

# IM ABELIAN THREEFOLDS

PIP GOODMAN, IN JOINT WORK WITH FRANCESC FITÉ

$M$  imaginary quadratic field.

$A$  an abelian threefold defined over a number field  $K$ .

We say  $A$  has potential imaginary multiplication (PIM) by  $M$ , if  $\text{End}^0(A_{\bar{K}}) := \text{End}(A_{\bar{K}}) \otimes \mathbb{Q} \cong M$ .

If in addition  $\text{End}^0(A_K) \cong M$ , we say  $A$  has imaginary multiplication (IM) by  $M$ .

**Theorem** (Fité, G. '25). *Let  $A/K$  be an abelian threefold with PIM by  $M$ , then  $\text{Cl}(M) \leq [K : \mathbb{Q}]$ .*

**$\lambda$ -adic representations.**  $B/K$  ab. var.,  $F$  number field,  $F \hookrightarrow \text{End}^0(B_L)$  for some  $L/K$ .  $\ell$  a prime number.

$V_\ell(B)$  is a free  $F \otimes \mathbb{Q}_\ell$ -module of rank  $h := \frac{2 \dim(B)}{[F : \mathbb{Q}]}$ .

The decomposition  $F \otimes \mathbb{Q}_\ell \cong \prod_{\lambda|\ell} F_\lambda$  induces

$$V_\ell(B_L) \cong \prod_{\lambda|\ell} V_\lambda(B_L),$$

with each  $V_\lambda(B_L)$  a  $F_\lambda$  v.s. of dim  $h$  and an action of  $G_L := \text{Gal}(\bar{L}/L)$ .

**Theorem** (Fité, G. '25). *Let  $A/K$  be an abelian threefold with PIM by  $M$ , then there exists an elliptic curve  $E/K$  with potential CM by  $M$  such that, for every prime  $\lambda$  of  $M$  over a rational prime  $\ell$ , the following is satisfied:*

i) *If  $\text{End}^0(A_K) \cong M$ , then  $M \subseteq K$  and*

$$(\wedge^3 V_\lambda(A) \oplus \wedge^3 V_{\bar{\lambda}}(A))(-1) \simeq V_\ell(E) \otimes_{\mathbb{Q}_\ell} M_\lambda$$

*as  $M_\lambda[G_K]$ -modules.*

ii) *If  $\text{End}^0(A_K) \not\cong M$ , then  $\text{End}^0(A_{MK}) \cong M$  and*

$$\text{Ind}_{KM}^K (\wedge^3 V_\lambda(A_{KM}))(-1) \simeq V_\ell(E) \otimes_{\mathbb{Q}_\ell} M_\lambda$$

*as  $M_\lambda[G_K]$ -modules.*

**Upcoming work.** Fité, Guitart and Pedret: Given  $A$ , explicitly determine  $E$ . Application: Compute L-polynomials of IM threefolds in average polynomial time.

Chidambaram, G.: Fast algorithm to compute Galois images of IM threefolds given the L-polynomials.

**Question.** There are nine imaginary quadratic fields of class number 9. Do all occur as the endomorphism algebras of PIM threefolds over  $\mathbb{Q}$ ?

$\mathbb{Q}(\sqrt{-d})$  is known to occur for  $d = 1, 2, 3, 7$ .

## PROOF OVERVIEW

$B/K$  ab. var.

We say  $B/K$  has CM by  $F$  if  $F$  is a number field of degree  $[F : \mathbb{Q}] = 2 \dim(B)$  and  $F \cong \text{End}^0(B_{\bar{K}})$ .

The action of  $F$  on  $\Omega^1(B)$  is described by  $\dim(B)$  distinct, non-conjugate embeddings of  $F$  into  $\bar{K}$ .

**Example.**  $C: y^2 = x^5 - 2$ ,  $[\zeta_5](x, y) = (\zeta_5 x, y)$ . Basis of  $\Omega^1(C)$ :  $\frac{dx}{y}, \frac{x dx}{y}$ .  $[\zeta_5]^* \frac{dx}{y} = \zeta_5 \frac{dx}{y}$  and  $[\zeta_5]^* \frac{x dx}{y} = \zeta_5^2 \frac{x dx}{y}$ .

Such a collection of embeddings  $\Phi$  is called a CM-type.

Modularity of CM abelian varieties:

$$\{\text{CM ab.var. } /K \text{ of CM-type } \Phi\} \longleftrightarrow \left\{ \begin{array}{l} \text{Algebraic Hecke character } \chi \text{ of } K \text{ of infinity type } \Phi^{-1} \\ \text{and a lattice } \Lambda \subseteq \mathbb{C} \text{ s.t. } \chi(s)\Lambda = N_{\Phi}(s)\Lambda \end{array} \right\}$$

“ $\rightarrow$ ” Shimura-Taniyama and “ $\leftarrow$ ” Casselman. The above both have natural  $\lambda$ -adic reps of  $G_K$  attached to them and the correspondence preserves these.

**Theorem** (Fité).  *$A/K$  abelian threefold with IM by  $M$ . There exists an algebraic Hecke character  $\psi$  of  $K$  which gives rise to the  $\wedge^3 V_{\lambda}(B)$  representations. Its infinity type is determined by the action of  $M$  on  $\Omega^1(A)$ .*

**Example.**  $C: y^3 = f(x)$ ,  $f$  degree 4,  $[\zeta_3](x, y) = (x, \zeta_3 y)$ . Basis of  $\Omega^1(C)$  given by  $\frac{dx}{y^2}, \frac{x dx}{y^2}, \frac{dy}{y}$ , eigenvalues of  $[\zeta_3]$  are  $\zeta_3, \zeta_3, \zeta_3^2$ .

$C: y^2 = f(x)$ , s.t.  $f(-x) = -f(x)$  and  $f$  degree 7,  $[i](x, y) = (-x, iy)$ . Basis of  $\Omega^1(C)$  given by  $\frac{dx}{y}, \frac{x dx}{y}, \frac{x^2 dx}{y}$ , eigenvalues of  $[i]$  are  $i, -i, i$ .

**Theorem** (Fité, G. '25). *If  $A/K$  has PIM by  $M$ , then  $A/KM$  has IM by  $M$ . The infinity type of  $\psi$  is  $(2, 1)$ .*

Twisting by the norm $^{-1}$  gives us an algebraic Hecke character whose infinity type is a CM-type. From here we obtain the CM elliptic curve in the IM case.

For the PIM case, we base change to  $MK$  and then use descent to prove the result over  $K$ .