IM ABELIAN THREEFOLDS

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M imaginary quadratic field.
A an abelian threefold defined over a number field K.
We say A has potential imaginary multiplication (PIM) by M, if $\mathrm{End}^0(A_{\bar{K}}) := \mathrm{End}(A_{\bar{K}}) \otimes \mathbb{Q} \cong M$.
If $\mathrm{End}^0(A_K) \cong M$, we say A has imaginary multiplication (IM) by M.

Theorem (Fité, G. ' 25)**.** *Let A/K be an abelian threefold with PIM by M, then* $\mathrm{Cl}(M) \leq [K\colon \mathbb{Q}]$.

λ-adic representations. B/K ab. var., F number field, $F \hookrightarrow \mathrm{End}^0(B_L)$ for some L/K. ℓ a prime number.
$V_\ell(B)$ is a free $F \otimes \mathbb{Q}_\ell$-module of rank $h := \frac{2\dim(B)}{[F\colon \mathbb{Q}]}$.
The decomposition $F \otimes \mathbb{Q}_\ell \cong \prod_{\lambda|\ell} F_\lambda$ induces

$$V_\ell(B_L) \cong \prod_{\lambda|\ell} V_\lambda(B_L),$$

with each $V_\lambda(B_L)$ a F_λ v.s. of dim h and an action of $G_L := \mathrm{Gal}(\bar{L}/L)$.

Theorem (Fité, G. '25)**.** *Let A/K be an abelian threefold with PIM by M, then there exists an elliptic curve E/K with potential CM by M such that, for every prime λ of M over a rational prime ℓ, the following is satisfied:*

i) If $\mathrm{End}^0(A_K) \cong M$, then $M \subseteq K$ and

$$\big(\wedge^3 V_\lambda(A) \oplus \wedge^3 V_{\bar{\lambda}}(A)\big)(-1) \simeq V_\ell(E) \otimes_{\mathbb{Q}_\ell} M_\lambda$$

as $M_\lambda[G_K]$-modules.

ii) If $\mathrm{End}^0(A_K) \ncong M$, then $\mathrm{End}^0(A_{MK}) \cong M$ and

$$\mathrm{Ind}_{KM}^{K}\big(\wedge^3 V_\lambda(A_{KM})\big)(-1) \simeq V_\ell(E) \otimes_{\mathbb{Q}_\ell} M_\lambda$$

as $M_\lambda[G_K]$-modules.

Upcoming work. Fité, Guitart and Pedret: Given A, explicitly determine E. Application: Compute L-polynomials of IM threefolds in average polynomial time.
Chidambaram, G.: Fast algorithm to compute Galois images of IM threefolds given the L-polynomials.

Question. There are nine imaginary quadratic fields of class number 9. Do all occur as the endomorphism algebras of PIM threefolds over $\mathbb{Q}$?
$\mathbb{Q}(\sqrt{-d})$ is known to occur for $d = 1, 2, 3, 7$.

Proof overview

B/K ab. var.
We say B/K has CM by F if F is a number field of degree $[F\colon \mathbb{Q}] = 2\dim(B)$ and $F \cong \mathrm{End}^0(B_K)$.
The action of F on $\Omega^1(B)$ is described by $\dim(B)$ distinct, non-conjugate embeddings of F into $\bar{K}$.

Example. $C\colon y^2 = x^5 - 2$, $[\zeta_5](x, y) = (\zeta_5 x, y)$. Basis of $\Omega^1(C)$: $\frac{dx}{y}, \frac{xdx}{y}$. $[\zeta_5]^* \frac{dx}{y} = \zeta_5 \frac{dx}{y}$ and $[\zeta_5]^* \frac{xdx}{y} = \zeta_5^2 \frac{xdx}{y}$.

Such a collection of embeddings Φ is called a CM-type.

Modularity of CM abelian varieties:

$$\{\text{CM ab.var. } /K \text{ of CM-type } \Phi\} \longleftrightarrow \left\{\substack{\text{Algebraic Hecke character } \chi \text{ of } K \text{ of infinity type } \Phi^{-1} \\ \text{and a lattice } \Lambda \subseteq \mathbb{C} \text{ s.t. } \chi(s)\Lambda = N_\Phi(s)\Lambda}\right\}$$

"$\to$" Shimura-Taniyama and "$\leftarrow$" Casselman. The above both have natural λ-adic reps of G_K attached to them and the correspondence preserves these.

Theorem (Fité). *A/K abelian threefold with IM by M. There exists an algebraic Hecke character ψ of K which gives rise to the $\wedge^3 V_\lambda(B)$ representations. Its infinity type is determined by the action of M on $\Omega^1(A)$.*

Example. $C\colon y^3 = f(x)$, f degree 4, $[\zeta_3](x,y) = (x, \zeta_3 y)$. Basis of $\Omega^1(C)$ given by $\frac{dx}{y^2}, \frac{xdx}{y^2}, \frac{dx}{y}$, eigenvalues of $[\zeta_3]$ are $\zeta_3, \zeta_3, \zeta_3^2$.

$C\colon y^2 = f(x)$, s.t. $f(-x) = -f(x)$ and f degree 7, $[i](x,y) = (-x, iy)$. Basis of $\Omega^1(C)$ given by $\frac{dx}{y}, \frac{xdx}{y}, \frac{x^2dx}{y}$, eigenvalues of $[i]$ are $i, -i, i$.

Theorem (Fité, G. '25). *If A/K has PIM by M, then A/KM has IM by M. The signature of ψ is $(2,1)$.*

Twisting by the norm^{-1} gives us an algebraic Hecke character whose infinity type is a CM-type. From here we obtain the CM elliptic curve.

We then use descent to prove the result over the ground field.