COMP1021 Mathematics for Computer Science
Linear Algebra (Part 2)
Practical - Week 14
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Instructions: Work on these problems in the practical sessions for the week specified. First try them on your own. If you're stuck, try discussing things with others. If you get the answer, still discuss with others to see if maybe you missed something. If you run into major roadblocks, ask the demonstrators for hints.

Solutions will be posted on Learn Ultra at the end of the week. Make sure you're all set with the solutions and understand them before the next practical.

Purpose of this practical: In this practical, we review eigenvalues, eigenvectors, complex numbers, and complex vector spaces. We look at not only how to directly use these concepts, but also useful properties related to them.

1. (Warm up) Find a basis and the dimension of the solution space of the following linear system:

$$\begin{array}{rrrrrl} 2x_1 & +2x_2 & -x_3 & & +x_5 & =0 \\ -x_1 & -x_2 & +2x_3 & -3x_4 & +x_5 & =0 \\ x_1 & +x_2 & -2x_3 & & -x_5 & =0 \\ & & x_3 & +x_4 & +x_5 & =0 \end{array}$$

2. For the the following matrix, find the basis of the eigenspace corresponding to the eigenvalue $\lambda_0 = 2$. Hint: Use the previous question.

$$A = \begin{pmatrix} 0 & -2 & 1 & 0 & -1 \\ 1 & 3 & -2 & 3 & -1 \\ -1 & -1 & 4 & 0 & 1 \\ 0 & 0 & -1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix}$$

3. Find the eigenvalues and eigenvectors of the following matrix: $M = \begin{pmatrix} 1 & -1 \\ 2 & 4 \end{pmatrix}$.

4. Suppose that λ is an eigenvalue of a matrix A, and $\mathbf{x}$ is a corresponding eigenvector.
   (a) Show that $\mathbf{x}$ is also an eigenvector of A^k ($k \geq 0$) and find the corresponding eigenvalue.
   (b) Show that $\mathbf{x}$ is also an eigenvector of $B = A - 7I$ and find the corresponding eigenvalue.
   (c) Assuming that A is invertible, show that $\mathbf{x}$ is also an eigenvector of A^{-1} and find the corresponding eigenvalue.

5. Prove that A and A^T have the same eigenvalues. Do they always have the same eigenvectors?

6. Assume that the characteristic polynomial of a matrix A can be factored as $\lambda^2(\lambda+5)^3(\lambda-2)^4$.
   (a) What is the size of A? Is there enough information to say something about this?
   (b) Is A invertible? Is there enough information to say something about this?

7. Find all complex scalars k, if any, for which $\mathbf{u}$ and $\mathbf{v}$ are orthogonal in $\mathbb{C}^3$:
   (a) $\mathbf{u} = (3i, 1, i), \mathbf{v} = (-i, -5, k)$
   (b) $\mathbf{u} = (k, k, 1+i), \mathbf{v} = (1, -1, 1-i)$

8. Prove the following theorem stated in the lecture: The complex eigenvalues of the real matrix $C = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ are $\lambda = a \pm bi$. If a, b are not both zero, then C can be factored as

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix} = \begin{pmatrix} |\lambda| & 0 \\ 0 & |\lambda| \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

   where θ is the argument of $\lambda = a + bi$.

9. Find the (complex) eigenvalues and eigenspaces of the following matrix: $M = \begin{pmatrix} 4 & -5 \\ 1 & 0 \end{pmatrix}$.

10. (Optional) Consider the statement "For any square matrices A and B of the same size, if λ is an eigenvalue of A and μ is an eigenvalue of B then $\mu\lambda$ is an eigenvalue of AB." Find a flaw in the following proof of this statement: If λ is an eigenvalue of A and μ is an eigenvalue of B then, for some non-zero vector $\mathbf{x}$,

$$AB\mathbf{x} = A\mu\mathbf{x} = \mu A\mathbf{x} = \mu\lambda\mathbf{x}.$$

    Is the statement true at all?

11. (Optional)

(a) Choose the third row in the following matrix A so that its characteristic polynomial $det(\lambda I - A)$ is $\lambda^3 - 4\lambda^2 - 5\lambda - 6$.

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ * & * & * \end{pmatrix}.$$

(b) (hard) Show that any polynomial $p(\lambda) = \lambda^n + c_1\lambda^{n-1} + c_2\lambda^{n-2} + \ldots + c_{n-1}\lambda + c_n$ is the characteristic polynomial of a suitably constructed matrix A. (Use part (a) as an inspiration).