Examination Paper

Examination Session:	Year:	Exam Code:
May/June	2021	COMP1021-WE01

Title: Mathematics for Computer Science

Release Date/Time	9.30am 28/05/2021
Deadline Date/Time	9.30am 29/05/2021
Format of Exam	Take home exam
Duration:	2 hours
Word/Page Limit:	None
Additional Material provided:	None
Expected form of Submission	A SINGLE PDF file (handwritten or typed) submitted onto Gradescope. Upload your submission with a file name comprising of your student ID and the Exam Code e.g 00001234532 COMP1021-WE01.
Submission method	Gradescope
Instructions to Candidates:	Answer ALL questions.

Section A Linear Algebra
(Prof. Andrei Krokhin)

Question 1

(a) Find the rank of the following matrix for all values of a

$$\begin{pmatrix} 3 & 1 & 1 & 4 \\ a & 4 & 10 & 1 \\ 1 & 7 & 17 & 3 \\ 2 & 2 & 4 & 3 \end{pmatrix}.$$

Show your working. **[4 Marks]**

(b) Let A and B be matrices of size 2×3 and 3×2, respectively.

i. Prove or disprove: AB cannot be invertible. **[3 Marks]**

ii. Prove or disprove: BA cannot be invertible. **[3 Marks]**

(c) Let A be an $n \times n$ matrix and let $\mathbf{v} \in \mathbb{R}^n$ be a column vector such that $A^{k-1}\mathbf{v} \neq \mathbf{0}$, but $A^k\mathbf{v} = \mathbf{0}$ for some $k > 0$. Prove that the column vectors $\mathbf{v}, A\mathbf{v}, \ldots, A^{k-1}\mathbf{v}$ are linearly independent. **[5 Marks]**

(d) Consider the vector space $\mathbb{R}^\infty$ of all infinite sequences $\mathbf{x} = (x_1, x_2, \ldots)$ of real numbers. Let $f : \mathbb{R}^\infty \to \mathbb{R}^\infty$ be the linear map defined as follows:

$$f(\mathbf{x}) = (x_1 + x_2, x_2 + x_3, x_3 + x_4, \ldots).$$

i. Find a basis for the kernel of f. **[3 Marks]**

ii. What is the range of f? **[3 Marks]**

iii. Compute the eigenvalues of f and an eigenvector corresponding to each eigenvalue. **[4 Marks]**

Justify your answer.

continued

Question 2

(a) Consider the vector space $\mathbb{R}^5$ equipped with the weighted Euclidean inner product with weights $w_1 = 2, w_2 = 1, w_3 = 3, w_4 = 1, w_5 = 1$. Find a basis for the orthogonal complement (with respect to this inner product) of the subspace $span(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ where $\mathbf{v}_1 = (1, 2, 3, -1, 0)$, $\mathbf{v}_2 = (0, -3, 1, -4, 6)$, $\mathbf{v}_3 = (-2, -7, -5, -2, 6)$. Show your working. **[6 Marks]**

(b) Prove or disprove: if $\{\mathbf{u}, \mathbf{v}\}$ is a basis in $\mathbb{R}^2$ then there exist positive weights $w_1, w_2 \in \mathbb{R}$ such that $\mathbf{u}$ and $\mathbf{v}$ are orthogonal with respect to the weighted Euclidean inner product with weights w_1, w_2. **[4 Marks]**

(c) Consider the vector space $C[0, \pi]$ of all continuous functions on the interval $[0, \pi]$, equipped with the inner product

$$\langle \mathbf{f}, \mathbf{g} \rangle = \int_0^\pi f(x)g(x)\,dx$$

where $\mathbf{f} = f(x), \mathbf{g} = g(x)$. Find an orthonormal basis of the subspace of $C[0, \pi]$ spanned by the vectors $\mathbf{v}_1 = 1$, $\mathbf{v}_2 = x$, $\mathbf{v}_3 = \sin(x)$. Show your working. **[7 Marks]**

(d) Do there exist symmetric 3×3 matrices A and B such that

i. A has eigenvalues $\lambda_1 = -3$, $\lambda_2 = 0$, and $\lambda_3 = 7$ and corresponding eigenvectors $\mathbf{v}_1 = (0, -1, 1)$, $\mathbf{v}_2 = (1, 0, 0)$, $\mathbf{v}_3 = (1, 1, 1)$?

ii. B has eigenvalues $\lambda_1 = -2$, $\lambda_2 = 5$, and $\lambda_3 = 6$ and corresponding eigenvectors $\mathbf{v}_1 = (0, -1, 1)$, $\mathbf{v}_2 = (1, 0, 0)$, $\mathbf{v}_3 = (0, 1, 1)$?

In each case, either find such a matrix or explain why it does not exist. **[8 Marks]**

End of Section A

Section B Calculus (Prof. Magnus Bordewich)

Question 3

Let $f(x, y, z) = x^3 - x + \frac{y^2 z}{2} - 2y - 8z^2$.

(a) Calculate the gradient $\nabla f(x, y, z)$ and use it to determine the critical points of f. Show your working. **[10 Marks]**

(b) Determine the gradient of f in the direction of the vector $(2, 3, 1)^T$ at point $(3, 4, 2)$. Show your working. **[4 Marks]**

(c) Calculate the Hessian Matrix $H_f(x, y, z)$ and use it to determine the nature of each critical point you have identified. Show your working. **[9 Marks]**

Question 4

Let $e(x_1, x_2) = \tan^{-1}(2x_2 - 2x_1) + \max\{0, x_1^2 - x_2\} + x_1 x_2$.

(a) Draw a computation graph for e, identifying intermediate variables. List the intermediate variables and the operation by which each is obtained from its inputs. **[5 Marks]**

(b) Compute the partial derivatives of each intermediate variable with respect to its inputs. **[5 Marks]**

(c) Use a forward pass of your computation graph to compute $e(2, 3)$. Show your working. **[3 Marks]**

(d) Use forward mode Automatic Differentiation (AD) to compute the directional derivative of e at point $(2, 3)$ in the direction $(4, 3)$. Show your working. **[7 Marks]**

(e) Use reverse mode AD to compute a vector pointing in the direction of greatest decrease in e from point $(2, 3)$. Show your working. **[7 Marks]**

END OF PAPER