Examination Paper

Examination Session:	Year:	Exam Code:
May/June	2024	COMP1021-WE01

Title: Mathematics for Computer Science

Time Allowed:	2 hours	
Additional Material provided:	None	
Materials Permitted:	None	
Calculators Permitted:	Yes	Models Permitted: Casio fx-83GT range and Casio fx-85GT range
Visiting Students may use dictionaries:	Yes	

Instructions to Candidates:	Answer ALL questions. Students must use the Computer Science answer booklet.

Section A Linear Algebra (Dr. Karl Southern)

Question 1

(a) Perform the following calculations on the matrix M, show all your working.

$$M = \begin{bmatrix} 2 & 0 & 2 \\ 4 & 3 & 3 \\ 8 & -6 & 0 \end{bmatrix}$$

i. Perform an LU Decomposition on the matrix M. **[6 Marks]**

ii. Use your decomposition to calculate the determinant of M. **[2 Marks]**

(b) Let

$$A = \begin{bmatrix} 4 & 0 & 0 \\ 4 & 2 & 0 \\ 0 & 2 & 1 \end{bmatrix}, B = \begin{bmatrix} 3 & -1 & 0 \\ 1 & 1 & 0 \\ 5 & 0 & 4 \end{bmatrix}, C = \begin{bmatrix} 5 & 0 & 6 \\ 2 & 4 & 0 \\ -2 & 0 & -2 \end{bmatrix}.$$

i. Diagonalise each matrix. For each of A, B and C it suffices to give matrices D and P or D and P^{-1} such that $M = PDP^{-1}$, you do not need to find both P and P^{-1}. **[11 Marks]**

ii. Which pairs among A, B and C are similar and which pairs are not? Justify your answer for each pair. **[6 Marks]**

Question 2

(a) Give the definition of an orthonormal set of vectors. **[2 Marks]**

(b) Let S be the set of vectors $\left\{ \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 9 \\ 9 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix} \right\}$.

Use the Gram-Schmidt process to construct an orthonormal basis of $span(S)$.

[6 Marks]

(c) Find the QR decomposition of S. **[6 Marks]**

(d) Given the following points, use the least squares method to find the straight line and quadratic polynomial that best approximate them.

$$P = (0, -1), (0, 0), (1, -1), (1, -2), (2, 0), (2, 2).$$

You may find the following information useful.

The QR decomposition of $\begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 1 \\ 1 & 1 \\ 1 & 2 \\ 1 & 2 \end{bmatrix}$ is: $Q = \begin{bmatrix} \sqrt{6}/6 & -1/2 \\ \sqrt{6}/6 & -1/2 \\ \sqrt{6}/6 & 0 \\ \sqrt{6}/6 & 0 \\ \sqrt{6}/6 & 1/2 \\ \sqrt{6}/6 & 1/2 \end{bmatrix}$,

$R = \begin{bmatrix} \sqrt{6} & 0 \\ 0 & 2 \end{bmatrix}$

and the QR decomposition of $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 2 & 4 \end{bmatrix}$ is: $Q = \begin{bmatrix} \sqrt{6}/6 & -1/2 & \sqrt{3}/6 \\ \sqrt{6}/6 & -1/2 & \sqrt{3}/6 \\ \sqrt{6}/6 & 0 & -\sqrt{3}/3 \\ \sqrt{6}/6 & 0 & -\sqrt{3}/3 \\ \sqrt{6}/6 & 1/2 & \sqrt{3}/6 \\ \sqrt{6}/6 & 1/2 & \sqrt{3}/6 \end{bmatrix}$,

$R = \begin{bmatrix} \sqrt{6} & \sqrt{6} & 5\sqrt{6}/3 \\ 0 & 2 & 4 \\ 0 & 0 & 2\sqrt{3}/3 \end{bmatrix}$ **[8 Marks]**

(e) Which of the two polynomials is the best approximation? Justify your answer. **[3 Marks]**

Section B Calculus (Dr Eleni Akrida)

Question 3

(a) Identify the maxima and minima of the function $f(x, y) = x^2 + y^2$ that lie on the curve $g(x, y) = x^2 - 2y = 0,\ x, y \in \mathbb{R}$.
Show all your working. **[12 Marks]**

(b) Determine the Fourier series for the function $f(x) = x,\ \ x \in [-\pi, \pi]$, as follows:

i. Give the generic form of the Fourier series and discrete integrals which evaluate to the Fourier coefficients for f. **[2 Marks]**

ii. Evaluate the integrals and give the Fourier coefficients for f. Show all your working. **[9 Marks]**

iii. Examine whether the Fourier series for f converges for all $x \in [-\pi, \pi]$. Show all your working. **[2 Marks]**

Question 4

(a) Prove or disprove each of the following statements. Show all your working.

i. If $a_k > 0$ for all $k \in \mathbb{N}$ and $\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = 1$, then the series $\sum_{k=1}^{\infty} a_k$ diverges. **[5 Marks]**

ii. If $a_k > 0$ for all $k \in \mathbb{N}$ and the series $\sum_{k=1}^{\infty} a_k$ converges, then the series $\sum_{k=1}^{\infty} \sqrt{a_k}$ converges. **[5 Marks]**

iii. If the series $\sum_{k=1}^{\infty} a_k$ converges, then the series $\sum_{k=1}^{\infty} a_k^2$ converges. **[5 Marks]**

(b) Apply the ratio or root test on the following series to examine their convergence for all values of $x \in \mathbb{R}$. Show all your working.

i. $\sum_{k=1}^{\infty} k^k x^k$. **[5 Marks]**

ii. $\sum_{k=1}^{\infty} \frac{2^k x^k}{k^2}$. **[5 Marks]**

END OF PAPER