Durham University

Examination Paper

Examination Session:	Year:	Exam Code:
May/June	2020	COMP1021-WE01

Title: **Mathematics for Computer Science**

Time Allowed:	2 hours	
Additional Material provided:		
Materials Permitted:		
Calculators Permitted:	Yes	Models Permitted: Casio FX-83 GTPLUS, Casio FX-85GTPLUS, Casio FX83-GTX or Casio FX85-GTX
Visiting Students may use dictionaries: Yes		

Instructions to Candidates:	Answer TWO out of three questions from section A and TWO out of three questions from section B. Please answer each question in a separate answer booklet.

Revision:	

Section A Logic and Discrete Structures (Prof Daniël Paulusma)

Question 1

(a) In Propositional Logic, when do we say that a formula is a tautology and when do we say that it is a contradiction? **[3 Marks]**

(b) For a variable p, give an example of a formula in Propositional Logic that contains p and that is a tautology. Also give an example of a formula in Propositional Logic that contains p and that is a contradiction. **[4 Marks]**

(c) Prove, by using a truth table, that the following rule of inference is a correct rule: **[6 Marks]**

$$\frac{a \qquad a \Rightarrow b \qquad b \Rightarrow c}{c}$$

(d) In Propositional Logic, when do we say that a formula is in conjunctive normal form (c.n.f.)? **[2 Marks]**

(e) Is the formula $\neg(\neg z \wedge (\neg x \vee z) \wedge (\neg x \vee v) \wedge (x \vee y) \wedge (\neg y \vee z))$ a theorem? Use the proof system of resolution to prove or disprove this. **[10 Marks]**

Question 2

(a) Let $S = \{\emptyset, \{\emptyset\}, \{0, \emptyset\}\}$. Determine the power set $P(S)$. **[3 Marks]**

(b) Let S be a finite set and let $P(S)$ be its power set. How many elements does the power set of $P(S)$ have? Justify your answer. **[3 Marks]**

(c) Is it possible to construct two sets A and B such that $A \in B$ and $A \cap B = \emptyset$? Justify your answer. **[4 Marks]**.

(d) Let $A = \{1, 2\}$, $B = \{1, 2\}$ and $C = \{\emptyset\}$. Does the Cartesian product $A \times B \times C$ exist? If so, determine $A \times B \times C$. **[3 Marks]**

(e) Let f be a function from a set A to a set B.

i. When do we say that f is injective? **[1 Mark]**

ii. When do we say that f is surjective? **[1 Mark]**

iii. When do we say that f is bijective? **[1 Mark]**

(f) When do we say that a set is countable? **[3 Marks]**

(g) Is the set $\mathbb{Z}$ of integers countable? Justify your answer. **[6 Marks]**

continued

Question 3

(a) Let $A = \{0, 1, 2, 3\}$. Is the relation $R = \{(0, 0), (1, 1), (2, 2), (3, 2)\}$ a function from A to A? Justify your answer. **[2 Marks]**

(b) Let R be a binary relation on a set A.

i. When do we say that R is reflexive? **[1 Mark]**

ii. When do we say that R is irreflexive? **[1 Mark]**

iii. When do we say that R is transitive? **[1 Mark]**

(c) Are the following statements true for every set A and every two binary relations R and S on A:

i. If R is reflexive and S is transitive, then $R \cup S$ is reflexive. **[3 Marks]**

ii. If R and S are irreflexive, then $R \cup S$ is irreflexive. **[3 Marks]**

iii. If R is reflexive and S is irreflexive, then $R \cup S$ is not reflexive. **[3 Marks]**

Justify your answers.

(d) Consider a school with rooms, students and teachers. Let the domain of discourse D be the set of all rooms, students and teachers.

- Let R be the unary relation interpreted as the set of rooms.
- Let S be the unary relation interpreted as the set of students.
- Let T be the unary relation interpreted as the set of teachers.
- Let A be the 3-ary relation interpreted as the set of triples (r, s, t), where r is a room, s is a student and t is a teacher, such that student s is taught by teacher t in room r.

Express the following properties using an appropriate sentence of first-order logic:

i. Every student is taught by some teacher in some room **[3 Marks]**.

ii. There is a room where every teacher teaches to every student **[3 Marks]**.

(e) Show that the formula $\forall x(A(x) \Rightarrow \exists y \neg A(y))$ is not valid by giving a structure in which the formula is not satisfied. Justify your answer. **[5 Marks]**

End of Section A **continued**

Section B Discrete Mathematics and Linear Algebra (Prof Andrei Krokhin and Dr Ioannis Ivrissimtzis)

Question 4

(a) State Pascal's identity for binomial coefficients. Give a combinatorial proof of this identity. **[5 Marks]**

(b) Define the binomial distribution and explain why it is a probability distribution. **[3 Marks]**

(c) There are 11 players in team A. It is known that four of them (call them high-scoring) score a penalty with probability 0.8 each and the other seven (low-scoring) with probability 0.5 each.

i. A player is chosen randomly from team A (without knowing his scoring ability), and the player scores a penalty. Use Bayes' Theorem to decide whether this player is more likely to be high-scoring than low-scoring. You should justify your answer. **[9 Marks]**

ii. There is a penalty shoot-out with another team, team B (of 11 players), in which every player from each team takes exactly one shot, so that 22 shots are to be taken. Every player in team B scores with probability 0.64. The team that scores the most goals wins. Which team is expected to win? Justify your answer. **[8 Marks]**

Question 5

(a) What is a spanning tree of a graph? Prove that every connected graph has a spanning tree. **[6 Marks]**

(b) In a shooting practice (where each shot uses up exactly one bullet), each time Jack hits the target, he gets three additional bullets. Jack started practice with exactly one bullet and took 43 shots when he ran out of bullets. How many times did Jack hit the target? You should justify your answer, modeling this situation as a tree. **[7 Marks]**

(c) i. State Fermat's Little Theorem. **[2 Marks]**

ii. Find a positive integer x such that $x^{122} \equiv 4 \bmod 13$. **[4 Marks]**

(d) Find the general solution for the following system of linear equations by transforming its augmented matrix to the row-echelon form

$$\begin{cases} x + y - z = 1 \\ x - y + 2z = 2 \\ 4x + 2y - z = 5. \end{cases}$$

[6 Marks]

continued

Question 6

(a) Describe the computation of the determinant of an $n \times n$ matrix by cofactor expansion. **[6 Marks]**

(b) Find the inverse of the matrix

$$A = \begin{pmatrix} 1 & 0 & -2 \\ 3 & 1 & -2 \\ 1 & 2 & 7 \end{pmatrix}.$$

[6 Marks]

(c) Let A and B be two $n \times n$ matrices of rank n. Show that AB is also a matrix of rank n. **[5 Marks]**

(d) Define eigenvectors and eigenvalues of a square matrix. Find the eigenvalues and describe all eigenvectors of the following matrix

$$A = \begin{pmatrix} 2 & 4 \\ 1 & -1 \end{pmatrix}.$$

[8 Marks]

END OF PAPER