MCS Calculus
Practical Exercises 7
(Week 17)

Epiphany Term 2025

Before starting on this week's work, you may find it beneficial to complete the questions on series convergence from Calculus Practical 6 (week 15). If any questions below are on material we have not yet covered in lectures, leave them for next time. If you wish, try typesetting your answers with LaTeX.

1. Determine the location and nature of the stationary points of the function

$$f(w,x,y,z) = \frac{w^2}{2} + 2x^2 + 3xy - 11x + 2y^2 - 10y - \frac{z^3}{6} + z$$

2. For the following power series determine the radius of convergence.

   (a) $\sum_{n=1}^{\infty} \frac{n!}{n^n} x^n$

   (b) $\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!} x^n$

   (c) $\sum_{n=1}^{\infty} \frac{5^n}{n!} x^n$

   (d) $\sum_{n=1}^{\infty} \frac{(x+9)^n}{(n+1)^2}$

   (e) $\sum_{n=1}^{\infty} \frac{(x+7)^{2n+1}}{n \cdot 9^n}$

3. Determine the MacLaurin series for $f(x) = (1+e^x)^3$.

4. Determine the MacLaurin series for $f(x) = \cos(4x)$.

5. Determine the Taylor series for $f(x) = \frac{7}{x^4}$ about $x_0 = -3$.

6. Determine the Taylor series for $f(x) = 5x^2 + 2x + 1$ about $x_0 = 1$.

7. Determine the Taylor series for $f(x) = 5x^2 + 2x + 1$ about $x_0 = 5$.

8. Determine the Taylor series for $f(x) = e^{-3x}$ about $x_0 = -2$.

9. Let f and g be n-times differentiable functions such that:

- $f(a) = g(a) = 0$,
- the derivatives $f^{(r)}(a) = g^{(r)}(a) = 0$ for $1 \leq r \leq n-1$,
- $f^{(n)}(a) \neq 0$ and $g^{(n)}(a) \neq 0$.

Use Taylor's Theorem to directly prove the extended L'Hôpital rule:

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f^{(n)}(x)}{\lim_{x \to a} g^{(n)}(x)}.$$

10. Determine the value of

$$\lim_{x \to 2} \frac{\sin^2 \pi x}{2e^{x/2} - xe}.$$

11. Let f be an n-times differentiable function such that for some $k < n$:

- the derivatives $f^{(r)}(a) = 0$ for $1 \leq r \leq k-1$,
- $f^{(k)}(a) \neq 0$.

Use Taylor's Theorem to directly prove necessary and sufficient conditions on k and $f^{(k)}(a)$ to classify $f(a)$ as a local minimum, maximum or point of inflection.