi_X^{(p)}(0) = \frac{2\delta}{\pi}(I_p+\beta I_{p+1})\theta^{1-p}(p-2)!$ for $p \geq 2$. $\square$

Let $\mathbb{S}kwX$ stand for skewness, $\mathbb{K}urX$ denote excess kurtosis of random variable X. For $\alpha \neq 1$ formulas (3.89-3.90) from Stuart and Ord (1994) combined with Corollary 1 imply

$$\mathbb{E}X = \mu,\ \mathbb{V}arX = \alpha(1-\alpha)\Big(\cos\frac{\pi\alpha}{2}\Big)^{-1}\delta^\alpha\theta^{\alpha-2},$$

$$\mathbb{S}kwX = (2-\alpha)\beta\sqrt{\frac{\cos\frac{\pi\alpha}{2}}{\alpha(1-\alpha)(\delta\theta)^\alpha}},\ \mathbb{K}urX = \frac{(2-\alpha)(3-\alpha)\cos\frac{\pi\alpha}{2}}{\alpha(1-\alpha)(\delta\theta)^\alpha}.$$

If $\alpha = 1$ it holds that

$$\mathbb{E}X = \mu,\ \mathbb{V}arX = \frac{2\delta}{\pi\theta},\ \mathbb{S}kwX = \frac{\beta}{\sqrt{\delta\theta}}\sqrt{\frac{\pi}{2}},\ \mathbb{K}urX = \frac{\pi}{\delta\theta}.$$

It is possible to guarantee that the resulting distribution has unit variance by setting appropriate $\delta > 0$. If the density is standardised, skewness and excess kurtosis are its third and fourth cumulant.

By $\mu_p^{'}$ and μ_p denote, respectively, moments about the origin and about the mean. Given the cumulants, recursion formula from pp. 88–91 in Stuart and Ord (1994) yields moments

$$\mu_p^{'} = \kappa_p + \sum_{j=1}^{p-1}\binom{p-1}{j-1}\kappa_j\mu_{p-j}^{'}.$$

Moments about the origin of order p are polynomials of order p of the first p cumulants

$$\mu_1^{'} = \kappa_1,\ \mu_2^{'} = \kappa_2+\kappa_1^2,\ \mu_3^{'} = \kappa_3+3\kappa_2\kappa_1+\kappa_1^3,$$

$$\mu_4^{'} = \kappa_4+4\kappa_3\kappa_1+3\kappa_2^2+6\kappa_2\kappa_1^2+\kappa_1^4,$$

$$\mu_5^{'} = \kappa_5+5\kappa_4\kappa_1+10\kappa_3\kappa_2+10\kappa_3\kappa_1^2+15\kappa_2^2\kappa_1+10\kappa_2\kappa_1^3+\kappa_1^5,\ \dots\ .$$

The central moments fulfil the similar set of equations with $\kappa_1 = 0$.

B Inverse FT on asymmetric domain

Assume set $[a, b]$ is divided it into N disjoint sections of equal length. The aim of the algorithm presented in this appendix is to evaluate pdf of random variable with known characteristic function $\Phi_X(u)$ in the lower bounds of these sections.

For $k = 0, \dots, N-1$ set $x_k = a + hk$ with $h = (b-a)N^{-1}$. For N sufficiently large (h sufficiently small) constant $c = \pi/h$ is also large and

$$f(x_k) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-iux_k} \cdot \Phi_X(u)\, du \approx \frac{1}{2\pi}\int_{-c}^{c} e^{-iux_k} \cdot \Phi_X(u)\, du =$$

$$= \int_{-N/2(b-a)}^{N/2(b-a)} e^{-2\pi i \cdot \omega x_k} \cdot \Phi_X(2\pi\omega)\, d\omega.$$

Set $\omega_n = (n - N/2)s$ for $n = 0, \dots, N-1$ with $s = (hN)^{-1} = (b-a)^{-1}$ to obtain

$$\int_{-N/2(b-a)}^{N/2(b-a)} e^{-2\pi i \cdot \omega x_k} \cdot \Phi_X(2\pi\omega)\, d\omega = \int_{-Ns/2}^{Ns/2} e^{-2\pi i \cdot \omega x_k} \cdot \Phi_X(2\pi\omega)\, d\omega \approx$$

$$\approx s\sum_{n=0}^{N-1} \Phi_X(2\pi\omega_n) \cdot e^{-2\pi i \cdot \omega_n x_k} = s\sum_{n=0}^{N-1} \Phi_X\Big(2\pi s(n - \frac{N}{2})\Big) \cdot e^{-2\pi i \cdot (\frac{a}{h}+k)(n-\frac{N}{2})hs}.$$

As $e^{\pi i} = -1$, it follows that

$$e^{-2\pi i \cdot (\frac{a}{h}+k)(n-\frac{N}{2})hs} = (-1)^{\frac{a}{b-a}N+k} \cdot (-1)^{-\frac{2a}{b-a}n} \cdot e^{-2\pi i \cdot k\frac{n}{N}}.$$

Finally

$$f(x_k) \approx \frac{1}{b-a}(-1)^{\frac{a}{b-a}N+k} \sum_{n=0}^{N-1} (-1)^{-\frac{2a}{b-a}n} \cdot \Phi_X\Big(\frac{2\pi}{b-a}(n - \frac{N}{2})\Big) \cdot e^{-2\pi i \cdot k\frac{n}{N}}.$$

The sought result may be computed by evaluation of Inverse Fourier Transformation

$$\sum_{n=0}^{N-1} y_n \cdot e^{-2\pi i \cdot k\frac{n}{N}},\ \ k = 0, 1, \dots, N-1$$

via Fast Fourier Transform (FFT) algorithm applied to the sequence

$$y_n = (-1)^{-\frac{2a}{b-a}n} \cdot \Phi_X\Big(\frac{2\pi}{b-a}(n - \frac{N}{2})\Big),\ \ n = 0, 1, \dots, N-1.$$

An output of FFT procedure is a vector. In order to obtain it in MATLAB run *fft* procedure on $(y_0, \dots, y_{N-1})$. To get valid pdf values multiply the entries thus obtained by $\frac{1}{b-a}(-1)^{\frac{a}{b-a}N+k}$.

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