Algorithms & Data Structures 2024/25

Practical Week 8

I realise that model answers tend to be easily available. Please however try to come up with your own solutions. If you do get stuck then ask the demonstrators for help.

0. Finish anything that may be left over from last week.
1. Rank the following functions by order of growth; that is, find an arrangement $f_{i_1}, f_{i_2}, \ldots, f_{i_{25}}$ of the functions given below satisfying

$$f_{i_1} = \mathcal{O}(f_{i_2}),\ f_{i_2} = \mathcal{O}(f_{i_3}),\ \ldots,\ f_{i_{24}} = \mathcal{O}(f_{i_{25}})$$

for some permutation $(i_j)_{j=1,\ldots,25}$ of $\{1,\ldots,25\}$.

$f_1(n) = (3/2)^n$	$f_2(n) = (\sqrt{2})^{\log n}$	$f_3(n) = \log n$	$f_4(n) = n^2$
$f_5(n) = n^n$	$f_6(n) = \ln n$	$f_7(n) = \log^2 n$	$f_8(n) = n^{\log_3 3}$
$f_9(n) = 2^{2^n}$	$f_{10}(n) = n^{1/\log n}$	$f_{11}(n) = \log\log n$	$f_{12}(n) = n \cdot 2^n$
$f_{13}(n) = n^{\log\log n}$	$f_{14}(n) = \log(n^2)$	$f_{15}(n) = 2^n$	$f_{16}(n) = 2^{\log n}$
$f_{17}(n) = (\log n)^{\log n}$	$f_{18}(n) = 4^{\log n}$	$f_{19}(n) = \log_{(\log_5 25)} n$	$f_{20}(n) = \sqrt{\log n}$
$f_{21}(n) = \log(n!)$	$f_{22}(n) = 2^{\sqrt{2\log n}}$	$f_{23}(n) = n$	$f_{24}(n) = n\log n$
$f_{25}(n) = 1$			

Note: all logarithms are to base 2 unless explicitly specified otherwise. We always understand $\log^p n$ to mean $(\log(n))^p$. That's different from $\log(n^p)$.

Solution:

Much of the ranking is based on the facts that:

- exponential functions grow faster than polynomial functions, which grow faster than (poly)logarithmic functions, which grow faster than constant functions
- the base of a logarithm doesn't matter asymptotically (as long as it's constant), but the base of an exponential function and the degree of a polynomial do matter

In addition, several identities are helpful:

(a) $(\log n)^{\log n} = n^{\log\log n}$
(b) $2^{\log n} = n$
(c) $4^{\log n} = n^2$
(d) $n^{1/\log n} = 2$
(e) $2^{\sqrt{2\log n}} = n^{\sqrt{2/\log n}}$
(f) $(\sqrt{2})^{\log n} = \sqrt{n}$
(g) $n^{\log_3 3} = n^1 = n$
(h) $\log_{\log_5 25} n = \log_2 n$
(i) $\log(n^2) = 2\log(n)$
(j) $\log(n!) = \log(1 \cdot 2 \cdots (n/2) \cdot (n/2+1) \cdots n) > \log((n/2+1) \cdots n) > \log((n/2)^{n/2}) = (n/2) \cdot \log(n/2) = (n/2) \cdot (\log(n) - 1)$ and $\log(n!) < \log(n^n) = n\log n$

Here is the ordering in reverse order:

$$\begin{aligned}
2^{2^n} \\
n^n \\
n \cdot 2^n \\
2^n \\
(3/2)^n \\
(\log n)^{\log n}, \quad n^{\log \log n} \\
n^2, \quad 4^{\log n} \\
n \log n, \ \log(n!) \\
n, \ 2^{\log n}, \quad n^{\log_3 3} \\
(\sqrt{2})^{\log n} \\
2^{\sqrt{2 \log n}} \\
\log^2 n \\
\ln n, \quad \log n, \quad \log(n^2), \quad \log_{\log_5 25} n \\
\sqrt{\log n} \\
\log \log n \\
n^{1/\log n}, \quad 1
\end{aligned}$$

2. Sometimes we want to express lower bounds instead of upper bounds, as in "solving this problem for inputs of size n asymptotically takes at least n^2 steps", by which we, slightly more formally, actually mean "for any n large enough, and for any algorithm (past, present or future), there exists at least one input of size n that makes it take time n^2, asymptotically, to solve the problem".

   (a) Argue why saying "takes at least $\mathcal{O}(n^2)$ steps" is non-sensical.

   Solution:

   $\mathcal{O}$ is about upper bounds. There are lots of function in e.g. $\mathcal{O}(n^2)$ that are significantly slower-growing than n^2. What is "at least $O(n^2)$" supposed to mean? At least constant? At least logarithmic? At least linear, or maybe quadratic? All these types of functions are in $\mathcal{O}(n^2)$. Clearly the statement is supposed to express that it's at least quadratic (asymptotically), but that's what Ω is for, or Θ when asymptotic tightness is proved, but not $\mathcal{O}$.

   (b) Promise to *never* say "takes at least $\mathcal{O}(n^2)$ steps". Ever.

3. Download the PDF file https://tinyurl.com/lc6ao5x (this is a clickable link). Please note that this file has been created by nice colleagues from down under, at the University of Auckland in New Zealand. A local copy is available in Ultra, just in case the link goes dead.

   The file contains a set of exercises on asymptotic classes. Please work on those in Section 1.1. My general comment from above is even more valid here as the file not only contains the questions, but also the answers. Please note that the units in the answer to the third question in Section 1.1 are wrongly stated as milliseconds instead of seconds.

   If you don't manage everything this week then don't worry, you can continue next week.