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# Maths for Computer Science
# *Calculus*

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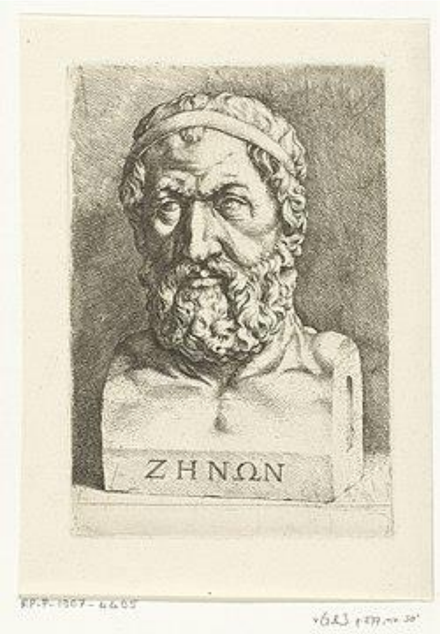
# Series

# Series

- Roughly speaking, a series is the operation of adding infinitely many quantities, one after the other, to a given starting quantity.
- Major part of calculus and mathematical analysis, and they are used widely in quantitative disciplines, including computer science.
- For a long time, the idea that such an infinite summation could produce a finite result was considered paradoxical!
- This paradox was resolved using the concept of a limit during the 17th century.
- This counterintuitive property of infinite sums, that they often sum up to finite quantities, is nicely illustrated by Zenon's Dichotomy paradox.
    - Zeno's Dichotomy paradox: *"That which is in locomotion must arrive at the half-way stage before it arrives at the goal."*

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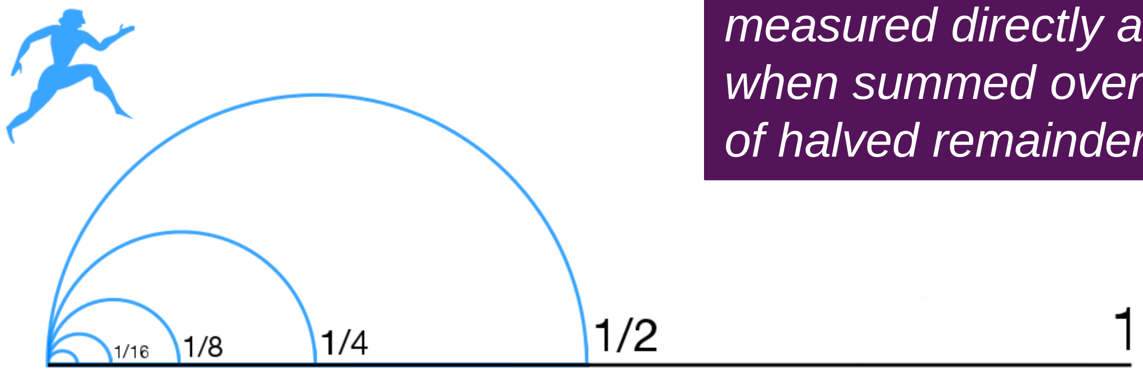
# Dichotomy

- Suppose a runner wishes to race to the end of a path.
- Before she can get there, she must get halfway there.
- Before she can get halfway there, she must get a quarter of the way there.
- Before traveling a quarter, she must travel one-eighth; before an eighth, one-sixteenth; and so on.
  - you continue halving the remaining distances *an infinite number of times*, because no matter how small the remaining distance is you still have to walk the first half of it.

The paradox revealed something was wrong with the assumption that an infinitely long list of numbers greater than zero summed to infinity.

The resolution of the paradox is that **certain series, although they have an infinite number of terms, have a finite sum, even if those terms are all positive**.

*How can a distance be short when measured directly and also infinite when summed over its infinite list of halved remainders?*



# Contents for today's lecture

- Series:
  - Definition
  - Convergence
  - Divergence
  - Tests for convergence / divergence

# Series vs Sequences

A **sequence** is an ordered countably infinite set of numbers $\{a_n\}$.

E.g. $a_0 = \frac{1}{2}, a_1 = \frac{1}{4}, a_2 = \frac{1}{8}, a_3 = \frac{1}{16}, \ldots$

A **series** is the sum of members of a sequence:

E.g. $\sum_{n=0}^{2} a_n = a_0 + a_1 + a_2 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = {}^7\!/_8$.

This is a **finite series.**

E.g. $\sum_{n=0}^{\infty} a_n = a_0 + a_1 + a_2 + a_3 + \cdots = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots$

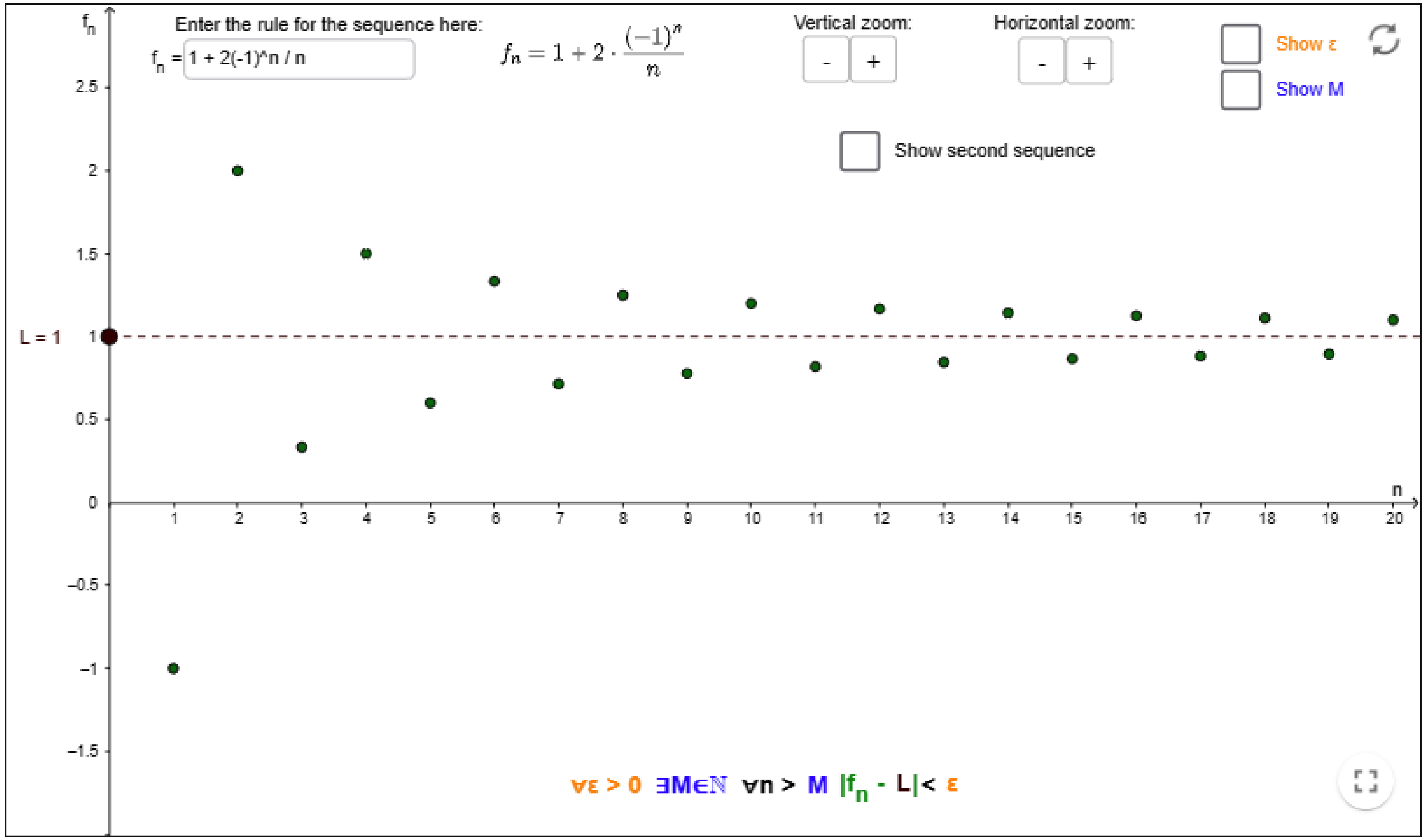
This in an **infinite series.**

The question is often whether a given series "has a value" or not.

# Limit of a Sequence

We can define the limit of a **sequence** $\{a_n\}$, as $n$ approaches infinity, similar to how we define limits of functions.

➢ Sequences can be viewed as functions of their indices.



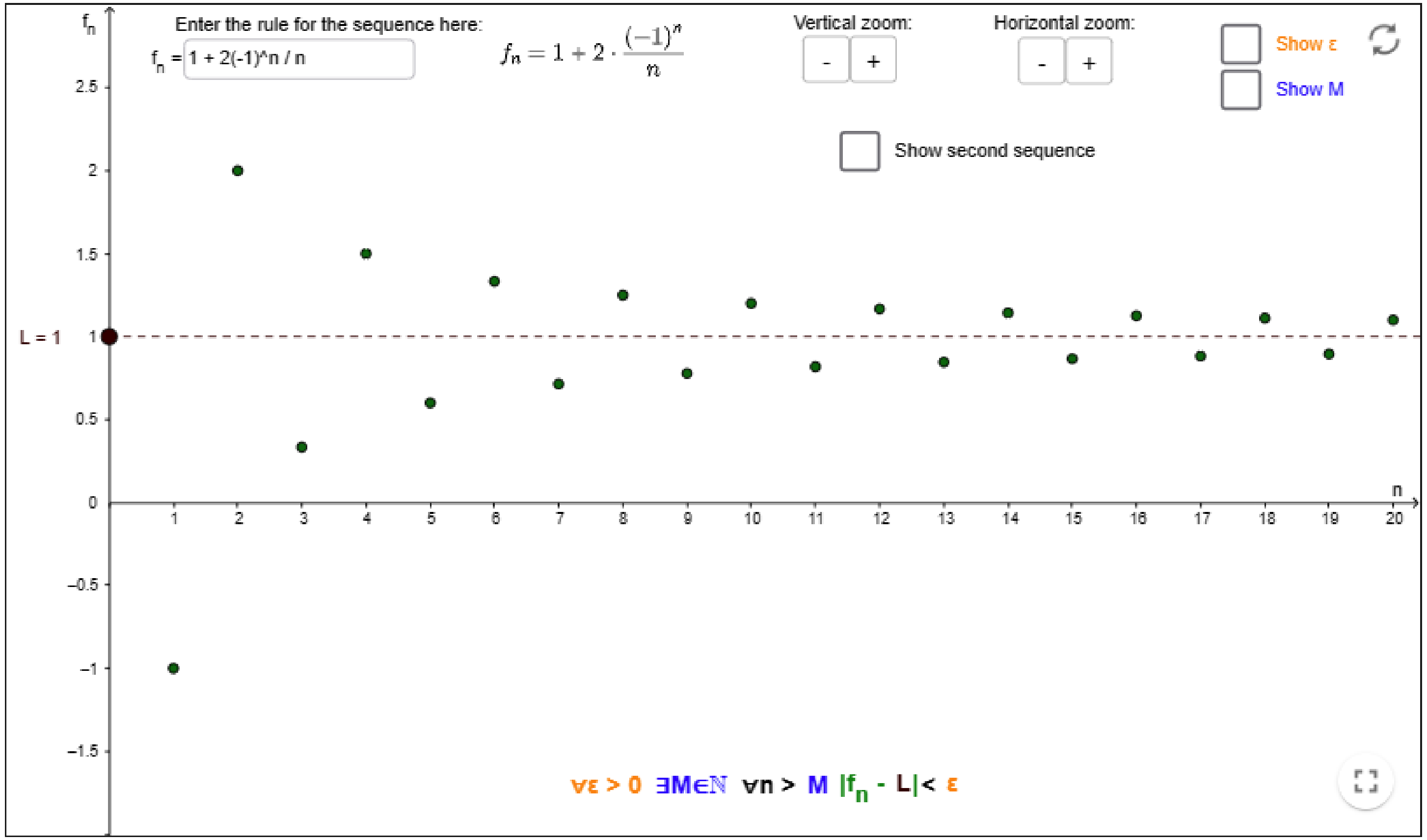

Convergence of a sequence – GeoGebra

# Limit of a Sequence

Symbolically, $\lim_{n\to\infty} a_n = L$

So what does it really mean for a sequence to converge to some value, $L$, i.e. for its limit to be $L$?

$$\forall \boldsymbol{\varepsilon} > \mathbf{0}, \quad \exists \boldsymbol{N} \in \mathbb{N}: \ \forall \boldsymbol{n} > \boldsymbol{N}, |\boldsymbol{a_n} - \boldsymbol{L}| < \boldsymbol{\varepsilon}$$

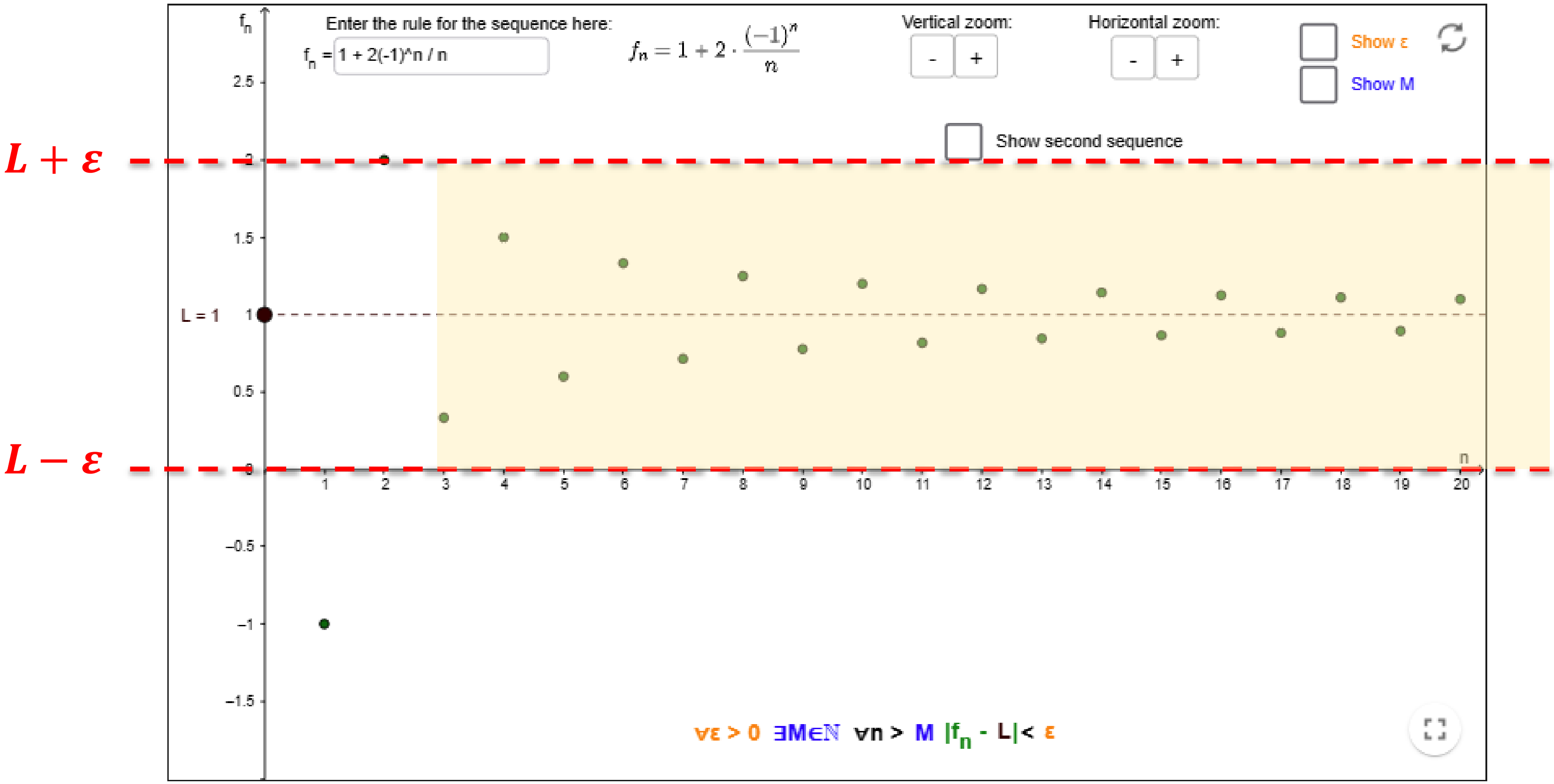



[Convergence of a sequence – GeoGebra]

# Limit of a Sequence

$$\forall \varepsilon > 0, \quad \exists N \in \mathbb{N}: \ \forall n > N, |a_n - L| < \varepsilon$$

$\varepsilon = 1$

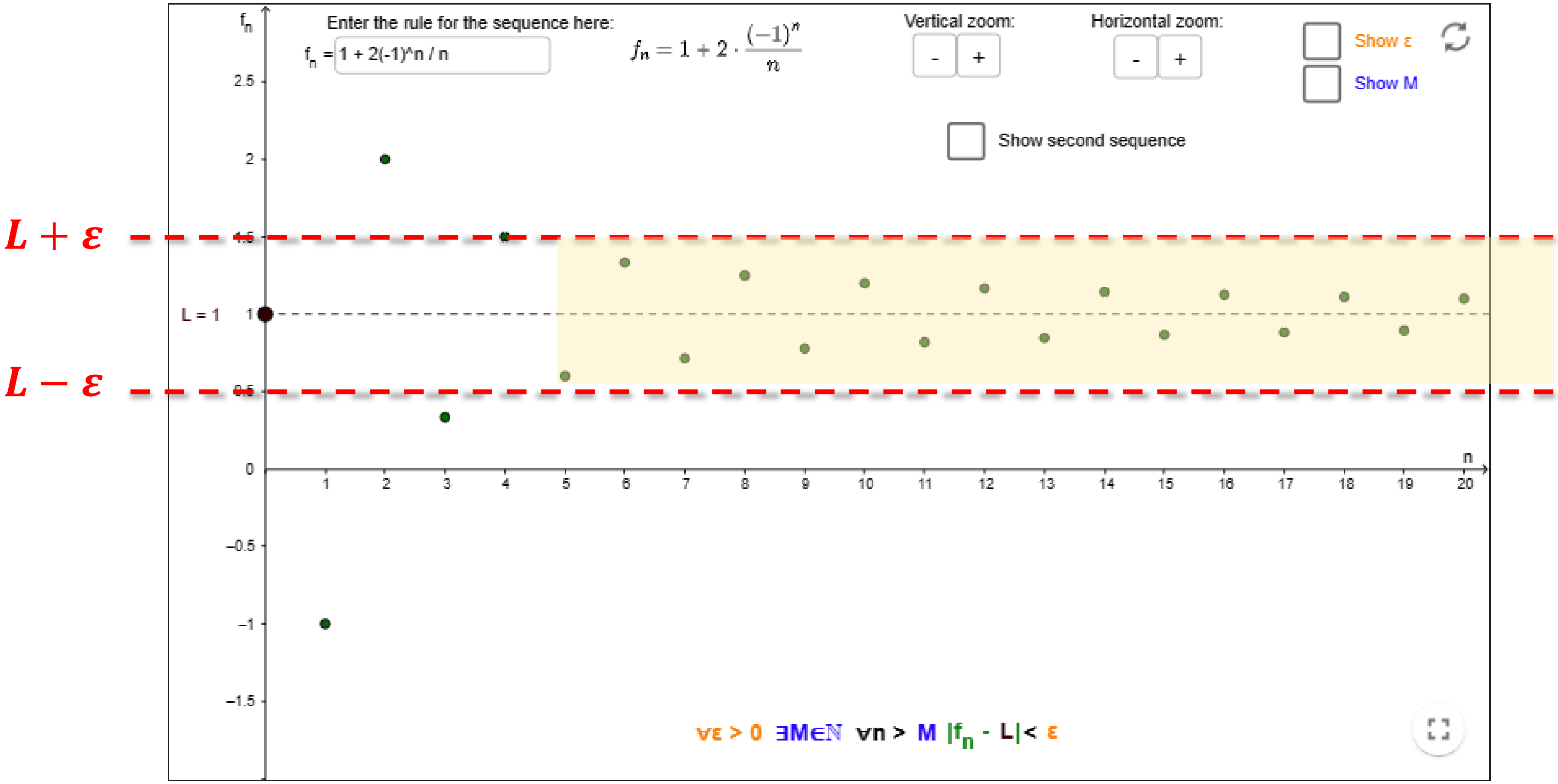



[Convergence of a sequence – GeoGebra]

# Limit of a Sequence

$$\forall \varepsilon > 0, \quad \exists N \in \mathbb{N}: \ \forall n > N, |a_n - L| < \varepsilon$$

$$\varepsilon = 0.5$$

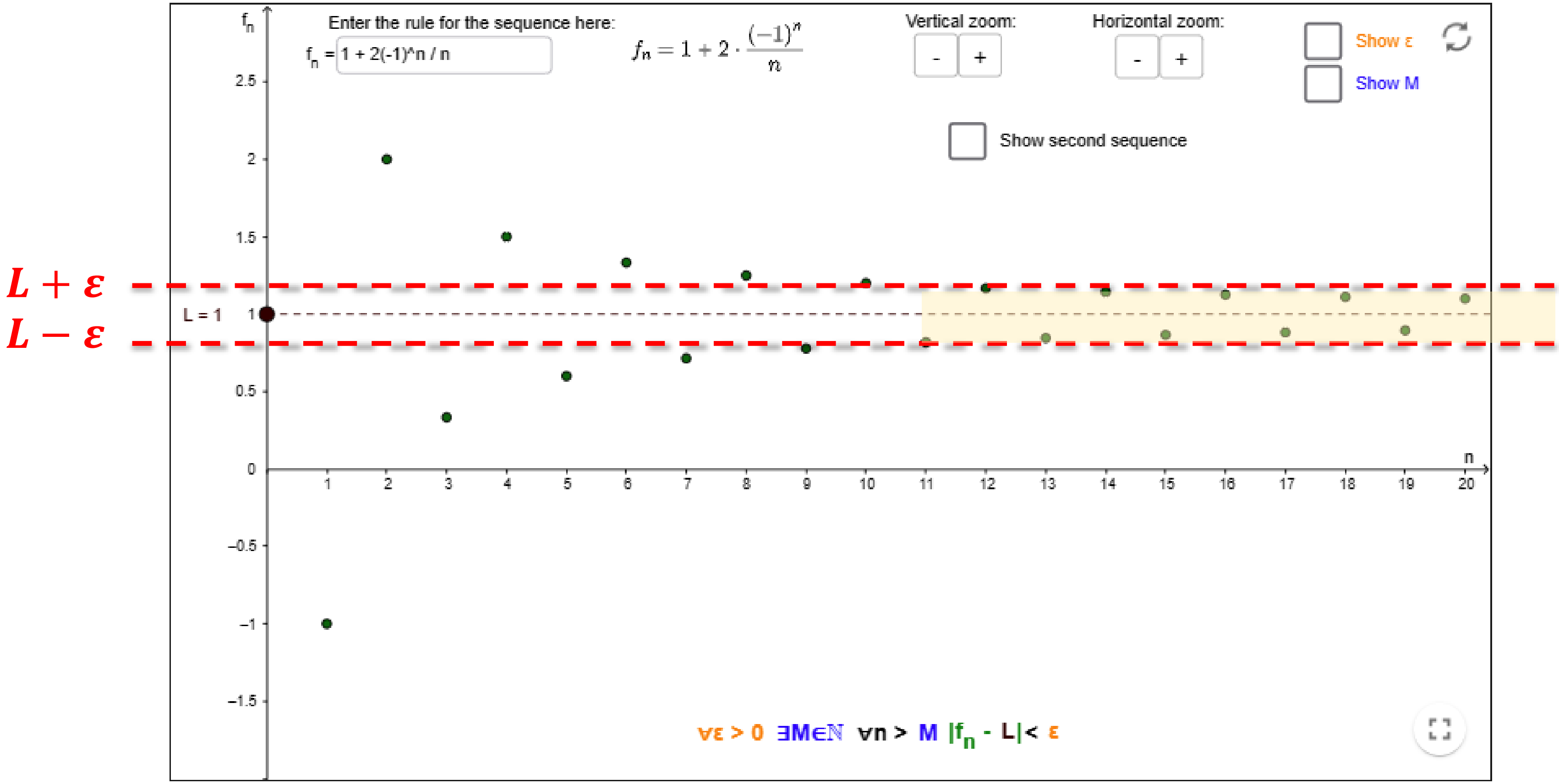



Convergence of a sequence – GeoGebra

# Limit of a Sequence

$$\forall \varepsilon > 0, \quad \exists N \in \mathbb{N}: \ \forall n > N, |a_n - L| < \varepsilon$$

$$\varepsilon = 0.2$$




Convergence of a sequence – GeoGebra

# Series value

Any **finite series** has a well-defined value:

E.g. $\sum_{n=1}^{5} n^2 = 1 + 4 + 9 + 16 + 25 = 55.$

But an **infinite series** may not:

E.g. $\sum_{n=0}^{\infty} \frac{1}{2^n} = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = 2$. This is "convergent"

But $\sum_{n=0}^{\infty} n^2 \to \infty$ is "divergent"

And $\sum_{n=0}^{\infty} (-1)^n$ is also "divergent"

Even series with bounded terms can diverge, e.g., $\sum_{n=0}^{\infty} 1 \to \infty$.

# Series value

For any infinite series $\sum_{n=0}^{\infty} a_n$, we define the partial sum

$$S_n = \sum_{r=0}^{n} a_r = a_0 + a_1 + \cdots + a_n$$

And the remainder

$$R_n = \sum_{r=n+1}^{\infty} a_r = a_{n+1} + a_{n+2} + \cdots$$

The set $\{S_n\}$ is now a sequence and we know that

$$\lim_{n\to\infty} S_n = S$$

means that

for any $\varepsilon > 0, \exists N$ s.t. $\forall n > N$,

$$|S_n - S| < \varepsilon$$

A **series** $\sum_{n=0}^{\infty} a_n$ is convergent to a finite value S if

$$\lim_{n\to\infty} S_n = S$$

Equivalently: if

$$\lim_{n\to\infty} R_n = 0 \quad \left(= \lim_{n\to\infty} (S - S_n)\right).$$

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Is the series $\sum_{n=0}^{\infty} \frac{1}{3^n}$ convergent or divergent?

- Convergent
- Divergent
- None of the above

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## Is the series $\sum_{n=0}^{\infty} \frac{1}{3^n}$ convergent or divergent?

Convergent

0%

Divergent

0%

None of the above

0%

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# Is the series $\sum_{n=0}^{\infty} \frac{1}{3^n}$ convergent or divergent?

Convergent

0%

Divergent

0%

None of the above

0%

# Example

$$\sum_{n=0}^{\infty} \frac{1}{3^n}$$

$\sum_{n=0}^{\infty} \frac{1}{3^n} = 1 + \frac{1}{3} + \frac{1}{9} + \cdots$ is a geometric series with initial value 1 and ratio 1/3.

Hence $S_n = \frac{1-\left(\frac{1}{3}\right)^{n+1}}{1-\frac{1}{3}} = \frac{3}{2}\left(1 - \left(\frac{1}{3}\right)^{n+1}\right)$.

So $\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{3}{2}\left(1 - \left(\frac{1}{3}\right)^{n+1}\right) = \frac{3}{2}$.

So this series is convergent to ${}^{3}/_{2}$.

Notice that $R_n = \frac{3}{2} - S_n = \frac{3}{2}\left(\frac{1}{3}\right)^{n+1} = \frac{1}{2}\left(\frac{1}{3}\right)^{n}$. Clearly $\lim_{n\to\infty} R_n = 0$.

# Example

$$\sum_{n=0}^{\infty} \frac{1}{3^n}$$

**Note:** analysis of $R_n$ can tell us how many terms of the series we need to take to get a sufficiently accurate approximation to the series' value.

E.g. here the sum of the **first five terms** is $1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81} = 1.493827\ldots$ which is **within 1%** of the true value.

You can come to the same conclusion by looking at the remainder of the partial sum of the first five terms:

$$R_4 = a_5 + a_6 + \cdots = \frac{1}{2}\left(\frac{1}{3}\right)^4 = \frac{1}{162} < 0.01.$$

# Sum and difference of convergent series

If we have some convergent series, we may wish to use them to help us when we analyse more complex series.

**Theorem:**

If the series $\sum_{n=0}^{\infty} a_n$ and the series $\sum_{n=0}^{\infty} b_n$ are convergent, with respective sums $\alpha$ and $\beta$, then

$$\sum_{n=0}^{\infty} (a_n + b_n) = \alpha + \beta$$

And

$$\sum_{n=0}^{\infty} (a_n - b_n) = \alpha - \beta$$

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**Proof:**

$\sum_{n=0}^{\infty} a_n$ converges to $\alpha$ and $\sum_{n=0}^{\infty} b_n$ converges to $\beta$.
Given arbitrary $\epsilon > 0$, let $\epsilon_a = \frac{\epsilon}{2} = \epsilon_b$.

Since $\sum_{n=0}^{\infty} a_n$ converges, $\exists N_a$ such that $|\sum_{r=0}^{n} a_r - \alpha| < \epsilon_a, \forall n > N_a$.
Similarly, since $\sum_{n=0}^{\infty} b_n$ converges, $\exists N_b$ s.t. $|\sum_{r=0}^{n} b_r - \beta| < \epsilon_b, \forall n > N_b$.

Take $N = \max\{N_a, N_b\}$. Then, $\forall n > N$,

$$\left|\sum_{r=0}^{n}(a_r + b_r) - (\alpha + \beta)\right| = \left|\left(\sum_{r=0}^{n} a_r - \alpha\right) + \left(\sum_{r=0}^{n} b_r - \beta\right)\right| \leq$$

$$\leq \left|\sum_{r=0}^{n} a_r - \alpha\right| + \left|\sum_{r=0}^{n} b_r - \beta\right| \leq \epsilon_a + \epsilon_b = \epsilon,$$

i.e. $\sum_{n=0}^{\infty}(a_n + b_n)$ converges to $\alpha + \beta$.

*Using triangle inequality: $|a + b| \leq |a| + |b|$.

Note: By taking $\epsilon_a = 2\epsilon$ and $\epsilon_b = \epsilon$, you can show that $\sum_{n=0}^{\infty}(a_n - b_n)$ converges to $\alpha - \beta$

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# First test for divergence

If $\lim\limits_{n\to\infty} a_n \neq 0$ then the series $\sum_{n=0}^{\infty} a_n$ diverges.

**Proof:**

Suppose $\sum_{n=0}^{\infty} a_n$ converges to some $S$. Then, $\lim\limits_{n\to\infty} S_n = S$, which means that given arbitrary $\epsilon > 0$ there exists some $N_\epsilon$ such that for all $n > N_\epsilon$,

$$|S_n - S| < \epsilon.$$

But also $|S_{n+1} - S| < \epsilon$, so combining these*:

$$2\epsilon > |S_n - S| + |S_{n+1} - S| = |S - S_n| + |S_{n+1} - S| \geq |S_{n+1} - S_n| = |a_{n+1}|$$

But $\epsilon$ was arbitrary, so we have shown for any $\epsilon' > 0$ there exists an $N_{\epsilon'}$ such that $|a_n| < \epsilon'$ for all $n > N_{\epsilon'}$.

I.e. $\lim\limits_{n\to\infty} |a_n| = 0$, and so $\lim\limits_{n\to\infty} a_n = 0$.

The contrapositive is the 1st divergence test.

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*Using triangle inequality: $|a + b| \leq |a| + |b|$.

# Absolute convergence implies convergence

If $\sum_{n=1}^{\infty} |a_n|$ converges then $\sum_{n=1}^{\infty} a_n$ converges.

**Proof:**

Let $S_n = \sum_{r=1}^{n} a_r$ and $T_n = \sum_{r=1}^{n} |a_r|$.

For each $r$ we have $0 \leq a_r + |a_r| \leq 2|a_r|$,

so $0 \leq S_n + T_n \leq 2T_n$.

But we know $\lim_{n\to\infty} T_n = T$ exists, so

$$0 \leq \lim_{n\to\infty}(S_n + T_n) \leq 2T$$

and so the series $\sum_{n=1}^{\infty}(a_n + |a_n|)$ converges, too.

By the “Difference of Convergent Series” Theorem, the difference

$$\sum_{n=1}^{\infty}(a_n + |a_n|) - \sum_{n=1}^{\infty}|a_n| = \sum_{n=1}^{\infty} a_n$$

also converges.

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# Comparison test

Fundamental 'rules' used to also prove the rest of the convergence tests we will see:

**Convergence:**

Let $\sum_{n=1}^{\infty} b_n$ be a convergent series of positive terms.

Let $\sum_{n=1}^{\infty} a_n$ be a series and $N$ be an integer such that

$$|a_n| \le b_n \text{ for all } n > N$$

then $\sum_{n=1}^{\infty} a_n$ is an absolutely convergent series.

**Divergence:**

Let $\sum_{n=1}^{\infty} b_n$ be a divergent series of positive terms.

Let $\sum_{n=1}^{\infty} a_n$ be a series and $N$ be an integer such that

$$0 \le b_n \le a_n \text{ for all } n > N$$

then $\sum_{n=1}^{\infty} a_n$ is a divergent series.

# The Ratio test

**Convergence:**

Let $\sum_{n=1}^{\infty} a_n$ be a series where $a_n \neq 0$ and such that $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L$ then

a) If $\mathrm{L} < 1$, $\sum_{n=1}^{\infty} a_n$ is an absolutely convergent series.

b) If $\mathrm{L} > 1$, $\sum_{n=1}^{\infty} a_n$ is a divergent series.

c) If $\mathrm{L} = 1$, the test fails.

# The Ratio test

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c) If $\mathrm{L} = 1$, the test fails.

**Example:**

$\sum_{n=1}^{\infty} \frac{(-1)^n n!}{n^n}$ ?

We have $\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)!n^n}{n!(n+1)^{n+1}} = \frac{n^n}{(n+1)^n} = \frac{1}{\left(1+\frac{1}{n}\right)^n}$,

so $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{1}{\left(1+\frac{1}{n}\right)^n} = \frac{1}{e}$ .

Since $\frac{1}{e} < 1$, the series absolutely converges.

# The Ratio test

**Convergence:**

Let $\sum_{n=1}^{\infty} a_n$ be a series where $a_n \neq 0$ and such that $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L$ then

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c) If $\mathrm{L} = 1$, the test fails.

**Proof a):**

If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L < 1$ then $\exists\, r$, with $L < r < 1$, and $N$ s.t. $\forall\, n > N$:

$$\left|\frac{a_{n+1}}{a_n}\right| < r \Rightarrow |a_{n+1}| < r|a_n|.$$

Then $|a_{n+2}| < r|a_{n+1}| < r^2|a_n|$, and $|a_{n+3}| < r|a_{n+2}| < r^3|a_n|$, etc.

So the remainder is:

$$R_{n+m} = |a_{n+m+1}| + |a_{n+m+2}| + \cdots \leq r^{m+1}|a_n| + r^{m+2}|a_n| + \cdots = |a_n| r^{m+1}(1 + r + r^2 + \cdots)$$

$$= |a_n| r^{m+1}\frac{1}{1-r} \to 0 \quad \text{as} \quad m \to \infty$$

Hence the series converges.

# The $n^{\text{th}}$ root test

Consider the series $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \left(\frac{nk}{3n+1}\right)^n$, where $k$ is a constant.

First observe that $\lim\limits_{n\to\infty} \sqrt[n]{a_n} = \lim\limits_{n\to\infty} \left(\frac{nk}{3n+1}\right) = \frac{k}{3}$.

- **If $\boldsymbol{k < 3}$**, then there is an $r \in \left(\frac{k}{3}, 1\right)$ and $N_r$ so that for all $n > N_r$, $\sqrt[n]{a_n} < r$.

  Thus $R_n = a_{n+1} + a_{n+2} + \cdots < r^{n+1} + r^{n+2} + \cdots = r^{n+1}\frac{1}{1-r} \to 0$ as $n \to \infty$.

  So the series is convergent.

- **If $\boldsymbol{k > 3}$**, then there is an $r \in \left(1, \frac{k}{3}\right)$ and $N_r$ so that for all $n > N_r$, $\sqrt[n]{a_n} > r$.

  Thus $R_n = a_{n+1} + a_{n+2} + \cdots > r^{n+1} + r^{n+2} + \cdots = r^{n+1}\frac{1}{1-r} \to \infty$ as $n \to \infty$.

  So the series is divergent.

# The $n^{\text{th}}$ root test

Consider the series $\sum_{n=1}^{\infty} a_n$ such that $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L$.

a) If $\mathrm{L} < 1$, $\sum_{n=1}^{\infty} a_n$ is an absolutely convergent series.

b) If $\mathrm{L} > 1$, $\sum_{n=1}^{\infty} a_n$ is a divergent series.

c) If $\mathrm{L} = 1$, the test fails.

# Alternating series test

The series $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$ converges if

- $a_n > 0$, and
- $a_{n+1} \leq a_n$ for all $n$, and
- $\lim\limits_{n\to\infty} a_n = 0$.

**Proof:**

Consider the partial sum

$$S_{2r} = a_1 - a_2 + a_3 - a_4 + a_5 \ldots - a_{2r-2} + a_{2r-1} - a_{2r}.$$

First note that

$$S_{2r} = (a_1 - a_2) + (a_3 - a_4) + \cdots + (a_{2r-1} - a_{2r}) \geq 0$$

and $\{S_{2r}\}$ is **monotonically increasing**.

Also

$$S_{2r} = a_1 - (a_2 - a_3) - (a_4 - a_5) \ldots - (a_{2r-2} - a_{2r-1}) - a_{2r} < a_1$$

So $\{S_{2r}\}$ is also **bounded**; hence converges to $S$.

But $S_{2r+1} = S_{2r} + a_{2r+1}$, and so $\lim\limits_{r\to\infty} S_{2r+1} = \lim\limits_{r\to\infty} S_{2r} + \lim\limits_{r\to\infty} a_{2r+1} = S + 0 = S$.

And as both the odd and even partial sums $\{S_{2r}\}$ and $\{S_{2r+1}\}$ tend to $S$, so does $\{S_n\}$.

# Grouping and rearrangement of series

For any finite sum we can group and rearrange terms as much as we like.

$$1+3-5-6=(1-5)-(6-3)$$

For an infinite sum we need to be careful!

Consider $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots$

By the alternating series test this is convergent, to some value $S$.

So

$$S=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\cdots$$

Rearranging

$$S=\left(1-\frac{1}{2}\right)-\frac{1}{4}+\left(\frac{1}{3}-\frac{1}{6}\right)-\frac{1}{8}+\left(\frac{1}{5}-\frac{1}{10}\right)-\frac{1}{12}\ldots+\left(\frac{1}{2k+1}-\frac{1}{4k+2}\right)-\frac{1}{4k+4}+\cdots$$

$$S=\frac{1}{2}\left(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\cdots\right)=\frac{1}{2}S$$

So as $S=\frac{1}{2}S$ we must have $S=0$, but actually $S>\frac{1}{2}$. (Exercise).

# Before we look at the next theorem...

Every subsequence of a convergent sequence converges to the same limit as the original sequence.

**A subsequence of $\{a_n\}$ is a sequence that can be derived from $\{a_n\}$ by deleting some or no elements without changing the order of the remaining elements.**

Formally, a subsequence of $\{a_n\}, n \in \mathbb{N}$, is a sequence of the form $\{a_{n_k}\}, k \in \mathbb{N}$, where $n_k$ is a strictly increasing sequence of positive integers.

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# Before we look at the next theorem...

Every subsequence of a convergent sequence converges to the same limit as the original sequence.

**Proof.**

Let $\{a_{n_k}\}$ be a subsequence of $\{a_n\}$.

First note that $n_k \geq k$ for all $k$; easy to prove by induction: $n_1 \geq 1$ and $n_k \geq k$ implies $n_{k+1} > n_k \geq k$ and so $n_{k+1} \geq k+1$.

Let $\lim_{n\to\infty} a_n = L$. Then for any $\epsilon > 0$, there exists $N_\epsilon$ such that for all $n > N_\epsilon$ it holds that $|a_n - L| < \epsilon$.

Let $k > N_\epsilon$. Then, $n_k > N_\epsilon$ and so it must be that $\left|a_{n_k} - L\right| < \epsilon$.

# Grouping and rearrangement of series

Without reordering

If $\sum_{n=1}^{\infty} a_n$ converges, then we can insert brackets/groupings without altering the sum.

**Proof:**

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + \cdots$$

Consider bracketing $(a_1 + a_2) + (a_3 + a_4 + a_5) + (a_6 + \cdots$ etc.

Define new series $\sum_{n=1}^{\infty} b_n$, where $b_n$ is the $n^{\text{th}}$ bracketed term of $\sum_{n=1}^{\infty} a_n$.

Now $\sum_{n=1}^{\infty} b_n = (a_1 + a_2) + (a_3 + a_4 + a_5) + (a_6 + \cdots$ etc.

But the partial sums $T_n$ of this series are a subsequence of the partial sums $S_n$ of $\sum_{n=1}^{\infty} a_n$.

Since $\sum_{n=1}^{\infty} a_n$ converges to some $S$, we have $\lim_{n\to\infty} S_n = S$, and all subsequences must also converge to $S$.

Therefore $\lim_{n\to\infty} T_n = S$ and so $\sum_{n=1}^{\infty} b_n$ converges.

# Grouping and rearrangement of series

If $\sum_{n=1}^{\infty} a_n$ absolutely converges, then we can reorder the terms without altering the sum.

**Proof:**

Let $\sum_{n=1}^{\infty} b_n$ be a reordering of $\sum_{n=1}^{\infty} a_n$.

Since $\sum_{n=1}^{\infty} |a_n|$ converges, say to $S$, and the partial sums $T_n = \sum_{r=1}^{n} |b_r|$ are monotonic increasing and bounded above by $S$, $\sum_{r=1}^{n} |b_r|$ also converges.

Since $\sum_{r=1}^{n} |b_r|$ converges, $\sum_{r=1}^{n} b_r$ also converges.

Let $\sum_{r=1}^{n} a_r = S_n'$ and $\sum_{n=1}^{\infty} a_n = S'$.

Let $\sum_{r=1}^{n} b_r = T_n'$ and $\sum_{n=1}^{\infty} b_n = T'$.

Then given $\epsilon > 0$, $\exists\, N$: $\forall n > N$, $|S_n' - S'| < \epsilon$.

Then for large enough $m$, $T_m' = S_n' + a_p + a_q + \cdots + a_r$ for some $a_p, a_q, \dots, a_t$, where $n < p, q, \dots, t$.

Then $|T_m' - S'| \le |S_n' - S'| + \underbrace{|a_p| + |a_q| + \cdots + |a_t|} < 2\epsilon$. So $T_m' \to S'$ as $n \to \infty$.

All these terms occur after $a_n$ in the series $\sum_{k=1}^{\infty} a_k$ and since $|S_n{}' - S'| < \epsilon$ their total contribution cannot exceed $\epsilon$.

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# What we learnt today

- A series is the sum of members of a sequence.
  - $\sum_{n=0}^{\infty} a_n$ converges if it takes a finite value, say $S$.
  - Equivalently, the series converges if $\lim_{n\to\infty} \sum_{r=0}^{n} a_r = S$.
- The sum and the difference of two convergent series converge.
- Tests for convergence / divergence:
  - If $\lim_{n\to\infty} a_n \neq 0$ then the series $\sum_{n=0}^{\infty} a_n$ diverges.
  - If $\sum_{n=1}^{\infty} |a_n|$ converges then $\sum_{n=1}^{\infty} a_n$ converges.
  - Comparison test
  - Ratio test
  - Root test