Mathematics for Computer Science
Linear Algebra (Part 2)
Complex Matrices

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Thanks to Andrei Krokhin and William Moses for use of some slides.

Outline

Reminder: Last Two Lectures

Let A be an $n \times n$ matrix.

- A non-zero vector $\mathbf{x} \in \mathbb{R}^n$ is called an eigenvector of A if $A\mathbf{x} = \lambda\mathbf{x}$.
- In this case, λ is called an eigenvalue of A, and $\mathbf{x}$ is an eigenvector corresponding to λ.

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- The polynomial $det(\lambda I - A)$ is called the characteristic polynomial of A and the equation $det(\lambda I - A) = 0$ the characteristic equation of A.

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- The polynomial $det(\lambda I - A)$ is called the characteristic polynomial of A and the equation $det(\lambda I - A) = 0$ the characteristic equation of A.
- The eigenvalues of A are the solutions of $det(\lambda I - A) = 0$.
  In particular, A is singular (non-invertible) iff 0 is an eigenvalue of A

Question from last week: algebraic and geometric multiplicity

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ -1 & 0 & 2 \end{pmatrix}$$

Computing the characteristic equation gives $(\lambda - 2)^2(\lambda - 1) = 0 \rightarrow \lambda_1 = 2, \lambda_2 = 1$.

Eigenvectors corresponding to $\lambda_1 = 2$ are: $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

Eigenvectors corresponding to $\lambda_2 = 1$ are: $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

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The algebraic multiplicity of $\lambda_1 = 2$ is 2 and the geometric multiplicity of $\lambda_1 = 2$ is 2.

The algebraic multiplicity of $\lambda_2 = 1$ is 1 and the geometric multiplicity of $\lambda_2 = 1$ is 1.

Question from last week: algebraic and geometric multiplicity

$$B = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 3 & 2 \end{pmatrix}$$

Computing the characteristic equation gives $(\lambda - 2)^2(\lambda - 5) = 0 \rightarrow \lambda_1 = 2, \lambda_2 = 5$.

Eigenvectors corresponding to $\lambda_1 = 2$ are: $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

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Content for Today

- Complex numbers
- Complex vector spaces
- Eigenvalues of symmetric real matrices

Outline

Complex Numbers: Motivation

Why Should You Care?

1. Discrete Fourier Transform (DFT)
2. Quantum mechanics
3. Finding all eigenvalues of some matrices

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Assume that we want to analyse the “eigen”-properties of the following matrix

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- This equation has no real roots, so we know that A has no real eigenvalues, but this is all we can say at the moment. Can we do more?

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- This equation has no real roots, so we know that A has no real eigenvalues, but this is all we can say at the moment. Can we do more?
- It would (probably) be useful to work with some number set that extends $\mathbb{R}$ and where every polynomial can be factorised into linear polynomials, i.e.

$$\lambda^n + c_1\lambda^{n-1} + \ldots + c_{n-1}\lambda + c_n = (\lambda - \lambda_1)(\lambda - \lambda_2)\cdots(\lambda - \lambda_n),$$

where the λ_i's are not necessarily distinct.

Complex Numbers: Reminder

A complex number is a number of the form $z = a + bi$ where $a, b \in \mathbb{R}$ and i is the imaginary unit: the number such that $i^2 = -1$.

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- $Re(z) = a$ is the real part of z and $Im(z) = b$ is the imaginary part of z
- $|z| = \sqrt{a^2 + b^2}$ is the modulus (or absolute value) of z (note that $|z| \in \mathbb{R}$)
- The number $\overline{z} = a - bi$ is the complex conjugate of z (and $z\overline{z} = |z|^2$)

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The set of all complex numbers is denoted by $\mathbb{C}$.

The arithmetic operations on $\mathbb{C}$ work as follows:

- $(a + bi) + (c + di) = (a + c) + (b + d)i$
- $(a + bi)(c + di) = ac + adi + bci + bdi^2 = (ac - bd) + (ad + bc)i$

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It is easy to check that $\overline{\overline{z}} = z$, $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$ and $\overline{z_1 z_2} = \overline{z_1}\,\overline{z_2}$.

Complex Numbers Geometrically: Reminder

- Each complex number $z = a + bi$ can be viewed as a vector $(a, b) \in \mathbb{R}^2$
- Addition and multiplication by a real number are the same in $\mathbb{C}$ and in $\mathbb{R}^2$

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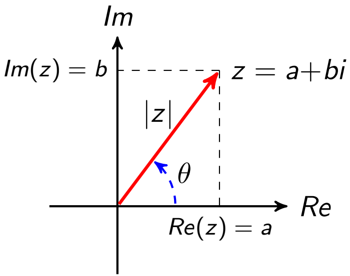

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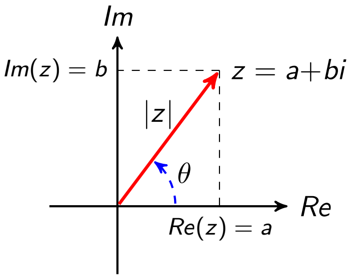

- The angle $\theta = \arctan(b/a)$ in the diagram is called the argument of z.
- The expression $z = |z|(\cos\theta + i\sin\theta)$ is the polar form of z.

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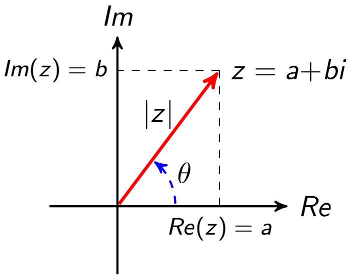

- The angle $\theta = \arctan(b/a)$ in the diagram is called the argument of z.
- The expression $z = |z|(\cos\theta + i\sin\theta)$ is the polar form of z.
  - Example: $\sqrt{2} - \sqrt{2}i = 2(\cos(-\frac{\pi}{4}) + i\sin(-\frac{\pi}{4}))$.

The Fundamental Theorem of Algebra

Theorem

Each polynomial of degree $n \geq 1$ with complex coefficients has n complex roots (counting with multiplicities). That is, each such polynomial can be factored into linear polynomials,

$$\lambda^n + c_1\lambda^{n-1} + \ldots + c_{n-1}\lambda + c_n = (\lambda - \lambda_1)(\lambda - \lambda_2)\cdots(\lambda - \lambda_n),$$

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For example,

$$\lambda^2 + 1 = (\lambda - i)(\lambda + i) \text{ and } \lambda^4 + 2\lambda^2 + 1 = (\lambda^2 + 1)^2 = (\lambda - i)^2(\lambda + i)^2.$$

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All quadratic polynomials can now be factorised by using the standard formula for solving quadratic equations and the fact that, for $D < 0$, we have $\sqrt{D} = i\sqrt{|D|}$.

Outline

The Vector Space $\mathbb{C}^n$ and Complex Matrices

- Similarly to $\mathbb{R}^n$, the vector space $\mathbb{C}^n$ is defined to consist of all n-tuples $(v_1, \ldots, v_n)$, where each $v_i \in \mathbb{C}$.
- Each vector $\mathbf{v} = (v_1, \ldots, v_n) \in \mathbb{C}^n$, where $v_i = a_i + b_i i$, can be represented as

$$\mathbf{v} = (v_1, \ldots, v_n) = (a_1 + b_1 i, \ldots, a_n + b_n i) = (a_1, \ldots, a_n) + i(b_1, \ldots, b_n) = \mathbf{a} + i\mathbf{b},$$

  where $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$. Then $\mathbf{a} = Re(\mathbf{v})$ and $\mathbf{b} = Im(\mathbf{v})$.
- Can extend the complex conjugate to $\mathbb{C}^n$: If $\mathbf{v} = \mathbf{a} + i\mathbf{b}$ then $\bar{\mathbf{v}} = \mathbf{a} - i\mathbf{b}$.

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- Example: if $\mathbf{v} = (3 + i, -2i, 5)$ then

$$Re(\mathbf{v}) = (3, 0, 5), \quad Im(\mathbf{v}) = (1, -2, 0), \quad \bar{\mathbf{v}} = (3 - i, 2i, 5).$$

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One can also consider complex matrices, i.e. matrices with complex entries.

All the above notions extend to complex matrices in a natural way.

We will call a matrix a real matrix to emphasize that all its entries are real.

Algebraic Properties of the Complex Conjugate

The facts that $\overline{\overline{z}} = z$, $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$ and $\overline{z_1 z_2} = \overline{z_1}\,\overline{z_2}$ immediately imply

Theorem

For any vectors $\mathbf{u}, \mathbf{v} \in \mathbb{C}^n$ and a scalar $k \in \mathbb{C}$, the following holds:

- $\overline{\overline{\mathbf{u}}} = \mathbf{u}$
- $\overline{k\mathbf{u}} = \overline{k}\,\overline{\mathbf{u}}$
- $\overline{\mathbf{u} + \mathbf{v}} = \overline{\mathbf{u}} + \overline{\mathbf{v}}$
- $\overline{\mathbf{u} - \mathbf{v}} = \overline{\mathbf{u}} - \overline{\mathbf{v}}$

Algebraic Properties of the Complex Conjugate (Continued)

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Theorem

If A is an $m \times k$ complex matrix and B is a $k \times n$ complex matrix, then

- $\overline{\overline{A}} = A$
- $\overline{(A^T)} = (\overline{A})^T$
- $\overline{AB} = \overline{A}\,\overline{B}$

Complex Dot Product

The complex dot product in $\mathbb{C}^n$ is defined as follows: if $\mathbf{u} = (u_1, \ldots, u_n)$ and $\mathbf{v} = (v_1, \ldots, v_n) \in \mathbb{C}^n$ then

$$\mathbf{u} \cdot \mathbf{v} = u_1\overline{v_1} + \ldots + u_n\overline{v_n}.$$

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Example: let $\mathbf{u} = (1 + i, i, 3 - i)$ and $\mathbf{v} = (1 + i, 2, 4i)$. Then

$$\begin{aligned}
\mathbf{u} \cdot \mathbf{v} &= (1 + i)(1 - i) + (i)(2) + (3 - i)(-4i) = -2 - 10i \\
\mathbf{v} \cdot \mathbf{u} &= (1 + i)(1 - i) + 2(-i) + 4i(3 + i) = -2 + 10i \\
||\mathbf{u}|| &= \sqrt{|1 + i|^2 + |i|^2 + |3 - i|^2} = \sqrt{2 + 1 + 10} = \sqrt{13}
\end{aligned}$$

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Properties of Complex Dot Product

For vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$, viewed as columns (i.e. $n \times 1$ matrices), we have

$$\mathbf{u} \cdot \mathbf{v} = \mathbf{u}^T\mathbf{v} = \mathbf{v}^T\mathbf{u} \quad \text{and} \quad ||\mathbf{u}||^2 = \mathbf{u} \cdot \mathbf{u} = \mathbf{u}^T\mathbf{u} = \mathbf{u}^T\mathbf{u}.$$

(The first product is the dot product and the other two are matrix products.)

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For complex vectors $\mathbf{u}, \mathbf{v} \in \mathbb{C}^n$, this becomes

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Theorem

For any vectors $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{C}^n$ and a scalar $k \in \mathbb{C}$, the following holds:

- $\mathbf{u} \cdot \mathbf{v} = \overline{\mathbf{v} \cdot \mathbf{u}}$
- $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$
- $(k\mathbf{u}) \cdot \mathbf{v} = k(\mathbf{u} \cdot \mathbf{v})$ *and* $\mathbf{u} \cdot (k\mathbf{v}) = \overline{k}(\mathbf{u} \cdot \mathbf{v})$
- $\mathbf{v} \cdot \mathbf{v} \geq 0$, *and* $\mathbf{v} \cdot \mathbf{v} = 0$ *iff* $\mathbf{v} = \mathbf{0}$

Complex Eigenvalues and Eigenvectors

Let A be an $n \times n$ matrix with complex entries.

As in the real case, $\lambda \in \mathbb{C}$ is an eigenvalue of A if $A\mathbf{x} = \lambda\mathbf{x}$ for a non-zero $\mathbf{x} \in \mathbb{C}^n$. Then $\mathbf{x}$ is a complex eigenvector corresponding to λ.

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As in the real case,

- the eigenvalues of A are the complex roots of $det(\lambda I - A) = 0$.
- the eigenspace of A corrresponding to an eigenvalue λ_0 is the solution space of the linear system $(\lambda_0 I - A)\mathbf{x} = \mathbf{0}$ (considered over $\mathbb{C}$).

Example 13.1

Example: find eigenvalues of the matrix $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

Example 13.2

Example: find eigenvalues of the matrix $A = \begin{pmatrix} 3 & 0 & 0 \\ 2 & 1 & -1 \\ 0 & 2 & 3 \end{pmatrix}$.

Complex Eigenvalues and Eigenvectors

Theorem (13.1)

If λ is an eigenvalue of a real $n \times n$ matrix A and $\mathbf{x}$ is a corresponding eigenvector, then $\overline{\lambda}$ is also an eigenvalue of A and $\overline{\mathbf{x}}$ is a corresponding eigenvector.

Complex Eigenvalues and Eigenvectors

Theorem (13.1)

If λ is an eigenvalue of a real $n \times n$ matrix A and $\mathbf{x}$ is a corresponding eigenvector, then $\overline{\lambda}$ is also an eigenvalue of A and $\overline{\mathbf{x}}$ is a corresponding eigenvector.

Proof.

Since A is real, i.e. $\overline{A} = A$, we have $A\overline{\mathbf{x}} = \overline{A}\overline{\mathbf{x}} = \overline{A\mathbf{x}} = \overline{\lambda\mathbf{x}} = \overline{\lambda}\,\overline{\mathbf{x}}$. (And $\overline{\mathbf{x}} \neq \mathbf{0}$.) $\square$

Outline

Eigenvalues of Real Symmetric Matrices

Theorem (13.2)

If A is a real symmetric matrix then all (complex) eigenvalues of A are real.

Eigenvalues of Real Symmetric Matrices

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Proof.

- Let $\lambda \in \mathbb{C}$ be an eigenvalue of A and $\mathbf{x} \in \mathbb{C}^n$ a corresponding eigenvector.
- Take the complex conjugate of both sides of the equation $A\mathbf{x} = \lambda\mathbf{x}$.
- We get $\overline{A}\,\overline{\mathbf{x}} = \overline{\lambda}\,\overline{\mathbf{x}}$, and, since $A = \overline{A}$ (A is real), it follows that $A\,\overline{\mathbf{x}} = \overline{\lambda}\,\overline{\mathbf{x}}$.
- Then, using $A = A^T$, we compute the number $\overline{\mathbf{x}}^T A\mathbf{x}$ in two different ways:

$$\overline{\mathbf{x}}^T A\mathbf{x} = \overline{\mathbf{x}}^T(A\mathbf{x}) = \overline{\mathbf{x}}^T(\lambda\mathbf{x}) = \lambda(\overline{\mathbf{x}}^T\mathbf{x}) = \lambda(\mathbf{x}\cdot\mathbf{x}) = \lambda||\mathbf{x}||^2,$$
$$\overline{\mathbf{x}}^T A\mathbf{x} = (A\overline{\mathbf{x}})^T\mathbf{x} = (\overline{\lambda}\,\overline{\mathbf{x}})^T\mathbf{x} = \overline{\lambda}(\overline{\mathbf{x}}^T\mathbf{x}) = \overline{\lambda}(\mathbf{x}\cdot\mathbf{x}) = \overline{\lambda}||\mathbf{x}||^2.$$

- Since $\mathbf{x} \neq \mathbf{0}$, have $||\mathbf{x}|| \neq 0$. So $\lambda||\mathbf{x}||^2 = \overline{\lambda}||\mathbf{x}||^2$ implies $\lambda = \overline{\lambda}$, i.e. $\lambda \in \mathbb{R}$.

Real 2×2 Matrices with Complex Eigenvalues

Theorem

The complex eigenvalues of the real matrix $C = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ *are* $\lambda = a \pm bi$.

If a, b *are not both zero, then* C *can be factored as*

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix} = \begin{pmatrix} |\lambda| & 0 \\ 0 & |\lambda| \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

where θ *is the argument of* $\lambda = a + bi$.

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where θ *is the argument of* $\lambda = a + bi$.

Geometrically, the operator T_C is equal to rotation by θ followed by scaling by $|\lambda|$.

Real 2×2 Matrices with Complex Eigenvalues (Continued)

Theorem

The complex eigenvalues of the real matrix $C = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ *are* $\lambda = a \pm bi$.

If a, b *are not both zero, then* C *can be factored as*

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix} = \begin{pmatrix} |\lambda| & 0 \\ 0 & |\lambda| \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

where θ *is the argument of* $\lambda = a + bi$.

Theorem

Let A *be a real* 2×2 *matrix with complex eigenvalues* $\lambda = a \pm bi$, *where* $b \neq 0$. *Then* A *is similar to* $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$.

Outline

What we learnt today

- Complex numbers
- Complex vector spaces
- All (complex) eigenvalues of real symmetric matrices are real
- Real 2×2 matrices with complex eigenvalues

Example Exam question

Let A be a matrix, with characteristic polynomial $\lambda^3 + a\lambda^2 + b\lambda + 1$. A has i as an eigenvalue, and has an eigenvalue that is an integer. For each of the following questions, either provide an answer or justify that there is not enough information to answer the questions.

1. What is the degree of the matrix?
2. What are the eigenvalues of the matrix?
3. Is the matrix invertible?

Next time

- Diagonalisation of matrices.
- Similar matrices

Example 13.1

Example: find eigenvalues of the matrix $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

$\det(\lambda I - A) = \lambda^2 + 1 = 0$

$\lambda_1 = i, \lambda_2 = -i$ are the eigenvalues.

$(iI - A)\mathbf{x} = \begin{pmatrix} i & 1 \\ -1 & i \end{pmatrix} \mathbf{x} = 0$

Giving an eigenvector of $\begin{pmatrix} i \\ -1 \end{pmatrix}$

With $\begin{pmatrix} -i \\ 1 \end{pmatrix}$ as the corresponding eigenvector for $\lambda_2 = -i$.

Example 13.2

Example: find eigenvalues of the matrix $A = \begin{pmatrix} 3 & 0 & 0 \\ 2 & 1 & -1 \\ 0 & 2 & 3 \end{pmatrix}$.

Characteristic equation is $(\lambda - 3)(\lambda^2 - 4\lambda + 5)$

Giving $\lambda_1 = 3, \lambda_2 = 2 + i, \lambda_3 = 2 - i$

For $\lambda_2 = 2 + i$, $\begin{pmatrix} -1+i & 0 & 0 \\ 2 & 1+i & 1 \\ 0 & -2 & -1+i \end{pmatrix} \mathbf{x} = 0$

Giving an eigenvector of $\begin{pmatrix} 0 \\ -1+i \\ 2 \end{pmatrix}$.

The End