COMP1021 Mathematics for Computer Science
Linear Algebra (Part 2)
Practical - Week 18
March 2025

Karl Southern

Instructions: Work on these problems in the practical sessions for the week specified. First try them on your own. If you're stuck, try discussing things with others. If you get the answer, still discuss with others to see if maybe you missed something. If you run into major roadblocks, ask the demonstrators for hints.

Solutions will be posted on Learn Ultra at the end of the week. Make sure you're all set with the solutions and understand them before the next practical.

Purpose of this practical: This practical will help you build your competency with the Gram-Schmidt process, QR decomposition, and the least squares method. Use the slides from Lecture 5 as a reference for different inner product definitions.

1. Use the Gram-Schmidt process to find a QR decomposition for each of the following matrices:

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 0 \\ 1 & 2 \\ 1 & 3 \\ 1 & 3 \end{pmatrix}.$$

(*Hint: For matrix A, check the example on slide 12 in lecture 6.*)

Solution:

<u>Matrix A</u>. Apply the Gram-Schmidt process to columns vectors of A, i.e., to $\mathbf{u}_1 = (1,1,1), \mathbf{u}_2 = (0,1,1), \mathbf{u}_3 = (0,0,1)$ – actually, this was done in that example on slide 12. We obtained the orthonormal basis

$$\{\mathbf{q}_1 = (\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}),\ \mathbf{q}_2 = (-\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}),\ \mathbf{q}_3 = (0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})\}.$$

These vectors are the columns vectors of Q in the required QR decomposition. To find R, recall that

$$R = \begin{pmatrix} \langle \mathbf{u}_1, \mathbf{q}_1 \rangle & \langle \mathbf{u}_2, \mathbf{q}_1 \rangle & \langle \mathbf{u}_3, \mathbf{q}_1 \rangle \\ 0 & \langle \mathbf{u}_2, \mathbf{q}_2 \rangle & \langle \mathbf{u}_3, \mathbf{q}_2 \rangle \\ 0 & 0 & \langle \mathbf{u}_3, \mathbf{q}_3 \rangle \end{pmatrix} = \begin{pmatrix} 3/\sqrt{3} & 2/\sqrt{3} & 1/\sqrt{3} \\ 0 & 2/\sqrt{6} & 1/\sqrt{6} \\ 0 & 0 & 1/\sqrt{2} \end{pmatrix}$$

Thus,

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} = A = QR = \begin{pmatrix} 1/\sqrt{3} & -2/\sqrt{6} & 0 \\ 1/\sqrt{3} & 1/\sqrt{6} & -1/\sqrt{2} \\ 1/\sqrt{3} & 1/\sqrt{6} & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 3/\sqrt{3} & 2/\sqrt{3} & 1/\sqrt{3} \\ 0 & 2/\sqrt{6} & 1/\sqrt{6} \\ 0 & 0 & 1/\sqrt{2} \end{pmatrix}$$

<u>Matrix B</u>. Apply Gram-Schmidt to the column vectors of B: $\mathbf{u}_1 = (1,1,1,1)$ and $\mathbf{u}_2 = (0,2,3,3)$.

Step 1. $\mathbf{v}_1 = \mathbf{u}_1 = (1,1,1,1)$.

Step 2. $\mathbf{v}_2 = \mathbf{u}_2 - \frac{\langle \mathbf{u}_2, \mathbf{v}_1 \rangle}{||\mathbf{v}_1||^2} \mathbf{v}_1 = (0,2,3,3) - \frac{8}{4}(1,1,1,1) = (-2,0,1,1)$.

We have $||\mathbf{v}_1|| = 2$ and $||\mathbf{v}_2|| = \sqrt{6}$. Normalising $\mathbf{v}_1$ and $\mathbf{v}_2$, we get

$$\mathbf{q}_1 = \frac{1}{||\mathbf{v}_1||} \mathbf{v}_1 = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}),\ \ \mathbf{q}_2 = \frac{1}{||\mathbf{v}_2||} \mathbf{v}_2 = (-\frac{2}{\sqrt{6}}, 0, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}).$$

These are the column vectors of matrix Q in the decomposition. To find R, recall that

$$R = \begin{pmatrix} \langle \mathbf{u}_1, \mathbf{q}_1 \rangle & \langle \mathbf{u}_2, \mathbf{q}_1 \rangle \\ 0 & \langle \mathbf{u}_2, \mathbf{q}_2 \rangle \end{pmatrix} = \begin{pmatrix} 2 & 4 \\ 0 & \sqrt{6} \end{pmatrix}$$

Thus,

$$\begin{pmatrix} 1 & 0 \\ 1 & 2 \\ 1 & 3 \\ 1 & 3 \end{pmatrix} = B = QR = \begin{pmatrix} 1/2 & -2/\sqrt{6} \\ 1/2 & 0 \\ 1/2 & 1/\sqrt{6} \\ 1/2 & 1/\sqrt{6} \end{pmatrix} \begin{pmatrix} 2 & 4 \\ 0 & \sqrt{6} \end{pmatrix}.$$

2. Consider $\mathbb{R}^3$ with the weighted Euclidean inner product with weights $w_1 = 1, w_2 = 2, w_3 = 3$. Use the Gram-Schimdt process to transform $\mathbf{u}_1 = (1,1,1), \mathbf{u}_2 = (1,1,0), \mathbf{u}_3 = (1,0,0)$ to an orthonormal basis.

Solution:

We have $\langle \mathbf{u}, \mathbf{v} \rangle = u_1 v_1 + 2u_2 v_2 + 3u_3 v_3$. Apply Gram-Schmidt to $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$.

Step 1. $\mathbf{v}_1 = \mathbf{u}_1 = (1,1,1)$. (We'll later need $||\mathbf{v}_1||^2 = 6$).

Step 2. $\mathbf{v}_2 = \mathbf{u}_2 - \frac{\langle \mathbf{u}_2, \mathbf{v}_1 \rangle}{||\mathbf{v}_1||^2} \mathbf{v}_1 = (1,1,0) - \frac{3}{6}(1,1,1) = (\frac{1}{2}, \frac{1}{2}, -\frac{1}{2})$. (We'll later need $||\mathbf{v}_2||^2 = \frac{3}{2}$).

Step 3. $\mathbf{v}_3 = \mathbf{u}_3 - \frac{\langle \mathbf{u}_3, \mathbf{v}_1 \rangle}{||\mathbf{v}_1||^2} \mathbf{v}_1 - \frac{\langle \mathbf{u}_3, \mathbf{v}_2 \rangle}{||\mathbf{v}_2||^2} \mathbf{v}_2 = (1,0,0) - \frac{1}{6}(1,1,1) - \frac{1/2}{3/2}(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}) = (\frac{2}{3}, -\frac{1}{3}, 0)$.

Compute $||\mathbf{v}_3||^2 = \frac{2}{3}$. Normalising the vectors $\mathbf{v}_i$ (scaling by $\frac{1}{||\mathbf{v}_i||}$, not by $\frac{1}{||\mathbf{v}_i||^2}$!), we get

$$\mathbf{q}_1 = (\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}),\ \mathbf{q}_2 = (\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}}),\ \mathbf{q}_3 = (\frac{2}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, 0).$$

3. Consider the inner product space $C[0,1]$. Find an orthogonal basis of $W = span(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3)$ where $\mathbf{u}_1 = 1, \mathbf{u}_2 = x, \mathbf{u}_3 = x^2$.

Solution:

Recall the definition of inner product in $C[0,1]$:

$$\langle \mathbf{f}, \mathbf{g} \rangle = \int_0^1 f(x)g(x)\, dx.$$

Apply Gram-Schmidt to $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$.

Step 1. $\mathbf{v}_1 = \mathbf{u}_1 = 1$. (We'll later need $||\mathbf{v}_1||^2 = 1$).

Step 2. $\mathbf{v}_2 = \mathbf{u}_2 - \frac{\langle \mathbf{u}_2, \mathbf{v}_1 \rangle}{||\mathbf{v}_1||^2} \mathbf{v}_1 = x - \frac{1/2}{1} 1 = x - \frac{1}{2}$. We used $\langle \mathbf{u}_2, \mathbf{v}_1 \rangle = \int_0^1 1x\, dx = \frac{1}{2}$.
We'll later need $||\mathbf{v}_2||^2 = \int_0^1 (x - \frac{1}{2})^2\, dx = \frac{1}{12}$.

Step 3. $\mathbf{v}_3 = \mathbf{u}_3 - \frac{\langle \mathbf{u}_3, \mathbf{v}_1 \rangle}{||\mathbf{v}_1||^2} \mathbf{v}_1 - \frac{\langle \mathbf{u}_3, \mathbf{v}_2 \rangle}{||\mathbf{v}_2||^2} \mathbf{v}_2$. We have

$$\langle \mathbf{u}_3, \mathbf{v}_1 \rangle = \int_0^1 x^2\, dx = \frac{1}{3} \qquad \text{and} \qquad \langle \mathbf{u}_3, \mathbf{v}_2 \rangle = \int_0^1 x^2(x - \frac{1}{2})\, dx = \frac{1}{12},$$

so $\mathbf{v}_3 = x^2 - \frac{1/3}{1} 1 - \frac{1/12}{1/12}(x - \frac{1}{2}) = x^2 - x + \frac{1}{6}$.

The orthogonal basis is $\mathbf{v}_1 = 1,\ \mathbf{v}_2 = x - \frac{1}{2},\ \mathbf{v}_3 = x^2 - x + \frac{1}{6}$.

4. Find all least squares solutions to the linear system $A\mathbf{x} = \mathbf{b}$ where

$$A = \begin{pmatrix} 1 & -2 \\ 2 & -4 \\ 3 & -6 \end{pmatrix} \text{ and } \mathbf{b} = \begin{pmatrix} 4 \\ 2 \\ -1 \end{pmatrix}$$

Solution:

The columns of A are linearly dependent, so there will be multiple solutions. We have

$$A^T A = \begin{pmatrix} 1 & 2 & 3 \\ -2 & -4 & -6 \end{pmatrix} \begin{pmatrix} 1 & -2 \\ 2 & -4 \\ 3 & -6 \end{pmatrix} = \begin{pmatrix} 14 & -28 \\ -28 & 56 \end{pmatrix}$$

$$A^T \mathbf{b} = \begin{pmatrix} 1 & 2 & 3 \\ -2 & -4 & -6 \end{pmatrix} \begin{pmatrix} 4 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 5 \\ -10 \end{pmatrix}$$

The normal system associated with is $A\mathbf{x} = \mathbf{b}$ is

$$\begin{pmatrix} 14 & -28 \\ -28 & 56 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 5 \\ -10 \end{pmatrix}$$

Clearly, the two linear equations $14x_1 - 28x_2 = 5$ and $-28x_1 + 56x_2 = -10$ are proportional, so we can take only the first of them, and the solutions are all pairs $(2x + \frac{5}{14}, x)$.

5. Find the least squares solutions to $A\mathbf{x} = b$ where

$$A = \begin{pmatrix} 1 & 0 \\ 1 & 2 \\ 1 & 3 \\ 1 & 3 \end{pmatrix} \text{ and } b = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 2 \end{pmatrix}.$$

Solution:

We found a QR decomposition of the coefficient matrix of this system in Q1:

$$\begin{pmatrix} 1 & 0 \\ 1 & 2 \\ 1 & 3 \\ 1 & 3 \end{pmatrix} = QR = \begin{pmatrix} 1/2 & -2/\sqrt{6} \\ 1/2 & 0 \\ 1/2 & 1/\sqrt{6} \\ 1/2 & 1/\sqrt{6} \end{pmatrix} \begin{pmatrix} 2 & 4 \\ 0 & \sqrt{6} \end{pmatrix}.$$

Hence, we can find the least squares solution from the system

$$\begin{pmatrix} 2 & 4 \\ 0 & \sqrt{6} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1/2 & 1/2 & 1/2 & 1/2 \\ -2/\sqrt{6} & 0 & 1/\sqrt{6} & 1/\sqrt{6} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1/\sqrt{6} \end{pmatrix}$$

From this, we have $x_2 = \frac{1}{6}$ and $x_1 = \frac{2}{3}$.

6. (Optional) (Hard) Define the least squares approximation problem in the inner product space $C[a, b]$ as follows: given a vector $\mathbf{f} \in C[a, b]$ and a finite-dimensional subspace W in $C[a, b]$, find a vector $\mathbf{g} \in W$ that minimises $||\mathbf{f} - \mathbf{g}||$. The best approximation theorem from lecture 17 applies in this case, so $\mathbf{g} = \text{proj}_W \mathbf{f}$ is the least squares approximation to $\mathbf{f}$ from W.

A trigonometric polynomial of order $\leq n$ is a function of the form

$$c_0 + c_1 \cos x + c_2 \cos(2x) + \ldots + c_n \cos(nx) + d_1 \sin x + d_2 \sin(2x) + \ldots + d_n \sin(nx).$$

Such polynomials form a subspace in $C[0, 2\pi]$, let's call it W_n.

(a) Check that the following functions form an orthonormal basis in W_n:

$$\mathbf{g}_0 = \frac{1}{\sqrt{2\pi}}, \mathbf{g}_1 = \frac{1}{\sqrt{\pi}} \cos x, \ldots, \mathbf{g}_n = \frac{1}{\sqrt{\pi}} \cos(nx), \mathbf{g}_{n+1} = \frac{1}{\sqrt{\pi}} \sin x, \ldots, \mathbf{g}_{2n} = \frac{1}{\sqrt{\pi}} \sin(nx).$$

Solution:

It is obvious that $W_n = span(\mathbf{g}_0, \mathbf{g}_1, \dots, \mathbf{g}_{2n})$. We need to check that $\langle \mathbf{g}_i, \mathbf{g}_j \rangle = 0$ if $i \neq j$ and $\langle \mathbf{g}_i, \mathbf{g}_j \rangle = 1$ if $i = j$. The following product-to-sum trigonometric identities are useful:

$$\begin{aligned}
\cos(ix)\cos(jx) &= \frac{1}{2}(\cos((i-j)x) + \cos((i+j)x)) \\
\sin(ix)\sin(jx) &= \frac{1}{2}(\sin((i-j)x) - \sin((i+j)x)) \\
\cos(ix)\sin(jx) &= \frac{1}{2}(\sin((i+j)x) - \sin((i-j)x)).
\end{aligned}$$

Using the above identities, checking that $\langle \mathbf{g}_i, \mathbf{g}_j \rangle = 0$ if $i \neq j$ reduces to showing that, for any integer $k \neq 0$,

$$\int_0^{2\pi} \cos(kx)\,dx = \int_0^{2\pi} \sin(kx)\,dx = 0,$$

which is a routine calculation. To prove that $\langle \mathbf{g}_i, \mathbf{g}_i \rangle = 1$ for all i, we need to check that

$$\int_0^{2\pi} \frac{1}{2\pi}\,dx = \int_0^{2\pi} \frac{1}{\pi}\cos^2(kx)\,dx = \int_0^{2\pi} \frac{1}{\pi}\sin^2(kx)\,dx = 1,$$

which again is a routine calculation (e.g., using the above identities with $i = j$).

(b) Find the least squares approximation to the function $f(x) = x$ in W_n (try $n = 2$ first).

Solution:

We know how to compute an orthogonal projection of a function $\mathbf{f} \in C[0, 2\pi]$ onto W_n if we know an orthonormal basis $\{\mathbf{g}_0, \mathbf{g}_1, \dots, \mathbf{g}_{2n}\}$ in W_n:

$$\text{proj}_{W_n}\mathbf{f} = \langle \mathbf{f}, \mathbf{g}_0 \rangle \mathbf{g}_0 + \langle \mathbf{f}, \mathbf{g}_1 \rangle \mathbf{g}_1 + \ldots + \langle \mathbf{f}, \mathbf{g}_{2n} \rangle \mathbf{g}_{2n}.$$

To apply this for $f(x) = x$, we need to compute the following integrals (e.g. by parts):

$$\langle x, \cos(kx) \rangle = \int_0^{2\pi} x\cos(kx)\,dx = 0 \;\text{ and }\; \langle x, \sin(kx) \rangle = \int_0^{2\pi} x\sin(kx)\,dx = -\frac{2\pi}{k}.$$

This gives us the following least squares approximation to the function $f(x) = x$:

$$x \approx \text{proj}_{W_n} x = \pi - 2\sin x - \sin(2x) - \frac{2}{3}\sin(3x) - \cdots - \frac{2}{n}\sin(nx) = \pi - 2\sum_{i=1}^{n} \frac{\sin(ix)}{i}.$$

Note: it can be shown that, as n increases and approaches infinity in the above formula, $\text{proj}_{W_n} x$ converges to x, and we get the *Fourier* series $x = \pi - 2\sum_{i=1}^{\infty} \frac{\sin(ix)}{i}$. Fourier series for continuous functions play an important role in applications.