Mathematics for Computer Science
Linear Algebra (Part 2)

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Durham University

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Thanks to Andrei Krokhin and Billy Moses for use of some slides.

Outline

What You've Learned & What You'll Learn

Last Term

1. Basics of matrices
2. Vector spaces, linear dependence, basis and dimension
3. Determinants
4. Solving linear systems & Gaussian elimination
5. Matrix inverse, rank, & kernel
6. Norms & dot product

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All about matrix decomposition & related useful concepts

Overview of This Term

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Matrix Decompositions

1. LU decomposition
2. Eigendecomposition
3. QR decomposition
4. Spectral decomposition
5. Singular Value Decomposition (SVD)

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Why so many?

1. Different decompositions, different uses
2. For arbitrary matrix A, not all decompositions possible

Organisation

Linear algebra lectures (1h per week)

- 10 weeks in total
- Every Monday at 4pm in CLC 202
- Slides uploaded on Ultra
- Examples will be on the Linear Algebra 2 Teams whiteboard
- Lectures are stream-captured

Practicals (2h every other week)

- every even week (so starting next week)
- 5 weeks in total

Office hour

- Thursday 12-1pm in MCS2007
- Open to meeting at other times/online but please email me first to arrange

Overview of This Term (contd.)

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1. **Learning:** Introduce useful concepts like eigenvalue and eigenvector.

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Why Care?

1. **Learning:** Introduce useful concepts like eigenvalue and eigenvector.
2. **Applications:** fast multiplication, fast solving of multiple systems of equations with same linear mapping, linear regression, and more

Today's Class

1. Recap: Gaussian elimination
2. What is an LU decomposition and what is it good for
3. When does it exist
4. How to find it
5. Improved version

Outline

Recap: Gaussian Elimination

Elementary row operations

1. Swapping two rows
2. Multiply a row by a nonzero number
3. Adding a multiple of one row to another row

LU decomposition: Definition

Definition

An LU decomposition (or LU factorisation) of a square matrix A is a (product) representation $A = LU$ where L is lower triangular and U is upper triangular.

Example: one can check that

$$A = \begin{pmatrix} 2 & 6 & 2 \\ -3 & -8 & 0 \\ 4 & 9 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 \\ -3 & 1 & 0 \\ 4 & -3 & 7 \end{pmatrix} \begin{pmatrix} 1 & 3 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix} = LU$$

Application: an algorithm for solving linear systems

Assume that we know an LU-decomposition $A = LU$.

Consider the following algorithm for solving the linear system $A\mathbf{x} = \mathbf{b}$:

1. re-write $A\mathbf{x} = \mathbf{b}$ as $LU\mathbf{x} = \mathbf{b}$,
2. denote $U\mathbf{x} = \mathbf{y}$ and substitute it in $LU\mathbf{x} = \mathbf{b}$ to obtain $L\mathbf{y} = \mathbf{b}$,
3. solve the triangular linear system $L\mathbf{y} = \mathbf{b}$ for $\mathbf{y}$,
4. now we know $\mathbf{y}$ and solve the triangular linear system $U\mathbf{x} = \mathbf{y}$ for $\mathbf{x}$.

Example 10.1

Solve the following linear system $A\mathbf{x} = \mathbf{b}$

$$\begin{pmatrix} 2 & 6 & 2 \\ -3 & -8 & 0 \\ 4 & 9 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix}$$

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First get its LU decomposition. We know an LU decomposition for A (see previous slides):

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Step 1. Rewrite the system as

$$\begin{pmatrix} 2 & 0 & 0 \\ -3 & 1 & 0 \\ 4 & -3 & 7 \end{pmatrix} \begin{pmatrix} 1 & 3 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix}$$

Discussion

LU method: Reduce solving a linear system to solving two triangular systems.

This method is widely used in Scientific Computing to solve mid-size linear system
It is incorporated into many numerical libraries (e.g. Python's numpy and scipy)

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Question: How is this better then solving $A\mathbf{x} = \mathbf{b}$ by Gauss-Jordan elimination or, if A is invertible, by finding A^{-1} and computing $\mathbf{x} = A^{-1}\mathbf{b}$?

Answer: Solving triangular linear systems is easy and fast (and can be done in parallel).

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Answer: Solving triangular linear systems is easy and fast (and can be done in parallel).

Next question(s): OK, but we need to know an LU-decomposition before we start. When / how / how quickly can we find one?

Answer: Let's find out ...

LU decomposition: Sufficient condition for existence

Let A be a square matrix and let U be its (non-reduced) row echelon form, obtained by Gaussian elimination. Note that U is always <u>upper triangular</u>.

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Theorem

If A and U are as above and no row exchanges were performed while obtaining U from A, then A can be factored $A = LU$, where L is lower triangular.

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We exchange rows while computing U only when we get 0 in a pivot position.
The condition "no row exchanges" means that we never get this situation.

LU decomposition: How to find it

- **Elementary matrix:** matrix different from identity matrix by 1 elementary row operation
  1. Row switching
  2. Row multiplication
  3. Row addition
- **LU decomposition process:**
  1. Keep track of row operations used to compute U by Gaussian elimination
  2. Let $E_1, \ldots, E_k$ be the corresponding elementary matrices (E_1 corresponding to the first row operation and E_k to the last)
  3. Then the inverse (elementary) matrices $E_1^{-1}, \ldots, E_k^{-1}$ are easy to find
  4. Compute $L = E_1^{-1} \cdots E_k^{-1}$, e.g., by applying the corresponding row operations (starting from E_k^{-1}) to the identity matrix I.

Example 10.2

$$A = \begin{pmatrix} 2 & -4 \\ 3 & -2 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix} = LU$$

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Observation: entries in L can in fact be computed in parallel to computing U – in the same order as we create 1s and 0s in U. (This is a general rule.)

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \dashrightarrow \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \dashrightarrow \begin{pmatrix} 2 & 0 \\ 3 & 1 \end{pmatrix} \dashrightarrow \begin{pmatrix} 2 & 0 \\ 3 & 4 \end{pmatrix} = L$$

Permutation matrices

Q: When does the LU method fail? A: When row exchanges must be used

Q: What can be done about this? A: Permute rows/equations in advance

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Definition

A permutation matrix is a square matrix P obtained from I by permuting its rows.

Example:

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

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Facts:

- If P has size $n \times n$, then, for any $n \times m$ matrix A, the product PA is the matrix obtained from A by permuting its rows in the same way (as P from I).
- P is invertible and $P^{-1} = P^T$ (which is also a permutation matrix)

PLU-decomposition

Definition

A PLU-decomposition of a square matrix A is a representation $A = PLU$ where P is a permutation matrix, L is lower triangular and U is upper triangular.

Note: $A = PLU$ is equivalent to $P^T A = LU$ (because $P^{-1} = P^T$).

Theorem

Every square matrix has a PLU-decomposition. *(Proof omitted).*

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How to use it:

- Since P^T is invertible, $A\mathbf{x} = \mathbf{b}$ has the same solutions as $P^T A\mathbf{x} = P^T\mathbf{b}$
- Compute $\mathbf{b}' = P^T\mathbf{b}$, write the above system as $LU\mathbf{x} = \mathbf{b}'$ and solve as before

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Beyond our scope:

- Algorithms for finding a PLU-decomposition

Outline

Wrapping Up

LU Decomposition

1. $A = LU$ where L is lower triangular and U is upper triangular.
2. Application: solving linear equations.
3. Requirements: square matrix, no row exchanges.

PLU Decomposition

1. $A = PLU$ where P is a permutation L is lower triangular and U is upper triangular.
2. Requirements: square matrix

Example Exam Question

(a) Perform the following calculations on the matrix M, show all your working.

$$M = \begin{bmatrix} 2 & 0 & 2 \\ 4 & 3 & 3 \\ 8 & -6 & 0 \end{bmatrix}$$

i. Perform an LU Decomposition on the matrix M. **[6 Marks]**

ii. Use your decomposition to calculate the determinant of M. **[2 Marks]**

Next Class

1. Briefly mentioned last term:
   for fast matrix multiplication, if we get $A = PDP^{-1}$, we can get $A^i, i \geq 2$ quickly.
2. Eigenvalues - diagonal elements in D
3. Eigenvectors - columns of P
4. A cool use of them (Principal Component Analysis)

The End

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We know an LU-decomposition for A (see previous slides):

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Example 10.1 cont'd

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Step 2. Denote $U\mathbf{x} = \mathbf{y}$ and substitute into the above equation to get

$$\begin{pmatrix} 2 & 0 & 0 \\ -3 & 1 & 0 \\ 4 & -3 & 7 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix}$$

In equations, this is

$$\begin{array}{rrrl} 2y_1 & & & = 2 \\ -3y_1 & +y_2 & & = 2 \\ 4y_1 & -3y_2 & +7y_3 & = 3 \end{array}$$

Example 10.1 cont'd

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Step 3. Solve the above (lower triangular) system by *forward substitution*:
Find $y_1 = 1$ from the 1st equation, then substitute it into the 2nd equation and find $y_2 = 5$, then substitute both values into the 3rd equation and find $y_3 = 2$.

Example 10.1 cont'd

Step 4. Now we know $\mathbf{y}$, solve the linear system $U\mathbf{x} = \mathbf{y}$:

$$\begin{pmatrix} 1 & 3 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 5 \\ 2 \end{pmatrix}$$

In equations, this is

$$\begin{array}{rrrl} x_1 & +3x_2 & +x_3 & = 1 \\ & x_2 & +3x_3 & = 5 \\ & & x_3 & = 2 \end{array}$$

This is an upper triangular system, can solve it by *backward substitution*:
Find $x_3 = 2$ from the 3rd equation, then substitute it into the 2nd equation and find $x_2 = -1$, then substitute both values into the 1st equation and find $x_1 = 2$.

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$$E_1 = \begin{pmatrix} 1/2 & 0 \\ 0 & 1 \end{pmatrix} \quad E_2 = \begin{pmatrix} 1 & 0 \\ -3 & 1 \end{pmatrix} \quad E_3 = \begin{pmatrix} 1 & 0 \\ 0 & 1/4 \end{pmatrix}$$

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$$L = E_1^{-1} E_2^{-1} E_3^{-1} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 3 & 4 \end{pmatrix}$$

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$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \dashrightarrow \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \dashrightarrow \begin{pmatrix} 2 & 0 \\ 3 & 1 \end{pmatrix} \dashrightarrow \begin{pmatrix} 2 & 0 \\ 3 & 4 \end{pmatrix} = L$$