Welcome to COMP1081 - Algorithms and Data Structures

The lecture will begin at 5 past. While you wait, join respond to the PollEverywhere question by SMS or on the web at `https://pollev.com/eamonn`

Consider the sequence of Fibonacci numbers $F(n)$ defined by

$$F(n) = \begin{cases} 0 & \text{for } n = 0 \\ 1 & \text{for } n = 1 \\ F(n-1) + F(n-2) & \text{for } n \geq 2 \end{cases}$$

Calculate $F(10)$.

Topic 5: Recursion and Backtracking

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Recursive Algorithms: Factorials

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  - ...and multiplying the result of that, by 5
- In turn, the problem of computing 4! can be broken down in a similar way.
- This makes the factorial function a candidate for implementation as a **recursive** algorithm.

Recursive Factorial: factorial(n)

```
if n=1 then
    return 1
else
    return n × factorial(n-1)
end if
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Iterative Factorial

```
total = 1
for i=1 to n do
    total = total × i
end for
return total
```

Recursive Algorithms

- A recursive algorithm is an algorithm that calls itself to do part of its work.
- A recursive algorithm must have a base case.
- A recursive algorithm must change its state and move toward the base case.
- A recursive algorithm must call itself, recursively.

Iterative Sum of a list L

```
sum = 0
for i in L do
    sum = sum + i
end for
return  sum
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Recursive Sum of a list L: listsum(L)

```
if len(L) = 1 then
    return L[0]
else
    return L[0] + listsum(L[1:])
end if
```

Memoization

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- We could store the result of these intermediate function calls, because these results do not change over time.
- **Storing the result of a computation so that it can be subsequently retrieved without repeating the computation** is called memoization.
- Hash tables are a good choice of data structure to implement memoization.

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Consider an implementation of the function **floodfill** given below, where *x* and *y* represent pixel locations in a 2D screen

floodfill(x, y)

```
if (x,y) is out of range or already filled then
    return
end if
colour in pixel at location (x,y)
floodfill(x+1,y)
floodfill(x-1,y)
floodfill(x,y+1)
floodfill(x,y-1)
```

A Recursive Technique: Backtracking

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- For example: there are 6,670,903,752,021,072,936,960 ways to fill in a sudoku grid.

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- A technique for problems with many candidate solutions but too many to try.
- For example: there are 6,670,903,752,021,072,936,960 ways to fill in a sudoku grid.
- General idea: build up the solution one step at a time, backtracking when unable to continue.

Generic algorithm (informal)

1. Do I have a solution yet?
2. No. Can I extend my solution by one “step”?
3. If yes, do that.
4. Do I have a solution now? If yes, I’m done.
5. If not, try and extend again.
6. When I can’t extend, take one step back and try a different way.
7. If no other extension available, then give up — no solution can be found.

▸ Sudoku demo from Wikimedia Commons

Generic algorithm

```
extend_solution(current solution)
  if current solution is valid then
      if current solution is complete then
          return  current solution
      else
          for each extension of the current solution do
              extend_solution(extension)
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```
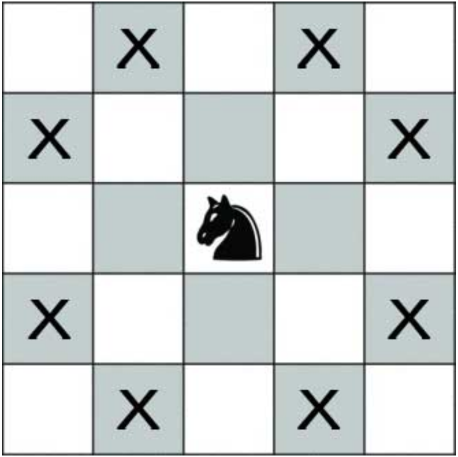

For sudoku, start by calling extend_solution with the partially filled grid that is given to you.

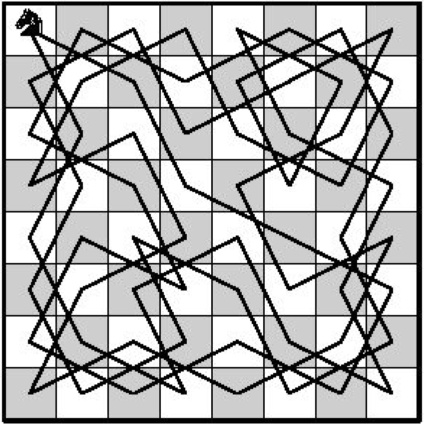
Knights

A knight is a chess piece that can move by moving one square in one direction and two squares in a perpendicular direction.



A Knight's Tour

A Knight's Tour: to move a knight around a chessboard such that each square is visited exactly once.



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- When is it complete?

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- How can the current solution be extended?

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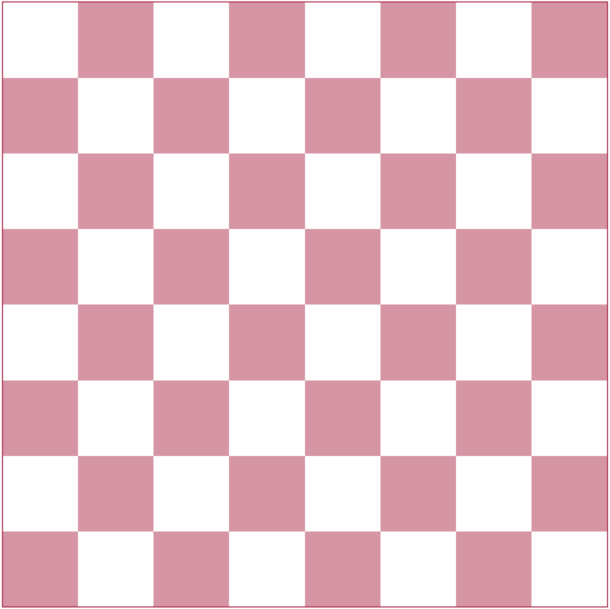
- When is it complete?
  - Every square visited: 64 items in the list.
- How can the current solution be extended?
  - Consider each of the eight possible moves.



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    if current solution is complete then
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        for each of eight possible moves do
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        end for
    end if
end if
```

Using the generic algorithm

```
extend_solution(current solution)
  if current solution is valid then
      if current solution is complete then
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          for each of eight possible moves do
              extend_solution(current solution with move added)
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extend_solution(current solution)
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So we have an algorithm for Knight's Tour . . .

Implementing Knight's Tour

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The algorithm is practical for a 6×6 board, but rather slow for an 8×8 board and impractical for much larger boards. What additional ideas could we add to the algorithm?