COMP1021 Mathematics for Computer Science
Linear Algebra (Part 2)
Practical - Week 18
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Instructions: Work on these problems in the practical sessions for the week specified. First try them on your own. If you're stuck, try discussing things with others. If you get the answer, still discuss with others to see if maybe you missed something. If you run into major roadblocks, ask the demonstrators for hints.

Solutions will be posted on Learn Ultra at the end of the week. Make sure you're all set with the solutions and understand them before the next practical.

Purpose of this practical: This practical will help you build your competency with the Gram-Schmidt process, QR decomposition, and the least squares method. Use the slides from Lecture 5 as a reference for different inner product definitions.

1. Use the Gram-Schmidt process to find a QR decomposition for each of the following matrices:

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 0 \\ 1 & 2 \\ 1 & 3 \\ 1 & 3 \end{pmatrix}.$$

(*Hint: For matrix A, check the example on slide 12 in lecture 6.*)

2. Consider $\mathbb{R}^3$ with the weighted Euclidean inner product with weights $w_1 = 1, w_2 = 2, w_3 = 3$. Use the Gram-Schimdt process to transform $\mathbf{u}_1 = (1,1,1), \mathbf{u}_2 = (1,1,0), \mathbf{u}_3 = (1,0,0)$ to an orthonormal basis.

3. Consider the inner product space $C[0,1]$. Find an orthogonal basis of $W = span(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3)$ where $\mathbf{u}_1 = 1, \mathbf{u}_2 = x, \mathbf{u}_3 = x^2$.

4. Find all least squares solutions to the linear system $A\mathbf{x} = \mathbf{b}$ where

$$A = \begin{pmatrix} 1 & -2 \\ 2 & -4 \\ 3 & -6 \end{pmatrix} \text{ and } \mathbf{b} = \begin{pmatrix} 4 \\ 2 \\ -1 \end{pmatrix}$$

5. Find the least squares solutions to $A\mathbf{x} = b$ where

$$A = \begin{pmatrix} 1 & 0 \\ 1 & 2 \\ 1 & 3 \\ 1 & 3 \end{pmatrix} \text{ and } b = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 2 \end{pmatrix}.$$

6. (Optional) (Hard) Define the least squares approximation problem in the inner product space $C[a,b]$ as follows: given a vector $\mathbf{f} \in C[a,b]$ and a finite-dimensional subspace W in $C[a,b]$, find a vector $\mathbf{g} \in W$ that minimises $||\mathbf{f} - \mathbf{g}||$. The best approximation theorem from lecture 17 applies in this case, so $\mathbf{g} = \text{proj}_W \mathbf{f}$ is the least squares approximation to $\mathbf{f}$ from W.

   A trigonometric polynomial of order $\leq n$ is a function of the form

$$c_0 + c_1 \cos x + c_2 \cos(2x) + \ldots + c_n \cos(nx) + d_1 \sin x + d_2 \sin(2x) + \ldots + d_n \sin(nx).$$

   Such polynomials form a subspace in $C[0, 2\pi]$, let's call it W_n.

   (a) Check that the following functions form an orthonormal basis in W_n:

$$\mathbf{g}_0 = \frac{1}{\sqrt{2\pi}}, \mathbf{g}_1 = \frac{1}{\sqrt{\pi}} \cos x, \ldots, \mathbf{g}_n = \frac{1}{\sqrt{\pi}} \cos(nx), \mathbf{g}_{n+1} = \frac{1}{\sqrt{\pi}} \sin x, \ldots, \mathbf{g}_{2n} = \frac{1}{\sqrt{\pi}} \sin(nx).$$

   (b) Find the least squares approximation to the function $f(x) = x$ in W_n (try $n = 2$ first).