Maths for Computer Science
Calculus

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Integration II

Contents for today's lecture

- Integration
  - How to deal with tricky cases (improper integrals)
- Systematic Integration
  - Methodologies for integrating in practice

Recall: The Fundamental Theorem of Calculus

integrand

Let f be a continuous function on $[a, b]$, and let

$$F(x) = \int_a^x f(t)\,dt$$

then F is continuous and differentiable on (a, b) and

$$F'(x) = f(x) \text{ for all } x \in (a, b).$$

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Recall: Integrals

A function F such that $F'(x) = f(x)$ is called an antiderivative of f.

An **indefinite integral** is one with no specific bounds:

$$\int f(t)\,dt = F(x) + C, \qquad or \qquad \int_a^x f(t)\,dt = F(x) + C,$$

where F is an antiderivative of f, to denote an indefinite integral.

A **definite integral** is one with specific bounds, and therefore a value:

$$\int_a^b f(x)\,dx = [F(x)]_a^b = F(b) - F(a)$$

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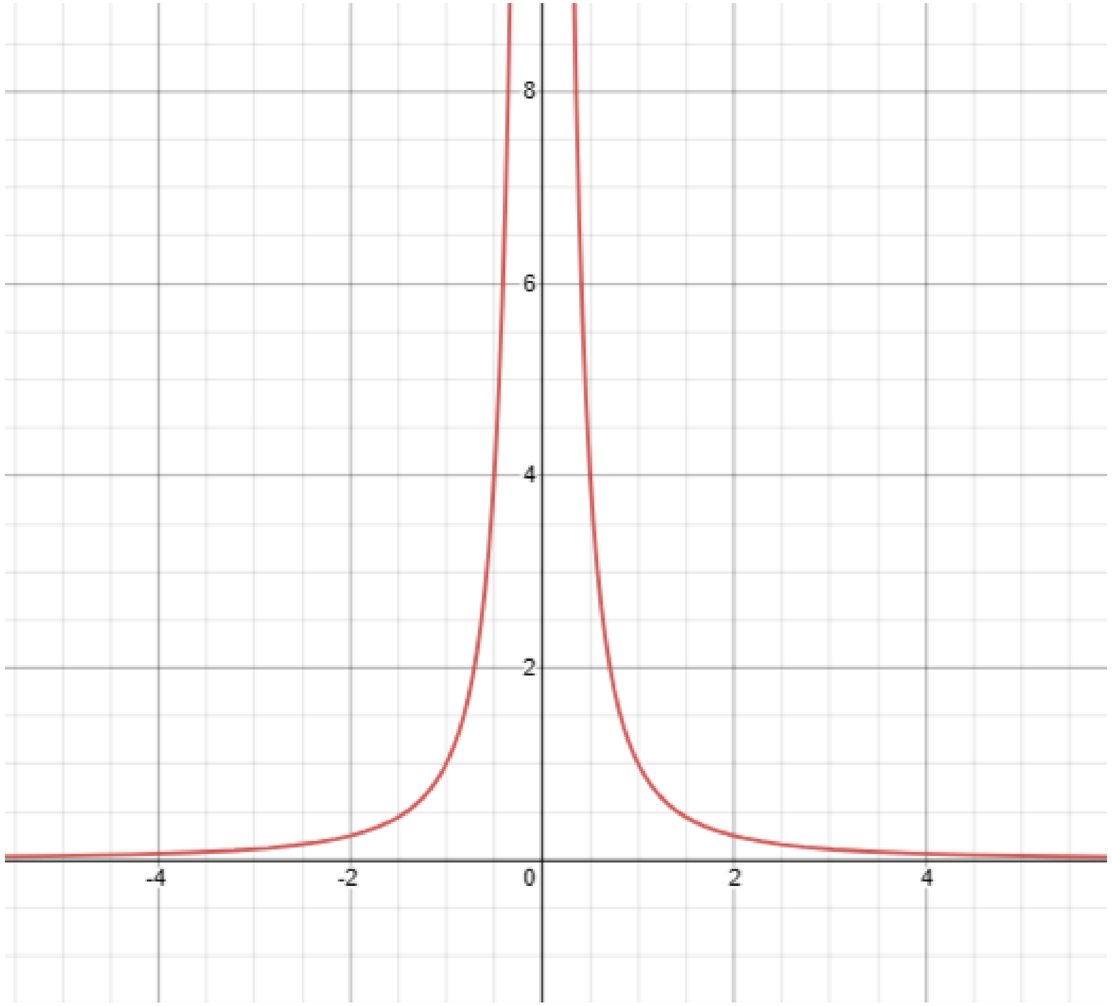
Example

How can we evaluate $\int_0^5 \frac{1}{x^2}\,dx$?

The integrand is infinite at $x = 0$.



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Improper integrals: infinity of integrand

We can deal with a function with a **singularity** – i.e. an isolated point at which the function tends to $\pm$infinity, the same way as discontinuities:

If f is continuous **and finite** on $[a, b]$ except at some point c where $\lim_{x \to c} f(x) \to \infty$, then we define

$$\int_a^b f(x)\,dx = \lim_{\epsilon \to 0} \int_a^{c-\epsilon} f(x)\,dx + \lim_{\epsilon' \to 0} \int_{c+\epsilon'}^b f(x)\,dx$$

where both limits exist.

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Or if f is continuous and finite on $[a, b]$ except at b where $\lim_{x\to b} f(x) \to \infty$, then we define

$$\int_a^b f(x)\,dx = \lim_{\epsilon\to 0}\int_a^{b-\epsilon} f(x)\,dx$$

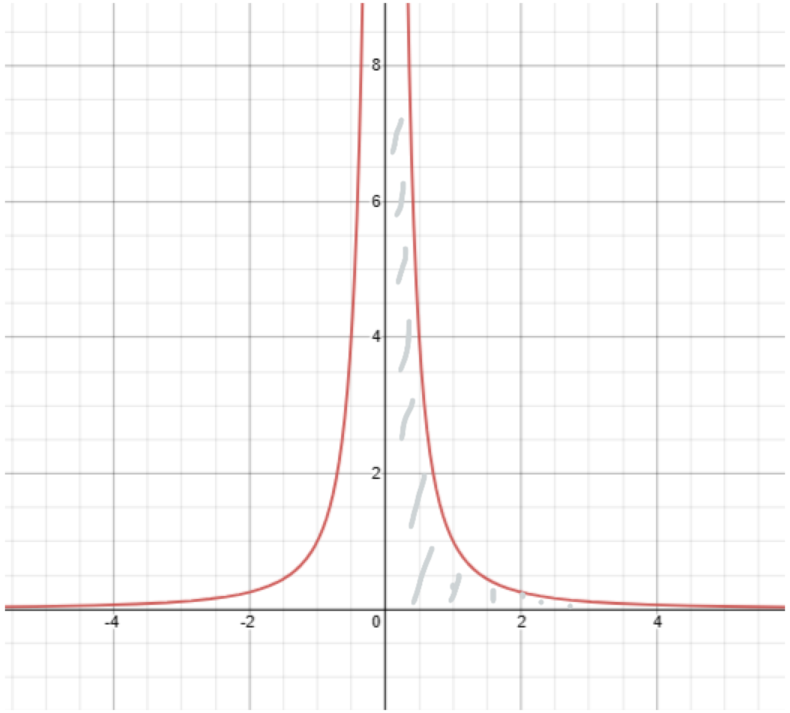
We can deal with a singularity at a similarly.

Divergent example

Now

$$\int_0^5 \frac{1}{x^2}\,dx = \lim_{\epsilon\to 0^+} \int_\epsilon^5 \frac{1}{x^2}\,dx = \lim_{\epsilon\to 0^+} \left[-\frac{1}{x}\right]_\epsilon^5 = \lim_{\epsilon\to 0^+} \left(-\frac{1}{5}+\frac{1}{\epsilon}\right) = +\infty$$

as $\epsilon \to 0$, so the integral **diverges**.



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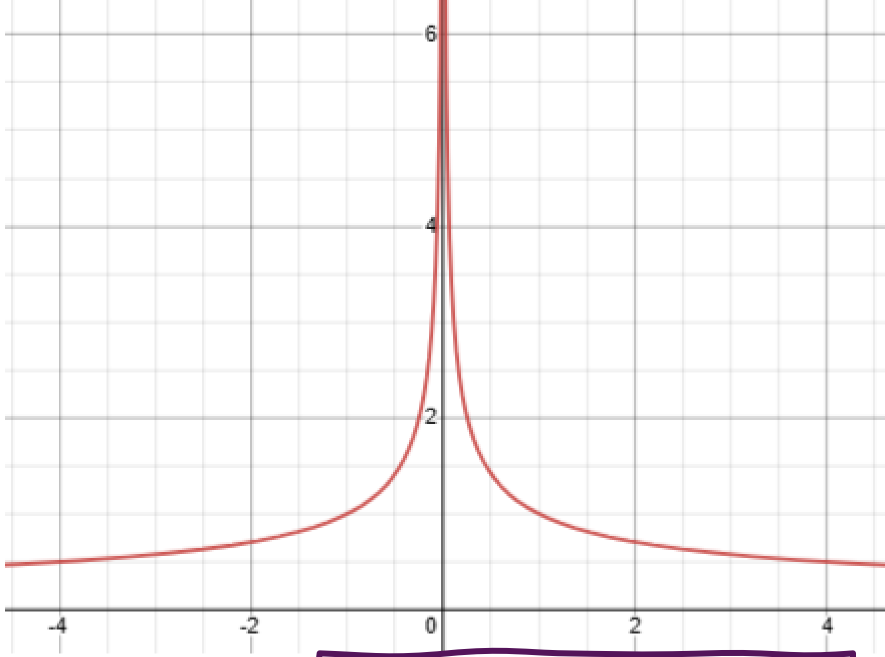
Convergent example

The following integrand is undefined when $x = 0$:

$$\int_{-1}^{2} \frac{1}{\sqrt{|x|}}\, dx$$

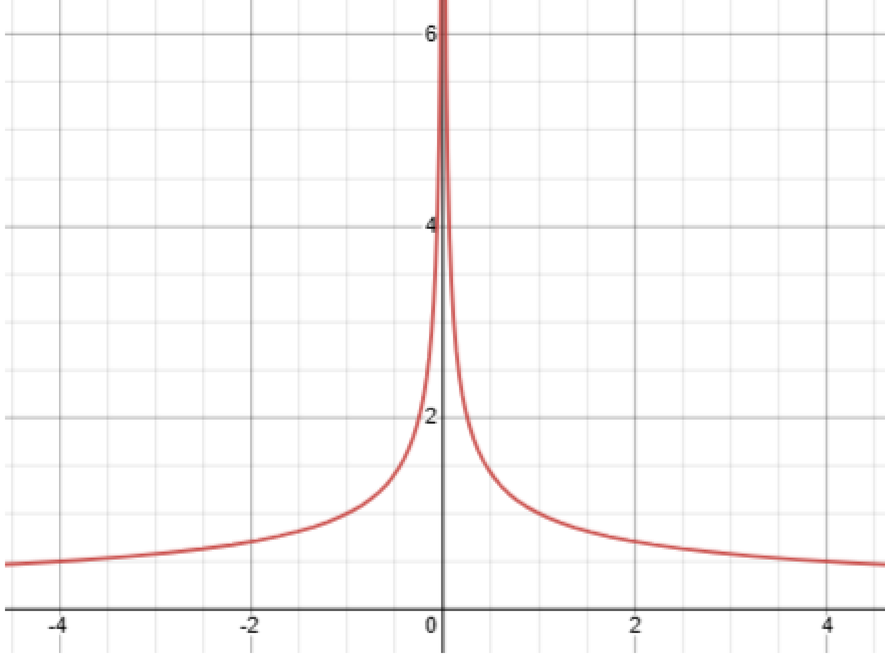


So we split the interval at 0 and take both limits:

$$\int_{-1}^{2} \frac{1}{\sqrt{|x|}} dx = \lim_{\epsilon \to 0} \int_{-1}^{-\epsilon} \frac{1}{\sqrt{-x}} dx + \lim_{\epsilon' \to 0} \int_{+\epsilon'}^{2} \frac{1}{\sqrt{x}} dx$$

We use a different ϵ and ϵ' because these things have to converge independently.

Convergent example



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Now guessing an antiderivative $F(x) = -2\sqrt{-x}$ for the first integral and $G(x) = 2\sqrt{x}$ for the second, we obtain

$$\int_{-1}^{2} \frac{1}{\sqrt{|x|}} dx = \lim_{\epsilon \to 0} [-2\sqrt{-x}]_{-1}^{-\epsilon} + \lim_{\epsilon' \to 0} [2\sqrt{x}]_{\epsilon'}^{2} =$$

$$\lim_{\epsilon \to 0}\left[-2\sqrt{\epsilon} + 2\sqrt{1}\right] + \lim_{\epsilon\prime \to 0}\left[2\sqrt{2} - 2\sqrt{\epsilon'}\right] = 2 + 2\sqrt{2}$$

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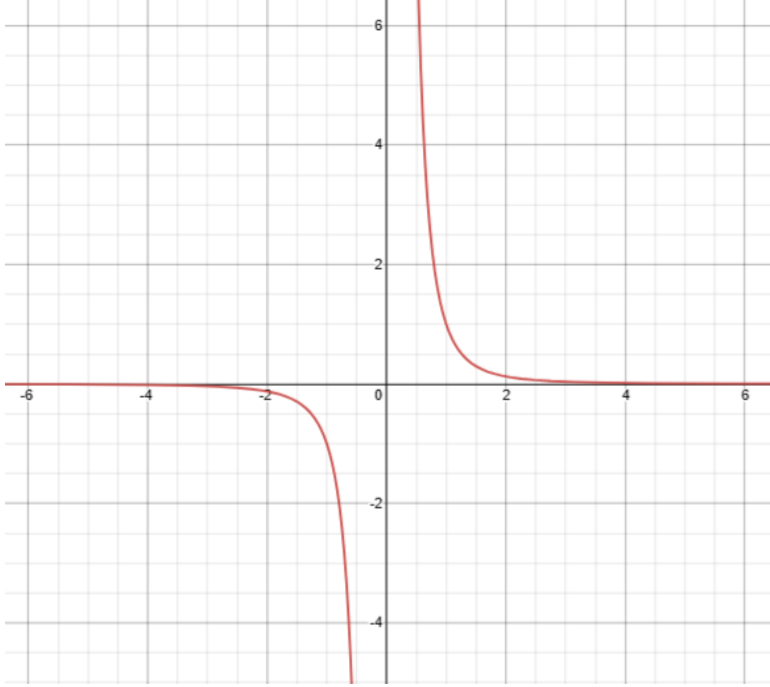
Troubling example



Consider $\int_{-2}^{3} \frac{1}{x^3}\,dx$

$$\int_{-2}^{3} \frac{1}{x^3}\,dx = \lim_{\epsilon\to 0}\left[\int_{-2}^{-\epsilon} \frac{1}{x^3}\,dx + \int_{\epsilon}^{3} \frac{1}{x^3}\,dx\right]$$

Guess the antiderivative $F(x) = -\frac{1}{2x^2}$, then

$$\int_{-2}^{3} \frac{1}{x^3}\,dx = \lim_{\epsilon\to 0}\left[\left[\frac{-1}{2x^2}\right]_{-2}^{-\epsilon} + \left[\frac{-1}{2x^2}\right]_{\epsilon}^{3}\right] = \lim_{\epsilon\to 0}\left[\frac{-1}{2\epsilon^2} - \frac{-1}{8} + \frac{-1}{18} - \frac{-1}{2\epsilon^2}\right] = \lim_{\epsilon\to 0}\left[\frac{1}{8} - \frac{1}{18}\right] = \frac{1}{8} - \frac{1}{18}.$$

Not considered appropriate! **The limits must independently exist**:

$$\int_{-2}^{3} \frac{1}{x^3}\,dx = \lim_{\epsilon\to 0}\left[\left[\frac{-1}{2x^2}\right]_{-2}^{-\epsilon}\right] + \lim_{\epsilon'\to 0}\left[\left[\frac{-1}{2x^2}\right]_{\epsilon'}^{3}\right] = \lim_{\epsilon\to 0}\left[\frac{-1}{2\epsilon^2} - \frac{-1}{8}\right] + \lim_{\epsilon'\to 0}\left[\frac{-1}{18} - \frac{-1}{2\epsilon'^2}\right]$$

neither of which exist.

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Improper integrals: infinity of range

We can deal also with integrals where the bounds are infinite by also taking limits:

If f is piecewise continuous on $[a, \infty]$ then we define

$$\int_a^\infty f(x)\,dx = \lim_{N\to\infty} \int_a^N f(x)\,dx$$

where the limit exists.

Likewise

$$\int_{-\infty}^b f(x)\,dx = \lim_{N\to\infty} \int_{-N}^b f(x)\,dx$$

and

$$\int_{-\infty}^\infty f(x)\,dx = \lim_{N\to\infty} \int_{-N}^a f(x)\,dx + \lim_{N'\to\infty} \int_a^{N'} f(x)\,dx$$

where the limits exist.

Examples

$$\int_2^{\infty} \frac{1}{x^2}\, dx =$$

$$\int_2^{\infty} \frac{1}{x}\, dx =$$

Examples

$$\int_2^\infty \frac{1}{x^2}\,dx = \lim_{N\to\infty}\int_2^N \frac{1}{x^2}\,dx = \lim_{N\to\infty}\left[-\frac{1}{x}\right]_2^N = \lim_{N\to\infty}\left(-\frac{1}{N}+\frac{1}{2}\right) = \frac{1}{2}.$$

$$\int_2^\infty \frac{1}{x}\,dx =$$

Examples

$$\int_2^\infty \frac{1}{x^2}dx = \lim_{N\to\infty} \int_2^N \frac{1}{x^2}dx = \lim_{N\to\infty} \left[-\frac{1}{x}\right]_2^N = \lim_{N\to\infty} \left(-\frac{1}{N} + \frac{1}{2}\right) = \frac{1}{2}.$$

$$\int_2^\infty \frac{1}{x}dx = \lim_{N\to\infty} \int_2^N \frac{1}{x}dx = \lim_{N\to\infty} [\ln x]_2^N = \lim_{N\to\infty} (\ln N - \ln 2) \to \infty.$$

Examples

$$\int_2^\infty \frac{1}{x^2}\,dx = \lim_{N\to\infty} \int_2^N \frac{1}{x^2}\,dx = \lim_{N\to\infty}\left[-\frac{1}{x}\right]_2^N = \lim_{N\to\infty}\left(-\frac{1}{N}+\frac{1}{2}\right) = \frac{1}{2}.$$

$$\int_2^\infty \frac{1}{x}\,dx = \lim_{N\to\infty} \int_2^N \frac{1}{x}\,dx = \lim_{N\to\infty}[\ln x]_2^N = \lim_{N\to\infty}(\ln N - \ln 2) \to \infty.$$

Seems familiar?

$\sum_{n=1}^\infty \frac{1}{n^2}$ converges, but $\sum_{n=1}^\infty \frac{1}{n}$ diverges.

Integral test for series convergence:

If f is a continuous, decreasing, positive function on $[m,\infty)$ then

- if $\int_m^\infty f(x)\,dx$ is convergent, then so is $\sum_{n=m}^\infty f(n)$
- if $\int_m^\infty f(x)\,dx$ is divergent, then so is $\sum_{n=m}^\infty f(n)$

Systematic Integration

Integration of elementary functions

The approach we have taken so far is to guess an antiderivative; deduced from knowledge of derivatives:

$f(x)$	$F(x)$
a (constant)	ax
$x^n \quad (n \neq -1)$	$\dfrac{x^{n+1}}{n+1}$
x^{-1}	$\ln\lvert x\rvert$
e^{ax}	$\dfrac{1}{a}e^{ax}$
$\sin(ax)$	$-\dfrac{1}{a}\cos(ax)$
$\cos(ax)$	$\dfrac{1}{a}\sin(ax)$
$\sinh(ax)$	$\dfrac{1}{a}\cosh(ax)$
$\cosh(ax)$	$\dfrac{1}{a}\sinh(ax)$

Integration of elementary functions

A lot of it is about knowing and recognising derivatives of functions. Several calculus books contain tables of functions and their derivatives, which can help when it comes to integrating.

Many more complex examples can be deduced from knowledge of derivatives:

E.g.:

$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a} + C,$ since $(\sin^{-1} x)' = (\arcsin x)' = \frac{1}{\sqrt{1-x^2}}$

$\int \sec x \tan x\, dx = \sec x + C,$ since $\sec x = \frac{1}{\cos x}$ and $\tan x = \frac{\sin x}{\cos x}$

We will look at two more structured approaches to finding an antiderivative:

- Integration by substitution, and
- Integration by parts.

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Integration by substitution

Recall the chain rule: if u is a function of x, $u(x)$, and g is a function of u, $g(u)$, then

$$\frac{dg}{dx} = \frac{dg}{du}\frac{du}{dx}.$$

If we spot an integrand of the form $f\big(u(x)\big)u'(x)$ and take $F(u)$ to be the anti-derivative of $f(u)$ wrt u, then

$$\frac{dF}{dx} = \frac{dF}{du}\frac{du}{dx} = f(u)u'(x)$$

i.e. our original integrand! Hence $F(u(x))$ is the antiderivative of $f\big(u(x)\big)u'(x)$ wrt x.

Therefore

$$\int f\big(u(x)\big)u'(x)\,dx = F\big(u(x)\big) + C = F(u) + C = \int f(u)\,du$$

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Integration by substitution: Example

Reminder:

$$\int f\big(u(x)\big)u'(x)\,dx = \int f(u)\frac{du}{dx}dx = \int f(u)\,du$$

$$\int \frac{\mathbf{4x}}{\sqrt{\mathbf{2x^2+1}}}\,\boldsymbol{dx}$$

Recognise that

$$\frac{4x}{\sqrt{2x^2+1}} = \frac{1}{\sqrt{2x^2+1}}4x = f(u)\frac{du}{dx}\ ,$$

where $u = u(x) = 2x^2 + 1$ and $f(u) = \frac{1}{\sqrt{u}}$.

So

$$\int \frac{4x}{\sqrt{2x^2+1}}dx = \int \frac{1}{\sqrt{u}}du = 2\sqrt{u} + C = 2\sqrt{2x^2+1} + C.$$

Integration by parts

Recall the product rule: if u and v are functions of x then

$$\frac{duv}{dx} = \frac{du}{dx}v + u\frac{dv}{dx}.$$

You may not necessarily spot this integrand

Thus uv is the anti-derivative of $u'v + uv'$. I.e.

$$\int u'v\,dx + \int uv'\,dx = uv + C$$

Rearranging

$$\int uv'\,dx = uv - \int u'v\,dx$$

You may spot this instead

Integration by parts: Example

Reminder:

$$\int uv'\,dx = uv - \int u'v\,dx$$

$$\int \boldsymbol{x^k \ln x\, dx}, \qquad \textbf{for some } \boldsymbol{k} \in \mathbb{N}.$$

We see a product of terms and wonder if we can select u, v in a way that will make things simpler.

Here if we select $u = x^k, v' = \ln x$, then it is hard to proceed: we need $v = \int \ln x$

Instead select $u = \ln x$, $v' = x^k$, then $v = \int x^k = \frac{x^{k+1}}{k+1}$

So

$$\int x^k \ln x\,dx = \int uv'\,dx = uv - \int u'v\,dx = \frac{x^{k+1}}{k+1}\ln x - \int \frac{1}{x}\frac{x^{k+1}}{k+1}\,dx$$

Which simplifies to

$$\frac{x^{k+1}}{k+1}\ln x - \int \frac{x^k}{k+1}\,dx = \frac{x^{k+1}}{k+1}\ln x - \frac{x^{k+1}}{(k+1)^2} + C.$$

Other techniques

Integration by partial fractions:

$$\int \frac{x^3}{x^2-4}\,dx = \int \frac{x(x^2-4)}{x^2-4} + \frac{4x}{x^2-4}\,dx$$

$$= \int x + \frac{2}{x-2} + \frac{2}{x+2}\,dx$$

$$= \frac{x^2}{2} + 2\ln|x-2| + 2\ln|x+2| + C$$

Useful for quotients of polynomials.

Other techniques

Reduction formulae:

$I_m = \int \cos^m \theta \, d\theta =$

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Other techniques

Reduction formulae:

$$I_m = \int \cos^m \theta \, d\theta = \int \cos^{m-1} \theta \frac{d \sin \theta}{d\theta} d\theta$$

$$= \cos^{m-1} \theta \sin \theta - \int \sin \theta \, . (m-1) \cos^{m-2} \theta \, (-\sin \theta) \, d\theta$$

$$= \cos^{m-1} \theta \sin \theta + (m-1) \int \cos^{m-2} \theta \, d\theta - (m-1) \int \cos^m \theta \, d\theta$$

$$= \cos^{m-1} \theta \sin \theta + (m-1) I_{m-2} - (m-1) I_m$$

So

$$I_m = \frac{1}{m} \cos^{m-1} \theta \sin \theta + \frac{(m-1)}{m} I_{m-2}$$

E.g. $I_7 = \frac{1}{7} \cos^6 \theta \sin \theta + \frac{6}{7} I_5 \; = \; \frac{1}{7} \cos^6 \theta \sin \theta + \frac{6}{35} \cos^4 \theta \sin \theta + \frac{24}{35} I_3$

$$= \frac{1}{7} \cos^6 \theta \sin \theta + \frac{6}{35} \cos^4 \theta \sin \theta + \frac{8}{35} \cos^2 \theta \sin \theta + \frac{16}{35} I_1$$

$$= \frac{1}{7} \cos^6 \theta \sin \theta + \frac{6}{35} \cos^4 \theta \sin \theta + \frac{8}{35} \cos^2 \theta \sin \theta + \frac{16}{35} \sin \theta + C$$

What we learnt today

- Evaluating improper integrals:
  - For f continuous and finite on $[a, b]$ except at b where $\lim_{x\to b} f(x) \to \infty$, $\int_a^b f(x)\,dx = \lim_{\epsilon\to 0} \int_a^{b-\epsilon} f(x)\,dx$
    - *Similarly for infinity at* a
  - For f continuous and finite on $[a, b]$ except at c where $\lim_{x\to c} f(x) \to \infty$, $\int_a^b f(x)\,dx = \lim_{\epsilon\to 0} \int_a^{c-\epsilon} f(x)\,dx + \lim_{\epsilon'\to 0} \int_{c+\epsilon'}^{b} f(x)\,dx$
  - For f piecewise continuous,

$$\int_{-\infty}^{\infty} f(x)\,dx = \lim_{N\to\infty} \int_{-N}^{a} f(x)\,dx + \lim_{N'\to\infty} \int_{a}^{N'} f(x)\,dx$$

- Systematic Integration
  - Various practical methodologies for integrating
  - Relying on knowledge of derivatives & differentiation rules
  - Practice, practice, practice!