IM ABELIAN THREEFOLDS

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M imaginary quadratic field.
A an abelian threefold defined over a number field K.
We say A has potential imaginary multiplication (PIM) by M, if $\mathrm{End}^0(A_{\bar{K}}) \coloneqq \mathrm{End}(A_{\bar{K}}) \otimes \mathbb{Q} \cong M$.
If $\mathrm{End}^0(A_K) \cong M$, we say A has imaginary multiplication (IM) by M.

Theorem (Fité, G. ' 25)**.** *Let A/K be an abelian threefold with PIM by M, then* $\mathrm{Cl}(M) \leq [K\colon \mathbb{Q}]$.

λ-adic representations. B/K ab. var., F number field, $F \hookrightarrow \mathrm{End}^0(B_L)$ for some L/K. ℓ a prime number.
$V_\ell(B)$ is a free $F \otimes \mathbb{Q}_\ell$-module of rank $h \coloneqq \frac{2\dim(B)}{[F\colon \mathbb{Q}]}$.
The decomposition $F \otimes \mathbb{Q}_\ell \cong \prod_{\lambda|\ell} F_\lambda$ induces

$$V_\ell(B_L) \cong \prod_{\lambda|\ell} V_\lambda(B_L),$$

with each $V_\lambda(B_L)$ a F_λ v.s. of dim h and an action of $G_L \coloneqq \mathrm{Gal}(\bar{L}/L)$.

Theorem (Fité, G. '25)**.** *Let A/K be an abelian threefold with PIM by M, then there exists an elliptic curve E/K with potential CM by M such that, for every prime λ of M over a rational prime ℓ, the following is satisfied:*

i) If $\mathrm{End}^0(A_K) \cong M$, *then* $M \subseteq K$ *and*

$$\big(\wedge^3 V_\lambda(A) \oplus \wedge^3 V_{\bar{\lambda}}(A)\big)(-1) \simeq V_\ell(E) \otimes_{\mathbb{Q}_\ell} M_\lambda$$

as $M_\lambda[G_K]$-modules.

ii) If $\mathrm{End}^0(A_K) \ncong M$, *then* $\mathrm{End}^0(A_{MK}) \cong M$ *and*

$$\mathrm{Ind}_{KM}^K \big(\wedge^3 V_\lambda(A_{KM})\big)(-1) \simeq V_\ell(E) \otimes_{\mathbb{Q}_\ell} M_\lambda$$

as $M_\lambda[G_K]$-modules.

Upcoming work. Fité, Guitart and Pedret: Given A, explicitly determine E. Application: Compute L-polynomials of IM threefolds in average polynomial time.
Chidambaram, G.: Fast algorithm to compute Galois images of IM threefolds given the L-polynomials.

Question. There are nine imaginary quadratic fields of class number 9. Do all occur as the endomorphism algebras of PIM threefolds over $\mathbb{Q}$?
$\mathbb{Q}(\sqrt{-d})$ is known to occur for $d = 1, 2, 3, 7$.

Proof overview

B/K ab. var.
We say B/K has CM by F if F is a number field of degree $[F\colon \mathbb{Q}] = 2\dim(B)$ and $F \cong \mathrm{End}^0(B_K)$.
The action of F on $\Omega^1(B)$ is described by $\dim(B)$ distinct, non-conjugate embeddings of F into $\bar{K}$.

Example. $C\colon y^2 = x^5 - 2$, $[\zeta_5](x, y) = (\zeta_5 x, y)$. Basis of $\Omega^1(C)$: $\frac{dx}{y}, \frac{xdx}{y}$. $[\zeta_5]^* \frac{dx}{y} = \zeta_5 \frac{dx}{y}$ and $[\zeta_5]^* \frac{xdx}{y} = \zeta_5^2 \frac{xdx}{y}$.

Such a collection of embeddings Φ is called a CM-type.

Modularity of CM abelian varieties:

$$\{\text{CM ab.var. } /K \text{ of CM-type } \Phi\} \longleftrightarrow \left\{\substack{\text{Algebraic Hecke character } \chi \text{ of } K \text{ of infinity type } \Phi^{-1} \\ \text{and a lattice } \Lambda \subseteq \mathbb{C} \text{ s.t. } \chi(s)\Lambda = N_\Phi(s)\Lambda}\right\}$$

"$\rightarrow$" Shimura-Taniyama and "$\leftarrow$" Casselman. The above both have natural λ-adic reps of G_K attached to them and the correspondence preserves these.

Theorem (Fité)**.** *A/K abelian threefold with IM by M. There exists an algebraic Hecke character ψ of K which gives rise to the $\wedge^3 V_\lambda(B)$ representations. Its infinity type is determined by the action of M on $\Omega^1(A)$.*

Example. $C\colon y^3 = f(x)$, f degree 4, $[\zeta_3](x,y) = (x, \zeta_3 y)$. Basis of $\Omega^1(C)$ given by $\frac{dx}{y^2}, \frac{xdx}{y^2}, \frac{dx}{y}$, eigenvalues of $[\zeta_3]$ are $\zeta_3, \zeta_3, \zeta_3^2$.

$C\colon y^2 = f(x)$, s.t. $f(-x) = -f(x)$ and f degree 7, $[i](x,y) = (-x, iy)$. Basis of $\Omega^1(C)$ given by $\frac{dx}{y}, \frac{xdx}{y}, \frac{x^2dx}{y}$, eigenvalues of $[i]$ are $i, -i, i$.

Theorem (Fité, G. '25)**.** *If A/K has PIM by M, then A/KM has IM by M. The infinity type of ψ is $(2,1)$.*

Twisting by the norm^{-1} gives us an algebraic Hecke character whose infinity type is a CM-type. From here we obtain the CM elliptic curve in the IM case.

For the PIM case, we base change to MK and then use descent to prove the result over K.