COMP1021 Mathematics for Computer Science
Linear Algebra (Part 2)
Practical - Week 16
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Instructions:

Work on these problems in the practical sessions for the week specified. First try them on your own. If you're stuck, try discussing things with others. If you get the answer, still discuss with others to see if maybe you missed something. If you run into major roadblocks, ask the demonstrators for hints.

Solutions will be posted on Learn Ultra at the end of the week. Make sure you're all set with the solutions and understand them before the next practical.

Note: If I don't mention some particular inner product when asking a question, assume it's the dot product.

Purpose of this practical: In this practical we review similarity of matrices, diagonilisation, eigendecomposition and inner product spaces.

1. Suppose a 3×3 upper triangular matrix has diagonal entries $1, 2$, and 7. Can we say whether it is diagonalisable with the given information? Why or why not?

Solution:

Hint: For an upper triangular matrix, what would the characteristic equation look like?

It is easy to check that $1, 2$, and 7 are the solutions to the characteristic equation and therefore the eigenvalues of this matrix. It's a 3×3 matrix with 3 distinct eigenvalues. By choosing an eigenvector, corresponding to each eigenvalue, we'll get 3 linearly independent eigenvectors. A 3×3 matrix with 3 linearly independent eigenvectors is diagonalisable.

2. Find a matrix A whose eigenvalues are 1 and 4 and whose corresponding eigenvectors are $(3, 1)$ and $(2, 1)$. Is this matrix unique?

Solution:

Such a matrix A would have size 2×2. Since A has two linearly independent eigenvectors (the vectors $(3, 1)$ and $(2, 1)$), A is diagonalisable and has an eigendecomposition

$$A = PDP^{-1} = \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix} \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix}^{-1}.$$

In particular, A is unique. Performing the above computation, get $A = \begin{pmatrix} -5 & 18 \\ -3 & 10 \end{pmatrix}$.

3. We saw in class that a matrix A of size $n \times n$ that has n distinct eigenvalues will have an eigendecomposition. In other words, if all the eigenvalues of A have algebraic multiplicity 1, then the eigendecomposition of A exists. Can you say something about the relationship between the geometric multiplicity of eigevalues of A and the existence of an eigendecomposition of A?

Solution:

Hint 1: For an eigenvalue λ of A, geometric multiplicity of $\lambda \leq$ algebraic multiplicity of λ.

Hint 2: Recall that for an $n \times n$ matrix to be diagonalisable, it must have n linearly independent eigenvectors.

Remember that the geometric multiplicity of an eigenvalue λ is the dimension of the nullspace of $\lambda I - A$, i.e., it is the number of linearly independent eigenvectors associated with a given eigenvalue. So, if we want n linearly independent eigenvectors associated with a given $n \times n$ matrix, we need that for each eigenvalue with some algebraic multiplicity x, the geometric multiplicity of that eigenvalue is also x. Since the geometric multiplicity of an eigenvalue cannot exceed its algebraic multiplicity, we can write the condition another way: the sum of the geometric multiplicities of the eigenvalues is n.

4. Prove the following properties about two similar matrices A and B.

   (a) A and B have the same rank.

   (b) A and B have the same trace.

   (c) A and B have the same eigenvalues.

Solution:

Refer to `https://www.statlect.com/matrix-algebra/similar-matrix` for solutions.

5. Show that for a diagonalisable $n \times n$ matrix A with eigenvalues $\lambda_1, \ldots, \lambda_n$ counted with multiplicity, then $\det(A) = \prod_i \lambda_i$. Hint: consider the Eigendecomposition of A.

Solution:

A can be decomposed into $A = PDP^{-1}$, where the columns of P are the eigenvectors of A and P is invertible, and $D = diag(\lambda_1, \ldots, \lambda_n)$. $\det(A) = \det(P) \cdot \det(D) \cdot \frac{1}{\det P} = \det(D) = \prod_i \lambda_i$

6. For each of the following expressions either prove that it defines an inner product on $\mathbb{R}^4$ or list all if the inner product axioms that fail to hold:

   (a) $\langle \mathbf{u}, \mathbf{b} \rangle = \mathbf{u}_1\mathbf{v}_1 - \mathbf{u}_2\mathbf{v}_2 + \mathbf{u}_3\mathbf{v}_3 - \mathbf{u}_4\mathbf{v}_4$
   (b) $\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u}_1^2\mathbf{v}_1^2 + \mathbf{u}_2^2\mathbf{v}_2^2 + \mathbf{u}_3^2\mathbf{v}_3^2 + \mathbf{u}_4^2\mathbf{v}_4^2$
   (c) $\langle \mathbf{u}, \mathbf{v} \rangle = 2\mathbf{u}_1 3\mathbf{v}_1 + 2\mathbf{u}_2 3\mathbf{v}_2 + 2\mathbf{u}_3 3\mathbf{v}_3 + 2\mathbf{u}_4 3\mathbf{v}_4$
   (d) $\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u}_1\mathbf{v}_4 + \mathbf{u}_2\mathbf{v}_3 + \mathbf{u}_3\mathbf{v}_2 + \mathbf{u}_4\mathbf{v}_1$

Solution:

(a) Positivity axiom does not hold (consider $\mathbf{u} = [0, 2, 0, 2]$).
(b) Additivity and homogeneity axioms do not hold.
(c) This is just scaling the standard euclidean inner product by 6.
(d) Positivity axiom does not hold (consider $\mathbf{u} = [1, 1, 1, -1]$).

7. In each part, use the given inner product on $\mathbb{R}^2$ to find $||\mathbf{v}||$, where $\mathbf{v} = (-1, 3)$.

   (a) The Euclidean inner product.

Solution:
$||\mathbf{v}|| = \sqrt{(-1)^2 + 3^2} = \sqrt{10}$.

   (b) The weighted Euclidean inner product with $w_1 = 3$ and $w_2 = 2$.

Solution:
$||\mathbf{v}|| = \sqrt{3(-1)^2 + 2(3)^2} = \sqrt{21}$.

   (c) The inner product generated by the matrix
$\begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix}$.

Solution:
We have

$$\begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} -1 \\ 3 \end{pmatrix} = \begin{pmatrix} 5 \\ 10 \end{pmatrix},$$

so $||\mathbf{v}|| = \sqrt{5^2 + 10^2} = 5\sqrt{5}$.

8. In each part, use the given inner product on $\mathbb{P}^2$ to find $\langle \mathbf{p}, \mathbf{q} \rangle$ where $\mathbf{p} = 1 - 2x^2$ and $\mathbf{q} = 1 - x + x^2$.

   (a) The standard inner product in $\mathbb{P}^2$.

Solution:
$\langle \mathbf{p}, \mathbf{q} \rangle = (1)(1) + 0(-1) + (-2)(1) = -1$

   (b) The evaluation inner product with $x_0 = -1$, $x_1 = 0$ and $x_2 = 3$.

Solution:
$\langle \mathbf{p}, \mathbf{q} \rangle = (1 - 2(-1)^2)(1 - (-1) + (-1)^2) + (1 - 2(0)^2)(1 - 0 + 0^2) + (1 - 2(3)^2)(1 - 3 + 3^2) = -3 + 1 + (-17)7 = -121$

   (c) The inner product on $C[0, 1]$.

Solution:

$$\langle \mathbf{p}, \mathbf{q} \rangle = \int_0^1 (1 - 2x^2)(1 - x + x^2)\, dx = \int_0^1 (1 - x - x^2 + 2x^3 - 2x^4)\, dx = \frac{4}{15}.$$

9. Consider $\mathbb{R}^4$ equipped with the weighted Euclidean inner product with following weights: $w_1 = 1, w_2 = 2, w_3 = 2, w_4 = 4$. Let $W = span(\mathbf{u}, \mathbf{v})$ where $\mathbf{u} = (1, 1, -1, 2)$ and $\mathbf{v} = (2, 2, 3, -1)$. Find a basis of $W^\perp$.

Solution:

As in the lecture, $W^\perp$ is the solution space of the linear system

$$\begin{aligned} x_1 + 2x_2 - 2x_3 + 8x_4 &= 0 \qquad (\langle \mathbf{u}, \mathbf{x} \rangle = 0) \\ 2x_1 + 4x_2 + 6x_3 - 4x_4 &= 0 \qquad (\langle \mathbf{v}, \mathbf{x} \rangle = 0) \end{aligned}$$

Use the algorithm for finding a basis in the solution space of a linear system. You can find, e.g., a basis of $W^\perp$ consisting of $(-2, 1, 0, 0)$ and $(-4, 0, 2, 1)$.

10. (Optional) Find all values of a, for which the following matrix A is diagonalisable. For each such value, find an eigendecomposition of A.

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & a \\ 0 & 0 & 0 \end{pmatrix}.$$

Solution:

Use the diagonalisation algorithm given in the lectures. First compute the characteristic polynomial:

$$det(\lambda I - A) = \begin{vmatrix} \lambda - 1 & -1 & 0 \\ 0 & \lambda & -a \\ 0 & 0 & \lambda \end{vmatrix}$$

This is a triangular matrix, so the polynomial is $(\lambda - 1)\lambda^2$. Hence, (for any value of a) A has two eigenvalues: 1 and 0.

To find the eigenspace of A corresponding to $\lambda_1 = 1$, form the equation $(I - A)\mathbf{x} = \mathbf{0}$, or

$$\begin{pmatrix} 0 & -1 & 0 \\ 0 & 1 & -a \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{or} \quad \begin{aligned} x_2 &= 0 \\ x_3 &= 0 \end{aligned}$$

One basis of the solution space of this system is $(1, 0, 0)$.

To find the eigenspace of A corresponding to $\lambda_2 = 0$, form the equation $(0I - A)\mathbf{x} = \mathbf{0}$, or

$$\begin{pmatrix} -1 & -1 & 0 \\ 0 & 0 & -a \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{or} \quad \begin{aligned} x_1 + x_2 &= 0 \\ ax_3 &= 0 \end{aligned}$$

The solution space of this system depends on whether $a = 0$, so we have two cases:

Case 1, $a \neq 0$. Then the linear system can be written as

$$\begin{aligned} x_1 + x_2 &= 0 \\ x_3 &= 0 \end{aligned}$$

It has one free variable x_2, and hence the solution space has a basis consisting of a single vector, say $(-1, 1, 0)$. By putting together the bases of the eigenspaces, we get only 2 vectors (which is less than 3, the size of the matrix). Therefore, A is **not** diagonalisable.

Case 2, $a = 0$. Then the linear system can be written as a single equation $x_1 + x_2 = 0$ This time we have two free variables: x_2 and x_3, and can find (e.g.) this basis of solution space of the system: $(-1, 1, 0)$ and $(0, 0, 1)$. Hence, in this case A is diagonalisable, and one matrix P that diagonalises it is

$$P = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

One eigendecomposition of A is then

$$A = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1}.$$

11. (optional) (hard) Consider the Maclaurin series for e^x as the definition of this function. By extending this definition to allow imaginary numbers, prove Euler's formula $e^{ix} = \cos(x) + i\sin(x)$, where $x \in \mathbb{R}$. Derive the equation $e^{i\pi} + 1 = 0$ connecting the five most important constants in mathematics.
(*Hint: The Maclaurin series* $\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$ *and* $\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$ *may be useful.*)

Solution:

The Maclaurin series for e^x is

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

Replace x by ix to get

$$e^{ix} = \sum_{n=0}^{\infty} \frac{(ix)^n}{n!}.$$

Observe that, for even n, i.e. $n = 2k$, we have $(ix)^{2k} = (-1)^k x^{2k}$. For odd n, i.e. $n = 2k+1$, we have $(ix)^{2k+1} = i(-1)^k x^{2k+1}$. Hence,

$$e^{ix} = \sum_{n=0}^{\infty} \frac{(ix)^n}{n!} = \sum_{k=0}^{\infty} \Big(\frac{(-1)^k x^{2k}}{(2k)!} + i\frac{(-1)^k}{(2k+1)!}x^{2k+1}\Big) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!} + i\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!}x^{2k+1}$$

By using the above Maclaurin series for $\cos(x)$ and $\sin(x)$, we can re-write the last expression as $\cos(x) + i\sin(x)$. By setting $x = \pi$ in $e^{ix} = \cos(x) + i\sin(x)$, we get $e^{i\pi} = -1$, or $e^{i\pi} + 1 = 0$.

12. (optional) (hard) Prove the following properties of the orthogonal complement in $\mathbb{R}^n$ (with the dot product): If W is a subspace of $\mathbb{R}^n$ then

(a) $W^\perp$ is a subspace of $\mathbb{R}^n$.

Solution:
Recall that, in $\mathbb{R}^n$, $W^\perp = \{\mathbf{x} \mid \mathbf{u}\cdot\mathbf{x} = 0 \text{ for all } \mathbf{u} \in W\}$.

Let $\mathbf{x}, \mathbf{y} \in W^\perp$, i.e. $\mathbf{u}\cdot\mathbf{x} = 0$ and $\mathbf{u}\cdot\mathbf{y} = 0$ for all $\mathbf{u} \in W$. Then $\mathbf{u}\cdot(\mathbf{x}+\mathbf{y}) = \mathbf{u}\cdot\mathbf{x} + \mathbf{u}\cdot\mathbf{y} = 0$ and, for any $k \in \mathbb{R}$, $\mathbf{u}\cdot(k\mathbf{x}) = k(\mathbf{u}\cdot\mathbf{x}) = 0$.

(b) $dim(W) + dim(W^\perp) = n$.

Solution:
Let $B = \{\mathbf{v}_1, \ldots \mathbf{v}_\ell\}$ be a basis in W and let $B' = \{\mathbf{v}_{\ell+1}, \ldots \mathbf{v}_{\ell+m}\}$ be a basis in $W^\perp$. First we prove that the set $B \cup B'$ is linearly independent - indeed let

$$k_1\mathbf{v}_1 + \ldots + k_\ell\mathbf{v}_\ell + k_{\ell+1}\mathbf{v}_{\ell+1} + \ldots + k_{\ell+m}\mathbf{v}_{\ell+m} = \mathbf{0}.$$

Re-write this as

$$k_1\mathbf{v}_1 + \ldots + k_\ell\mathbf{v}_\ell = -k_{\ell+1}\mathbf{v}_{\ell+1} - \ldots - k_{\ell+m}\mathbf{v}_{\ell+m}.$$

The vector on both sides of the above equation belongs both to W (because of the LHS) and to $W^\perp$ (because of the RHS). So, it is orthogonal to itself, and hence is equal to $\mathbf{0}$. Since both B and B' are linearly independent, it follows that $k_1 = \ldots = k_{\ell+m} = 0$. Thus, $B \cup B'$ is linearly independent.
If $\ell + m = n$ then we are done, so assume, for contradiction, that $\ell + m < n$. Consider the linear system whose equations are $\mathbf{v}_i \cdot \mathbf{x} = 0$, $i = 1, \ldots, \ell + m$. This is a system with more variables (n) than equations ($\ell + m$), so it has a a non-trivial solution, – call this vector $\mathbf{x}_0$. Since $\mathbf{x}_0$ is orthogonal to all vectors in B, we conclude that $\mathbf{x}_0 \in W^\perp$. On the other hand , $\mathbf{x}_0$ is also orthogonal to all vectors in B', and hence $\mathbf{x}_0 \in (W^\perp)^\perp$. Since $\mathbf{x}_0 \in W^\perp \cap (W^\perp)^\perp$, we conclude that $\mathbf{x}_0 = \mathbf{0}$, which contradicts the choice of $\mathbf{x}_0$.

(c) $(W^\perp)^\perp = W$.

Solution:
Clearly, $W \subseteq (W^\perp)^\perp$. To prove that these subspaces are equal, we prove that their dimensions are equal (and use the fact the the only subspace of a k-dimensional space V that has dimension k is V itself). Let $dim(W) = \ell$. From item (b), we have $dim(W) + dim(W^\perp) = n$, so $dim(W^\perp) = n - \ell$. Similarly, $dim(W^\perp) + dim((W^\perp)^\perp) = n$, so $dim((W^\perp)^\perp) = n - (n - \ell) = \ell\ (= dim(W))$.

Do your proofs generalise to an arbitrary (real) finite-dimensional inner product space?

Solution:

All proofs do generalise to an arbitrary finite-dimensional inner product space. The only part whose generalisation is not immediate is the fact that the system of equations $\langle \mathbf{v}_i, \mathbf{x} \rangle = 0$, $i = 1, \ldots, \ell + m$, can always be viewed as a linear system – this is true, though – just choose a basis in V and use co-ordinates of the vectors in this basis.