Durham University

Maths for Computer Science
Calculus

Dr Eleni Akrida

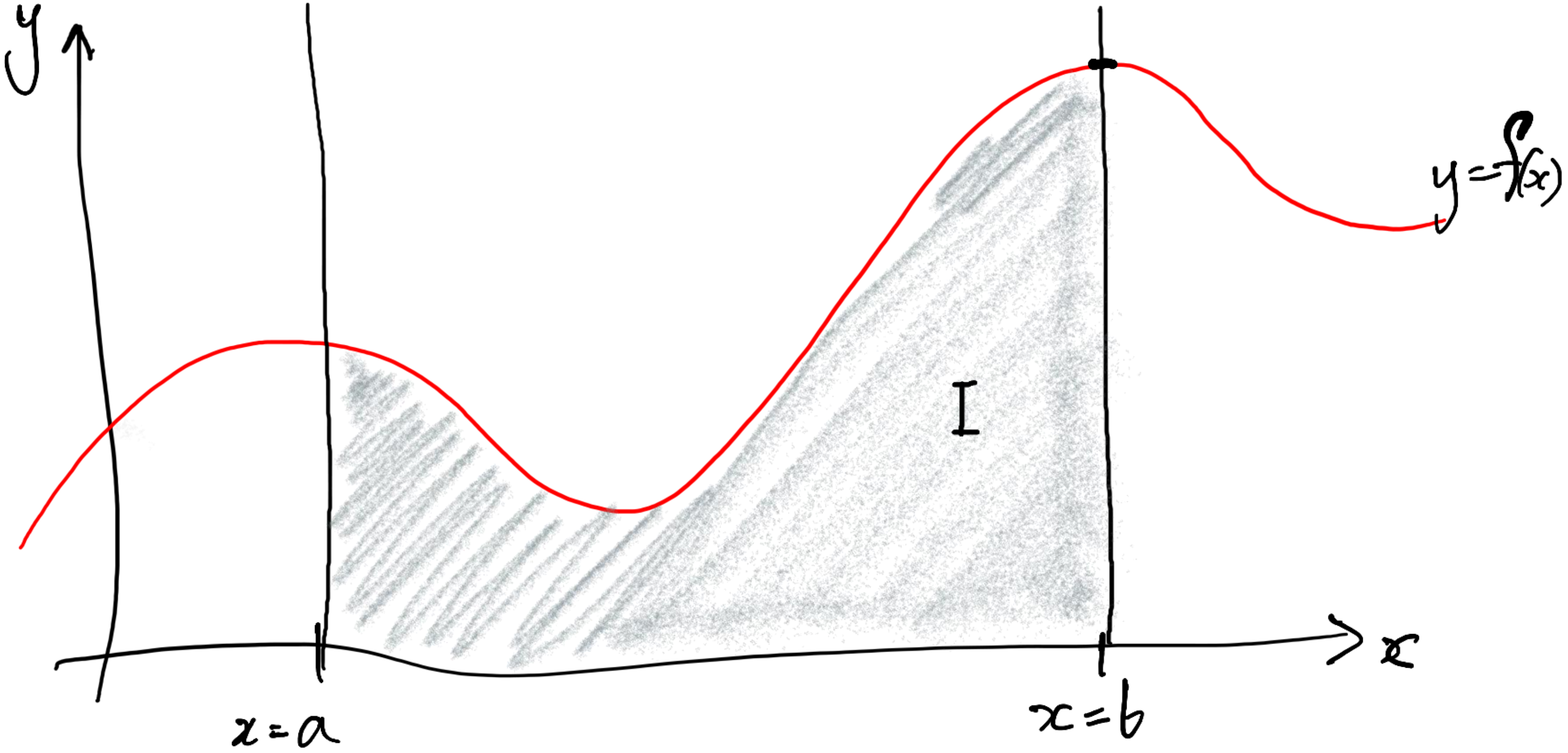
Integrals

Contents for today's lecture

- Integration
- Definite integrals:
  - Specifying the area under a curve within a fixed interval
  - Coping with discontinuities and negative function values
  - Reimann integral
  - Nice properties
- Mean value theorem for integrals
- The Fundamental Theorem of Calculus:
  - Indefinite integrals
  - Antiderivatives

Area under a curve

Given a non-negative, continuous function $f(x)$ can we mathematically compute the area I of the region bounded by the curve of f, $y = f(x)$, the vertical lines $x = a$, $x = b$, and the x-axis:



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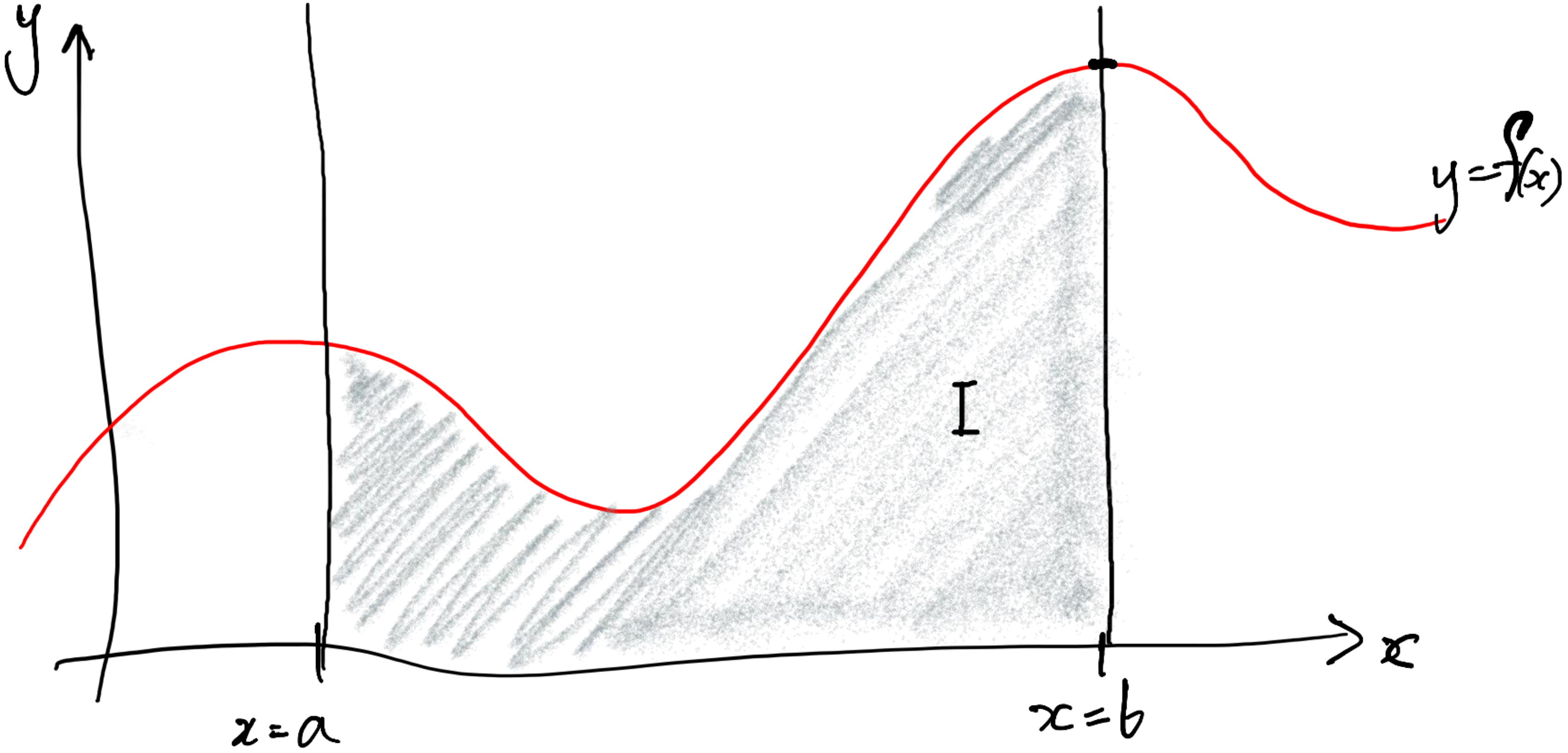
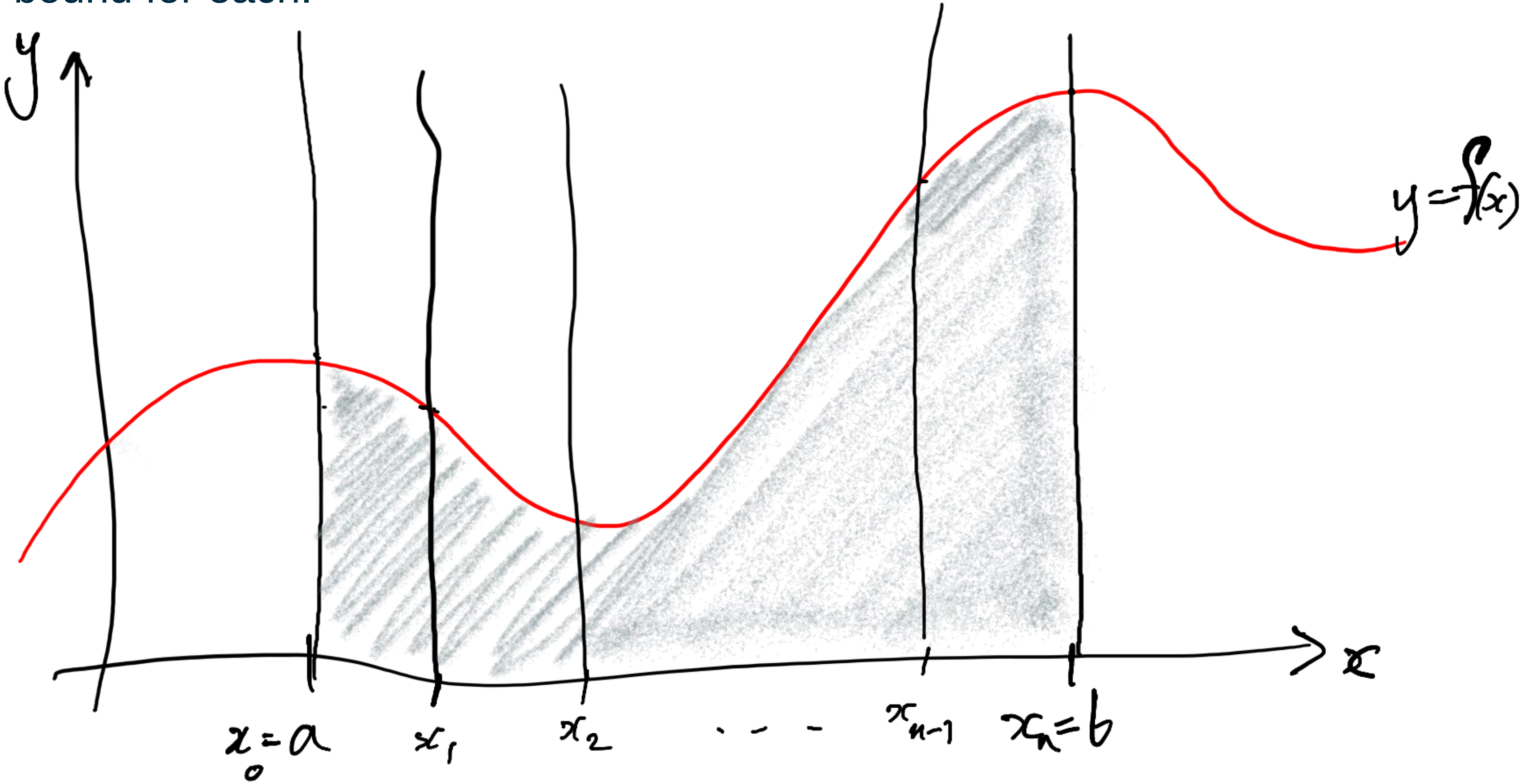


Let us take $m = \min[f(x): a \leq x \leq b]$ and $M = \max[f(x): a \leq x \leq b]$. Then:

$$m(b-a) \leq I \leq M(b-a).$$

Area under a curve

We can do better by refining the min and max values, through splitting the range $[a, b]$ into n strips, and using a separate bound for each.



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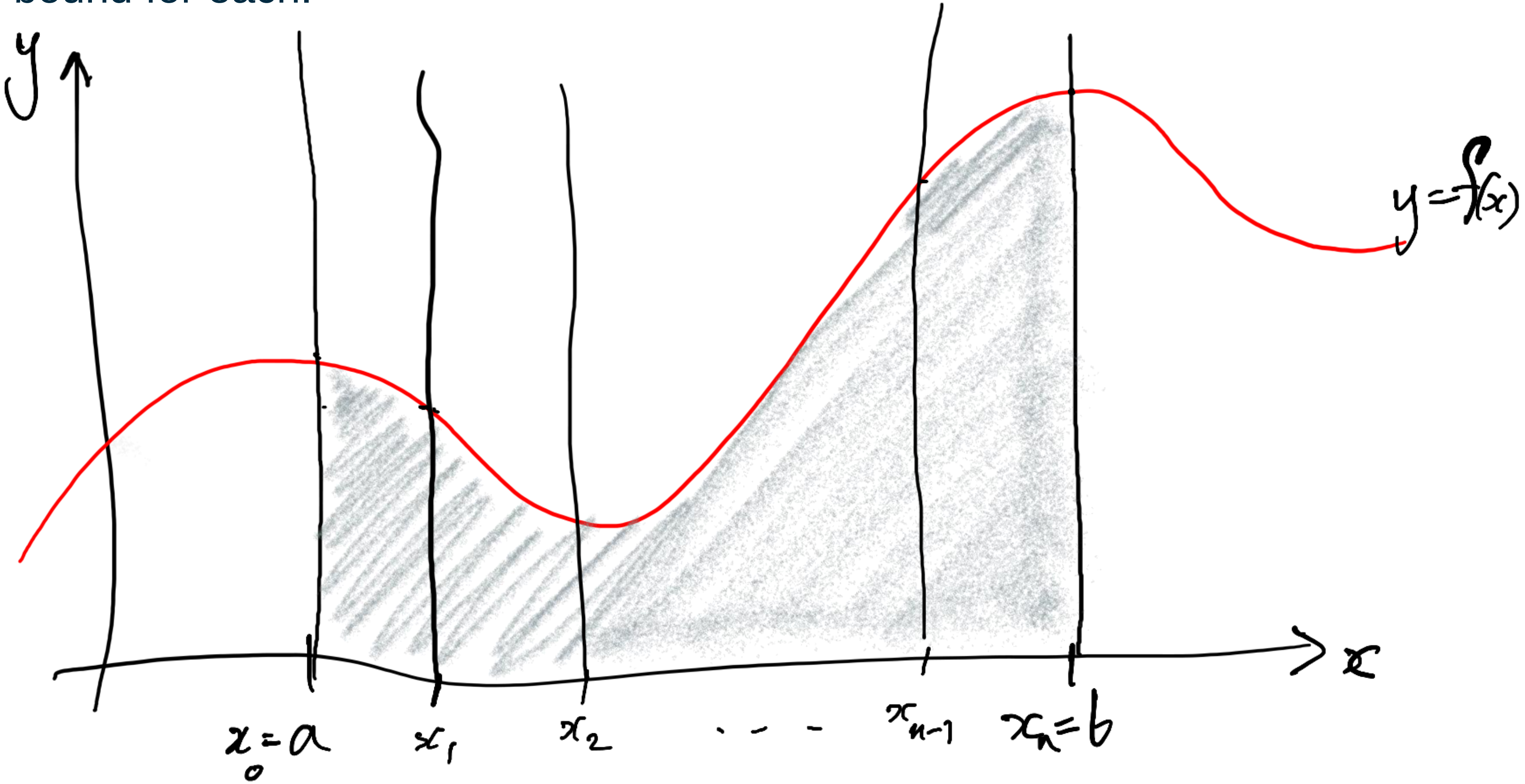

Let $m_i = \min[f(x): x_{i-1} \leq x \leq x_i]$ and $M_i = \max[f(x): x_{i-1} \leq x \leq x_i]$. Then:

$$\sum_{i=1}^{n} m_i \,(x_i - x_{i-1}) \leq I \leq \sum_{i=1}^{n} M_i \,(x_i - x_{i-1})\,.$$

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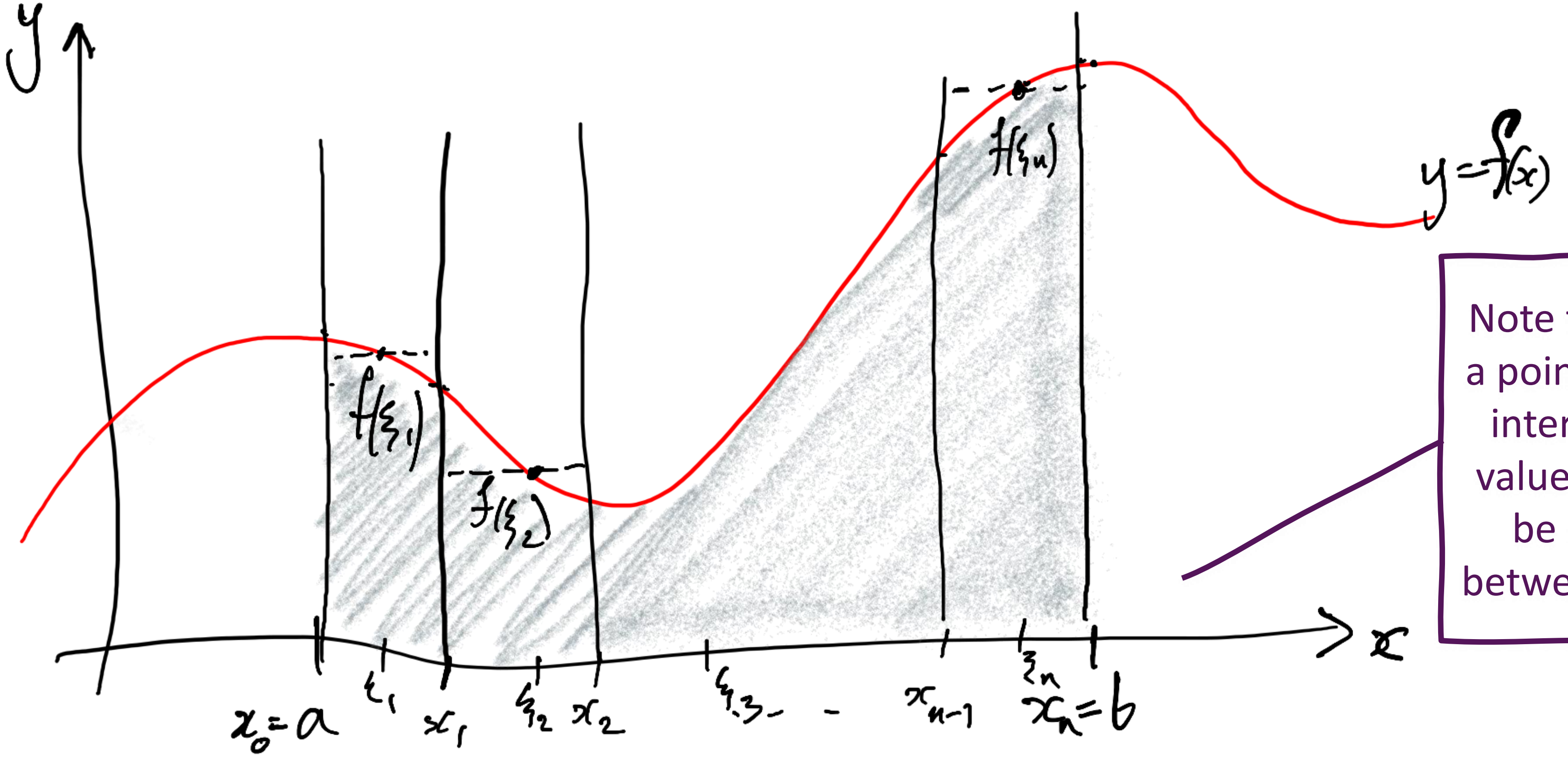
Area under a curve

Let $\Delta_i = x_i - x_{i-1}$, then if $\Delta_i \to 0$, as f is continuous, we also get $|M_i - m_i| \to 0$.

If we let ξ_i be any point in $[x_{i-1}, x_i]$, then $m_i \leq f(\xi_i) \leq M_i$.

$$\sum_{i=1}^{n} m_i\,(x_i - x_{i-1}) \leq \sum_{i=1}^{n} f(\xi_i)(x_i - x_{i-1}) \leq \sum_{i=1}^{n} M_i\,(x_i - x_{i-1})$$

and $m_i, f(\xi_i), M_i$ converge in value as $\Delta_i \to 0$.

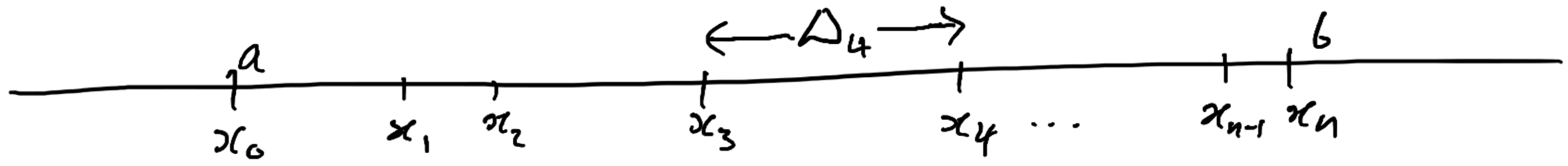


Note that if we take a point ξ_i in the i-th interval, then the value of f at ξ_i will be somewhere between m_i and M_i.

Area under a curve

Consider a partition $P_n = \{x_0 = a, x_1, x_2, \dots, x_n = b\}$ of $[a, b]$ into n subintervals, $[x_{i-1}, x_i]$, $i = 1 \dots n$.

Define the width of the subintervals $\Delta_i = x_i - x_{i-1}$, and the max width of the partition $\Delta_{P_n} = \max_i\{\Delta_i\}$.



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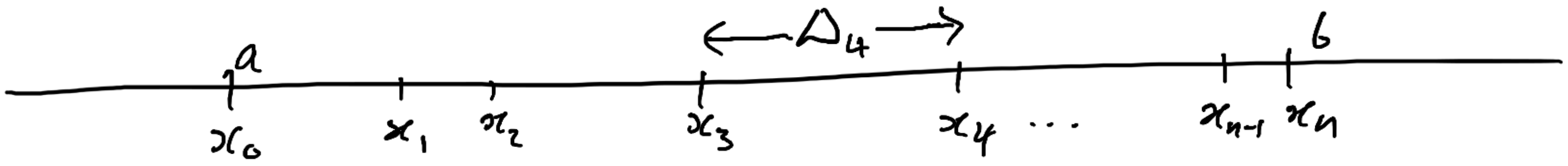


We say a partition $P_{n'}$ of $[a, b]$ is finer than P_n if $n' > n$ and $\Delta_{P_{n'}} < \Delta_{P_n}$.

Now we take an infinite sequence $\{P_n\}$ of partitions of $[a, b]$ such that P_{n+1} is finer than P_n for all n, and such that $\Delta_{P_n} \to 0$, as $n \to \infty$.

This gives us enough to define a definite integral of a non-negative continuous function on an interval [a,b]. We take this sum $\sum_{i=1}^{n} f(\xi_i)\Delta_i$, of all these rectangles, and we look at the limit of the value of the sum as n tends to infinity.

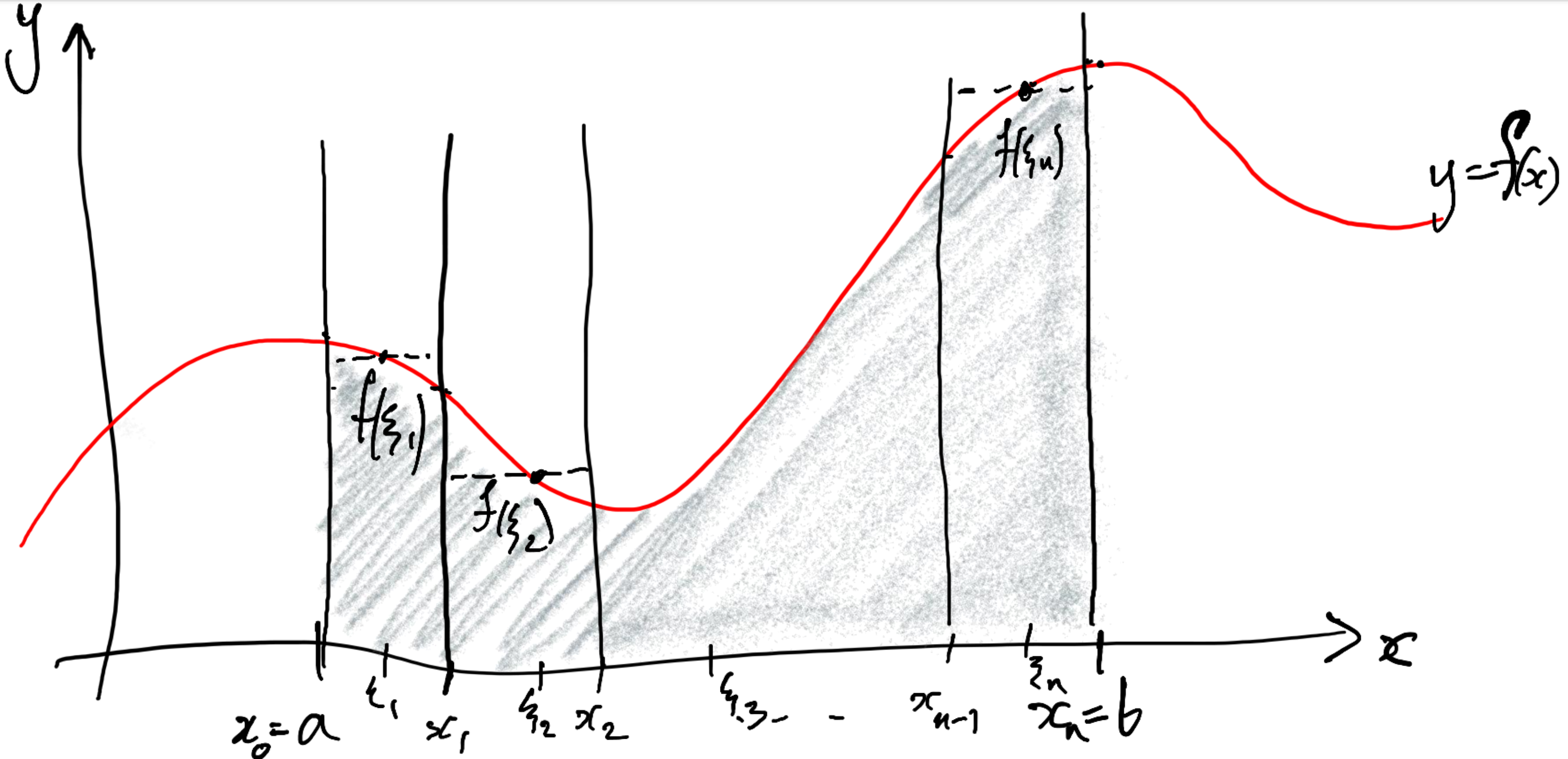
Definite integral

Let f be a non-negative continuous function on $[a, b]$, and let $\{P_n\}$ be a sequence of finer partitions of $[a, b]$ such that $\Delta_i \to 0$ and ξ_i be an arbitrary point in the i^{th} subinterval.

We define the definite integral of f, where the limit exists, by:

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(\xi_i)\Delta_i$$

You can think of the $\int$ as a “stretched out” S, signifying a kind of a sum of an infinite set of continuous values, whereas Σ signifies the discrete sum of those strips.



Example

Take $f(x) = x^2$ on the interval $[a, b]$. What is the value of $\int_a^b f(x)\, dx$?

We will use a convenient partition P_n which subdivides $[a, b]$ in to n even strips of width $\Delta = \frac{b-a}{n}$, and define $\xi_i = a + i\Delta$, i.e. the right-hand end of the subinterval.

Then, to find the definite integral of f on $[a, b]$, we need to evaluate the limit of the following sum:

$$\begin{aligned}\sum_{i=1}^{n} f(\xi_i)\Delta_i &= \sum_{i=1}^{n}(a + i\Delta)^2\,\Delta \\ &= na^2\Delta + 2a\Delta^2(1 + 2 + 3 + \cdots + n) + \Delta^3(1^2 + 2^2 + 3^2 + \cdots + n^2) \\ &= na^2\Delta + 2a\Delta^2\frac{n(n+1)}{2} + \Delta^3\frac{n(n+1)(2n+1)}{6} \\ &= a^2(b-a) + a(b-a)^2\frac{(n+1)}{n} + (b-a)^3\frac{(n+1)(2n+1)}{6n^2}\end{aligned}$$

So

$$\begin{aligned}\int_a^b f(x)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(\xi_i)\Delta_i &= a^2(b-a) + a(b-a)^2 + (b-a)^3\frac{2}{6} \\ &= a^2b - a^3 + ab^2 - 2a^2b + a^3 + \frac{b^3}{3} - b^2a + ba^2 - \frac{a^3}{3} = \frac{1}{3}(b^3 - a^3).\end{aligned}$$

Coping with discontinuities

Suppose f is continuous on $[a, b]$ except at some point c at which there is a discontinuity.

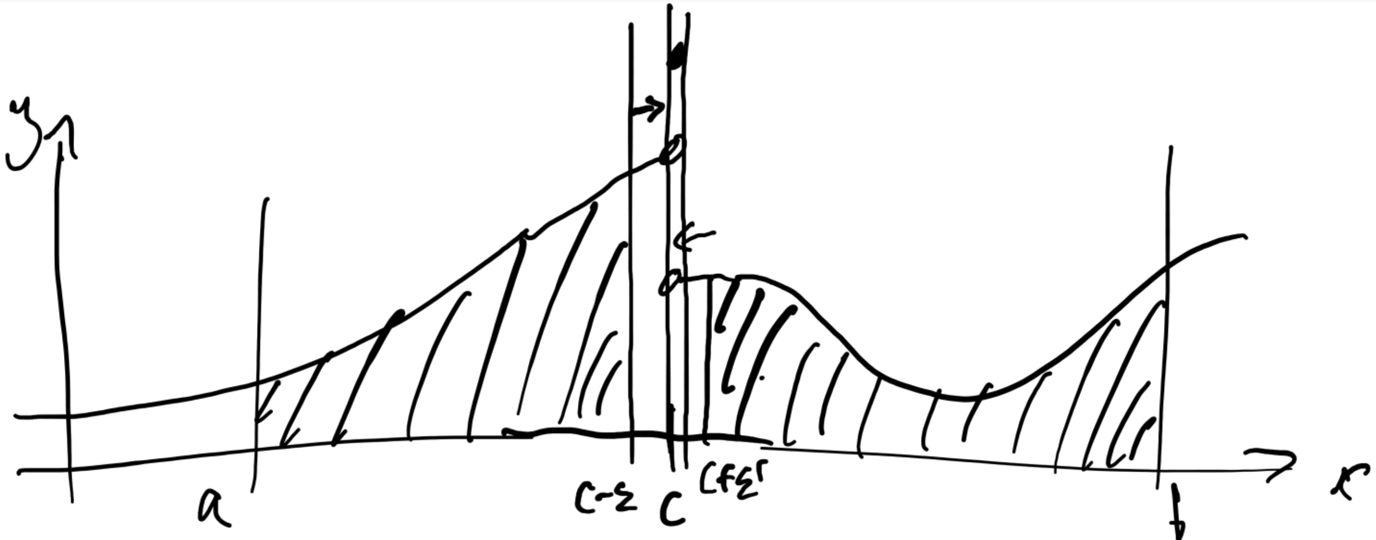
We can no longer conclude that for the interval containing c, m_i, M_i and $f(\xi_i)$ all converge to the same value.

Instead we can consider the integrals $\int_a^{c-\epsilon} f(x)\,dx$ and $\int_{c+\epsilon'}^{b} f(x)\,dx$, which are well defined.

Then define

$$\int_a^b f(x)\,dx = \lim_{\epsilon\to 0} \int_a^{c-\epsilon} f(x)\,dx + \lim_{\epsilon'\to 0} \int_{c+\epsilon'}^{b} f(x)\,dx$$

when **both** limits exist.



Coping with negative valued functions

Suppose f goes negative on the interval $[a, b]$.

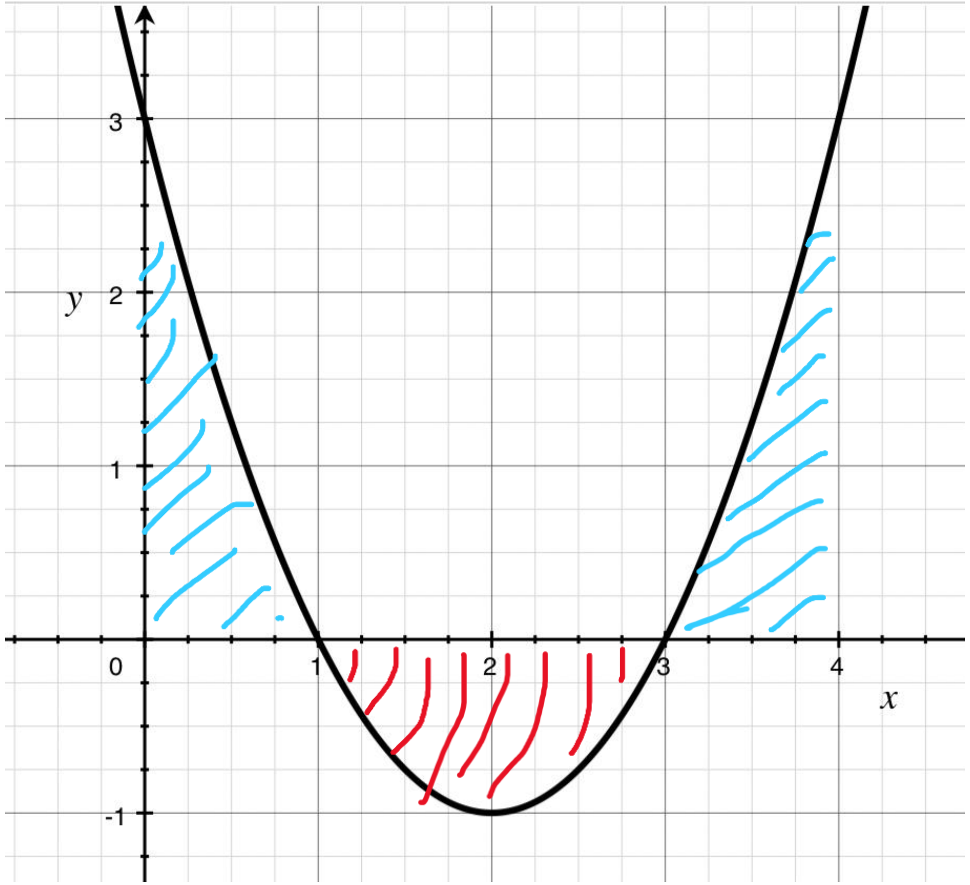
If we define the area below the x-axis as being a **negative area**, then everything we have done so far goes through with no issues.

E.g. If $f(x) = (x-2)^2 - 1$

Then

$I = \int_0^4 f(x)\,dx$ can be rewritten as:

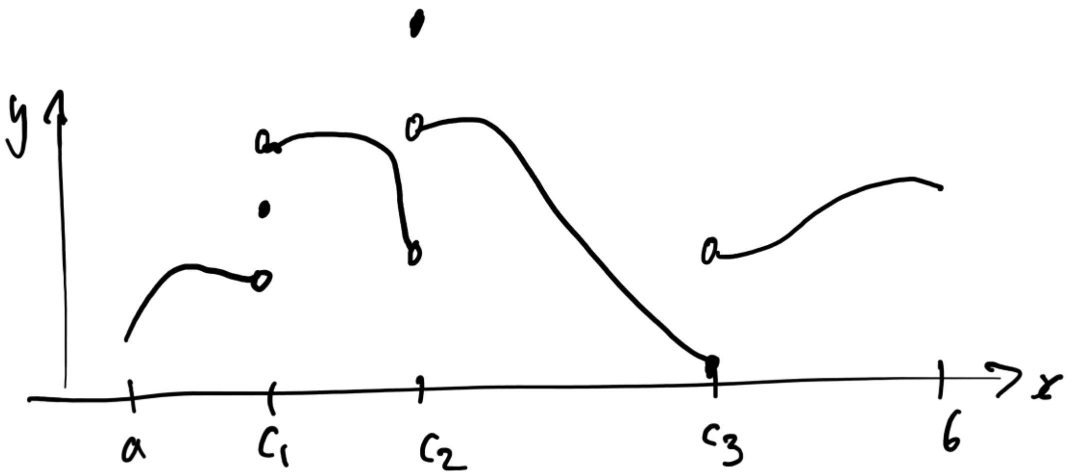
$$I = \int_0^1 f(x)\,dx - \int_1^3 |f(x)| dx + \int_3^4 f(x)\,dx$$



Piecewise continuous

We can now fully define our integrals; but first, how many discontinuities we can deal with? We can deal with a countably infinite set of discontinuities as long as they are somewhat far apart (isolated) from each other; we need that in order to take the limits below and above a point of discontinuity so that they make sense in our integral.

A (potentially infinite) countable set of points $\{c_1, \dots, c_k\} \subset \mathbb{R}$ is **isolated** if there is some $\epsilon > 0$ such that for all $i, j, 1 \leq i \leq j \leq k$ we have $|c_i - c_j| > \epsilon$.



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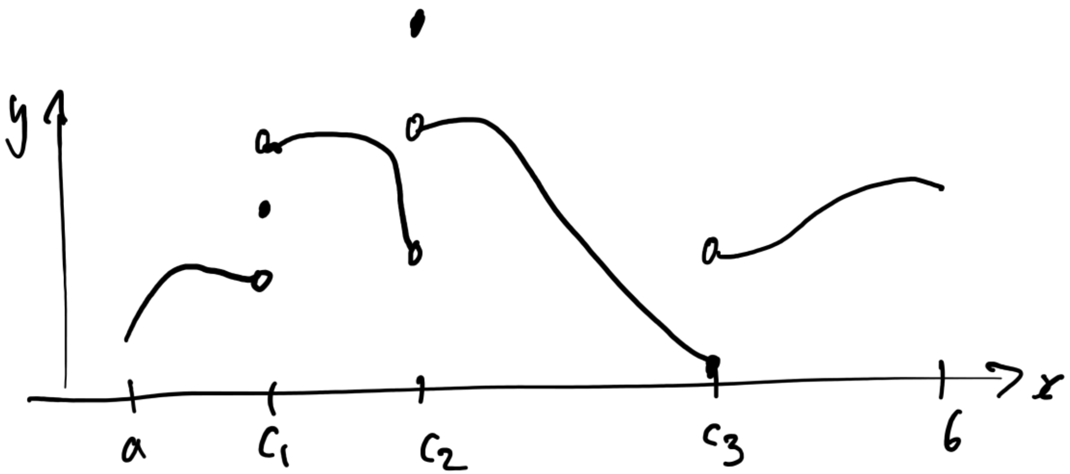
A function f is **piecewise continuous** on an interval $[a, b]$ if there is a set of isolated points $\{c_1, \ldots, c_k\}$, $c_1 < c_2 < \cdots < c_k$ such that f is continuous on $(a, c_1), (c_k, b)$ and $(c_i, c_{i+1}), i = 1 \ldots k - 1$.



Reimann Integral

Loosely speaking, the Riemann integral is the limit of the Riemann sums of a function as the partitions get finer.

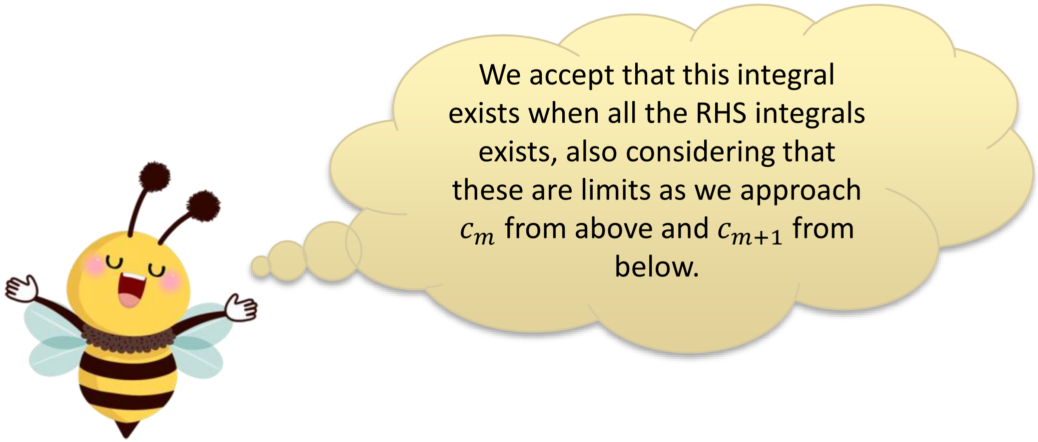
Let f be a piecewise continuous function on $[a, b]$, let $\{P_n\}$ be a sequence of ever-finer partitions of $[a, b]$ such that $\Delta_i \to 0$.

We define the definite integral of f:

$$\int_a^b f(x)\, dx = \sum_{m=0}^{k} \int_{c_m}^{c_{m+1}} f(x)\, dx$$

where the discontinuities of f are at $c_1 < c_2 < \cdots < c_k$ and $c_0 = a, c_{k+1} = b$,

as long as all the integrals on the right hand side exist.



Properties of definite integrals

From the definition of the integral as a limit, we can inherit the following properties from the arithmetic of limits. Let f, g be piecewise continuous on $[a, b]$, $c \in \mathbb{R}$ and $k \in (a, b)$:

Additivity with respect to range:

$$\int_a^b f(x)\,dx = \int_a^k f(x)\,dx + \int_k^b f(x)\,dx$$

Homogeneity:

$$\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$$

Linearity:

$$\int_a^b (f(x) + g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$

Inequality: if $f(x) \leq g(x)$ for all $x \in [a, b]$ then

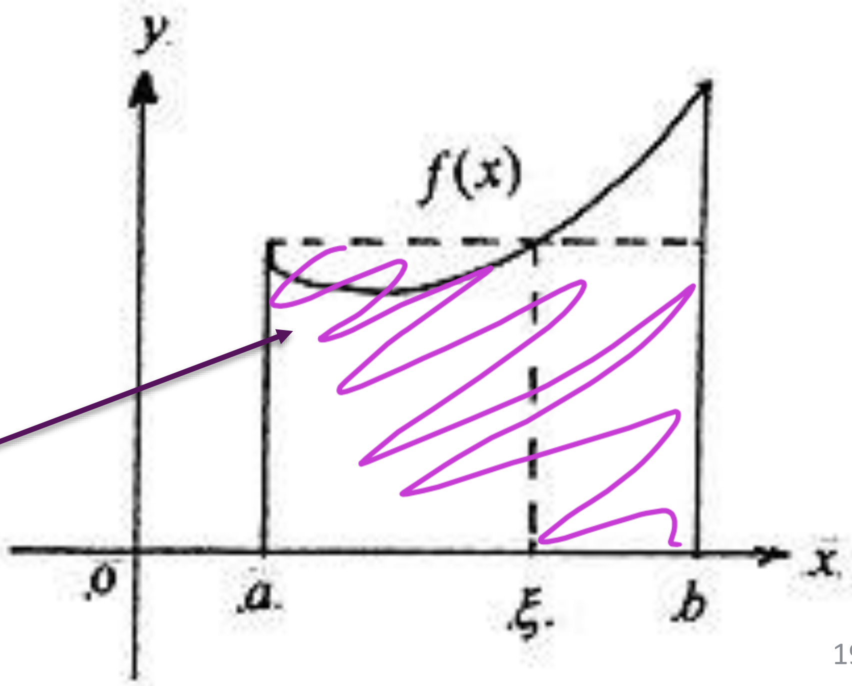
$$\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx$$

These relations are fairly easy to prove although not straightforward, as you would have to write out and follow the definitions of integrals and limits to derive them.

First mean value theorem for integrals

Let f be a continuous function on $[a, b]$. Then there exists $\xi \in (a, b)$ such that

$$\int_a^b f(x)\,dx = (b-a)f(\xi).$$



Geometrically: this rectangle has the same area as the region below the curve from a to b

First mean value theorem for integrals

Let f be a continuous function on $[a, b]$. Then there exists $\xi \in (a, b)$ such that
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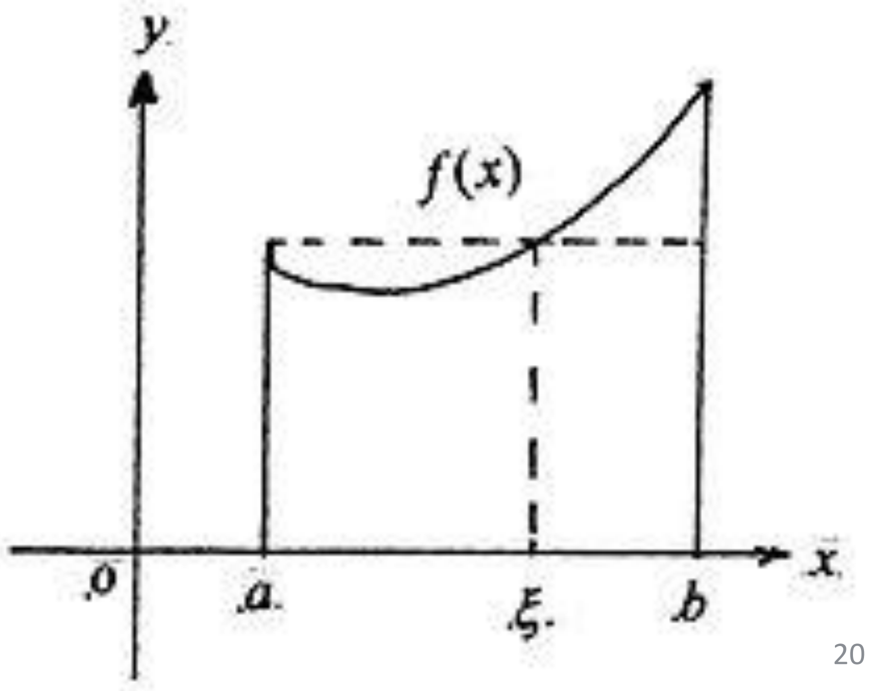
Proof:
Let m, M be the minimum and maximum of f on $[a, b]$.
Then clearly:
$$m(b-a) \leq \int_a^b f(x) \leq M(b-a) \quad \Rightarrow$$
$$m \leq \frac{1}{b-a}\int_a^b f(x) \leq M$$

Since f is continuous we can apply the intermediate value theorem, which states that "if f is continuous and ranges from m to M in $[a, b]$, then for any $y \in [m, M]$, there exists $\xi \in (a, b)$ such that $y = f(\xi)$".

That is, there is a $\xi \in (a, b)$ such that:
$$f(\xi) = \frac{1}{b-a}\int_a^b f(x)$$



Definite integrals

Definite integrals calculate the area of the region bounded by $f,\ x = a,\ x = b,\ y = 0$, so they **take a specific value**.

Note that the variable x in the definite integrals plays no role in the value of the integral, which is just a number.

We could replace x with any dummy variable we choose, which then ranges over the interval $[a, b]$:

$$\int_a^b f(x)\,dx = \int_a^b f(y)\,dy = \int_a^b f(t)\,dt$$

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Let us now define a function $F(x)$ which is the integral from a up to a variable point x, i.e. we replace the upper end of the range of integration with a true variable x:

$$F(x) = \int_a^x f(t)\,dt$$

Now the value of this integral ***does*** depend on the value of x, and the dummy variable ranges over $a \leq t \leq x$.

Definite integrals

Consider the difference:

$$F(x+h) - F(x) = \int_a^{x+h} f(t)\,dt - \int_a^{x} f(t)\,dt = \int_x^{x+h} f(t)\,dt.$$

By the mean value theorem there is some $\xi \in (x, x+h)$ such that:

$$F(x+h) - F(x) = hf(\xi) \qquad \Rightarrow \qquad \frac{F(x+h) - F(x)}{h} = f(\xi).$$

Taking the limit as $h \to 0$ we get:

$$F'(x) = \lim_{h \to 0} \frac{F(x+h) - F(x)}{h} = \lim_{h \to 0} f(\xi) = f(x),$$

i.e. the derivative of the value of the integral up to x is the integrand, f, evaluated at x.

The Fundamental Theorem of Calculus

integrand

Let f be a continuous function on $[a, b]$, and let

$$F(x) = \int_a^x f(t)\,dt$$

then F is continuous and differentiable on (a, b) and

$$F'(x) = f(x) \text{ for all } x \in (a, b).$$

A function F such that $F'(x) = f(x)$ is called an **antiderivative** of f.

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Indefinite integrals

Now suppose we take any antiderivative G of f such that $G'(x) = f(x)$. Define $H = G - F$.

Then $H'(x) = G'(x) - F'(x) = 0$, so it must be $H(x) = C$ for some constant C, i.e.

$$G(x) = F(x) + C = \int_a^x f(t)\, dt + C.$$

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$$G(x) = F(x) + C = \int_a^x f(t)\,dt + C.$$

In general, we write

$$\int f(t)\,dt = F(x) + C,$$

where F is an antiderivative of f, to denote an indefinite integral.

[And often $\int f(x)\,dx = F(x) + C$ although we must remember that the x's on each side play different roles.]

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Evaluating definite integrals with antiderivatives

Let f be continuous on $[a, b]$ with antiderivative F. Consider **any** antiderivative G of f, i.e. $G(x) = F(x) + C$.

Then

$$G(b) - G(a) = F(b) - F(a) = \int_a^b f(x)\,dx - \int_a^a f(x)\,dx = \int_a^b f(x)\,dx.$$

So we can use any anti-derivative to evaluate a definite integral.

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So we can use any anti-derivative to evaluate a definite integral.

Restating this we get:

$$\int_a^b f(x)\,dx = [G(x)]_a^b = G(b) - G(a)$$

where G is any **antiderivative** of f.

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What is the value of $\int_1^5 \frac{1}{x^2}\,dx$?

- $\frac{5}{4}$
- $\frac{3}{4}$
- $\frac{4}{5}$
- $\frac{4}{3}$
- $\frac{3}{2}$
- None of the above

Total Results: 0

Example

How can we evaluate $\int_1^5 \frac{1}{x^2} dx$?

We can guess an anti-derivative: if $F(x) =$ **???** , then $F'(x) = \frac{1}{x^2}$.

So

$$\int_1^5 \frac{1}{x^2} dx =$$

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Example

How can we evaluate $\int_1^5 \frac{1}{x^2} dx$?

We can guess an anti-derivative: if $F(x) = -\frac{1}{x}$, then $F'(x) = \frac{1}{x^2}$.

So

$$\int_1^5 \frac{1}{x^2} dx =$$

$$= [F(x)]_1^5$$

$$= \left[-\frac{1}{x}\right]_1^5$$

$$= -\frac{1}{5} + 1 = \frac{4}{5}$$

Example

What about

$$\int_0^5 \frac{1}{x^2}\,dx \ ?$$

We are in trouble here since the integrand is infinite at $x = 0$.

We will see how to deal with this next week!

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What we learnt today

- $\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(\xi_i)\Delta_i$ where f is continuous and the limit exists
- $\int_a^b f(x)\,dx = \lim_{\epsilon\to 0} \int_a^{c-\epsilon} f(x)\,dx + \lim_{\epsilon'\to 0} \int_{c+\epsilon'}^{b} f(x)\,dx$ where f is not continuous at c and both limits exist
- **Reimann integral**, where f is not continuous at c_i's and the RHS integrals exist:

$$\int_a^b f(x)\,dx = \sum_{m=0}^{k} \int_{c_m}^{c_{m+1}} f(x)\,dx$$

- Mean value theorem for integrals
- **The Fundamental Theorem of Calculus:** Let f be continuous on $[a,b]$, with $F(x) = \int_a^x f(t)\,dt$. Then F is continuous and differentiable on (a,b) with:

$$F'(x) = f(x) \text{ for all } x \in (a,b).$$