Durham University

Examination Paper

Examination Session:	Year:	Exam Code:
May/June	2022	COMP1021-WE01

Title: Mathematics for Computer Science

Release Date/Time	18/05/2022 14:00
Latest Submission Date/Time	19/05/2022 14:00
Format of Exam	Online open book exam
Duration:	2 hours
Word/Page Limit:	None
Additional Material provided:	None
Expected form of Submission	A SINGLE PDF file submitted to Gradescope
Submission method	Gradescope

Instructions to Candidates:	Answer ALL questions.

Section A Linear Algebra (Prof. Andrei Krokhin)

Question 1

(a) A linear operator T on $\mathbb{R}^3$ maps $\mathbf{v}_1 = (1, -3, -4)$ to $\mathbf{v}_2 = (2, 7, -8)$ and $\mathbf{v}_3 = (-2, 5, 0)$ to $\mathbf{v}_4 = (-1, 1, -12)$.

i. Does the above information uniquely determine $T(\mathbf{v}_2)$? If so, find it. **[4 Marks]**

ii. Does the above information uniquely determine $T(\mathbf{v}_4)$? If so, find it. **[4 Marks]**

Justify your reasoning.

(b) Consider the following set S of points in the plane: $\{(1, -2), (2, 1), (4, 1)\}$. Find a value of α such that S and the set S' obtained by adding the point $(0, \alpha)$ to S have the same least squares straight line fit. Show your working. **[7 Marks]**

(c) For a polynomial $p = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n$, let $\hat{p}$ denote the polynomial obtained from p by removing the constant term, i.e. $\hat{p} = p - a_0$. Consider the vector space P_∞ of all polynomials and let $T : P_\infty \to P_\infty$ be the linear map defined as follows:

$$T(p) = \hat{p} + xp',$$

where p' is the derivative of p.

For example, if $p = 2 + 5x - x^2$ then $\hat{p} = 5x - x^2$, $p' = 5 - 2x$ and $T(p) = 5x - x^2 + x(5 - 2x) = 10x - 3x^2$.

i. Find a basis for the kernel of T. **[3 Marks]**

ii. What is the range of T? **[3 Marks]**

iii. Compute the eigenvalues of T and an eigenvector corresponding to each eigenvalue. **[4 Marks]**

Justify your answer.

Question 2

(a) Consider the space P_2 of all polynomials of degree at most two, equipped with the inner product $\langle f, g \rangle = \int_{-1}^{1} f(x)g(x)\,dx$. Find values of α, β, and γ which make the following set of polynomials orthogonal:

$$\{p_1 = 1,\ p_2 = \alpha + x,\ p_3 = \beta + \gamma x + x^2\}.$$

Show your working. **[6 Marks]**

(b) Consider the vector space P_3 of all polynomials of degree at most 3, equipped with the evaluation inner product at sample points $-2, -1, 1, 2$. Prove that all polynomials $f \in P_3$ satisfying $f(x) = f(-x)$ for all x form a subspace W of P_3 and find an orthonormal basis of W. Show your working. **[7 Marks]**

(c) Consider the following matrix

$$A = \begin{pmatrix} 1 & a \\ a & 1 \end{pmatrix}.$$

i. For each $a \neq 0$, find a spectral decomposition of A. **[6 Marks]**

ii. For each $a \neq 0$, find rank-1 approximation of A. **[6 Marks]**

Show your working.

Section B Calculus
(Dr Eleni Akrida)

Question 3

Consider $f(x, y, z, w) = x\sqrt{1+y} + y\sqrt{1+x} + z\sqrt{1+w} + w\sqrt{1+z}$, where $x, y, z, w \in \mathbb{R},\ x, y, z, w > -1$.

(a) Calculate the Hessian matrix for f. Show all your working. **[10 Marks]**

(b) Use your solution above to determine the location and nature of the stationary points of f. Show all your working. **[15 Marks]**

Question 4

(a) Let $\{a_k\}$ be a sequence of real numbers. Give either a proof or a counterexample to each of the following assertions:

i. Let $s_n = a_1 + \ldots + a_n$. If the sequence $\{s_n\}$ is bounded then the series $\sum_{k=1}^{\infty} a_k$ converges.

ii. If $a_k > 0$ for all $k \in \mathbb{N}$ and $0 < \frac{a_{k+1}}{a_k} < 1$ for all $k \in \mathbb{N}$, then the series $\sum_{k=1}^{\infty} a_k$ converges.

iii. If $a_k > 0$ for all $k \in \mathbb{N}$ and $\frac{a_{k+1}}{a_k} \to +\infty$, then the series $\sum_{k=1}^{\infty} a_k$ diverges.

iv. If $a_k \to 0$, then the series $\sum_{k=1}^{\infty} (-1)^k a_k$ converges.

v. If $a_k > 0$ for all $k \in \mathbb{N}$ and the series $\sum_{k=1}^{\infty} a_k$ converges, then the series $\sum_{k=1}^{\infty} a_k^2$ converges. **[10 Marks]**

(b) Determine the values of $a > 0$ for which the series below converges

$$\sum_{n=1}^{\infty} \frac{a^n}{\sqrt[n]{n!}}.$$

Show all your working. **[7 Marks]**

(c) Consider a continuous function $f : [0, a] \to \mathbb{R}$. Apply the Fundamental Theorem of Calculus to show that for all $x \in [0, a]$,

$$\int_0^x f(u)(x-u)du = \int_0^x \left(\int_0^u f(t)dt \right) du$$

[8 Marks]

END OF PAPER