MCS Calculus
Practical Exercises 7
(Week 17)

Epiphany Term 2025

Before starting on this week's work, you may find it beneficial to complete the questions on series convergence from Calculus Practical 6 (week 15). If any questions below are on material we have not yet covered in lectures, leave them for next time. If you wish, try typesetting your answers with LaTeX.

1. Determine the location and nature of the stationary points of the function

$$f(w,x,y,z) = \frac{w^2}{2} + 2x^2 + 3xy - 11x + 2y^2 - 10y - \frac{z^3}{6} + z$$

Answer: [Hint: Determine ∇f, look for zeros, determine H_f study eigenvalues.]

$$\nabla f = \begin{pmatrix} w \\ 4x+3y-11 \\ 4y+3x-10 \\ -\frac{z^2}{2}+1 \end{pmatrix}$$

So stationary points ($\nabla f = \vec{0}$) at

$$\begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 1 \\ \sqrt{2} \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 1 \\ -\sqrt{2} \end{pmatrix}$$

Now look at the second derivatives

$$H_f = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 4 & 3 & 0 \\ 0 & 3 & 4 & 0 \\ 0 & 0 & 0 & -z \end{pmatrix}$$

This has eigenvalues given by $det(H_f - \lambda I) = 0$, i.e.

$$(1-\lambda)(4-\lambda)^2(-z-\lambda) - (1-\lambda).3.3.(-z-\lambda) = 0$$

$$(1-\lambda)(-z-\lambda)[(4-\lambda)^2-9]=0$$

Whence the eigenvalues are $1, 1, 7$ and $-z$. These are all positive at $(0, 2, 1, -\sqrt{2})$, giving a minimum, and have differing signs at $(0, 2, 1, +\sqrt{2})$, giving a saddle point.

2. For the following power series determine the radius of convergence.

   (a) $\sum_{n=1}^{\infty} \frac{n!}{n^n} x^n$

Answer: $r = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty} \left|\frac{n!(n+1)^{n+1}}{n^n(n+1)!}\right| = \lim_{n\to\infty} \left|\frac{(n+1)^n}{n^n}\right| \to e.$

Hence the radius of convergence is e.

   (b) $\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!} x^n$

Answer: $r = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty} \left|\frac{(n!)^2(2(n+1))!}{(n+1)!^2(2n)!}\right| = \lim_{n\to\infty} \left|\frac{(2n+2)(2n+1)}{(n+1)^2}\right| \to 4.$

Hence the radius of convergence is 4.

   (c) $\sum_{n=1}^{\infty} \frac{5^n}{n!} x^n$

Answer: $r = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty} \left|\frac{5^n(n+1)!}{5^{n+1}n!}\right| = \lim_{n\to\infty} \left|\frac{(n+1)}{5}\right| \to \infty.$

Hence series converges for all x.

   (d) $\sum_{n=1}^{\infty} \frac{(x+9)^n}{(n+1)^2}$

Answer: $r = \lim_{n\to\infty} \left|\frac{(n+2)^2}{(n+1)^2}\right| = 1$. (I.e. convergence for $x \in (-10, -8)$.)

Hence the radius of convergence is 1.

   (e) $\sum_{n=1}^{\infty} \frac{(x+7)^{2n+1}}{n\cdot 9^n}$

Answer: This one is messed up by the power of x going up by 2 with n, Easiest to apply the series ratio test directly: $\lim_{n\to\infty} \left|\frac{(x+7)^{2n+3} n 9^n}{(n+1)9^{n+1}(x+7)^{2n+1}}\right| = \lim_{n\to\infty} \left|\frac{(x+7)^2 n}{(n+1)9}\right| = \left|\frac{(x+7)^2}{9}\right|$. This is less than 1 for $x \in (-10, -4)$.

Hence the radius of convergence is 3.

3. Determine the MacLaurin series for $f(x) = (1+e^x)^3$.

Answer: $f(0) = 8$

$f'(x) = 3(1+e^x)^2 e^x = 3e^x + 2.3e^{2x} + 3e^{3x}, \quad f'(0) = 12$

$f''(x) = 3e^x + 2^2.3e^{2x} + 3.3e^{3x}, \quad f''(0) = 24$

$f'''(x) = 3e^x + 2^3.3e^{2x} + 3^3e^{3x}, \quad f'''(0) = 3 + 2^3.3 + 3^3$

$f^{(n)}(x) = 3e^x + 2^n.3e^{2x} + 3^n e^{3x}, \quad f^{(n)}(0) = 3 + 2^n.3 + 3^n$

So $f(x) = 8 + \sum_{n=1}^{\infty} \frac{(3+2^n.3+3^n)}{n!} x^n$.

4. Determine the MacLaurin series for $f(x) = \cos(4x)$.

Answer: $f(0) = 1$

$f'(x) = -4^1 \sin(4x), \quad f'(0) = 0$

$f''(x) = -4^2 \cos(4x), \quad f''(0) = -4^2$

$f'''(x) = 4^3 \sin(4x), \quad f'''(0) = 0$

$f^{(4)}(x) = 4^4 \cos(4x), \quad f^{(4)}(0) = 4^4$

$f^{(5)}(x) = -4^5 \sin(4x), \quad f^{(5)}(0) = 0$
$f^{(6)}(x) = -4^6 \cos(4x), \quad f^{(6)}(0) = -4^6$
We notice that $f^{(2n+1)}(0) = 0$ and $f^{(2n)}(0) = (-1)^n 4^{2n}$, for all $n \in \mathbb{N}$. So $f(x) = 1+0+\frac{(-1)^1 4^2}{2!}x^2+0+\frac{(-1)^2 4^4}{4!}x^4+0+\frac{(-1)^3 4^6}{6!}x^6+\ldots = \sum_{n=1}^{\infty} \frac{(-1)^n 4^{2n}}{(2n)!}x^{2n}$.

5. Determine the Taylor series for $f(x) = \frac{7}{x^4}$ about $x_0 = -3$.

Answer: $f(-3) = \frac{7}{3^4}$
$f'(x) = -7 \cdot 4x^{-5}, \quad f'(-3) = \frac{7 \cdot 4}{3^5}$
$f''(x) = 7 \cdot 4 \cdot 5x^{-6}, \quad f''(-3) = \frac{7 \cdot 4 \cdot 5}{3^6}$
$f'''(x) = -7 \cdot 4 \cdot 5 \cdot 6x^{-7}, \quad f'''(-3) = \frac{7 \cdot 4 \cdot 5 \cdot 6}{3^7}$
$f^{(4)}(x) = 7 \cdot 4 \cdot 5 \cdot 6 \cdot 7x^{-8}, \quad f^{(4)}(-3) = \frac{7 \cdot 4 \cdot 5 \cdot 6 \cdot 7}{3^8}$
We notice that $f^{(n)}(-3) = \frac{7(n+3)!}{2 \cdot 3^{n+5}}$, for all $n \in \mathbb{N}$. So the Taylor series is:

$$f(x) = \sum_{n=1}^{\infty} \frac{7(n+3)!}{2 \cdot 3^{n+5} n!}(x+3)^n = \sum_{n=1}^{\infty} \frac{7(n+1)(n+2)(n+3)}{2 \cdot 3^{n+5}}(x+3)^n.$$

6. Determine the Taylor series for $f(x) = 5x^2 + 2x + 1$ about $x_0 = 1$.

Answer: $f(1) = 5 + 2 + 1 = 8$
$f'(x) = 10x + 2, \quad f'(1) = 12$
$f''(x) = 10, \quad f''(1) = 10$
$f'''(x) = 0, \quad f'''(1) = 0$
and for all $n > 2$, we notice that $f^{(n)}(1) = 0$. So the Taylor series is simply:

$$f(x) = 8 + \frac{12}{1!}(x-1) + \frac{10}{2!}(x-1)^2 = 8 + 12(x-1) + 5(x-1)^2.$$

Notice that in this case, we started already with a polynomial function $f(x)$. Indeed if you expand the Taylor series we got for f, you will see that it is exactly the polynomial $5x^2 + 2x + 1$.

7. Determine the Taylor series for $f(x) = 5x^2 + 2x + 1$ about $x_0 = 5$.

Answer: $f(5) = 125 + 10 + 1 = 136$
$f'(x) = 10x + 2, \quad f'(5) = 52$
$f''(x) = 10, \quad f''(5) = 10$
$f'''(x) = 0, \quad f'''(5) = 0$
and for all $n > 2$, we notice that $f^{(n)}(5) = 0$. So the Taylor series is simply:

$$f(x) = 136 + \frac{52}{1!}(x-5) + \frac{10}{2!}(x-5)^2 = 136 + 52(x-5) + 5(x-5)^2.$$

Again, we started with a polynomial function $f(x)$, and notice that indeed if you expand the Taylor series we got for f, you will get exactly the polynomial $5x^2 + 2x + 1$.

8. Determine the Taylor series for $f(x) = e^{-3x}$ about $x_0 = -2$.

Answer: $f(-2) = e^6$
$f'(x) = -3e^{-3x}, \quad f'(-2) = -3e^6$
$f''(x) = 3^2 e^{-3x}, \quad f''(-2) = 3^2 e^6$
$f'''(x) = -3^3 e^{-3x}, \quad f'''(-2) = -3^3 e^6$
and for all $n \in \mathbb{N}$, we notice that $f^{(n)}(-2) = (-3)^n e^6$. So the Taylor series is:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(-2)}{n!}(x+2)^n = \sum_{n=0}^{\infty} \frac{(-3)^n e^6}{n!}(x+2)^n.$$

9. Let f and g be n-times differentiable functions such that:

   - $f(a) = g(a) = 0$,
   - the derivatives $f^{(r)}(a) = g^{(r)}(a) = 0$ for $1 \leq r \leq n-1$,
   - $f^{(n)}(a) \neq 0$ and $g^{(n)}(a) \neq 0$.

   Use Taylor's Theorem to directly prove the extended L'Hôpital rule:

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f^{(n)}(x)}{\lim_{x \to a} g^{(n)}(x)}.$$

Answer: Consider $\frac{f(a+h)}{g(a+h)}$ for some small value h. By Taylors theorem there exists $\xi_1, \xi_2 \in (a, a+h)$ such that

$$f(a+h) = f(a) + f'(a)h + \ldots + f^{(n)}(\xi_1)\tfrac{h^n}{n!}$$

and

$$g(a+h) = g(a) + g'(a)h + \ldots + g^{(n)}(\xi_2)\tfrac{h^n}{n!}.$$

Therefore

$$\frac{f(a+h)}{g(a+h)} = \frac{f(a) + f'(a)h + \ldots + f^{(n)}(\xi_1)\frac{h^n}{n!}}{g(a) + g'(a)h + \ldots + g^{(n)}(\xi_2)\frac{h^n}{n!}} = \frac{f^{(n)}(\xi_1)}{g^{(n)}(\xi_2)}$$

Now as $h \to 0, \xi_1, \xi_2 \to a$, so

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{h \to 0} \frac{f(a+h)}{g(a+h)} = \frac{\lim_{x \to a} f^{(n)}(x)}{\lim_{x \to a} g^{(n)}(x)}.$$

10. Determine the value of

$$\lim_{x \to 2} \frac{\sin^2 \pi x}{2e^{x/2} - xe}.$$

Answer: Numerator and denominator tend to 0 as $x \to 2$, and still do after differentiating once, so differentiate again and apply extended L'Hôpital.

$$\lim_{x \to 2} \frac{\sin^2 \pi x}{2e^{x/2} - xe} = \lim_{x \to 2} \frac{2\pi \sin \pi x \cos \pi x}{e^{x/2} - e} = \lim_{x \to 2} \frac{2\pi^2(\cos^2 \pi x - \sin^2 \pi x)}{e^{x/2}/2} = \frac{2\pi^2}{e/2} = 4\pi^2 e^{-1}.$$

11. Let f be an n-times differentiable function such that for some $k < n$:

    - the derivatives $f^{(r)}(a) = 0$ for $1 \leq r \leq k-1$,
    - $f^{(k)}(a) \neq 0$.

    Use Taylor's Theorem to directly prove necessary and sufficient conditions on k and $f^{(k)}(a)$ to classify $f(a)$ as a local minimum, maximum or point of inflection.

Answer: By Taylor's Theorem there exists $\xi \in (a, a+h)$ such that

$$f(a+h) = f(a) + f'(a)h + \ldots + f^{(k)}(\xi)\tfrac{h^k}{k!} = f(a) + f^{(k)}(\xi)\tfrac{h^k}{k!}.$$

So

$$f(a+h) - f(a) = f^{(k)}(\xi)\tfrac{h^k}{k!}.$$

By continuity of $f^{(k)}$, for small enough h the sign of $f^{(k)}(\xi)$ and $f^{(k)}(a)$ are the same.

If k is even, then h^k is positive, so $f(a+h) - f(a)$ has the sign of $f^{(k)}(a)$, if k is odd then h^k is positive or negative with h, and so $f(a+h) - f(a)$ is positive/negative with h or $-h$. Thus:

- If k is even and $f^{(k)}(a) < 0$ then f has a local maximum at a
- If k is even and $f^{(k)}(a) > 0$ then f has a local minimum at a
- If k is odd then f has an inflection a.