MCS Calculus
Practical Exercises 6
(Week 15)

Epiphany Term 2025

Make sure you have completed all exercises from the previous Calculus practical. If you wish, try typesetting your answers with LaTeX.

1. Consider the harmonic series $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + ...$ By grouping 1 then 1 then 2 then 4 then 8 etc terms, obtain a underestimate for S_m where $m = 1+2^k-1$. Deduce that the series diverges.

Answer: Groupings are

(a) 1

(b) $\frac{1}{2} \geq \frac{1}{2}$

(c) $\frac{1}{3} + \frac{1}{4} \geq 2 \times \frac{1}{4} \geq \frac{1}{2}$

(d) $\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} \geq 4 \times \frac{1}{8} \geq \frac{1}{2}$

(e) $\geq 8 \times \frac{1}{16} \geq \frac{1}{2}$

(f) ...

Hence $S_{1+2^k-1} > 1 + k \times \frac{1}{2} \to \infty$ as $k \to \infty$.

2. Use the comparison test to determine if the following series converge.

(a) $1 + (\frac{2}{3}) + (\frac{2}{3})^4 + (\frac{2}{3})^9 + ... + (\frac{2}{3})^{n^2} + ...$

Answer: Compare with $\sum_{n=0}^{\infty} (\frac{2}{3})^n$. The latter is a geometric series, which converges, hence our series converges.

(b) $\frac{3}{4} + \frac{4}{7} + \frac{5}{12} + ... + \frac{n+2}{n^2+3} + ...$

Answer: Compare with $\sum_{n=0}^{\infty} \frac{1}{n}$, the harmonic series. Since $\frac{n+2}{n^2+3} > \frac{n}{n^2} = \frac{1}{n}$, and the harmonic series diverges, our series diverges.

(c) $1 + \frac{1}{3^2} + \frac{1}{5^3} + \frac{1}{7^4} + ... + \frac{1}{(2n-1)^n} + ...$

Answer: Compare with $\sum_{n=0}^{\infty} \frac{1}{n!}$. Since $\frac{1}{(2n-1)^n} < \frac{1}{n^n} < \frac{1}{n!}$, and comparator converges to e, our series converges.

(d) $\frac{1}{3-1} + \frac{1}{3^2-2} + \frac{1}{3^3-3} + ... + \frac{1}{3^n-n} + ...$

Answer: Compare with $\sum_{n=0}^{\infty} \frac{1}{3^{n-1}}$. Since $\frac{1}{3^n - n} = \frac{1}{3^{n-1}(3-n/3^{n-1})} < \frac{1}{3^{n-1}}$, and comparator is a geometric convergent series, our series converges.

3. Use the ratio test to determine if the following series converge. If the ratio test fails, identify another way of testing for convergence.

   (a) $\frac{2}{1} + \frac{2.5}{1.5} + \frac{2.5.8}{1.5.9} + ... + \frac{2.5.8...(3n-1)}{1.5.9...(4n-3)} + ...$

Answer: The ratio is $\frac{a_{n+1}}{a_n} = \frac{3(n+1)-1}{4(n+1)-3} \to \frac{3}{4} < 1$. Hence our series converges.

   (b) $\frac{1}{\sqrt{3}} + \frac{3}{3} + \frac{5}{(\sqrt{3})^3} + ... + \frac{2n-1}{(\sqrt{3})^n} + ...$

Answer: The ratio is $\frac{a_{n+1}}{a_n} = \frac{2(n+1)-1}{\sqrt{3}^{n+1}} \frac{\sqrt{3}^n}{2n-1} = \frac{1}{\sqrt{3}} \frac{2(n+1)-1}{2n-1} \to \frac{1}{\sqrt{3}} < 1$. Hence our series converges.

   (c) $\frac{2}{5} + \frac{5}{14} + \frac{10}{29} + ... + \frac{n^2+1}{3n^2+2} + ...$

Answer: The ratio is $\frac{a_{n+1}}{a_n} = \frac{(n+1)^2+1}{3(n+1)^2+2} \frac{3n^2+2}{n^2+1} = \to 1$. So the test fails!
However the series' terms themselves do not converge to zero: indeed, $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n^2+1}{3n^2+2} = \frac{1}{3}$. So, our series diverges.

   (d) $\frac{1}{10} + \frac{2!}{10^2} + \frac{3!}{10^3} + ... + \frac{n!}{10^n} + ...$

Answer: The ratio is $\frac{a_{n+1}}{a_n} = \frac{n+1}{10} \to \infty$. Hence our series diverges.

4. Consider the series

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \left(\frac{nk}{3n+1} \right)^n$$

where $k \in \mathbb{R}$ is some constant.

   (a) Determine the value of $\lim_{n\to\infty} \sqrt[n]{a_n}$.

Answer: $\sqrt[n]{a_n} = \frac{nk}{3n+1}$, so $\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \frac{nk}{3n+1} = \lim_{n\to\infty} \frac{k}{3+1/n} = \frac{k}{3}$.

   (b) Let $R_m = \sum_{n=m+1}^{\infty} a_n$. By comparison with a suitable geometric series, show that

      i. if $k < 3$ then $\lim_{m\to\infty} R_m = 0$, whereas
      ii. if $k > 3$ then $\lim_{m\to\infty} R_m \to \infty$.

Answer: If $k < 3$, then there is an r such that $\frac{k}{3} < r < 1$ and N_r so that for all $n > N_r$, $\sqrt[n]{a_n} < r$. Thus

$$R_n = a_{n+1} + a_{n+2} + \ldots < r^{n+1} + r^{n+2} + \ldots = \frac{r^{n+1}}{1-r} \to 0$$

as $n \to \infty$.
If $k > 3$, then there is an r such that $\frac{k}{3} > r > 1$ and N_r so that for all $n > N_r$, $\sqrt[n]{a_n} > r$. Thus

$$R_n = a_{n+1} + a_{n+2} + \ldots > r^{n+1} + r^{n+2} + \ldots \to \infty$$

as $n \to \infty$.

(c) For what values of k can you conclude that $\sum_{n=1}^{\infty} a_n$ converges or diverges and for what values of k can you reach no conclusion?

Answer: Suppose $\sum_{n=1}^{\infty} a_n = S$. Then $R_n = S - S_n$, and so as $S_n \to S$ we must have $R_n \to 0$. Conversely if $R_n \to 0$, then there is some S such that $S_n \to S$.

[To see this consider that there exists some N such that $|R_N| < 1/2$, thus $S_N - 1/2 < S_n + R_n < S_N + 1/2$ for $n > N$, and $S_N - 1 < S_n < S_N + 1$. Now S_n is an infinite bounded sequence, so must have a monotonic subsequence (see here) which therefore has limit S. Since the subsequence converges to S and $|S_n - S_{n'}| = |R_n - R_{n'}| \to 0$ as $\min\{n, n'\} \to \infty$, given $\epsilon > 0$ there is an N' such that for $n, n' > N'$ where $S_{n'}$ is in the subsequence, $|S_n - S_{n'}| < \epsilon/2$ and $|S_{n'} - S| < \epsilon/2$, hence $|S_n - S_{n'}| < \epsilon$.]

If $k < 3$, then $R_n \to 0$ as $n \to \infty$ and hence the series does.

If $k > 3$, then $R_n \to \infty$ as $n \to \infty$ and hence the series does not converge.

If $k = 3$, then we can deduce nothing directly from the comparisons so far.

5. After tackling the question above, recall 'The nth root test' and have a look at some examples on Math24.

6. Determine whether the following series converge.

(a) $\sum_{n=2}^{\infty} (-1)^n \frac{\sqrt{n}}{\ln n}$

Answer: Use the (alternating series) divergence test. Consider $\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \frac{\sqrt{n}}{\ln n}$. We must use L'Hôpital's rule. Hence $\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \frac{-1/2\sqrt{n}}{1/n} = \lim_{n\to\infty} \frac{-\sqrt{n}}{2} \to \infty$.

Hence this series diverges.

(b) $\sum_{n=2}^{\infty} (-1)^n \frac{\ln n}{\sqrt{n}}$

Answer: Use the alternating series test. Consider $\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \frac{\ln n}{\sqrt{n}}$. We must use L'Hôpital's rule. Hence $\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \frac{1/n}{-1/2\sqrt{n}} = \lim_{n\to\infty} \frac{-2}{\sqrt{n}} = 0$.

Need to check terms are decreasing in absolute value. Let $f(x) = \frac{\ln x}{\sqrt{x}}$, then $f'(x) = \frac{\sqrt{x}/x - \ln x/2\sqrt{x}}{x} = \frac{2 - \ln x}{2x\sqrt{x}}$ (using the quotient rule). This is less than zero when $\ln x > 2$, i.e. for $x > e^2$. Hence $a_n = f(n) > f(n+1) = a_{n+1}$ for $n \geq 8$. Hence this series converges.