COMP1021 Mathematics for Computer Science
Linear Algebra (Part 2)
Practical - Week 20
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Instructions: Work on these problems in the practical sessions for the week specified. First try them on your own. If you're stuck, try discussing things with others. If you get the answer, still discuss with others to see if maybe you missed something. If you run into major roadblocks, ask the demonstrators for hints.

Solutions will be posted on Learn Ultra at the end of the week. Make sure you're all set with the solutions and understand them before the next practical.

Purpose of this practical: This practical will be used to strengthen your understanding of the concepts of linear regression, orthogonal matrices, orthogonal diagonalisation.

1. Find the least squares straight line fit to the four points: $(0,1),(2,0),(3,1),(3,2)$.

Solution:

To find the coefficients of the line $y = a + bx$, find the least squares solution to the system

$$\begin{pmatrix} 1 & 0 \\ 1 & 2 \\ 1 & 3 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 2 \end{pmatrix}$$

We found a QR decomposition of the coefficient matrix of this system in Q1:

$$\begin{pmatrix} 1 & 0 \\ 1 & 2 \\ 1 & 3 \\ 1 & 3 \end{pmatrix} = QR = \begin{pmatrix} 1/2 & -2/\sqrt{6} \\ 1/2 & 0 \\ 1/2 & 1/\sqrt{6} \\ 1/2 & 1/\sqrt{6} \end{pmatrix} \begin{pmatrix} 2 & 4 \\ 0 & \sqrt{6} \end{pmatrix}.$$

Hence, we can find the least squares solution from the system

$$\begin{pmatrix} 2 & 4 \\ 0 & \sqrt{6} \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1/2 & 1/2 & 1/2 & 1/2 \\ -2/\sqrt{6} & 0 & 1/\sqrt{6} & 1/\sqrt{6} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1/\sqrt{6} \end{pmatrix}$$

From this, we have $b = \frac{1}{6}$ and $a = \frac{2}{3}$, so the least squares straight line fit is $y = \frac{2}{3} + \frac{1}{6}x$.

2. Find a, b, and c such that the following matrix is orthogonal.

$$\begin{pmatrix} a & 1/\sqrt{2} & -1/\sqrt{2} \\ b & 1/\sqrt{6} & 1/\sqrt{6} \\ c & 1/\sqrt{3} & 1/\sqrt{3} \end{pmatrix}$$

Are the values of a, b, and c unique?

Solution:

For this matrix to be orthogonal, its columns need to form an orthonormal set. It is easy to check the the last two columns are orthogonal unit vectors. Hence, we are looking for unit vectors (a,b,c) which are solutions to the linear system

$$\begin{aligned} \frac{1}{\sqrt{2}}a + \frac{1}{\sqrt{6}}b + \frac{1}{\sqrt{3}}c = 0 \\ -\frac{1}{\sqrt{2}}a + \frac{1}{\sqrt{6}}b + \frac{1}{\sqrt{3}}c = 0 \end{aligned}$$

One basis for the solution space of this system is $\{\mathbf{v} = (0, -\sqrt{2}, 1)\}$. Normalising $\mathbf{v}$, we get $\mathbf{v}' = (0, -\sqrt{2}/\sqrt{3}, 1/\sqrt{3})$. It is clear that $-\mathbf{v}' = (0, \sqrt{2}/\sqrt{3}, -1/\sqrt{3})$ will also be a unit vector which is a solution to the system. (In fact, $\mathbf{v}'$ and $-\mathbf{v}'$ are the only two such vectors).

Hence, (a,b,c) is either $(0, \sqrt{2}/\sqrt{3}, -1/\sqrt{3})$ or $(0, -\sqrt{2}/\sqrt{3}, 1/\sqrt{3})$.

3. Recall that in $\mathbb{R}^2$, the standard unit vectors are $\bar{i} = (1,0)$ and $\bar{j} = (0,1)$ and vectors are usually represented as $x\bar{i} + y\bar{j}$. Vectors can be represented in another way, called the polar form, where a vector is represented by its length from the origin, denoted by r, and the angle from the

positive x-axis taken counter-clockwise, denoted by θ. The components of the polar form (r, Θ) of a vector can be converted to rectangular form $x\bar{\imath} + y\bar{\jmath}$ as $x = r\cos\theta$ and $y = r\sin\theta$.

Prove that a 2×2 orthogonal matrix Q can have only one of two possible forms (not accounting for row exchanges and/or column exchanges):

$$Q = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \quad \text{or} \quad Q = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}.$$

where $0 \le \theta < 2\pi$. Describe the corresponding linear operators T_Q on $\mathbb{R}^2$ geometrically.

Solution:

If Q is a 2×2 orthogonal matrix then its columns form an orthonormal basis in $\mathbb{R}^2$. If (x, y) is the first column vector then it is a unit vector in $\mathbb{R}^2$, hence $x = \cos\theta$ and $y = \sin\theta$, where θ is the angle from the positive direction of the x-axis to vector (x, y), counted anti-clockwise. Then it is easy to see that there are only two unit vectors in $\mathbb{R}^2$ which are orthogonal to $(\cos\theta, \sin\theta)$ – these are the vectors in the second column of two possible forms of Q.

Geometrically, each operator T_Q is either a rotation anti-clockwise through the angle θ (first case), or such a rotation followed by a reflection about the x-axis (second case).

4. Orthogonally diagonalise the following matrix:

$$A = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}.$$

Solution:

Follow the orthogonal diagonalisation algorithm.

Step 1. First find the eigenvalues of A: compute the characteristic polynomial $det(\lambda I - A) = \lambda(\lambda - 3)^2$, so the eignvalues are $\lambda_1 = 0$ and $\lambda_2 = 3$.

To find bases in the eigenspaces of A, solve linear systems $(0I - A)\mathbf{x} = \mathbf{0}$ and $(3I - A)\mathbf{x} = \mathbf{0}$. For the first system, one basis of solution space is $\{\mathbf{u}_1 = (1, 1, 1)\}$. For the second system, one basis for the solution space is $\{\mathbf{u}_2 = (-1, 1, 0), \mathbf{u}_3 = (-1, 0, 1)\}$.

Step 2. Apply the Gram-Schmidt algorithm to each of these bases (separately). This produces $\{\mathbf{v}_1 = (\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})\}$ and $\{\mathbf{v}_2 = (-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0), \mathbf{v}_3 = (-\frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}})\}$.

Step 3. We now have

$$Q = \begin{pmatrix} 1/\sqrt{3} & -1/\sqrt{2} & -1/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & -1/\sqrt{6} \\ 1/\sqrt{3} & 0 & 2/\sqrt{6} \end{pmatrix} \quad \text{and} \quad Q^T A Q = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}.$$

5. Explain how the least squares straight line fit can be generalised from polynomials of degree 1 (i.e., straight lines) to polynomials of degree $n > 1$.

Solution:

If we have points $(x_1, y_1), \ldots, (x_k, y_k)$ and we are looking to fit a curve $y = a_0 + a_1 x + \ldots + a_n x^n$ to our points, this gives us the system (with unknowns $a_0, a_1, \ldots, a_n$)

$$\begin{aligned} a_0 + a_1 x_1 + \ldots + a_n x_1^n &= y_1 \\ a_0 + a_1 x_2 + \ldots + a_n x_2^n &= y_2 \\ &\vdots \\ a_0 + a_1 x_k + \ldots + a_n x_k^n &= y_k \end{aligned}$$

or, in matrix notation,

$$\begin{pmatrix} 1 & x_1 & x_1^2 & \ldots & x_1^n \\ 1 & x_2 & x_2^2 & \ldots & x_2^n \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_k & x_k^2 & \ldots & x_k^n \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_k \end{pmatrix}$$

All that remains is to find a least squares solution to this system.

6. Use your generalisation to find the quadratic polynomial that best fits the four points:

$$(1, 6), (2, 1), (-1, 5), (-2, 2).$$

Solution:

Apply this method to points $(1, 6), (2, 1), (-1, 5), (-2, 2)$ and $n = 2$. This gives us the system

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & -1 & 1 \\ 1 & -2 & 4 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 6 \\ 1 \\ 5 \\ 2 \end{pmatrix}$$

To form the normal system, compute $A^T A$ and $A^T \mathbf{b}$:

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & -1 & -2 \\ 1 & 4 & 1 & 4 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & -1 & 1 \\ 1 & -2 & 4 \end{pmatrix} = \begin{pmatrix} 4 & 0 & 10 \\ 0 & 10 & 0 \\ 10 & 0 & 34 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & -1 & -2 \\ 1 & 4 & 1 & 4 \end{pmatrix} \begin{pmatrix} 6 \\ 1 \\ 5 \\ 2 \end{pmatrix} = \begin{pmatrix} 14 \\ -1 \\ 23 \end{pmatrix}$$

So the normal system is

$$\begin{pmatrix} 4 & 0 & 10 \\ 0 & 10 & 0 \\ 10 & 0 & 34 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 14 \\ -1 \\ 23 \end{pmatrix}$$

Solving this system, we get $a_0 = \frac{41}{6}, a_1 = -\frac{1}{10}, a_2 = -\frac{4}{3}$, so the quadratic polynomial that best fits the points is $y = \frac{41}{6} - \frac{1}{10}x - \frac{4}{3}x^2$.

7. (hard) A symmetric matrix is called *positive definite* if all its eigenvalues are strictly positive. Use orthogonal diagonalisation and QR decomposition to prove that each symmetric positive

definite matrix A has a so-called *Cholesky* (pronounced with sh-) *decomposition*:

$$A = LL^T$$

for some lower triangular matrix L. (This is a special case of LU decomposition with $U = L^T$ - when it exists, it allows one to solve linear systems twice as fast as the general LU method).

Solution:

Let $A = UDU^T$, where U is orthogonal and D is diagonal. Since A is positive definite, we have $D = diag(\lambda_1, \ldots, \lambda_n)$ where all $\lambda_i > 0$. Let $\sqrt{D}$ be the matrix $diag(\sqrt{\lambda_1}, \ldots, \sqrt{\lambda_n})$. Clearly, $(\sqrt{D})^2 = D$, and we have $A = (U\sqrt{D})\sqrt{D}U^T$. The matrix $\sqrt{D}U^T$ is invertible, so it has a QR decomposition $\sqrt{D}U^T = QR$ where Q is orthogonal and R is upper triangular. Then we have

$$A = (U\sqrt{D})\sqrt{D}U^T = (\sqrt{D}U^T)^T\sqrt{D}U^T = (QR)^T(QR) = R^TQ^TQR = R^TR.$$

Letting $L = R^T$, we have the required decomposition.