COMP1021 Mathematics for Computer Science
Linear Algebra (Part 2)
Practical - Week 20
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Instructions: Work on these problems in the practical sessions for the week specified. First try them on your own. If you're stuck, try discussing things with others. If you get the answer, still discuss with others to see if maybe you missed something. If you run into major roadblocks, ask the demonstrators for hints.

Solutions will be posted on Learn Ultra at the end of the week. Make sure you're all set with the solutions and understand them before the next practical.

Purpose of this practical: This practical will be used to strengthen your understanding of the concepts of linear regression, orthogonal matrices, orthogonal diagonalisation.

1. Find the least squares straight line fit to the four points: $(0,1),(2,0),(3,1),(3,2)$.

2. Find a, b, and c such that the following matrix is orthogonal.

$$\begin{pmatrix} a & 1/\sqrt{2} & -1/\sqrt{2} \\ b & 1/\sqrt{6} & 1/\sqrt{6} \\ c & 1/\sqrt{3} & 1/\sqrt{3} \end{pmatrix}$$

   Are the values of a, b, and c unique?

3. Recall that in $\mathbb{R}^2$, the standard unit vectors are $\bar{i} = (1,0)$ and $\bar{j} = (0,1)$ and vectors are usually represented as $x\bar{i} + y\bar{j}$. Vectors can be represented in another way, called the polar form, where a vector is represented by its length from the origin, denoted by r, and the angle from the positive x-axis taken counter-clockwise, denoted by θ. The components of the polar form (r, Θ) of a vector can be converted to rectangular form $x\bar{i} + y\bar{j}$ as $x = r\cos\theta$ and $y = r\sin\theta$.

   Prove that a 2×2 orthogonal matrix Q can have only one of two possible forms (not accounting for row exchanges and/or column exchanges):

$$Q = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \quad \text{or} \quad Q = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}.$$

   where $0 \leq \theta < 2\pi$. Describe the corresponding linear operators T_Q on $\mathbb{R}^2$ geometrically.

4. Orthogonally diagonalise the following matrix:

$$A = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}.$$

5. Explain how the least squares straight line fit can be generalised from polynomials of degree 1 (i.e., straight lines) to polynomials of degree $n > 1$.

6. Use your generalisation to find the quadratic polynomial that best fits the four points:

$$(1,6),(2,1),(-1,5),(-2,2).$$

7. (hard) A symmetric matrix is called *positive definite* if all its eigenvalues are strictly positive. Use orthogonal diagonalisation and QR decomposition to prove that each symmetric positive definite matrix A has a so-called *Cholesky* (pronounced with sh-) *decomposition*:

$$A = LL^T$$

   for some lower triangular matrix L. (This is a special case of LU decomposition with $U = L^T$ - when it exists, it allows one to solve linear systems twice as fast as the general LU method).