Mathematics for Computer Science
Linear Algebra (Part 2)
Eigen Values & Eigen Vectors

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Durham University

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Thanks to Andrei Krokhin and Billy Moses for use of some slides.

Outline

Name of slide

Last Week

1. LU decomposition - what is it/how to use it

This Week

1. eigenvalues
2. eigenvectors
3. Characteristic equation/polynomial
4. Principal Component Analysis

Outline

Intuition

1. eigen - proper, characteristic, own
2. Consider some linear mapping A
3. When applied to a vector $\mathbf{v}$, it might cause rotation and scaling
4. All those vectors $\mathbf{v}$ that are only scaled: eigenvectors
5. How much a eigenvector $\mathbf{v}$ is scaled by: corresponding eigenvalue

Eigenvalues and eigenvectors

Definition

Let A be an $n \times n$ matrix. A non-zero vector $\mathbf{x} \in \mathbb{R}^n$ is called an eigenvector of A (or, equivalently, of the operator $T_A : \mathbb{R}^n \to \mathbb{R}^n$) if, for some scalar λ,

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- The assumption $\mathbf{x} \neq \mathbf{0}$ is necessary to avoid the case $A\mathbf{0} = \lambda\mathbf{0}$ which always holds.
- The meaning of the notion is that T_A does not change the direction of $\mathbf{x}$ (up to reversal), it only scales $\mathbf{x}$ by λ.

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Example: vector $\mathbf{x} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ is an eigenvector of $A = \begin{pmatrix} 3 & 0 \\ 8 & -1 \end{pmatrix}$ corr. to eigenvalue 3.

Indeed,

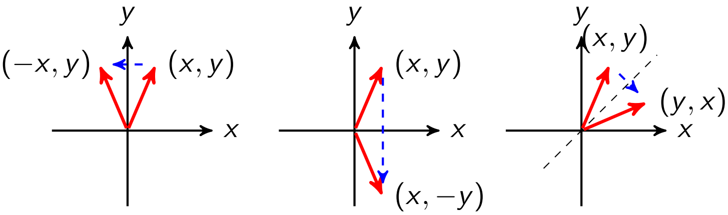
$$A\mathbf{x} = \begin{pmatrix} 3 & 0 \\ 8 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \end{pmatrix} = 3\mathbf{x}.$$

Example in $\mathbb{R}^2$

Consider linear operators T_A on $\mathbb{R}^2$ where A is one of the following matrices:

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

They correspond to reflections of $\mathbb{R}^2$ about y-axis, x-axis, and line $x = y$, resp.



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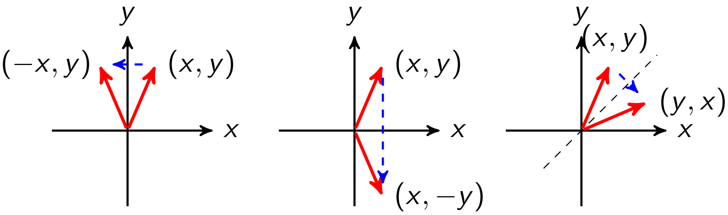

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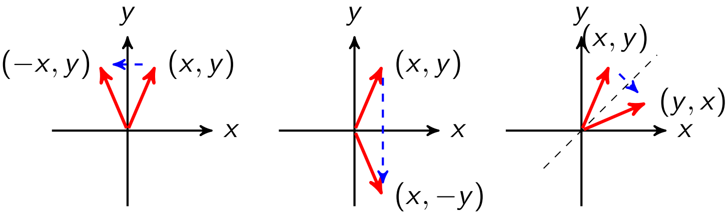

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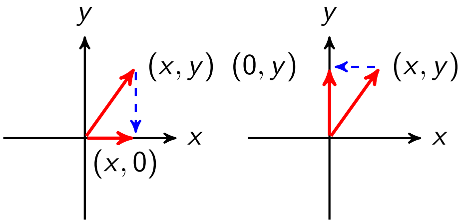

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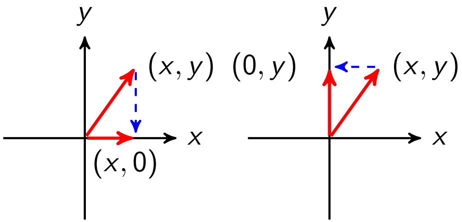

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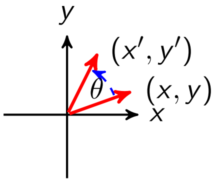
Consider the linear operator T_A on $\mathbb{R}^2$ where A is the following matrix:

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

The corresponding linear map T_A satisfies

$$T_A(x, y) = (x', y') = (x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta).$$

This corresponds to the rotation of $\mathbb{R}^2$ by angle θ counterclock-wise.



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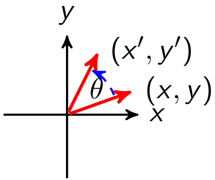

This linear map has no eigenvectors for any $0 < \theta < 180°$.

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Consider linear operators T_A on $\mathbb{R}^2$ where A is the following matrix:

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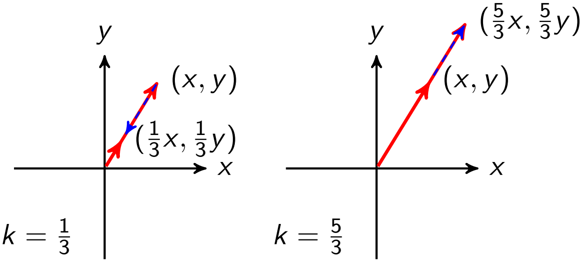

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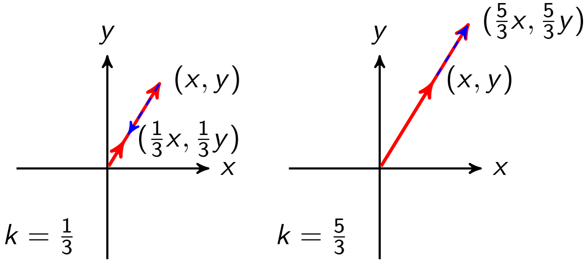

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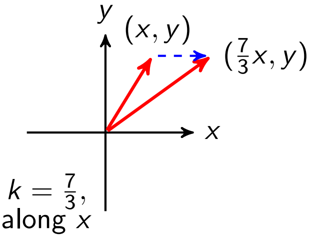

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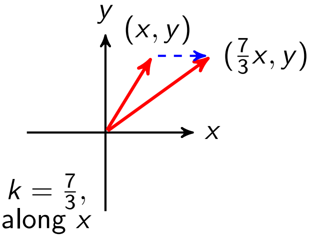

The eigenvectors are all non-zero vectors $(x, 0)$ and $(0, y)$, corresponding to eigenvalues k and 1, respectively.

Example in $\mathbb{R}^2$

Consider the transformation T_A on $\mathbb{R}^2$ where A is the following matrix:

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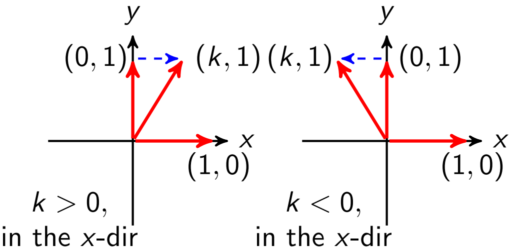
The transformation T_A satisfies $T_A(x, y) = (x + ky, y)$. For $k \neq 0$, it corresponds to shear of $\mathbb{R}^2$ in the x-direction with factor k.



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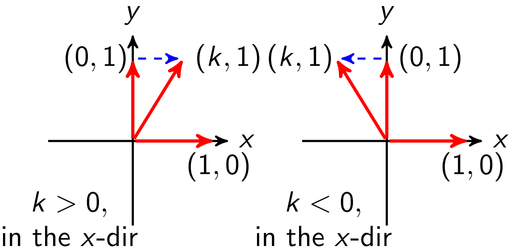

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Outline

Characteristic equation of a matrix

Theorem

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By theorem about invertible matrices, the last equation has a solution $\mathbf{x} \neq \mathbf{0}$ iff $det(\lambda I - A) = 0$. $\square$

Example 12.1

Example: find eigenvalues of the matrix $A = \begin{pmatrix} 2 & -1 \\ 10 & -9 \end{pmatrix}$.

Characteristic polynomial of a matrix

- In general, the expression $det(\lambda I - A)$ is a polynomial

$$p(\lambda) = \lambda^n + c_1\lambda^{n-1} + \ldots + c_{n-1}\lambda + c_n.$$

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- If all coefficients of $p(\lambda)$ are integers and the equation $p(\lambda) = 0$ has an integer solution $\lambda = k$ then $k | c_n$. This can be used to find some eigenvalues.

Example 12.2

Example: find eigenvalues of $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & -17 & 8 \end{pmatrix}$.

Eigenspaces and their bases

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Solution. Form the equation $(-8I - A)\mathbf{x} = \mathbf{0}$, or

$$\begin{pmatrix} -10 & 1 \\ -10 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{or} \quad \begin{array}{rcl} -10x_1 + x_2 & = & 0 \\ -10x_1 + x_2 & = & 0 \end{array}$$

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Exercise: Find the eigenspace of A corresponding to eigenvalue $\lambda = 1$.

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- The algebraic multiplicity of λ_0 is the power k with which $(\lambda - \lambda_0)$ appears as a factor of $det(\lambda I - A)$ - the characteristic polynomial of A.
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Theorem

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Exercise: Find an example of A and its eigenvalue where the inequality in the theorem is strict.

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1. PCA - Why study it?
    1. Useful applications - data compression
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2. What we care about:
   1. General idea of PCA
   2. Understanding how eigenvalues and eigenvectors are used
3. What we don't care about:
   1. Completely understanding PCA
   2. Learning concepts used in PCA that are beyond the scope of this module

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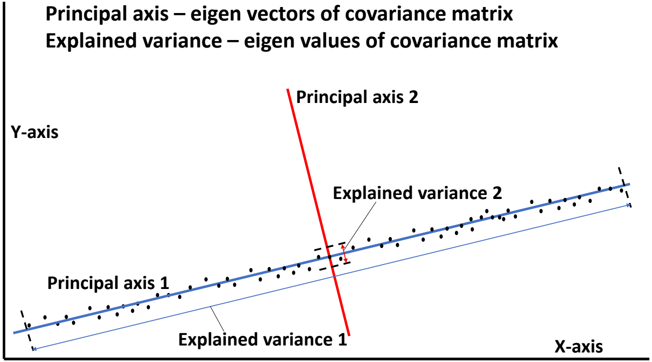
6. What does it do?
   - **Input:** data set in n dimensions (possibly m data points, $m \neq n$)
   - **Output:** $n \times n$ matrix, element e_{ij} shows how dimension i data varies with dimension j data.
   - Captures the **variance** of one dimension of the data with another.

Visual representation of principal components of data set (eigenvectors of its covariance matrix)

- Data set: m points of 2 dimensions (x & y); Covariance matrix COV_{2x2}
- Eigenvectors of COV_{2x2} (aka principal axes) - axes which best capture variance of data
- Eigenvalues of COV_{2x2} (aka explained variance) - amount of variance along eigenvectors



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7. Now we can represent original data (n dimensions) using less data (k dimensions) while possibly losing some information

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5. Put data into new form: $A_{mxn}P_{nxk} = B_{mxk}$
6. Can retrieve old data (possibly with some loss):
   $A'_{mxn} = B_{mxk}P^T_{kxn}$, then "un-preprocess A'"

Outline

What we learnt today

- Eigenvalues and eigenvectors of matrices
- Examples in $\mathbb{R}^2$
- Characteristic equation of a matrix – how to find eigenvalues
- Eigenspaces and how to find their bases
- Principal Component Analysis (PCA)

Next Week:

- Complex vector spaces.
- If you're unfamiliar with complex numbers, read up on them on this Wikipedia page and/or watch this video lecture by 3Blue1Brown in advance of the lecture.

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Divide $\lambda^3 - 8\lambda^2 + 17\lambda - 4$ by $\lambda - 4$ to get

$$\lambda^3 - 8\lambda^2 + 17\lambda - 4 = (\lambda - 4)(\lambda^2 - 4\lambda + 1).$$

Solving the equation $\lambda^2 - 4\lambda + 1 = 0$, we get that the eigenvalues of A are

$$\lambda_1 = 4, \lambda_2 = 2 + \sqrt{3}, \text{ and } \lambda_3 = 2 - \sqrt{3}.$$

The End