Equilibrium position of the mass

Modelling and Data Science

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1. Division of work

Piotr Imiński:

- Derivation of equations and checking their correctness
- Writing the program and comments, adding tests
- Help with the report
- Creating plots

Jan Gorzela:

- Derivation of equations and checking their correctness
- Helping with writing the program and comments
- Rewriting the equations to word
- Writing the report and drawing picture

2. Statement of the problem

Consider a system consisting of two massless springs (of different free lengths L1 , and L2 , and of different spring coefficients k1 and k2) and a rectangle of sides a and b and mass m (uniformly distributed within the rectangle). The springs are fastened to points (0,0), and (D,0), and to opposite corners of the top side of the mass (let's assume it's the longer side). The objective is to find the equilibrium position of the mass (i.e. coordinates of the mass centre and the angle of inclination of the top side).

3. Physics behind the problem

The situation which we want to consider is shown on the Figure 1:

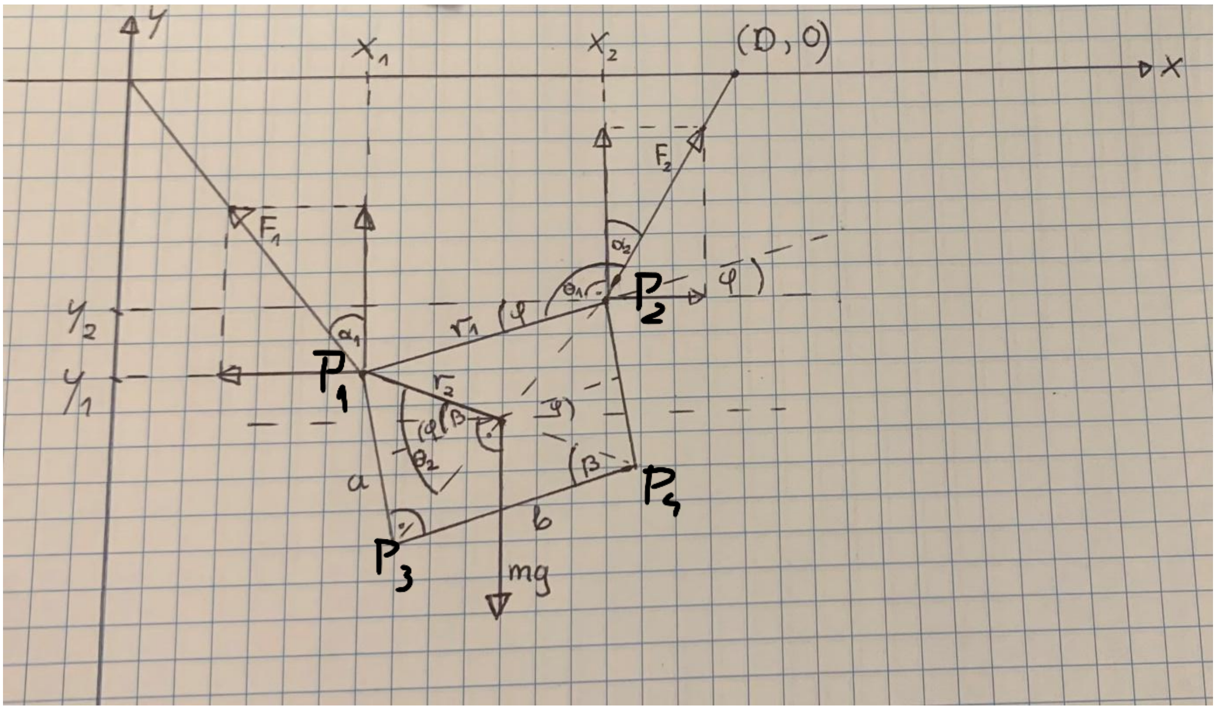

Figure 1. The system of rectangular mass and two springs.

The rectangular mass on the picture will be in equilibrium if and only if the forces acting on it will be equal in every direction and the moments of forces will equilibrate. Only the symbols not shown on the picture will be described.

The first two equations we get from the condition that the forces must be equal.

One can use the Cartesian coordinates system. From x-axis direction we get the equation:

$$0 = -F_1 \sin \alpha_1 + F_2 \sin \alpha_2 \quad (1)$$

And similarly, from y-axis direction:

$$0 = F_1 \cos \alpha_1 + F_2 \cos \alpha_2 - mg \quad (2)$$

To make the equations dependent on the coordinates we must use a few additional equations.

From trigonometry we have following properties:

$$\cos\alpha_1 = \frac{y_1}{l_1 + d_1} \tag{3}$$

$$\sin\alpha_1 = \frac{x_1}{l_1 + d_1} \tag{4}$$

$$\cos\alpha_2 = \frac{y_2}{l_2 + d_2} \tag{5}$$

$$\sin\alpha_2 = \frac{D - x_2}{l_2 + d_2} \tag{6}$$

Where l_1, l_2 are the free lengths of the springs and d_1, d_2 are their elongations.

We can make coordinates of one upper vertex depending on the other and the angle of inclination because we know the length of the rectangle *b*.

$$x_2 = x_1 + b\cos\varphi \tag{7}$$

$$y_2 = y_1 + b\sin\varphi \tag{8}$$

Where x_1, y_1 are the coordinates of the upper left vertex and x_2, y_2 coordinates of the upper right vertex.

To describe elastic forces, we must write:

$$F_1 = k_1 d_1 \tag{9}$$

$$F_2 = k_2 d_2 \tag{10}$$

Where k_1 is the spring coefficient of the left spring and k_2 of the right one.

One can also write the following relationship between coordinates, elongation, and free length of each spring.

$$d_1 = \sqrt{x_1^2 + y_1^2} - l_1 \tag{11}$$

$$d_2 = \sqrt{(D - x_2)^2 + y_2^2} - l_2 \tag{12}$$

The third equation one can obtain from equilibrium of moments of forces.

The axis of rotation can be chosen at every point because the aim is to find equilibrium point but to make one moment of force equal zero one can choose the upper left vertex and write:

$$0 = -r_1 F_2 \sin\theta_1 + r_2 mg \sin\theta_2 \tag{13}$$

Where r_1, r_2 are the arms of forces and the θ_1, θ_2 angles between the arm and the force.

To describe the arms of forces by known variables one can write:

$$r_1 = b \tag{14}$$

$$r_2 = \frac{1}{2}\sqrt{a^2 + b^2} \tag{15}$$

To describe the angles θ_1, θ_2 one can write:

$$\theta_1 = \varphi + \frac{\pi}{2} + \alpha_2 \tag{16}$$

$$\theta_2 = \frac{\pi}{2} + \beta - \varphi \tag{17}$$

Where β and α_2 are equal to:

$$\beta = \operatorname{arctg}\frac{a}{b} \tag{18}$$

$$\alpha_2 = \arccos\frac{y_2}{l_2 + d_2} \tag{19}$$

Substituting everything we get three main equations with three variables : x_1 , y_1, φ.

$$0 = k_1 d_1 \frac{y_1}{\sqrt{x_1^2 + y_1^2}} + k_2 d_2 \frac{y_1 + b\sin\varphi}{\sqrt{(D - x_1 - b\cos\varphi)^2 + (y_1 + b\sin\varphi)^2}} - mg \tag{20}$$

$$0 = -k_1 d_1 \frac{x_1}{\sqrt{x_1^2 + y_1^2}} + k_2 d_2 \frac{D - x_1 + b\cos\varphi}{\sqrt{(D - x_1 - b\cos\varphi)^2 + (y_1 + b\sin\varphi)^2}} \tag{21}$$

$$0 = -bk_2 d_2 \sin\theta_1 + \frac{1}{2}\sqrt{a^2 + b^2} mg \sin(\frac{\pi}{2} + \operatorname{arctg}\frac{a}{b} - \varphi) \tag{22}$$

$$\sin\theta_1 = \sin(\varphi + \frac{\pi}{2} + \arccos(\frac{y_1 + b\sin\varphi}{\sqrt{(D - x_1 - b\cos\varphi)^2 + (y_1 + b\sin\varphi)^2}}) \tag{23}$$

4. Numerical approach to solve the system of equation.

To compute the solution of system of equations we used numerical approach. We chose to use in the loop function scipy.lingalg.solve which takes as arguments Jacobian of values of variables being used to find better approximation and returns corrections which are used to compute better approximations of the solutions. The loop stops when the norm of the function at the calculated point is smaller than previously selected error approximation (cs).

5. Outcomes and conclusions

To conclude, after running the program we obtain the position of the centre of mass as well as the angle of inclination of the top side of the box.

In order to validate the correctness of the outcomes we calculated the forces and moments of forces on obtained values. After calculation we obtained that the forces add up to zero in x and y directions, as well as values of moments of forces. Knowing that we can state that our results are correct.

We also plot the figure with the use of obtained values. However, we also chose several different constants such as the springs lengths, stiffeness coefficients or displacement of spring 2 in order to check the phisical correctness of the program. As a result we created the plots which are shown on figures 2 – 6.

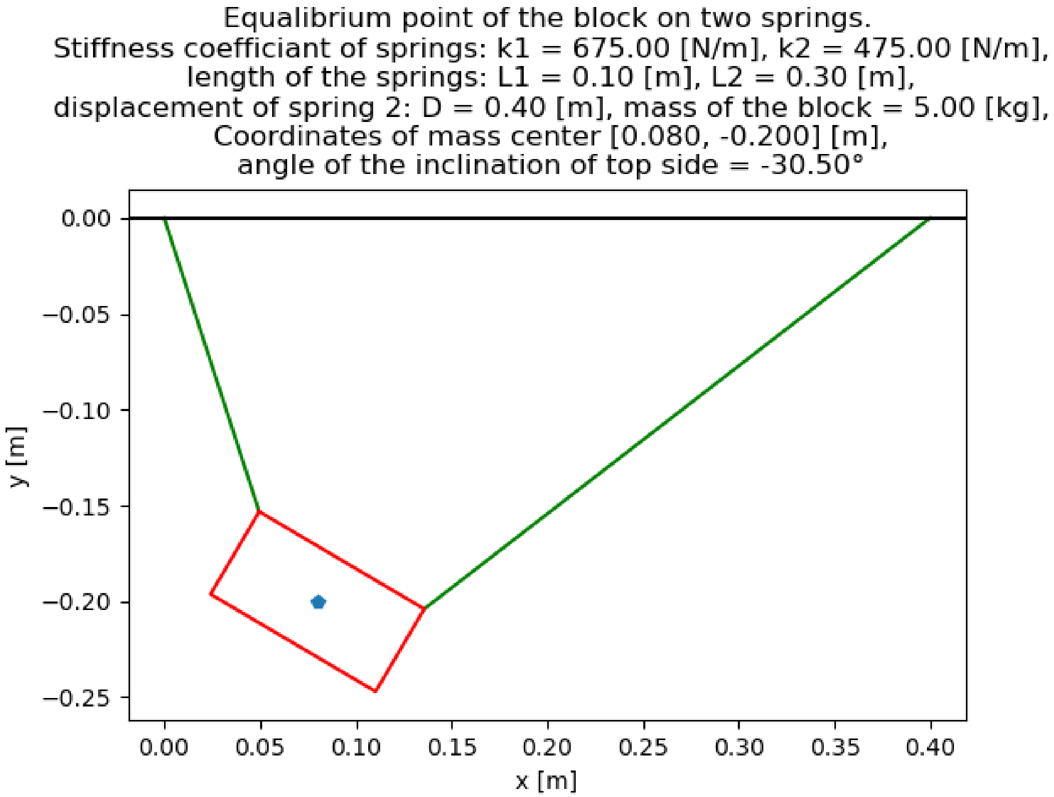

Figure 2. Visualization 1.

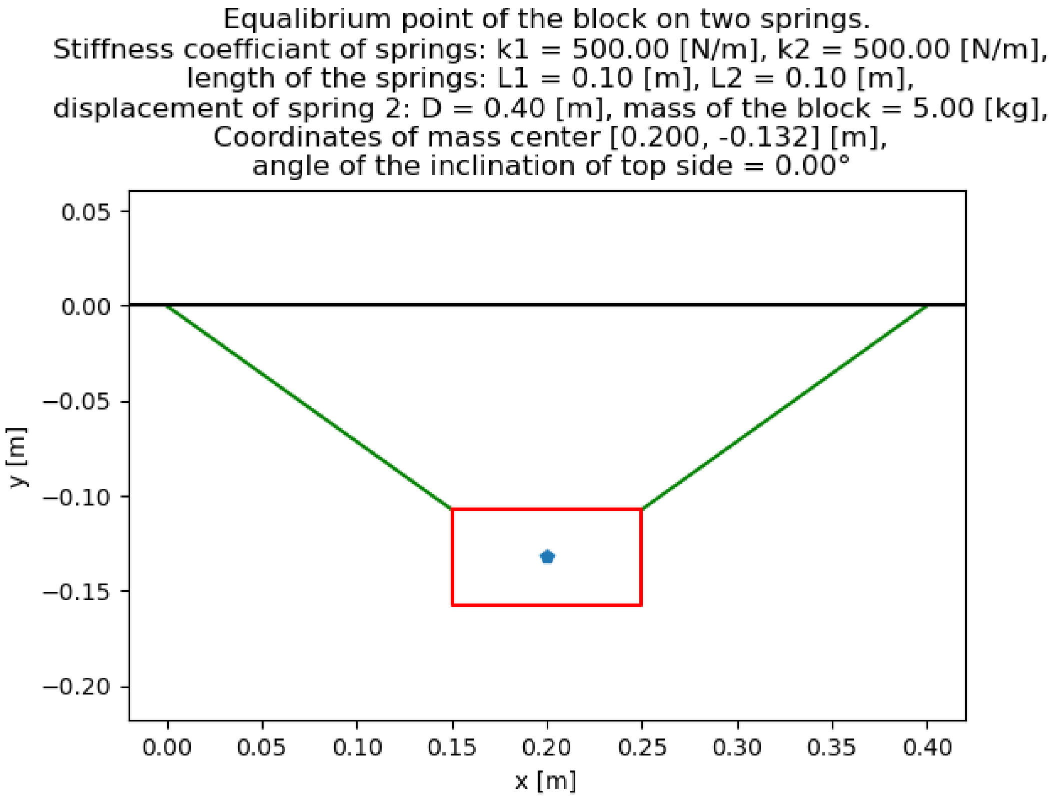

Figure 3. Visualization 2.

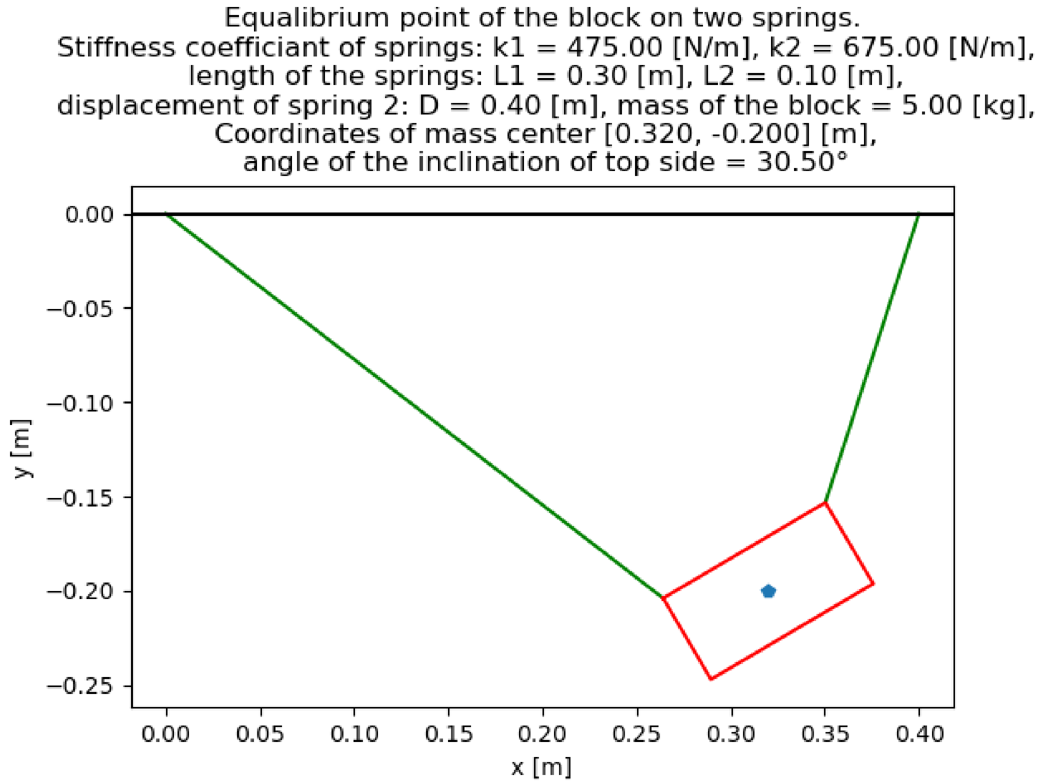

Figure 4. Visualization 3.

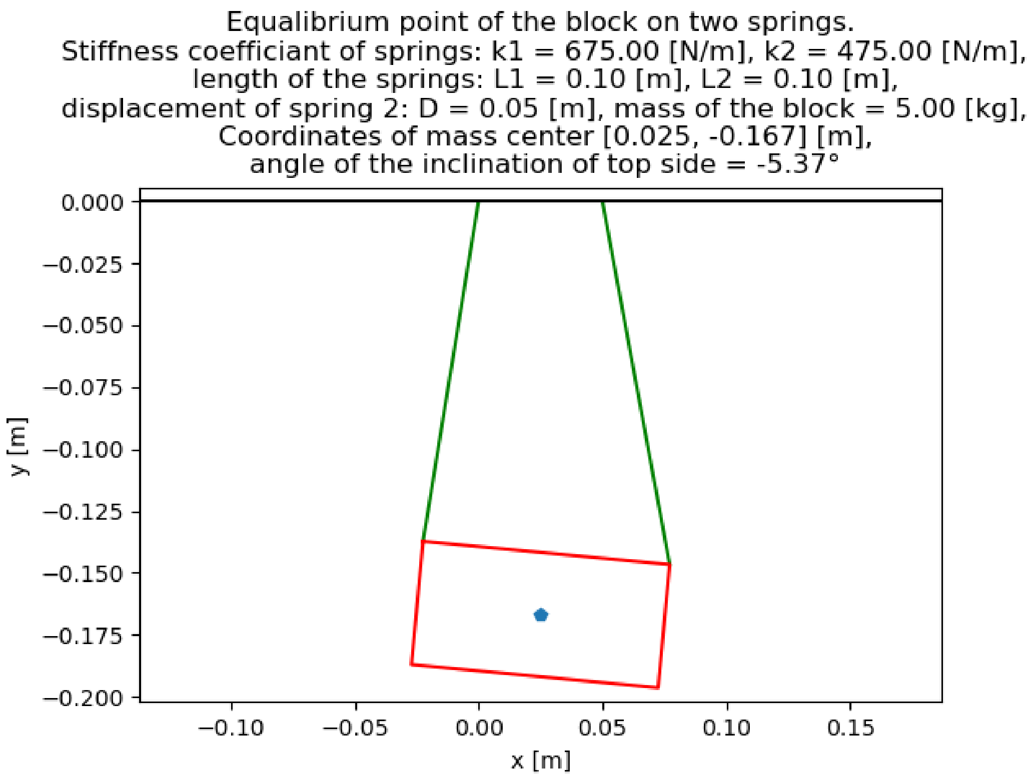

Figure 5. Visualization 4.

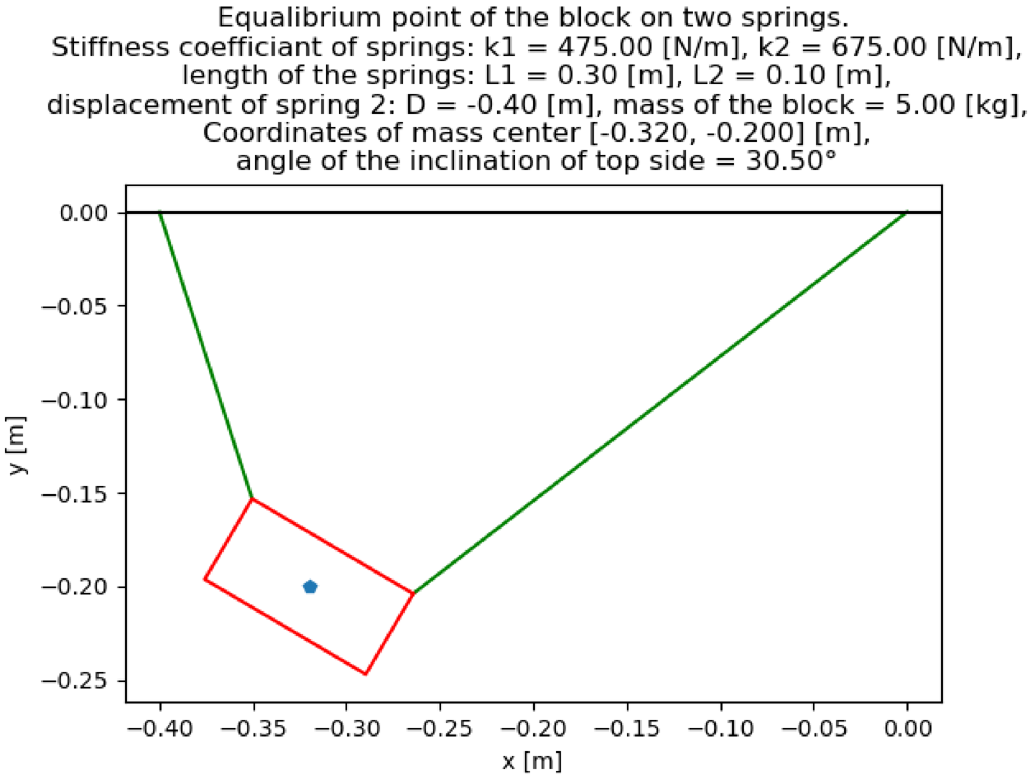

Figure 6. Vizualisation 5.