Position of satellite in time

Modelling and Data Science

Lodz University of Technology

Piotr Imiński 247088

Jan Gorzela 247086

1. Division of work

Piotr Imiński:

- Derivation of equations and checking their correctness
- Writing the program and comments
- Help with the report
- Creating plots

Jan Gorzela:

- Derivation of equations and checking their correctness
- Helping with writing the program and comments
- Rewriting the equations to word
- Writing the report and drawing picture

2. Statement of the problem

An artificial satellite (of Earth) consists of two different point masses (m_1 and m_2) joined together by a massless and perfectly rigid rod of length l (i.e. it has some moment of inertia, but cannot store energy in the form of oscillations). The center of mass of this system is placed on some elliptic orbit and given linear velocity appropriate for this orbit. There is no initial angular velocity. The task consists in description of the satellite position in time — not only the position of the mass center, but also its orientation (i.e. the angle of the line l)

3. Physics behind the problem

The situation which we want to consider is shown on the Figure 1:

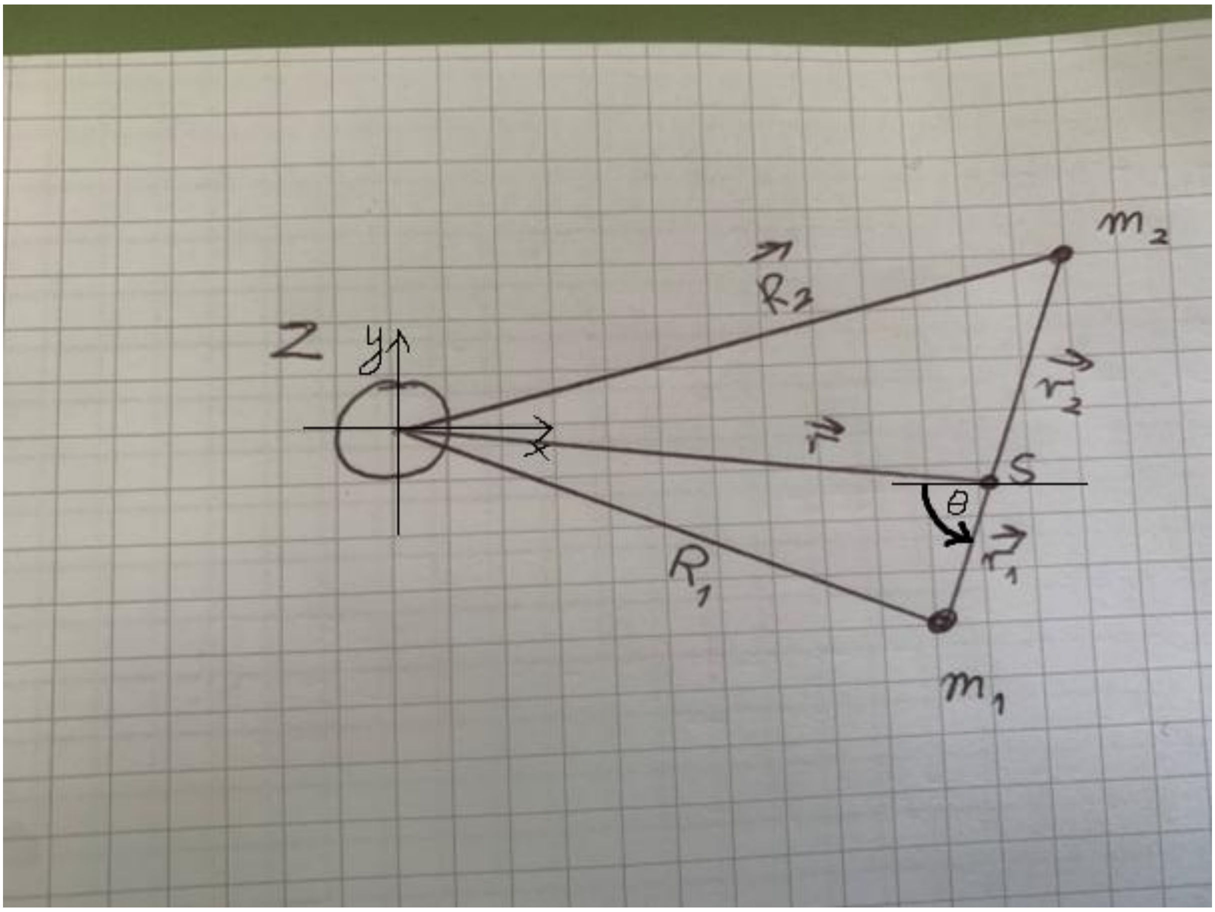

Figure 1. Visualisation of the problem.

To describe the position of the centre of mass in time and its orientation the linear acceleration must be determined as well as the angular one.

1. Centre of the mass and calculation of r_1 and r_2 which are distances between edges and the centre of the mass.

The centre of the mass is given by:

$$x_s(m_1 + m_2) = x_1 m_1 + x_2 m_2 \tag{1}$$

Where x_s, x_1, x_2 are coordinates of centre of the mass and two masses at the ends of the rigid. We have:

$$x_s = 0 \tag{2}$$

$$x_2 = l + x_1 \tag{3}$$

Where l is the length of the rod.

So, the distances r_1 and r_2 between the end of the rigid and the centre of the mass are:

$$0 = x_1 m_1 + l m_2 + x_1 m_2 \tag{4}$$

So, we have:

$$x_1 = \frac{-l m_2}{m_1 + m_2} \tag{5}$$

$$r_1 = \frac{l m_2}{m_1 + m_2} \tag{6}$$

$$r_2 = l - r_1 \tag{7}$$

2. Determination of linear acceleration

Let's introduce the 2D cartesian coordinate system such that its centre is in the Earth, and it is located in the plane of the orbit.

We are dealing with gravitational force so the formula for linear acceleration of centre of the mass is given by:

$$\vec{a} = -G\frac{M\vec{r}}{r^3} \tag{8}$$

Where M is the mass of Earth and $\vec{r}$ the radius between Earth and the centre of the mass. One can integrate this equation to determine velocity and position in time of the mass.

3. **Determination of angular acceleration**

Because we are dealing also with rotational movement, we have to consider the moment of inertia of the system:

$$I = m_1 r_1^2 + m_2 r_2^2 \tag{9}$$

Where I is the moment of inertia of the system.

To calculate angular acceleration, one must determine the moments of the forces (caused by gravitational force acting on masses on the edges of the rod). The formulas for the moments of the forces are:

$$\Gamma_1 = \overrightarrow{r_1} \times \overrightarrow{F_1} \tag{10}$$

$$\Gamma_2 = \overrightarrow{r_2} \times \overrightarrow{F_2} \tag{11}$$

Where:

$$\overrightarrow{r_1} = [-r_1 \cos\theta\,, -r_1 \sin\theta] \tag{12}$$

$$\overrightarrow{r_2} = [r_2 \cos\theta\,, r_1 \sin\theta] \tag{13}$$

$$\overrightarrow{F_1} = -G\frac{Mm_1}{R_1^3}\overrightarrow{R_1} \tag{14}$$

$$\overrightarrow{F_2} = -G\frac{Mm_2}{R_2^3}\overrightarrow{R_2} \tag{15}$$

θ is the angle of the line *l*. One can describe R_1 and R_2 as:

$$\overrightarrow{R_1} = \vec{r} + \overrightarrow{r_1} \tag{16}$$

$$\overrightarrow{R_2} = \vec{r} + \overrightarrow{r_2} \tag{17}$$

The formula connecting torque with angular acceleration and moment of inertia:

$$\Gamma = I\varepsilon \tag{18}$$

So finally, we have:

$$\varepsilon = \frac{\Gamma_1 - \Gamma_2}{I} \tag{19}$$

Substituting formulas for torques from (10) and (11) to (19) we get final formula for angular acceleration. To obtain angular velocity and position one can integrate this expression over time.

Summing up, we obtained three differential equations (two from linear acceleration – motion in 2D plane, because the force is central - and one from angular acceleration). Solving them up allows to obtain values of velocities and positions in time.

4. **Numerical approach to solve the problem**

To compute the solution of system of equations we used numerical approach. In order to solve ordinary differential equations we used solve_ivp from scipy.integrate. As input we provide the function which returns velocities and accelerations in the form of array. We also provide time span, initial state which is an array of initial coordinates and velocities, and the time vector. After running solver returns the coordinates and velocities in the corresponding time moments.

5. Outcomes and conclusions

In order to validate the correctness of the outcomes we calculated the kinetic and potential energy of mass centre as well as the sum of those two. After plotting them we are able to say that the sum of kinetic and potential energy is constant in time so the outcome can be regarded as correct (Figure 2).

To visualize the satellite motion we also plot the angle of the rod in time (Figure 3) as well as the orbit of the satellite (Figure 4).

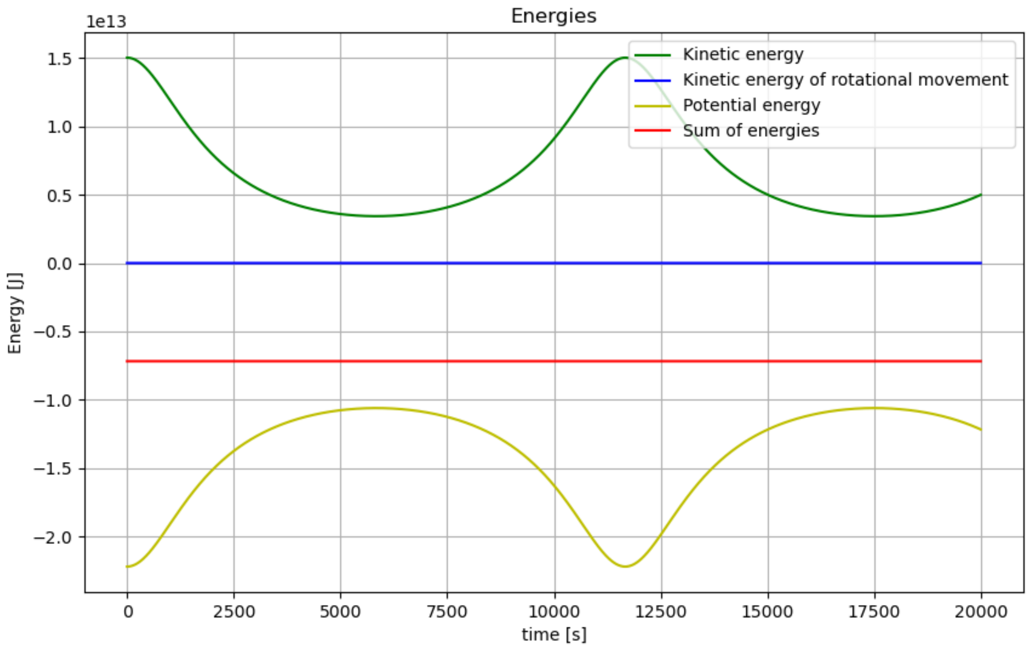

Figure 2. Visualisation of energies of centre of mass of satellite.

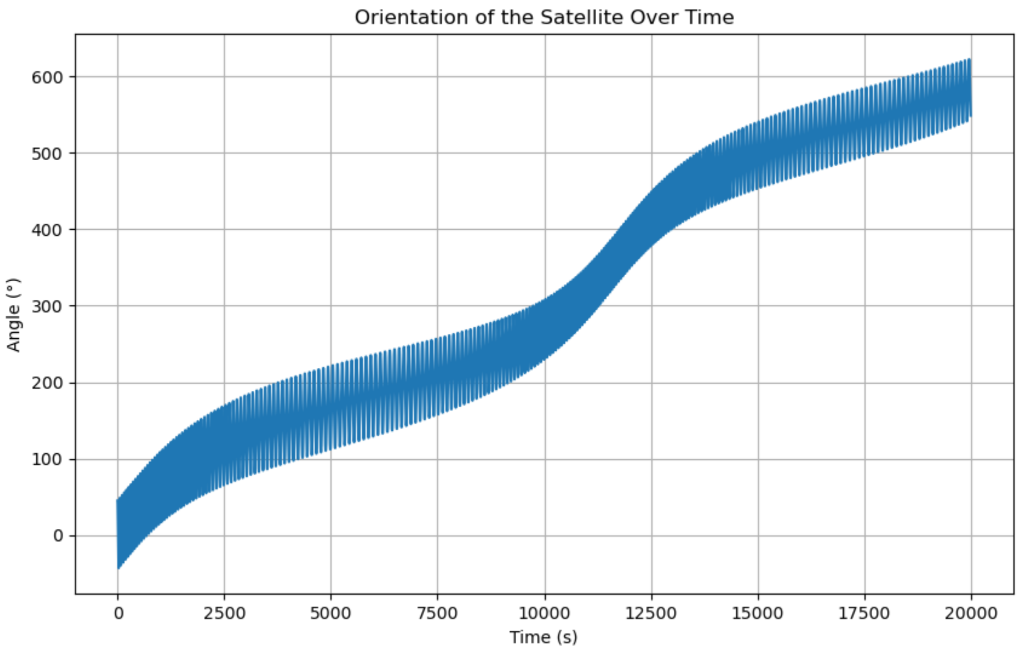

Figure 3. Angle θ over time.

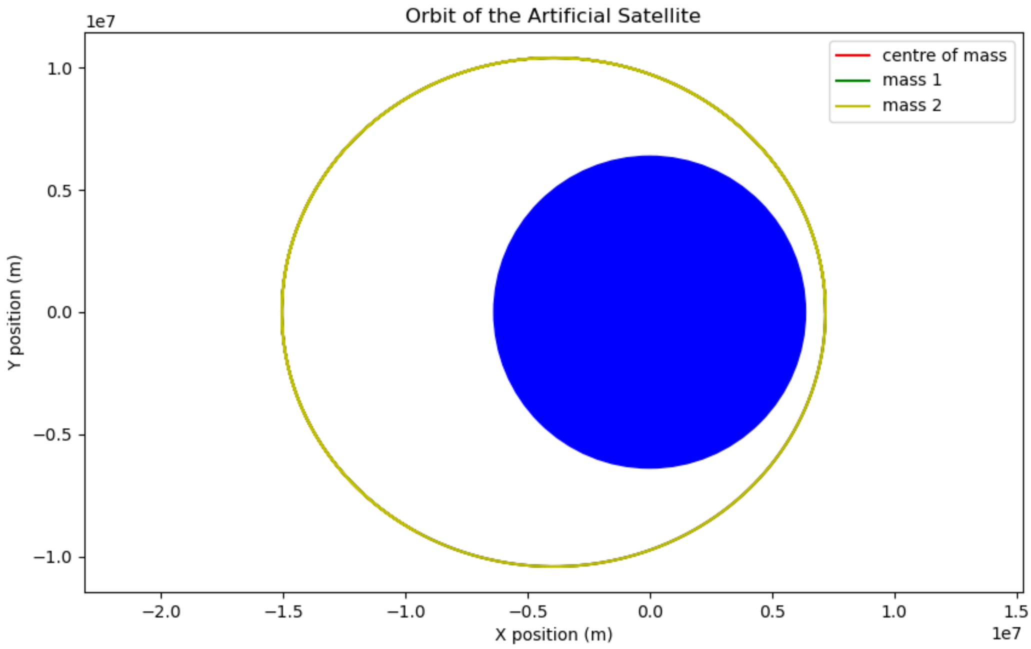

Figure 4. Visualisation of the orbit of the satellite.