Mathematics for Computer Science
Linear Algebra (Part 2)
Inner Product Spaces

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Thanks to Andrei Krokhin and William Moses for use of some slides.

Outline

Recap of Last Week

- Matrices A and B are similar if $A = PBP^{-1}$ for some invertible P.

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- Matrices A and B are similar if $A = PBP^{-1}$ for some invertible P.
- A is diagonalisable if it is similar to B and B is a diagonal matrix.
- $A = PDP^{-1}$ is an eigendecomposition of A if P is the eigenvectors of A and D is the diagonal matrix of eigenvalues.

Roadmap for Next Few Classes

- **End Goal:** Application - linear regression.

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- **End Goal:** Application - linear regression.
- **Using:** QR decomposition.
- Requires knowledge of some basics: Inner product spaces.

Contents for Today's Class

- Inner product spaces – definition, norm, orthogonality.
- Important examples.

Outline

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Inner Product: The Definition

Recall: The dot product (aka Euclidean inner product) of vectors $\mathbf{u} = (u_1, \ldots, u_n)$ and $\mathbf{v} = (v_1, \ldots, v_n)$ in $\mathbb{R}^n$ is defined as $\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + \ldots + u_n v_n$.

Using it, one can define norm (aka length), distance, angles, orthogonality in $\mathbb{R}^n$.

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Definition

Let V be a (real) vector space. An inner product on V is a function that

$$\text{associates to each pair } \mathbf{u}, \mathbf{v} \in V \text{ a real number } \langle \mathbf{u}, \mathbf{v} \rangle \in \mathbb{R},$$

satisfying the following properties for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$ and $k \in \mathbb{R}$.

1. $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle$ [Symmetry axiom]
2. $\langle \mathbf{u} + \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle$ [Additivity axiom]
3. $\langle k\mathbf{u}, \mathbf{v} \rangle = k\langle \mathbf{u}, \mathbf{v} \rangle$ [Homogeneity axiom]
4. $\langle \mathbf{v}, \mathbf{v} \rangle \geq 0$, and $\langle \mathbf{v}, \mathbf{v} \rangle = 0$ iff $\mathbf{v} = \mathbf{0}$ [Positivity axiom]

Outline

Norm and Distance

Generalising from $\mathbb{R}^n$ to an arbitrary inner product space (i.e. a vector space equipped with an inner product), we can define norm and distance as

$$||\mathbf{v}|| = \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle} \text{ and } d(\mathbf{u}, \mathbf{v}) = ||\mathbf{u} - \mathbf{v}||.$$

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The following properties of norm and distance follow directly from definitions:

- $||\mathbf{v}|| \geq 0$, and $||\mathbf{v}|| = 0$ iff $\mathbf{v} = \mathbf{0}$
- $||k\mathbf{v}|| = |k| \, ||\mathbf{v}||$
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A vector $\mathbf{v}$ with $||\mathbf{v}|| = 1$ is called a unit vector. Each non-zero vector can be normalised (scaled to become a unit vector): $\mathbf{v} \mapsto \frac{1}{||\mathbf{v}||}\mathbf{v}$.

Outline

Orthogonality and Orthogonal Complement

Definition

Vectors $\mathbf{u}$ and $\mathbf{v}$ in an inner product space V are called orthogonal if $\langle \mathbf{u}, \mathbf{v} \rangle = 0$.

In $\mathbb{R}^n$ with the dot product, this is the same notion as before.

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Definition

Let W be a subspace in an inner product space V. Then the set

$$W^{\perp} = \{\mathbf{x} \in V \mid \langle \mathbf{u}, \mathbf{x} \rangle = 0 \text{ for all } \mathbf{u} \in W\}$$

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Example: Take $\mathbf{u} = (2, -3, 5, 4)$ and $\mathbf{v} = (0, 1, -4, 7)$ in $\mathbb{R}^4$ (with the dot product) and let $W = span(\mathbf{u}, \mathbf{v})$. Then $W^{\perp}$ is the solution space of the linear system

$$\begin{array}{rcll} 2x_1 - 3x_2 + 5x_3 + 4x_4 & = & 0 & \qquad (\langle \mathbf{u}, \mathbf{x} \rangle = 0) \\ x_2 - 4x_3 + 7x_4 & = & 0 & \qquad (\langle \mathbf{v}, \mathbf{x} \rangle = 0) \end{array}$$

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For any subspace W in an inner product space V, the set $W^{\perp}$ is also a subspace.

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Finding a basis for $W^\perp$ = finding a basis in the solution space of linear system.

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Example: Weighted Euclidean Inner Product

- Let $w_1, \ldots, w_n \in \mathbb{R}$ be arbitrary *positive* numbers, which we'll call *weights*.
- The weighted Euclidean inner product (with weights $w_1, \ldots, w_n$) on $\mathbb{R}^n$ is defined as follows: for vectors $\mathbf{u} = (u_1, \ldots, u_n)$ and $\mathbf{v} = (v_1, \ldots, v_n)$,

$$\langle \mathbf{u}, \mathbf{v} \rangle = w_1 u_1 v_1 + w_2 u_2 v_2 + \ldots + w_n u_n v_n.$$

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Example: Consider $\mathbb{R}^2$ equipped with the weighted Euclidean inner product with weights $w_1 = 3, w_2 = 2$, i.e., define $\langle \mathbf{u}, \mathbf{v} \rangle = 3u_1 v_1 + 2u_2 v_2$.

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- The norm of $\mathbf{e}_1 = (1, 0)$ is $||\mathbf{e}_1|| = \sqrt{\langle \mathbf{e}_1, \mathbf{e}_1 \rangle} = \sqrt{3 \cdot 1^2 + 2 \cdot 0^2} = \sqrt{3}$.

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Norms, distances and orthogonality depend on the choice of inner product!

Example: Weighted Euclidean Inner Product - Orthogonal Complement

- **Dot Product:** Take $\mathbf{u} = (2, -3, 5, 4)$ and $\mathbf{v} = (0, 1, -4, 7)$ in $\mathbb{R}^4$ and let $W = span(\mathbf{u}, \mathbf{v})$. If our inner product on $\mathbb{R}^4$ is the dot product, $W^\perp$ is the solution space of

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$$\begin{aligned} 4x_1 - 3x_2 + 15x_3 + 4x_4 &= 0 \qquad (\langle \mathbf{u}, \mathbf{x} \rangle = 0) \\ x_2 - 12x_3 + 7x_4 &= 0 \qquad (\langle \mathbf{v}, \mathbf{x} \rangle = 0) \end{aligned}$$

Outline

Example: Matrix Inner Product on $\mathbb{R}^n$

Let A be an *invertible* $n \times n$ matrix.

Considering vectors in $\mathbb{R}^n$ as column vectors, define

$$\langle \mathbf{u}, \mathbf{v} \rangle = A\mathbf{u} \cdot A\mathbf{v}, \text{ (or, equivalently, } \langle \mathbf{u}, \mathbf{v} \rangle = (A\mathbf{v})^T A\mathbf{u} = \mathbf{v}^T A^T A\mathbf{u})$$

where the right-hand side uses the standard dot product in $\mathbb{R}^n$.

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  For the earlier example of a weighted inner product on $\mathbb{R}^2$,

$$\langle \mathbf{u}, \mathbf{v} \rangle = 3u_1v_1 + 2u_2v_2 = \begin{pmatrix} \sqrt{3} & 0 \\ 0 & \sqrt{2} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \cdot \begin{pmatrix} \sqrt{3} & 0 \\ 0 & \sqrt{2} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

Outline

Example: Standard Inner Product on $\mathbb{P}_n$

Recall: $\mathbb{P}_n$ is the space of all polynomials of degree at most n.

For vectors $\mathbf{p} = a_0 + a_1 x + \ldots + a_n x^n$ and $\mathbf{q} = b_0 + b_1 x + \ldots + b_n x^n$ in $\mathbb{P}_n$, define

$$\langle \mathbf{p}, \mathbf{q} \rangle = a_0 b_0 + a_1 b_1 + \ldots + a_n b_n.$$

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- It is easy to see that each vector $\mathbf{p} = a_0 + a_1 x + \ldots + a_n x^n \in \mathbb{P}_n$ can be identified with the corresponding vector $(a_0, a_1, \ldots, a_n) \in \mathbb{R}^{n+1}$.
- Then the standard inner product on $\mathbb{P}_n$ = the dot product on $\mathbb{R}^{n+1}$.

Outline

Example: Evaluation Inner Product on $\mathbb{P}_n$

Fix *distinct* points $x_0, x_1, \ldots, x_n \in \mathbb{R}$ (called *sample points*).

For vectors $\mathbf{p} = p(x)$ and $\mathbf{q} = q(x)$ in $\mathbb{P}_n$, define

$$\langle \mathbf{p}, \mathbf{q} \rangle = p(x_0)q(x_0) + p(x_1)q(x_1) + \ldots + p(x_n)q(x_n).$$

This inner product on $\mathbb{P}_n$ is called the evaluation inner product at $x_0, x_1, \ldots, x_n$.

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- It is easy to see that one can identify each vector $\mathbf{p} = p(x) \in \mathbb{P}_n$ with the corresponding vector $(p(x_0), p(x_1), \ldots, p(x_n)) \in \mathbb{R}^{n+1}$.
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- One subtlety: need to check that $\langle \mathbf{p}, \mathbf{p} \rangle = 0$ iff $\mathbf{p} = 0$. This holds because
  $\langle \mathbf{p}, \mathbf{p} \rangle = [p(x_0)]^2 + [p(x_1)]^2 + \ldots + [p(x_n)]^2 = 0$
  $\implies p(x_0) = p(x_1) = \ldots = p(x_n) = 0 \qquad \implies \qquad \mathbf{p} = 0$

Example: Evaluation Inner Product on $\mathbb{P}_n$

Fix <u>*distinct*</u> points $x_0, x_1, \dots, x_n \in \mathbb{R}$ (called *sample points*).

For vectors $\mathbf{p} = p(x)$ and $\mathbf{q} = q(x)$ in $\mathbb{P}_n$, define

$$\langle \mathbf{p}, \mathbf{q} \rangle = p(x_0)q(x_0) + p(x_1)q(x_1) + \dots + p(x_n)q(x_n).$$

This inner product on $\mathbb{P}_n$ is called the evaluation inner product at $x_0, x_1, \dots, x_n$.

- It is easy to see that one can identify each vector $\mathbf{p} = p(x) \in \mathbb{P}_n$ with the corresponding vector $(p(x_0), p(x_1), \dots, p(x_n)) \in \mathbb{R}^{n+1}$.
- Then the evaluation inner product on $\mathbb{P}_n$ = the dot product in $\mathbb{R}^{n+1}$.
- One subtlety: need to check that $\langle \mathbf{p}, \mathbf{p} \rangle = 0$ iff $\mathbf{p} = 0$. This holds because
  $\langle \mathbf{p}, \mathbf{p} \rangle = [p(x_0)]^2 + [p(x_1)]^2 + \dots + [p(x_n)]^2 = 0$
  $\implies p(x_0) = p(x_1) = \dots = p(x_n) = 0 \qquad \implies \qquad \mathbf{p} = 0$
  The last implication follows from the fundamental theorem of algebra:
  a non-0 polynomial of degree $\leq n$ can have at most n distinct roots.

Working with Evaluation Inner Product on $\mathbb{P}_n$

Consider $\mathbb{P}_2$ with evaluation inner product at $x_0 = -2, x_1 = 0, x_2 = 2$, i.e.

$$\langle \mathbf{p}, \mathbf{q} \rangle = p(x_0)q(x_0) + p(x_1)q(x_1) + p(x_2)q(x_2) = p(-2)q(-2) + p(0)q(0) + p(2)q(2)$$

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$$\sqrt{[p(-2)]^2 + [p(0)]^2 + [p(2)]^2} = \sqrt{4^2 + 0^2 + 4^2} = 4\sqrt{2}.$$

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If we normalise $\mathbf{p}$, we get vector $\mathbf{p}' = \frac{1}{||\mathbf{p}||}\mathbf{p} = \frac{1}{4\sqrt{2}}x^2 \in \mathbb{P}_2$.

Outline

Example: Inner Product on the Space $C[a, b]$

- **Recall:** $C[a, b]$ consists of all functions that are continuous on interval $[a, b]$.
- The operations in $C[a, b]$ are defined point-wise: if $\mathbf{f} = f(x)$ and $\mathbf{g} = g(x)$ then $(\mathbf{f} + \mathbf{g})(x) = f(x) + g(x)$ and $(k\mathbf{f})(x) = kf(x)$.

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- **Recall:** any function continuous on $[a, b]$ is integrable on $[a, b]$.
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- Each polynomial is a continuous function: so $\mathbb{P}_n$ is a subspace of $C[a, b]$, and this inner product works on $\mathbb{P}_n$ too.

Working with Inner Product on $C[a, b]$

Consider $\mathbb{P}_2$ or $C[-1, 1]$ with inner product

$$\langle \mathbf{f}, \mathbf{g} \rangle = \int_{-1}^{1} f(x)g(x)\,dx.$$

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Consider two vectors $\mathbf{p} = x$ and $\mathbf{q} = x^2$. Then

$$||\mathbf{p}|| = \sqrt{\langle \mathbf{p}, \mathbf{p} \rangle} = \sqrt{\int_{-1}^{1} xx\,dx} = \sqrt{\int_{-1}^{1} x^2\,dx} = \sqrt{\frac{2}{3}}$$

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$$\langle \mathbf{p}, \mathbf{q} \rangle = \int_{-1}^{1} xx^2\, dx = \int_{-1}^{1} x^3\, dx = 0$$

In particular, $\mathbf{p} = x$ and $\mathbf{q} = x^2$ are orthogonal w.r.t. this inner product.

Outline

Hermitian conjugation

For a complex vector $\mathbf{v}$, the Hermitian conjugation of $\mathbf{v}$,denoted $\mathbf{v}^\dagger$ is the conjugate transpose of the vector, i.e. $\mathbf{v}^\dagger = \overline{(\mathbf{v}^\top)}$.

A complex square matrix A is a Hermitian matrix if it is equal to its own conjugate transpose. i.e.

$$A = A^\dagger = \overline{A^\top}$$

or

$$a_{i,j} = \overline{a_{j,i}}$$

The matrix $A = \begin{pmatrix} 1 & 2-i \\ 2+i & 3 \end{pmatrix}$ is a Hermitian matrix.

Inner Product on $\mathbb{C}^n$

Considering vectors in $\mathbb{C}^n$ as column vectors, the Hermitian Inner product is defined as:

$$\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u}^\dagger \mathbf{v} \ .$$

Examples:

- If the vectors $\mathbf{u}$ and $\mathbf{v}$ are real, then this is the dot product.
- For $\mathbf{u} = (1+i, 5, 3+2i), \mathbf{v} = (2, 7, 3+4i)$

$$\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u}^\dagger \mathbf{v} = 2(1-i) + 5(7) + (3-2i)(3+4i) = 54 + 4i$$

Outline

Standard (In)Equalities

The standard (in)equalities for the dot product work for general inner products (and the proofs are the same):

Theorem (Pythogoras' theorem)

If $\mathbf{u}$ *and* $\mathbf{v}$ *are orthogonal vectors in an inner product space then* $||\mathbf{u}+\mathbf{v}||^2 = ||\mathbf{u}||^2 + ||\mathbf{v}||^2$.

Theorem (Cauchy-Schwarz inequality)

If $\mathbf{u}$ *and* $\mathbf{v}$ *are vectors in an inner product space then* $|\langle \mathbf{u}, \mathbf{v} \rangle| \leq ||\mathbf{u}|| \, ||\mathbf{v}||$.

Corollary (Triangle inequality)

If $\mathbf{u}$ *and* $\mathbf{v}$ *are vectors in an inner product space then* $||\mathbf{u}+\mathbf{v}|| \leq ||\mathbf{u}|| + ||\mathbf{v}||$.

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Example: Cauchy-Schwarz inequality in $C[a, b]$:

$$| \int_a^b f(x)g(x)\, dx| \leq \sqrt{\int_a^b f^2(x)\, dx} \sqrt{\int_a^b g^2(x)\, dx}.$$

Outline

Example Exam question

Consider the weighted Euclidean inner product with weights $(a, b, a+b)$. Let $v_1 = (1, 2, 3)$ and $v_2 = (3, 4, -2)$. Find values for a and b such that v_1 and v_2 are orthogonal, and $||v_1|| = \sqrt{59}$.

Wrapping Things Up

Today:

- Inner product spaces
- Norm and orthogonality in these spaces
- Important examples

Next time:

- The Gram-Schmidt orthogonalisation process
- QR decomposition of matrices

The End