COMP1021 Mathematics for Computer Science
Linear Algebra (Part 2)
Practical - Week 14
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Instructions: Work on these problems in the practical sessions for the week specified. First try them on your own. If you're stuck, try discussing things with others. If you get the answer, still discuss with others to see if maybe you missed something. If you run into major roadblocks, ask the demonstrators for hints.

Solutions will be posted on Learn Ultra at the end of the week. Make sure you're all set with the solutions and understand them before the next practical.

Purpose of this practical: In this practical, we review eigenvalues, eigenvectors, complex numbers, and complex vector spaces. We look at not only how to directly use these concepts, but also useful properties related to them.

1. (Warm up) Find a basis and the dimension of the solution space of the following linear system:

$$\begin{array}{rrrrrl} 2x_1 & +2x_2 & -x_3 & & +x_5 & = 0 \\ -x_1 & -x_2 & +2x_3 & -3x_4 & +x_5 & = 0 \\ x_1 & +x_2 & -2x_3 & & -x_5 & = 0 \\ & & x_3 & +x_4 & +x_5 & = 0 \end{array}$$

Solution:

The general solution of the system is $x_1 = -x_2 - x_5, x_3 = -x_5, x_4 = 0$, with free variables x_2 and x_5 (other choice fo free variables are possible). So, the dimension of the solution space is 2. To find a basis, draw the table (for 2 vectors).

	x_1	x_2	x_3	x_4	x_5
$\mathbf{v}_1$	-1	1	0	0	0
$\mathbf{v}_2$	-1	0	-1	0	1

2. For the the following matrix, find the basis of the eigenspace corresponding to the eigenvalue $\lambda_0 = 2$. Hint: Use the previous question.

$$A = \begin{pmatrix} 0 & -2 & 1 & 0 & -1 \\ 1 & 3 & -2 & 3 & -1 \\ -1 & -1 & 4 & 0 & 1 \\ 0 & 0 & -1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix}$$

Solution:

$(2I - A)\mathbf{x} = 0$

$$\begin{pmatrix} 2 & -2 & -1 & 0 & 1 \\ -1 & -1 & 2 & -3 & 1 \\ 1 & 1 & -2 & 0 & -1 \\ 1 & 1 & -2 & 0 & -1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \mathbf{x} = 0$$

The solution space is the same as that in question 1, so a basis of the eigenspace is the basis found above.

3. Find the eigenvalues and eigenvectors of the following matrix: $M = \begin{pmatrix} 1 & -1 \\ 2 & 4 \end{pmatrix}$.

Solution:

First find the characteristic polynomial of M:

$$det(\lambda I - A) = \begin{vmatrix} \lambda - 1 & 1 \\ -2 & \lambda - 4 \end{vmatrix} = (\lambda - 1) \cdot (\lambda - 4) - 1 \cdot (-2) = \lambda^2 - 5\lambda + 6.$$

So, the characteristic equation of M is $\lambda^2 - 5\lambda + 6 = 0$. Its solutions $\lambda_1 = 2$ and $\lambda_2 = 3$ are the eigenvalues of M.

To find the eigenspace of M corresponding to $\lambda_1 = 2$, form the equation $(2I - A)\mathbf{x} = \mathbf{0}$, or

$$\begin{pmatrix} 1 & 1 \\ -2 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{or} \quad \begin{array}{rcl} x_1 + x_2 & = & 0 \\ -2x_1 - 2x_2 & = & 0 \end{array}$$

One basis of the solution space of this system is $(1, -1)$.

To find the eigenspace of M corresponding to $\lambda_2 = 3$, form the equation $(3I - A)\mathbf{x} = \mathbf{0}$, or

$$\begin{pmatrix} 2 & 1 \\ -2 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{or} \quad \begin{array}{rcl} 2x_1 + x_2 & = & 0 \\ -2x_1 - x_2 & = & 0 \end{array}$$

One basis of the solution space of this system is $(1, -2)$.

4. Suppose that λ is an eigenvalue of a matrix A, and $\mathbf{x}$ is a corresponding eigenvector.

   (a) Show that $\mathbf{x}$ is also an eigenvector of A^k ($k \geq 0$) and find the corresponding eigenvalue.

Solution:
We have $A^k\mathbf{x} = A^{k-1}A\mathbf{x} = A^{k-1}\lambda\mathbf{x} = \lambda A^{k-1}\mathbf{x} = \lambda(A^{k-2}A\mathbf{x}) = \lambda(A^{k-2}\lambda\mathbf{x}) = \ldots = \lambda^k\mathbf{x}$, so λ^k is the corresponding eigenvalue.

   (b) Show that $\mathbf{x}$ is also an eigenvector of $B = A - 7I$ and find the corresponding eigenvalue.

Solution:
We have $B\mathbf{x} = (A - 7\lambda)\mathbf{x} = A\mathbf{x} - 7I\mathbf{x} = \lambda\mathbf{x} - 7\mathbf{x} = (\lambda - 7)\mathbf{x}$. So $\lambda - 7$ is the corresponding eigenvalue.

   (c) Assuming that A is invertible, show that $\mathbf{x}$ is also an eigenvector of A^{-1} and find the corresponding eigenvalue.

Solution:
Note that if A is invertible then $\lambda \neq 0$. We are given that $A\mathbf{x} = \lambda\mathbf{x}$. Left multiply with A^{-1} on both sides:

$$\begin{aligned} & A^{-1}A\mathbf{x} = A^{-1}(\lambda\mathbf{x}) \\ \Longrightarrow\ & \mathbf{x} = \lambda A^{-1}(\mathbf{x}) \\ \Longrightarrow\ & \frac{1}{\lambda}\mathbf{x} = A^{-1}\mathbf{x}. \end{aligned}$$

So $\frac{1}{\lambda}$ is the corresponding eigenvalue.

5. Prove that A and A^T have the same eigenvalues. Do they always have the same eigenvectors?

Solution:

Hint: Transposing a matrix does not change its determinant.

We have $(\lambda I - A)^T = \lambda I^T - A^T = \lambda I - A^T$ and hence $det(\lambda I - A) = det((\lambda I - A)^T) = det(\lambda I - A^T)$. Since the characteristic polynomials of A and A^T are the same, these matrices have the same eigenvalues.

To see that the eigenvectors of A and A^T are not necessarily the same, take, for example, any 2×2 matrix corresponding to a shear in the x-direction, e.g. matrix $A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$. Then A^T corresponds to a shear in the y-direction. We looked at the eigenvectors of A in the lecture, and you can easily check that A and A^T do not share eigenvectors at all.

6. Assume that the characteristic polynomial of a matrix A can be factored as $\lambda^2(\lambda+5)^3(\lambda-2)^4$.

   (a) What is the size of A? Is there enough information to say something about this?

Solution:
A is a 9×9 matrix, the size is the same as the degree of (i.e. the largest power of λ in) the polynomial.

   (b) Is A invertible? Is there enough information to say something about this?

Solution:
A is not invertible because one of its eigenvalues is 0.

7. Find all complex scalars k, if any, for which $\mathbf{u}$ and $\mathbf{v}$ are orthogonal in $\mathbb{C}^3$:

   (a) $\mathbf{u} = (3i, 1, i), \mathbf{v} = (-i, -5, k)$

Solution:
It's important not to forget complex conjugates in the definition of the dot product in $\mathbb{C}^3$.

$\mathbf{u} \cdot \mathbf{v} = 3i(i) + 1(-5) + i\overline{k} = -8 + i\overline{k}$. So we need to solve the equation $-8 + i\overline{k} = 0$. Multiplying both sides by $-i$, we get $8i + \overline{k} = 0$. $\overline{k} = -8i \implies k = 8i$.

   (b) $\mathbf{u} = (k, k, 1+i), \mathbf{v} = (1, -1, 1-i)$

Solution:
$\mathbf{u} \cdot \mathbf{v} = k(1) + k(-1) + (1+i)(1+i) = 2i$. So, in this case, there is no k such that $\mathbf{u}$ and $\mathbf{v}$ are orthogonal in $\mathbb{C}^3$.

8. Prove the following theorem stated in the lecture: The complex eigenvalues of the real matrix $C = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ are $\lambda = a \pm bi$. If a, b are not both zero, then C can be factored as

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix} = \begin{pmatrix} |\lambda| & 0 \\ 0 & |\lambda| \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

where θ is the argument of $\lambda = a + bi$.

Solution:
Performing the multiplication on the right-hand side of the above matrix equation, we get

$$\begin{pmatrix} |\lambda| & 0 \\ 0 & |\lambda| \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} |\lambda|\cos\theta & -|\lambda|\sin\theta \\ |\lambda|\sin\theta & |\lambda|\cos\theta \end{pmatrix}$$

So, all we need to prove is that $a = |\lambda|\cos\theta$ and $b = |\lambda|\sin\theta$, which follows immediately from $\lambda = a + bi$ and the definitions of the absolute value and argument of a complex number (recall the polar form of a vector mentioned in class).

9. Find the (complex) eigenvalues and eigenspaces of the following matrix: $M = \begin{pmatrix} 4 & -5 \\ 1 & 0 \end{pmatrix}$.

Solution:

First find the characteristic polynomial of M:

$$det(\lambda I - M) = \begin{vmatrix} \lambda - 4 & 5 \\ -1 & \lambda \end{vmatrix} = (\lambda - 4)\lambda + 5 = \lambda^2 - 4\lambda + 5.$$

So, the characteristic equation of M is $\lambda^2 - 4\lambda + 5 = 0$. Solving it by using the usual formula for quadratic equations (or by observing that $\lambda^2 - 4\lambda + 5 = (\lambda - 2)^2 + 1$), we get that $\lambda_1 = 2 - i$ and $\lambda_2 = 2 + i$ are the eigenvalues of M.

To find the eigenspace of M corresponding to $\lambda_1 = 2 - i$, form the linear system $((2 - i)I - A)\mathbf{x} = \mathbf{0}$, or

$$\begin{pmatrix} -2 - i & 5 \\ -1 & 2 - i \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{or} \quad \begin{array}{rcl} (-2 - i)x_1 + 5x_2 & = & 0 \\ -x_1 + (2 - i)x_2 & = & 0 \end{array}$$

The matrix of this system is singular (because $2 - i$ is an eigenvalue), so its rows are proportional — in other words, each of the equations in this linear system is a (complex) multiple of the other. By choosing the second equation, it's easy to find one solution for it, $(2 - i, 1)$. This vector forms a basis of the solution space of this system.

To find the eigenspace of M corresponding to $\lambda_2 = 2 + i$, we could do a similar calculation. Instead, we can use the theorem from the lectures stating that if M is a real matrix and $\mathbf{x}$ is its complex eigenvector corresponding to eigenvalue λ_0, then its complex conjugate, $\overline{\mathbf{x}}$, is also an eigenvector corresponding to eigenvalue $\overline{\lambda_0}$. The dimension of the eigenspace is 1 (it can be neither 0 nor 2), so any non-zero vector forms a basis in it. So we can take $\overline{(2 - i, 1)} = (2 + i, 1)$.

10. (Optional) Consider the statement "For any square matrices A and B of the same size, if λ is an eigenvalue of A and μ is an eigenvalue of B then $\mu\lambda$ is an eigenvalue of AB." Find a flaw in the following proof of this statement: If λ is an eigenvalue of A and μ is an eigenvalue of B then, for some non-zero vector $\mathbf{x}$,

$$AB\mathbf{x} = A\mu\mathbf{x} = \mu A\mathbf{x} = \mu\lambda\mathbf{x}.$$

Is the statement true at all?

Solution:

The flaw is in the implicit assumption that $\mathbf{x}$ is a common eigenvector for A and B: the proof uses both $B\mathbf{x} = \mu\mathbf{x}$ and $A\mathbf{x} = \lambda\mathbf{x}$. We know that it is possible that A and B do not share any eigenvectors (see, e.g., solution to Q3).

The statement is actually false, there are many counterexamples. For example, we can use the same A in the solution to Q3, $A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ and $B = A^T$. It is clear that 1 is the only eigenvalue of both A and B, but $AB = \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix}$ does not have 1 as an eigenvalue (as one can easily check).

11. (Optional)

(a) Choose the third row in the following matrix A so that its characteristic polynomial $det(\lambda I - A)$ is $\lambda^3 - 4\lambda^2 - 5\lambda - 6$.

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ * & * & * \end{pmatrix}.$$

Solution:
Let

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ a & b & c \end{pmatrix}.$$

Then

$$det(\lambda I - A) = \begin{vmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ -a & -b & \lambda - c \end{vmatrix}$$

Computing this determinant, by expanding along the first row or otherwise, we get that $det(\lambda I - A) = \lambda^3 - c\lambda^2 - b\lambda - a$. Hence, $a = 6$, $b = 5$, and $c = 4$.

(b) (hard) Show that any polynomial $p(\lambda) = \lambda^n + c_1\lambda^{n-1} + c_2\lambda^{n-2} + \ldots + c_{n-1}\lambda + c_n$ is the characteristic polynomial of a suitably constructed matrix A. (Use part (a) as an inspiration).

Solution:
One solution is the so-called companion matrix C_p defined as follows:

$$C_p = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ -c_n & -c_{n-1} & -c_{n-2} & \cdots & -c_2 & -c_1 \end{pmatrix}.$$

We will prove that $det(\lambda I - C_p) = p$. This claim can be proved by induction on n, the size of the matrix (or the degree of the polynomial). For $n = 2$,

$$det(\lambda I - C_p) = \begin{vmatrix} \lambda & -1 \\ c_2 & \lambda + c_1 \end{vmatrix} = \lambda(\lambda + c_1) + c_2 = \lambda^2 + c_1\lambda + c_2,$$

as required. Assume now that the claim holds for n and prove it for $n+1$. Take an arbitrary polynomial $p(\lambda) = \lambda^{n+1} + c_1\lambda^n + c_2\lambda^{n-1} + \ldots + c_n\lambda + c_{n+1}$. We have

$$det(\lambda I - C_p) = \begin{vmatrix} \lambda & -1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda & -1 \\ c_{n+1} & c_n & c_n & \cdots & c_2 & \lambda + c_1 \end{vmatrix}.$$

Expanding this determinant along the first column, we get

$$\lambda \cdot \begin{vmatrix} \lambda & -1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda & -1 \\ c_n & c_{n-1} & c_{n-2} & \cdots & c_2 & \lambda + c_1 \end{vmatrix} + (-1)^{n+2}c_{n+1} \cdot \begin{vmatrix} -1 & 0 & 0 & \cdots & 0 & 0 \\ \lambda & -1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -1 & 0 \\ 0 & 0 & 0 & \cdots & \lambda & -1 \end{vmatrix}$$

The first determinant in the above expression is the characteristic polynomial of the the companion matrix for the polynomial $\lambda^n + c_1\lambda^{n-1} + c_2\lambda^{n-2} + \ldots + c_{n-1}\lambda + c_n$ and is therefore equal to this polynomial, by the inductive assumption. The second determinant is triangular and is equal to $(-1)^n$. Thus,

$$\begin{aligned} det(\lambda I - C_p) &= \lambda(\lambda^n + c_1\lambda^{n-1} + c_2\lambda^{n-2} + \ldots + c_{n-1}\lambda + c_n) + (-1)^{n+2}c_{n+1}(-1)^n \\ &= \lambda^{n+1} + c_1\lambda^n + c_2\lambda^{n-1} + \ldots + c_n\lambda + c_{n+1}, \end{aligned}$$

as required.