Graphs and their products.

1 Basic definitions and theorems.

1.1 A graph

Definition 1.1. A graph is an ordered set $G = (V(G),\ E(G),\ \delta_G)$ comprising of a set of vertices $V(G)$ and a set of edges $E(G)$ and a function $\delta_G : E(G) \to V(G) \times V(G)$.

1.2 Edges and verices

Definition 1.2. Each edge $e \in E(G)$ starts at a vertex denoted by $i(e) \in V(G)$ and terminates at a vertex denoted by $t(e) \in V(G)$. $\delta_G(e) = (i(e),\ t(e))$

Definition 1.3. Let graph $G = (V(G), E(G))$. Vertices $a, b \in V(G)$ are incident to $e \in E(G)$ and are adjacent (each other's neighbors) if there exists an edge $e \in E(G)$ such that $e = ab$.

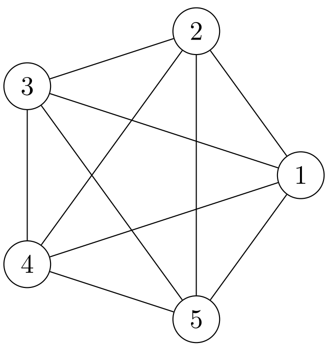
Definition 1.4. A simple graph has no multiple edges and no loops.

1.3 Basic facts

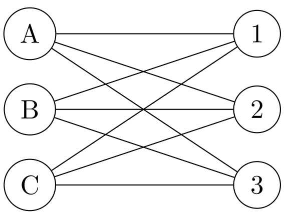
1. $G = (V(G),\ E(G))$ is finite if $V(G)$ is finite
2. $O = (V(O),\ E(O))$ is an empty graph is $V(O) = \emptyset$
3. $G = (V(G),\ E(G))$ is nontrivial if $|V(G)| > 1$
4. $|V(G)|$ is called order and $|E(G)|$ is called size

1.4 Types of graphs

- **Complete graph** K_n
  A graph with n vertices, where any two are connected by an edge.
  Order: $|V(K_n)| = n$, size: $|E(K_n)| = \binom{n}{2} = \frac{n(n-1)}{2}$
  - K_5

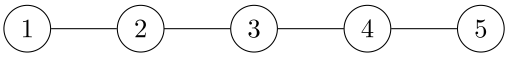


- **Complete bipartite graph** $K_{n,m}$
  A graph with n points in one part, m points in another part, where any two points from different parts are connected by an edge.
  Order: $|V(K_{n,m})| = n + m$, size: $|E(K_{n,m})| = n \cdot m$
  - $K_{3,3}$



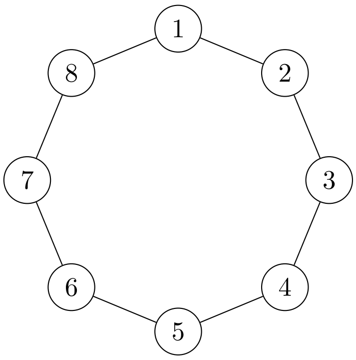
- **Path graph** P_n
  A graph with n vertices $\{v_1, v_2, \ldots, v_n\}$ and $n-1$ edges $\{v_1v_2, v_2v_3, ..., v_{n-1}v_n\}$.
  Order: $|V(P_n)| = n$, size: $|E(P_n)| = n-1$

  - P_5



- **Cycle graph** C_n
  A graph with n vertices $\{v_1, v_2, \ldots, v_n\}$ and n edges $\{v_1v_2, v_2v_3, \ldots, v_{n-1}v_n, v_nv_1\}$.
  Order: $|V(C_n)| = n$, size: $|E(C_n)| = \begin{cases} n & \text{for } n > 2 \\ n-1 & \text{for } n \leq 2 \end{cases}$

  - C_8



1.5 Degrees, subgraphs, etc.

Definition 1.5. A degree of a vertex $v \in V(G)$ is the number of edges incident with v.

Definition 1.6. A graph $G' = (V(G'), E(G'))$ is a subgraph of graph $G = (V(G), E(G))$ if $V(G') \subseteq V(G)$ and $E(G') \subseteq E(G)$. G' is a spanning subgraph of G if $V(G') = V(G)$.

Lemma 1.1 (Handshaking lemma)**.** Let $G = (V(G), E(G))$ be a graph. Let $V(G) = \{v_1, \ldots, v_n\}$. Then $\sum_{i=1}^{n} \deg v_i = 2 \cdot |E(G)|$ or $\sum_{i=1}^{n} \deg v_i = 2 \cdot (|E(G)| + |L(G)|)$ where $L(G)$ is the set of loops.

Definition 1.7. G is a regular graph if all its vertices have the same degree r. We can also say that G is r-regular. 3-regular graphs are called cubic graphs.

Definition 1.8. We can say that graphs G and H are isomorphic, or $G \cong H$, if there exists a bijection $\phi : V(G) \to V(H)$ such that $\phi(u)\phi(v) \in E(H) \iff uv \in E(G)$.

Definition 1.9. Let $G = (V(G), E(G))$ be a graph. The complement graph $\overline{G}$ is a graph such that $V(\overline{G}) = V(G)$ and $ab \in E(\overline{G}) \iff ab \notin E(G)$.

Definition 1.10. Let G and H be graphs. Homomorphism consists of a pair of maps $\Phi : V(G) \to V(H)$ and $\Psi : E(G) \to E(H)$ such that $i(\Psi(e)) = \Phi(i(e))$ and $t(\Psi(e)) = \Phi(t(e))$ for all edges $e \in E(G)$. We write $(\Phi, \Psi) : G \to H$.

- If both Φ and Ψ are $1-1$ it's called **graph embedding**.
- If both Φ and Ψ are bijective it's called **graph isomorphism**.

Definition 1.11 (Adjacency matrix)**.** Let G be a graph with a finite enumarated set $V(G)$. Let $M_{I,J}$ denote the number of edges in G with initial state I and terminal state J for vertices $I, J \in V(G)$. The adjacency matrix of G is $M = [M_{I,J}]$ and its formation from G is denoted by $M = M(G)$ or $M = M_G$.

1.6 Walks, paths, etc.

Definition 1.12. Let $(v_0, \ldots, v_n)$ be a sequence of vertices in G such that there exists $e_i = v_{i-1}v_i$ for $i = 1, \ldots, n$. The sequence is called **walk**.
If $v_0 = v_n$, it's called **closed walk**.

Definition 1.13. A walk for which all edges e_i are distinct is called **trail**.
If $v_0 = v_n$, it's called **closed trail** or **tour**.

Definition 1.14. If all vertices in a trail are distinct, it's called **path**.
A closed trail for $n \geq 3$ for which all vertices v_i are distinct (except $v_0 = v_n$) is called **cycle**.

Lemma 1.2. A connected graph of n vertices has at least $n - 1$ edges.

1.7 Trees

Definition 1.15. An acyclic graph (one not containing any cycles) is called **forest**.

Definition 1.16. A leaf is a vertex of degree 1 in a forest.

Definition 1.17. A **tree** is a connected forest.

The following statements are equivalent:

- T is a tree
- T is an acyclic graph with $n - 1$ edges
- T is a connected graph with $n - 1$ edges
- Any two vertices of T are linked by a unique path in T

1.8 Eulerian and Hamiltonian graphs

Definition 1.18. An **Eulerian trail** of graph G is a trail which contains each edge of G exactly once. If the trail is closed, it's called **Euler tour**.

Definition 1.19. A graph is Eulerian if it admits Euler tour.

Lemma 1.3. Let G be a graph such that $\deg v \geq 2$ for any vertex $v \in V(G)$. Then G contains a closed trail.

Theorem 1.4. A connected finite graph is Eulerian $\iff \forall v \in V(G): \ 2 | \deg v$.

Corollary 1.1. A connected finite graph G has an Euler trail from a vertex $x \in V(G)$ to a vertex $y \in V(G)$ $(x \neq y)$ $\iff$ x and y are the only vertices of odd degrees.

Definition 1.20. Hamiltonian (traceable) path is a path that visits each vertex exactly once. **Hamiltonian cycle** is a cycle which contains each vertex exactly once.

Definition 1.21. A graph is Hamiltonian if it admits Hamiltonian cycle.

1.9 Dirac, Ore and other theorems

Theorem 1.5 (Dirac's)**.** Let G be a graph with $n \geq 3$ vertices. If each vertex of G has a degree at least $\frac{n}{2}$, then G is Hamiltonian.

Theorem 1.6 (Ore's)**.** Let G be a graph with $n \geq 3$ vertices. If $\deg u + \deg v \geq n$ for any two non-adjacent vertices $u, v \in V(G)$, then G is Hamiltonian.

Definition 1.22 (Closure of a graph)**.** Let G be a graph with n vertices. If G contains non-adjacent vertices $u, v \in V(G)$ such that $\deg u + \deg v \geq n$, we add the edge uv to G. We continue until we get a graph $[G]$ in which for any two non-adjacent vertices $x, y \in V([G])$ we always have $\deg x + \deg y < n$. $[G]$ is the closure of G.

Theorem 1.7 (Bondy and Chvatal)**.** A graph G is Hamiltonian $\iff$ $[G]$ is Hamiltonian.

Theorem 1.8 (Rachman and Kaykobad)**.** A simple graph G with n vertices has a Hamiltonian path if for any non-adjacent vertex pairs the sum of their degrees and their shortest path length is greater than n.

2 Planar graphs, graph products and graph colouring.

2.1 Geometric graphs

Definition 2.1. Let E be the set of line segments in 3-dimensional euklidean space, V be the set of end points of those segments. A graph $G = (V, E)$ is called geometric if any two line segments in E are disjoint, or have one of their end points in common.

Lemma 2.1. Every graph is isomorphic to a geometric graph.

Definition 2.2. Geometric graph is a **plane** if all of its line segments lie in one plane. Any graph isomorphic to a plane graph is called **planar**.

Definition 2.3. Let $G = (V, E)$ be planar. The remainder $\mathbb{R}^2 \setminus G$ splits into a number of connected open regions. Closure of such a region is called a **face**.

Theorem 2.2 (Euler's formula)**.** Let G be a connected planar graph with n vertices, m edges and f faces. Then the following equation holds:

$$n - m + f = 2$$

Corollary 2.1. Let G be a connected planar graph with $n \geq 3$ vertices and m edges. Then $m \leq 3n - 6$.

2.2 Adjacency matrix and its usage

Definition 2.4. The matrix $M = [a_{ij}]_{n \times n}$ where a_{ij} is the number of edges between vertices v_i and v_j is called the **adjacency matrix**.

Proposition 2.1. Let G be a graph with adjacency matrix M. Let $k \geq 0$.

1. The number of walks of length k from v_i to v_j is M^k_{ij} - the (i, j)th entry of M^k.
2. The number of closed walks of length k is $\mathrm{tr}(M^k)$

2.3 Distances

Definition 2.5. Let G be a simple graph. The distance function $d_G : V(G) \times V(G) \to \mathbb{R}$ between two verices can be defined as follows:

$$d_G(u, v) = \begin{cases} \text{length of the shortest path between } u \text{ and } v & u \neq v \\ 0 & u = v \\ \infty & \text{no path between } u \text{ and } v \end{cases}$$

Definition 2.6. The diameter of a graph G is defined as follows: $\operatorname{diam} G = \max\limits_{u,v \in V(G)} d_G(u, v)$.

- $\operatorname{diam} K_n = 1$
- $\operatorname{diam} P_n = n - 1$
- $\operatorname{diam} H_n = n$
- $\operatorname{diam} C_n = \left\lfloor \frac{n}{2} \right\rfloor$

2.4 Hamming metric space and hamming graphs

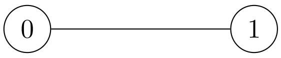
Definition 2.7. Let $A = \{a_1, \ldots, a_m\}$ be the alphabet set, $H_n(A) = \{(x_1, \ldots, x_n) : x_i \in A\}$. Let d_{H_n} be the metric, defined as the number of positions with different symbols. The pair $(H_n(A), d_{H_n})$ is called the **hamming metric space**.

Definition 2.8. A Hamming graph is the pair $H_n = (H_n(\{0, 1\}),\ E(H_n))$ with the edges defined as follows: $uv \in E(H_n) \iff d_{H_n}(u, v) = 1$.

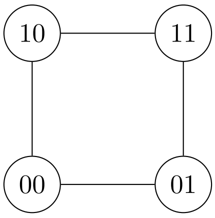
Proposition 2.2. For any $x \in H_n$ there exists exactly one point y such that $d_{H_n}(x, y) = \operatorname{diam} H_n$

2.4.1 Examples of hamming graphs

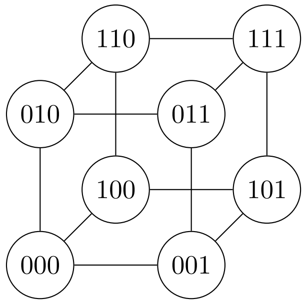
- H_1



- H_2



- H_3



2.5 Clique, independence and domination numbers

Definition 2.9. A clique is a subgraph of G isomorphic to a complete graph. The **clique number** $\omega(G)$ of G is the size of the largest clique in G.

- $\omega(H_n) = 2$
- $\omega(K_n) = n$
- $\omega(P_n) = 2$

Definition 2.10. Independent set of a graph G is a subset of its verices such that no two vertices in the subset are connected by an edge. The number of vertices in the maximum independent set is called the **independence number** $\alpha(G)$.

- $\alpha(C_n) = \lfloor \frac{n}{2} \rfloor$
- $\alpha(K_n) = 1$
- $\alpha(P_n) = \lfloor \frac{n+1}{2} \rfloor$

Definition 2.11. A domination set of a graph G is a subset S of vertices such that any vertex $v \in V(G)$ either belongs to S or is adjacent to one of its vertices. Number of vertices in a minimum domination set is called the **domination number** $\gamma(G)$ of G.

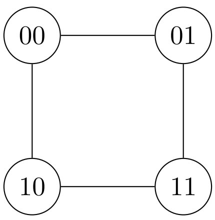
- $\gamma(C_n) = \lceil \frac{n}{3} \rceil$
- $\gamma(K_n) = 1$
- $\gamma(P_n) = \lceil \frac{n}{3} \rceil$

2.6 Graph products

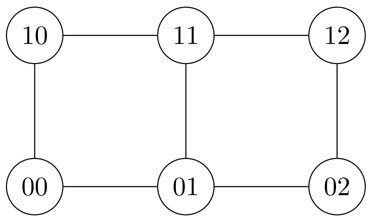
Definition 2.12. The carthesian product of G and H is the graph $G \,\Box\, H$ with vertex set $V(G \,\Box\, H) = V(G) \times V(H)$. Two vertices (g_1, h_1) and (g_2, h_2) are adjacent precisely if either $g_1 = g_2$ and $h_1 h_2 \in E(H)$, or $g_1 g_2 \in E(G)$ and $h_1 = h_2$.

1. $|V(G \,\Box\, H)| = |V(G)| \cdot |V(H)|$
2. $|E(G \,\Box\, H)| = |V(G)| \cdot |E(H)| + |V(H)| \cdot |E(G)|$
3. $\mathrm{diam}(G \,\Box\, H) = \mathrm{diam}(G) + \mathrm{diam}(H)$

- $P_2 \,\Box\, P_2$



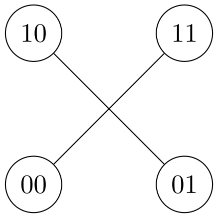
- $P_2 \,\Box\, P_3$



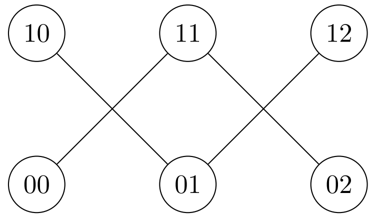
Definition 2.13. **The direct product** of G and H is the graph $G \times H$ with vertex set $V(G \times H) = V(G) \times V(H)$. Two vertices (g_1, h_1) and (g_2, h_2) are adjacent precisely if $g_1 g_2 \in E(G)$ and $h_1 h_2 \in E(H)$.

1. $|V(G \times H)| = |V(G)| \cdot |V(H)|$
2. $|E(G \times H)| = 2 \cdot |E(G)| \cdot |E(H)|$
3. $G_1 \times (G_2 + G_3) = G_1 \times G_2 + G_1 \times G_3$

- $P_2 \times P_2$



- $P_2 \times P_3$

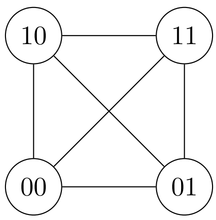


If $G_1, \ldots, G_k$ are finite non empty graphs, then their direct product is the graph $G_1 \times \cdots \times G_k = \times_{i=1}^{k} G_i$ with vertex set $V(\times_{i=1}^{k} G_i) = \{(x_1, \ldots, x_k) : x_i \in V(G_i)\}$ and for which vertices $(x_1, \ldots, x_k)$ and $(y_1, \ldots, y_k)$ are adjacent precisely if $\forall i \in \{1, \ldots, k\}$ $x_i y_i \in E(G_i)$.

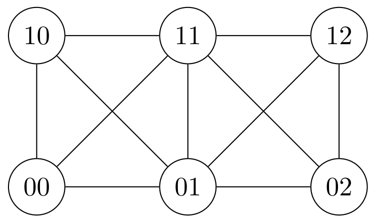
Definition 2.14. **The strong product** of G and H is the graph $G \boxtimes H$ with vertex set $V(G \boxtimes H) = V(G) \times V(H)$ and the edges set $E(G \boxtimes H) = E(G \,\square\, H) \cup E(G \times H)$.

1. $|V(G \boxtimes H)| = |V(G)| \cdot |V(H)|$
2. $|E(G \boxtimes H)| = |V(G)| \cdot |E(H)| + |V(H)| \cdot |E(G)| + 2 \cdot |E(G)| \cdot |E(H)|$
3. $K_n \boxtimes K_m \cong K_{nm}$
4. $K_1 \boxtimes G \cong G$

- $P_2 \boxtimes P_2$



- $P_2 \boxtimes P_3$



All products above are **commutative** and **associative**, meaning the following relations hold:

1. $G_1 \star G_2 \cong G_2 \star G_1$
2. $(G_1 \star G_2) \star G_3 \cong G_1 \star (G_2 \star G_3)$

2.7 Graph colouring

Definition 2.15. The minimum value k such that $V(G)$ can be positioned into k classes $V_1, V_2, \ldots, V_k$ for which $\forall u, v \in V_i(G) \iff uv \notin E(G)$ is called the (vertex) **chromatic number** of G and denoted $\chi(G)$.
It's the minimum number of colours in a vertex colouring of G - we colour each graph in such way that adjacent vertices have different colours.

- $\chi(G) \geq 2 \iff E(G) \neq \emptyset$
- $\chi(G) \geq 3 \iff G$ contains an odd cycle
- $\chi(G) = n \iff G \cong K_n$
- $\chi(G) = 2 \iff G$ is a bipartite graph

2.7.1 Greedy algorithm

1. Order vertices of a graph: $x_1, \dots, x_n$
2. Colour them one by one: $x_1 \mapsto 1$, $x_2 \mapsto \begin{cases} 1 & \text{if } x_1x_2 \notin E(G) \\ 2 & \text{otherwise} \end{cases}$, and so on...

Theorem 2.3. Let G be a graph and $\Delta(G) = \max\limits_{v \in V(G)} \deg v$, then: $\omega(G) \le \chi(G) \le \Delta(G) + 1$

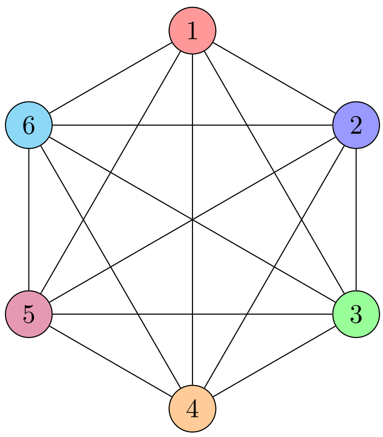
Proposition 2.3. If G is a connected, non-regular graph then $\chi(G) \le \Delta(G)$.

Proposition 2.4. Let G be a connected planar graph. Then $\exists v \in V(G) :\ \deg v \le 5$.

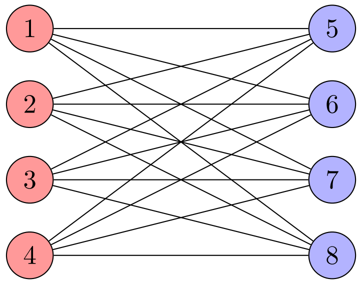
Proposition 2.5. Every planar graph is 4-colourable.

2.7.2 Examples

- K_6 with chromatic number $\chi(K_6) = 6$



- $K_{4,4}$ with chromatic number $\chi(K_{4,4}) = 2$



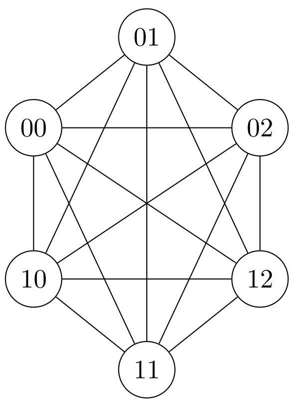
3 Lexicographic product, prime graphs and distances

3.1 Lexicographic (wreath) product

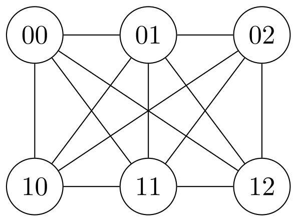
Definition 3.1. The lexicographic (or wreath) product of G and H is the graph $G \circ H$ with vertex set $V(G \circ H) = V(G) \times V(H)$. Two vertices (g_1, h_1) and (g_2, h_2) are adjacent precisely if either $g_1 g_2 \in E(G)$, or $h_1 h_2 \in E(H)$ and $g_1 = g_2$.

1. $|V(G \circ H)| = |V(G)| \cdot |V(H)|$
2. $|E(G \circ H)| = |E(G)| \cdot |V(H)|^2 + |V(G)| \cdot |E(H)|$
3. $K_n \circ K_m \cong K_{nm}$
4. $K_1 \circ G \cong G$
5. $G \circ K_1 \cong G$
6. $(G \cup H) \circ K \cong G \circ K \cup H \circ K$
7. $\overline{G} \circ \overline{H} \cong \overline{G \circ H}$
8. $\mathrm{diam}(G \circ H) = \max\big\{\mathrm{diam}(G), min\{\mathrm{diam}(H), 2\}\big\}$

- $P_2 \circ C_3$



- $P_2 \circ P_3$



3.1.1 Distances in the lexicographic product

Suppose that $(g_1, h_1), (g_2, h_2) \in V(G \circ H)$. Then:

$$d_{G \circ H}\big((g_1, h_1), (g_2, h_2)\big) = \begin{cases} d_G(g_1, g_2) & \text{if } g_1 \neq g_2 \\ d_H(h_1, h_2) & \text{if } g_1 = g_2 \text{ and } \deg_G(g_1) = 0 \\ \min\{d_H(h_1, h_2), 2\} & \text{if } g_1 = g_2 \text{ and } \deg_G(g_1) \neq 0 \end{cases}$$

Corollary 3.1. Suppose that $\mathbb{x} = (x_1, \ldots, x_k)$ and $\mathbb{y} = (y_1, \ldots, y_k)$ are vertices of graph $G = G_1 \circ \cdots \circ G_k$. Let i be the smallest index for which $x_i \neq y_i$. Then:

$$d_G(\mathbb{x}, \mathbb{y}) = \begin{cases} d_{G_i}(x_i, y_i) & \text{if } \forall l = 1, \ldots, i \quad \deg_{G_l}(x_l) = 0 \text{ and } x_1 = y_1, \ldots, x_{i-1} = y_{i-1} \\ \min\{d_{G_i}(x_i, y_i), 2\} & \text{if } \exists l = 1, \ldots, i \quad \deg_{G_l}(x_l) \neq 0 \text{ and } x_1 = y_1, \ldots, x_{i-1} = y_{i-1} \end{cases}$$

Corollary 3.2. The product $G = G_1 \circ \cdots \circ G_k$ of nontrivial graphs is connected $\iff$ G_1 is connected.

3.2 Prime factor decompositions

Definition 3.2. A graph is **prime** with respect to a given product if it's nontrivial and can't be represented as a product of two nontrivial graphs:
G is prime if $G = G_1 \,\square\, G_2 \implies G_1 \cong K_1 \vee G_2 \cong K_1$

Proposition 3.1. Every nontrivial graph G has a prime factor decomposition with respect to the carthesian product. The number of prime factors is at most $\log_2 |V(G)|$

Theorem 3.1. The prime factorization is not unique for the carthesian product in the class of possibly disconnected simple graphs.

Theorem 3.2 (Sabidussi-Vizing)**.** Every connected graph has a unique reprezentation as a product of prime graphs, up to isomorphism and the order of the factors.

Corollary 3.3. Suppose there is an isomorphism $\phi: G_1 \square \ldots \square G_k \to H_1 \square \ldots \square H_k$ where each G_i and H_i are prime. Then the vertices of the graph H_i can be relabeled such that $\phi(x_1, \ldots, x_k) = (x_{\sigma(1)}, \ldots, x_{\sigma(k)})$ for permutation σ of $\{1, \ldots, k\}$.

Theorem 3.3. Let G, H, K be finite simple graphs. Suppose that K is not empty. Then $G \square K \cong H \square K \implies G \cong H$.

Proposition 3.2. Suppose that $(g_1, h_1), (g_2, h_2) \in V(G \square H)$. Then:
$d_{G \square H}\big((g_1, h_1), (g_2, h_2)\big) = d_G(g_1, g_2) + d_H(h_1, h_2)$.

Corollary 3.4. Suppose that $\mathbb{x} = (x_1, \ldots, x_k)$ and $\mathbb{y} = (y_1, \ldots, y_k)$ are distinct vertices of graph $G = G_1 \square \ldots \square G_k$. Then:
$d_G(\mathbb{x}, \mathbb{y}) = \sum\limits_{i=1}^{k} d_{G_i}(x_i, y_i)$

Corollary 3.5. The carthesian product of graphs is connected $\iff$ every factor of the graph is connected.

3.3 Distances in the direct product

Proposition 3.3. Suppose (g_1, h_1) and (g_2, h_2) are vertices of a direct product $G \times H$ and n is an integer for which G has a g_1, g_2-walk of length n and H has an h_1, h_2-walk of length n. Then $G \times H$ has a walk of length n from (g_1, h_1) to (g_2, h_2). The smallest such n (if it exists) equals $d_{G \times H}\big((g_1, h_1), (g_2, h_2)\big)$. If no such n exists, then $d_{G \times H}\big((g_1, h_1), (g_2, h_2)\big) = \infty$.

Proposition 3.4. Suppose $\mathbb{x}$ and $\mathbb{y}$ are vertices of $G = \times_{i=1}^{k} G_i$. Then:
$d_G(\mathbb{x}, \mathbb{y}) = \min\{n \in \mathbb{N} :$ each factor G_i has a walk of length n from $p_i(\mathbb{x})$ to $p_i(\mathbb{y})\}$
Where it is understood that $d_G(\mathbb{x}, \mathbb{y}) = \infty$ if no such n exists.

Theorem 3.4 (Weichsel's theorem)**.** Suppose G and H are connected finite nontrivial graphs. If at least one of G or H has an odd cycle, then $G \times H$ is connected. If both G and H are bipartite, then $G \times H$ has exactly two components.

Corollary 3.6. A direct product of connected nontrivial finite graphs is connected $\iff$ at most one of the factors is bipartite.
The product has $2k - 1$ components, where k is the number of bipartite factors.