Maths for Computer Science
Calculus

Dr Eleni Akrida

Calculus in general … and MCS Calculus in term 2

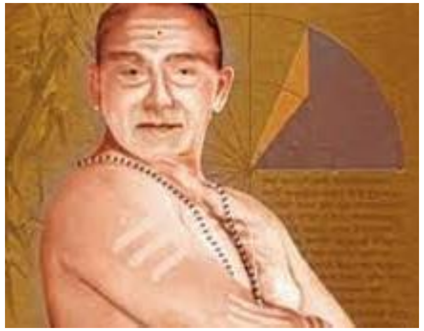
- Calculus is a mathematical discipline focused on **limits**, **continuity**, **derivatives**, **integrals**, and **infinite series**.
- Elements appeared in ancient Egypt, then ancient Greece, then China & Middle East, then again in medieval Europe and India.

The calculus … controversy!

The first one recorded to seriously consider dividing objects into an infinite number of cross-sections (4th century BC)

Zu Chongzhi (5th century AD) deduced the formula for the volume of a sphere

Ḥasan Ibn al-Haytham, Latinized as Alhazen (10-11th century AD), derived a formula for the sum of fourth powers and used it to carry out a form of integration



Mādhava of Sangamagrāma (14th century AD) and later mathematicians of the Kerala school of astronomy and mathematics stated components of calculus such as the Taylor series and infinite series approximations

Calculus in general … and MCS Calculus in term 2

- Lagrange multipliers for constrained optimisation
  (first-order derivatives, …)
- Multivariate Extrema (*revisited*)
  (first-order derivatives, second-order derivatives, …)
- Series
  (limits, …)
- Power series
  (series, limits, differentiation, …)
- Taylor's theorem
  (Taylor series, differentiation, limits, …)
- Integration
  (limits, series, …)
- Fourier series
  (trig identities, series, limits, …)

Organisation

Calculus lectures (1h per week)

- 10 weeks in total;
- Every Thursday at 2pm in CLC202
- Slide material uploaded on Ultra;
- Normally stream-captured (modulo technical problems).

Calculus practicals (2h every other week)

- every odd week during weeks 12-20
- 4 weeks in total

Assessment

- Examinable by an end-of-year exam
- Main Examination Period: Monday 12th May to Friday 6th June 2025 inclusive

- Formative exercises will be given in practicals
- Practice them in the practical and check against solutions released at the end of the week

Reading

- Slides;
- Formative exercises;
- Recommended Textbooks (check Ultra);
  - “Mathematics for Engineers and scientists” by Alan Jeffrey
  - For each topic we cover, check Ultra lecture pages for the textbook chapter
  - For each topic we cover, I will also refer to any relevant resources on "Paul's online notes"

Lagrange multipliers

Contents for today's lecture

- Constrained optimization
- Lagrange multipliers

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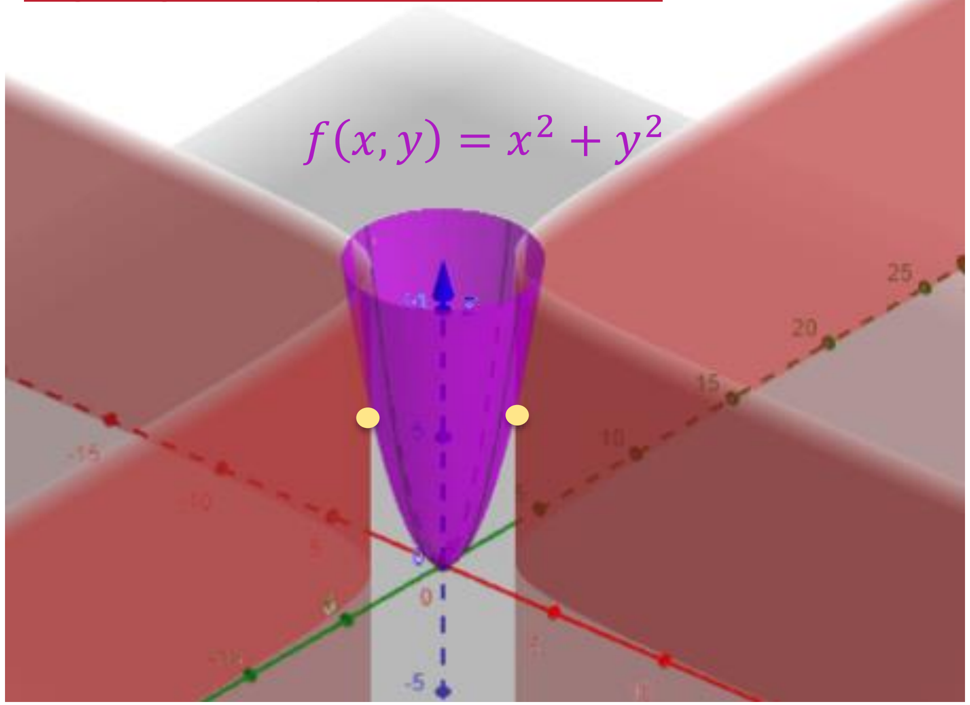
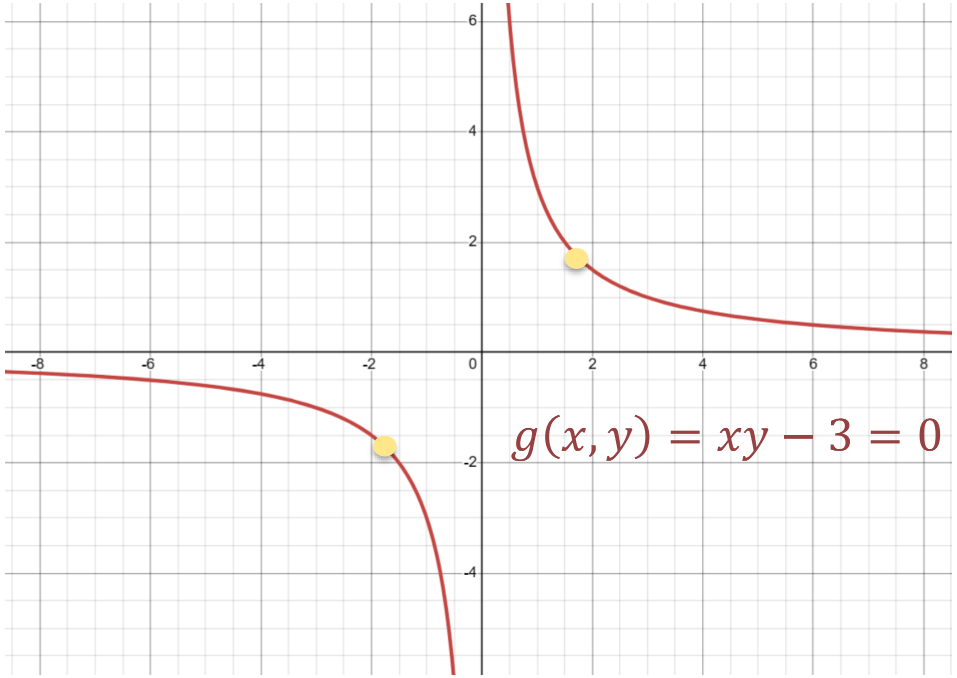
Constrained (???) optimization

Lagrange multipliers – GeoGebra

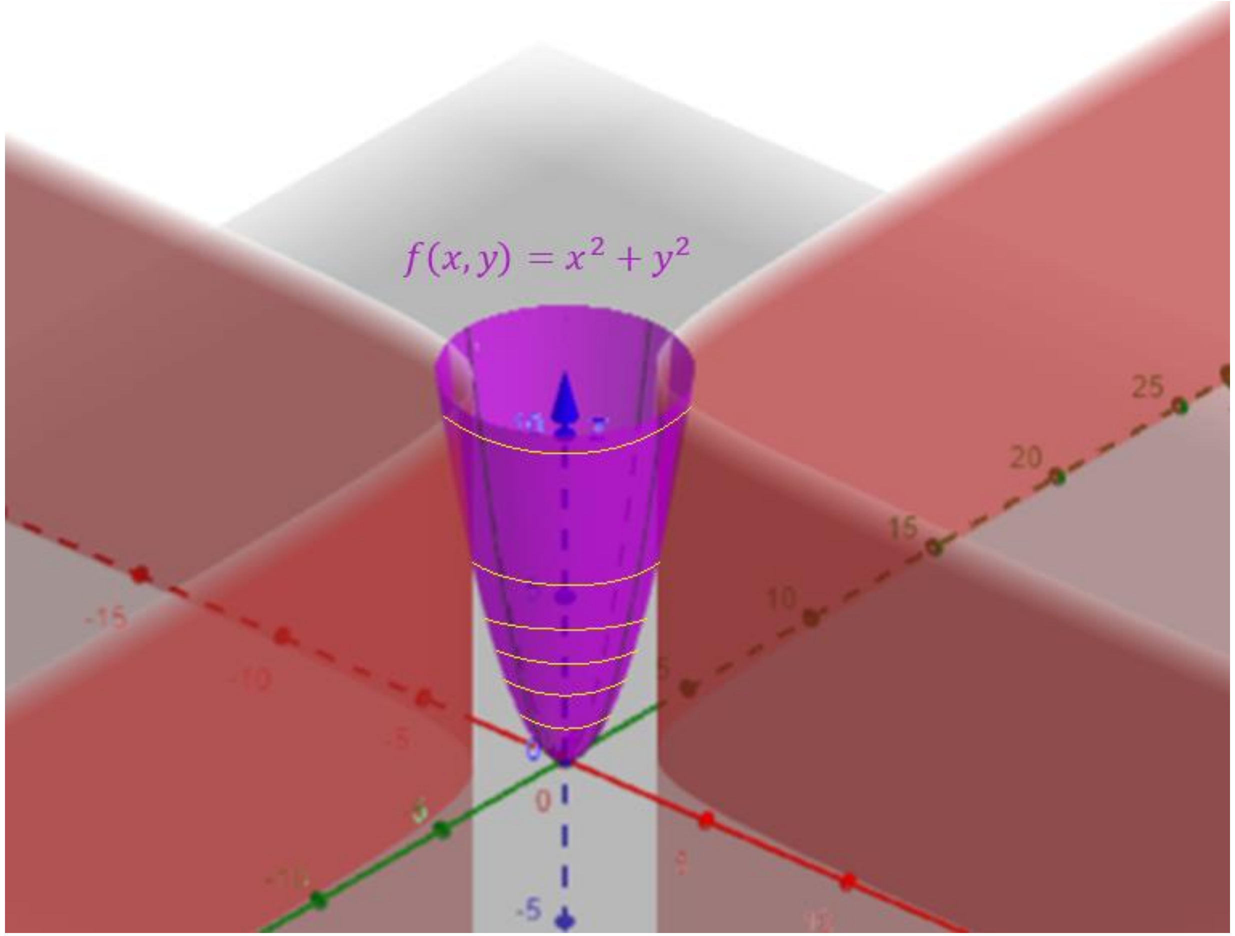




Constrained (???) optimization

Here, to help me understand better, I have put up contours of the graph of my function:

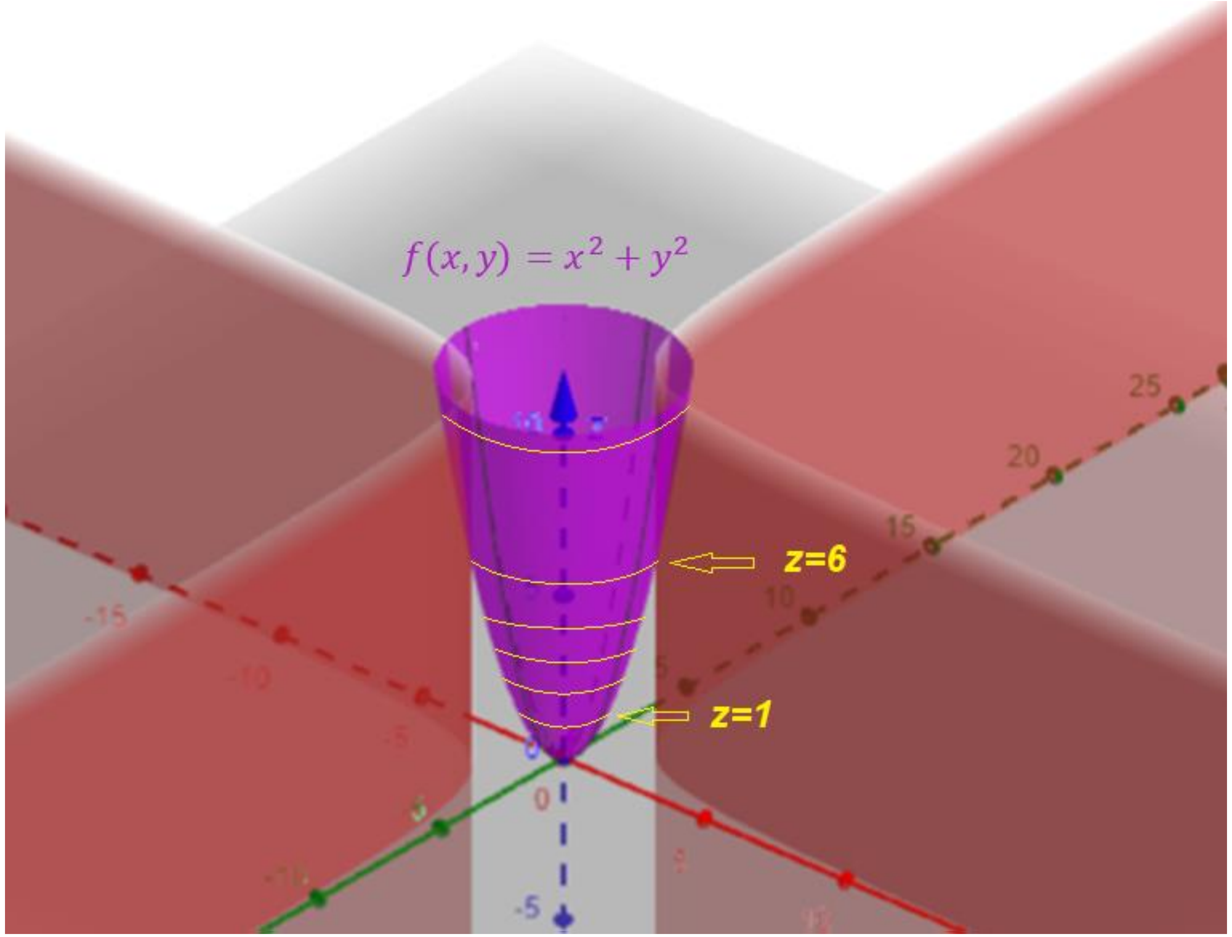
the contour is “telling me” that the height is constant.



Constrained (???) optimization

For example, if we set it equal to the value of $z = 1$, we get one curve; if we set it equal to 6, we get a different curve and so on.

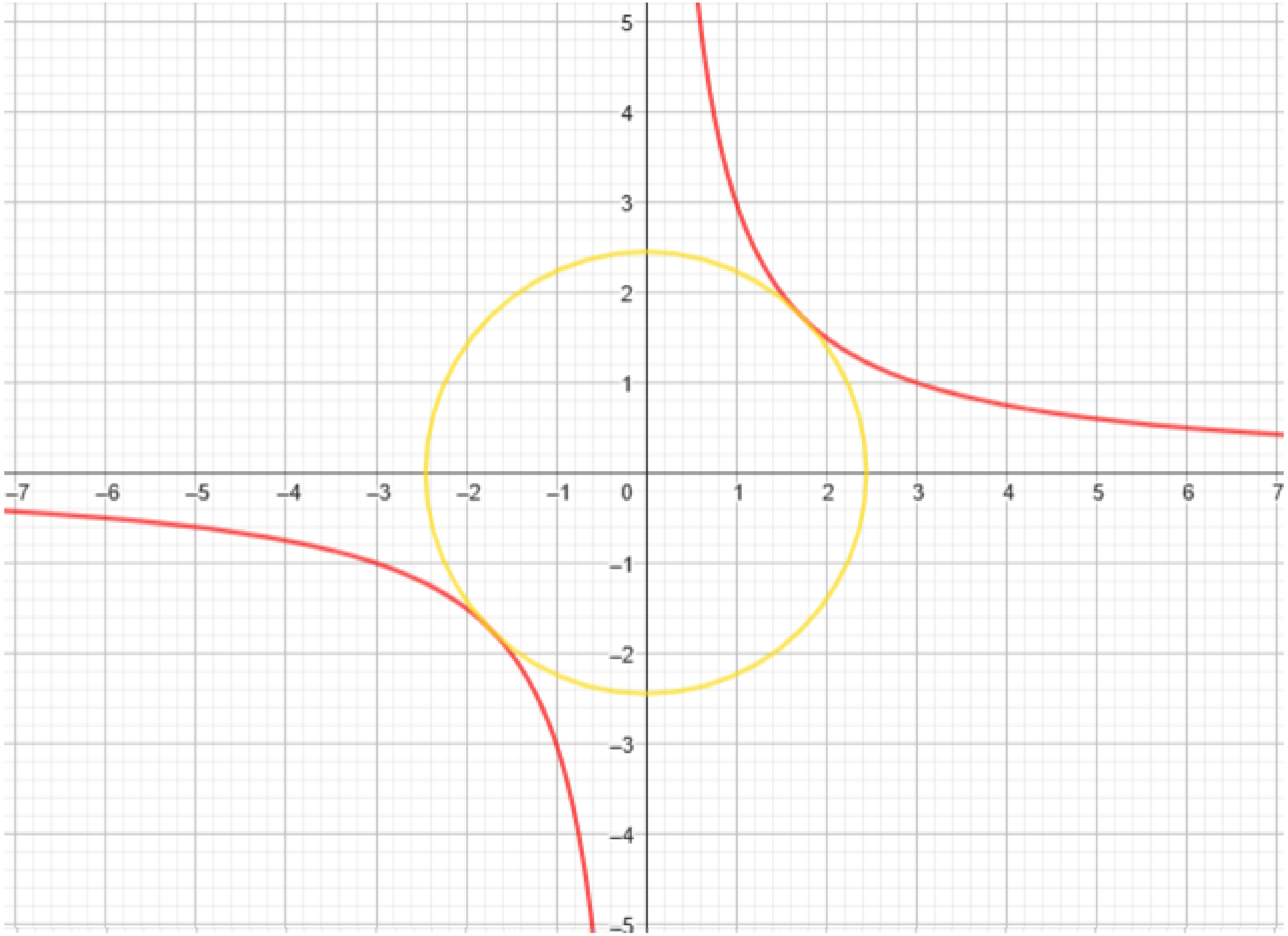
These are what we call *level curves*: the set of points where a function $f(x, y)$ equals a specific value, z.



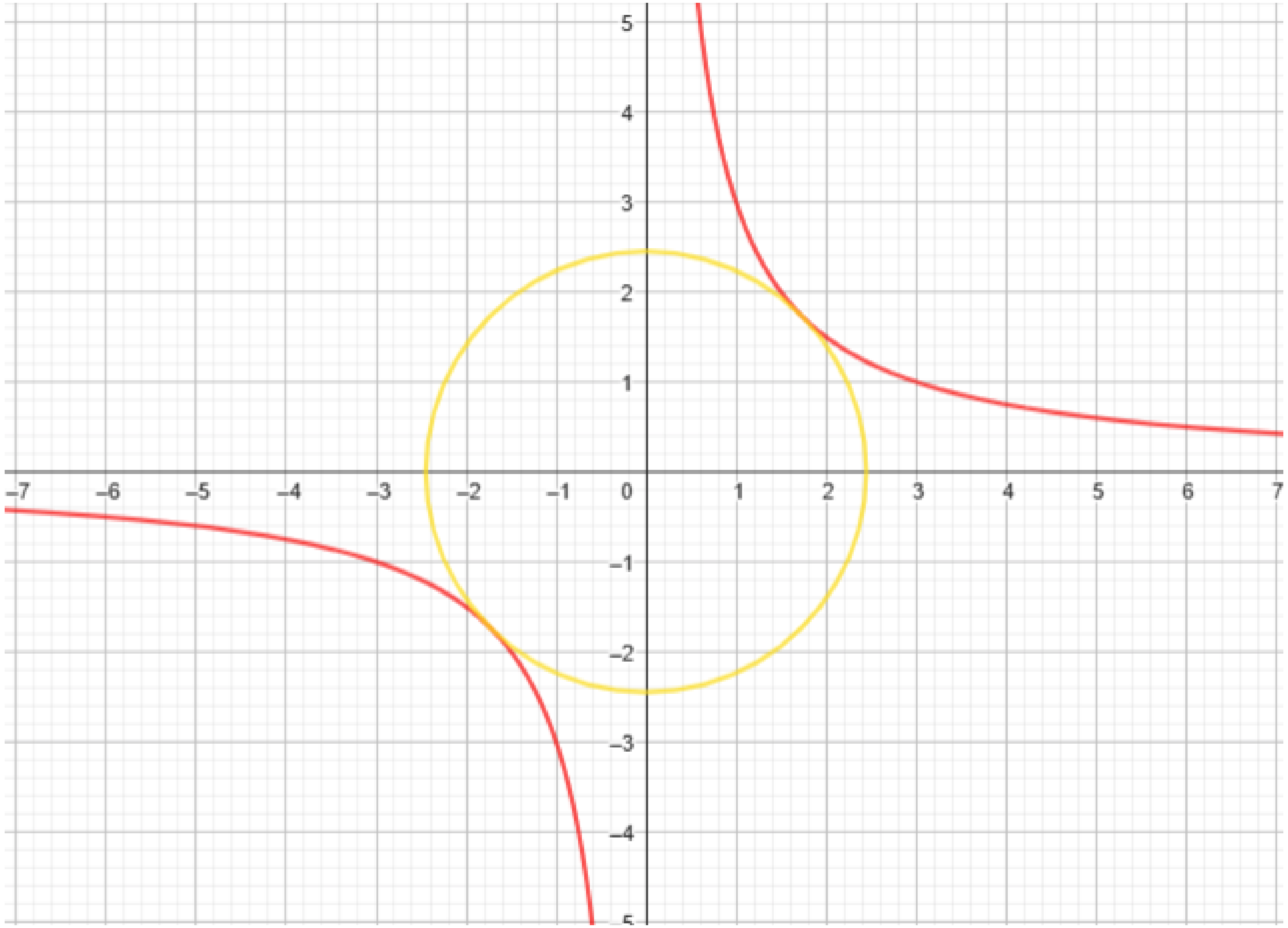
Constrained (???) optimization

We can also have a look at what these look like in the domain as well. If we take a “bird’s eye” view then we get a similar effect: we have the red hyperbola (the restriction / constraint) and the yellow curves are the level curves of $f(x, y)$.

Of these level curves, one is very special – **it ‘barely touches the restriction’**



Constrained (???) optimization



Constrained optimisation

- Recall min-max problems
- Now variables are **not independent**

Goal is to min / max a multivariate function $f(\boldsymbol{x})$ where $x = (x_1, \dots, x_n)$ and $x_1, \dots, x_n$ are not independent, i.e.

- Relation $g(\boldsymbol{x}) = 0$ (think of it as a constraint)

If the constraint is very simple, we may solve the constraint for one variable and plug back into f and then we have the usual min/max problem.

What is the minimum of $f(x, y) = x^2 + y^2$ subject to the constraint $x + y = 2$?

$(0, 2)$

$(1, 1)$

$(2, 0)$

None of the above

What is the minimum of $f(x, y) = x^2 + y^2$ subject to the constraint $x + y = 2$?

$(0, 2)$ 0%

$(1, 1)$ 0%

$(2, 0)$ 0%

None of the above 0%

What is the minimum of $f(x, y) = x^2 + y^2$ subject to the constraint $x + y = 2$?

$(0, 2)$	0%
$(1, 1)$	0%
$(2, 0)$	0%
None of the above	0%

Constrained optimisation

If the constraint is very simple, we may solve the constraint for one variable and plug back into f and then we have the usual min/max problem.

- $f(x, y) = x^2 + y^2$ subject to $x = 2 - y$
- $h(y) = (2 - y)^2 + y^2 = 2y^2 - 4y + 4$ - how do we minimise that?
- $h'(y) = 0 \Leftrightarrow 4y - 4 = 0 \Leftrightarrow y = 1$
- Therefore, based on our constraint $x = 1$
- f is minimised at $(1,1)$

Constrained optimisation

- Recall min-max problems
- Now variables are **not independent**

Goal is to min / max a multivariate function $f(\boldsymbol{x})$ where $x = (x_1, \dots, x_n)$ and $x_1, \dots, x_n$ are not independent, i.e.

- Relation $g(\boldsymbol{x}) = 0$ (think of it as a constraint)

If the constraint is not simple, we need a new method.

- Observe: cannot use usual method of looking for critical points of f as they will typically not satisfy this condition / constraint, so will not be good solutions!

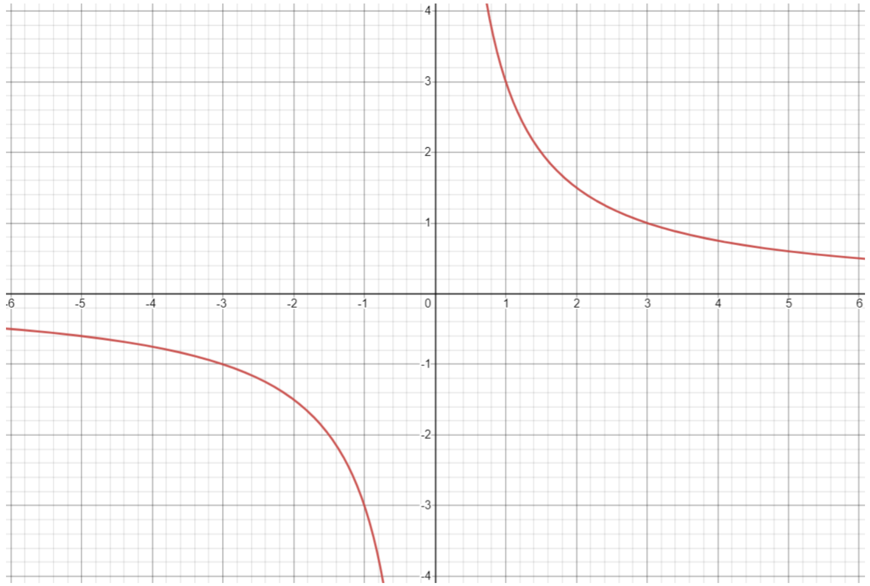
Simple example demonstration

Find point closest to origin on hyperbola $xy = 3$.

i.e. minimise the distance of a point of the hyperbola to the origin $(0,0)$,

i.e. minimise the function $f(x, y) = \sqrt{x^2 + y^2}$

subject to the constraint $xy = 3 \Rightarrow xy - 3 = 0$.



https://www.geogebra.org/m/cg9z7ay3
https://www.geogebra.org/m/zdnbhk4e

Durham University

Simple example demonstration

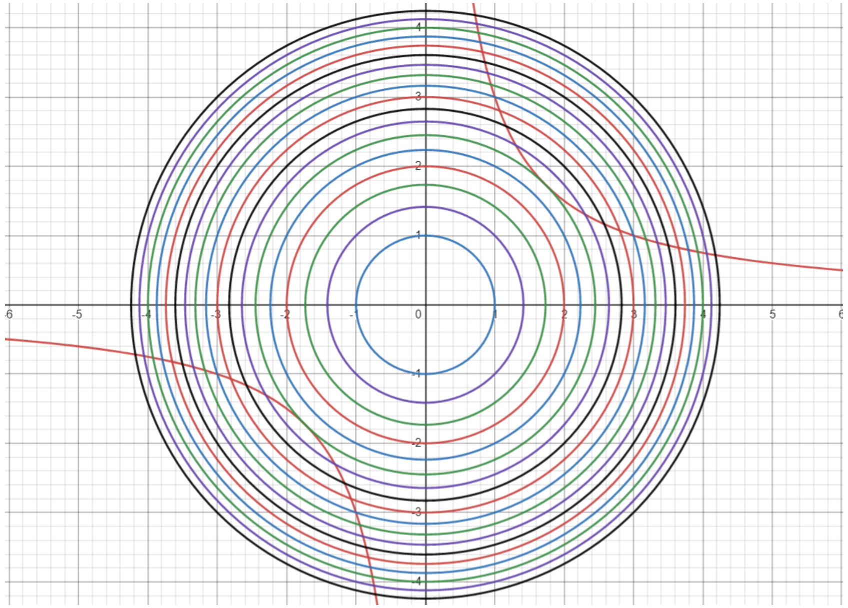
Find point closest to origin on hyperbola $xy = 3$.

i.e. minimise the distance of a point of the hyperbola to the origin $(0,0)$,

i.e. minimise the function $\cancel{f(x,y) = \sqrt{x^2+y^2}}$ or equivalently $f(x,y) = x^2 + y^2$

subject to the constraint $xy = 3 \Rightarrow \underbrace{xy - 3}_{g(x,y)} = 0$.

We can actually solve this with elemental geometry, but we will use it now as an example for Lagrange multipliers



https://www.geogebra.org/m/cg9z7ay3
https://www.geogebra.org/m/zdnbhk4e

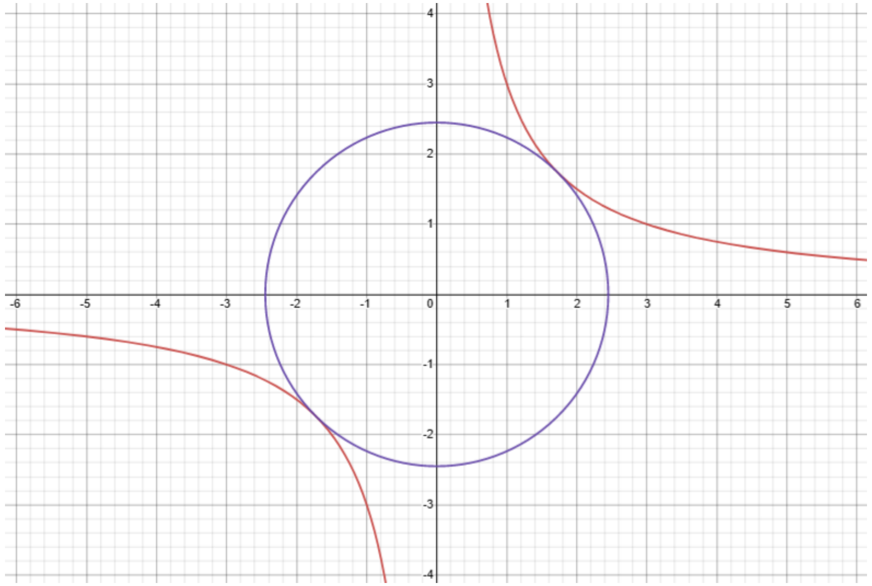
Simple example demonstration

$$\min f(x,y) = x^2 + y^2$$
$$\text{subject to} \quad g(x,y) = xy - 3 = 0$$

Observe: at the minimum, the level curve of f is tangent to the hyperbola $xy = 3$, i.e., **at the minimum, the level curve of f is tangent to the level curve of g**

- Claim: this is going to hold in general but we will come back to that!

How do we solve for points where this holds, i.e. find (x, y) where the level curves of f and g are tangent to each other?



Simple example demonstration

The level curves of f and g are tangent to each other

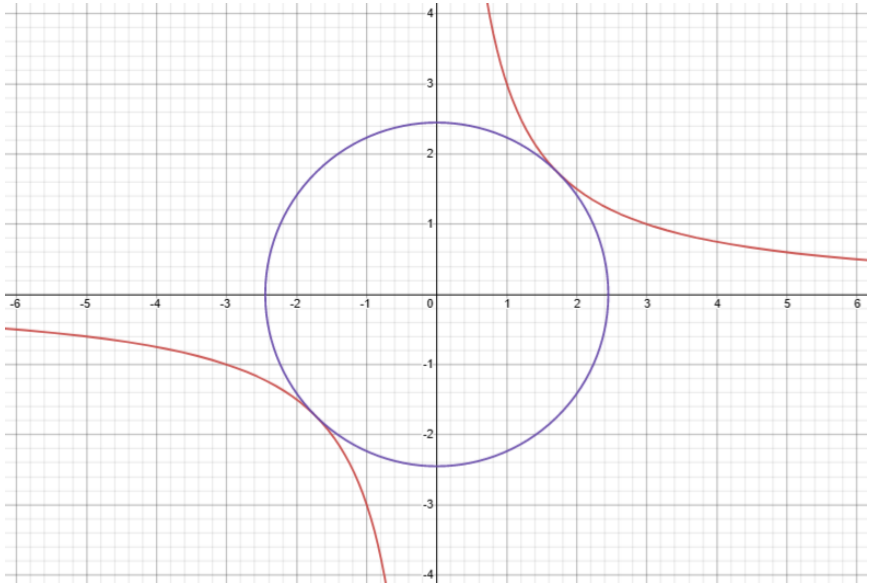
$\Rightarrow$ f and g have the same tangent line $\Rightarrow$ the gradient vectors ∇f and ∇g are parallel to each other

$$\Rightarrow \exists \lambda \in \mathbb{R}: \ \nabla f = \lambda \nabla g$$

So, our optimisation problem 'becomes' a system of equations!

$$\left.\begin{array}{l} \nabla f = \lambda \nabla g \quad \Rightarrow \begin{cases} f_x = \lambda g_x \\ f_y = \lambda g_y \end{cases} \\ \text{Also} \qquad g = 0 \end{array}\right\}$$

So, we have a system of three equations and three unknowns, namely x, y, λ.



Simple example demonstration

Recall $f(x,y) = x^2 + y^2$ and $g(x,y) = xy - 3$. So, we need to solve:

$$\begin{Bmatrix} f_x = \lambda g_x \\ f_y = \lambda g_y \\ g = 0 \end{Bmatrix} \Rightarrow \begin{Bmatrix} 2x = \lambda y \\ 2y = \lambda x \\ xy = 3 \end{Bmatrix} \Rightarrow \begin{Bmatrix} 2x = \dfrac{3\lambda}{x} \\ 2\dfrac{3}{x} = \lambda x \\ y = \dfrac{3}{x} \end{Bmatrix} \Rightarrow \begin{Bmatrix} x^2 = \dfrac{3}{2}\lambda \\ x^2 = \dfrac{6}{\lambda} \\ y = \dfrac{3}{x} \end{Bmatrix}$$

Note that $x \neq 0$

Note that $\lambda \neq 0$

Solving for λ, we get $\lambda = \pm 2$.

For $\lambda = 2$, we get two solutions, namely $(x,y) = (\sqrt{3}, \sqrt{3})$ or $(x,y) = (-\sqrt{3}, -\sqrt{3})$.

For $\lambda = -2$, we have no solutions.

Note: you can check that at both $(x,y) = (\sqrt{3}, \sqrt{3})$ and $(x,y) = (-\sqrt{3}, -\sqrt{3})$, $\nabla f = 2\nabla g$.

*λ is called the **Lagrange multiplier**; it is what you multiply the gradient of g to get the gradient of f at the constraint minimum.*

Lagrange multipliers method

The goal: to find the stationary points of an objective function $f(\boldsymbol{x}) \in C^1$ subject to the constraint $g(\boldsymbol{x}) = 0$.

The method:

1. Define a new objective function called the **Lagrangian**:
$$L(\boldsymbol{x}, \lambda) = f(\boldsymbol{x}) - \lambda g(\boldsymbol{x})$$
2. Find the stationary points of L with respect to both $\boldsymbol{x}$ and λ, i.e. solve the system of equations $\nabla f(\boldsymbol{x}) = \lambda \nabla g(\boldsymbol{x})$ and $g(\boldsymbol{x}) = 0$ for the unknowns $\boldsymbol{x}, \lambda$
3. Then the constrained extrema are found among the solutions to these equations. Identify their nature.

Observe: when trying to minimise L we will need to set $\nabla L = 0$, i.e. $\nabla f = \lambda \nabla g$ which is what we used to find a solution to our example; our reasoning was that **'at the minimum, the level curve of $\boldsymbol{f}$ is tangent to the level curve of $\boldsymbol{g}$'**

Why is this method valid?

i.e. why does $\nabla f = \lambda \nabla g$ hold at a constrained extremum

At the constrained min/max, in any direction along the level curve of $g = 0$ the rate of change of f must be 0.

So, at the constrained min/max, for any unit vector u tangent to $g = 0$, we must have that the directional derivative of f in the direction of u is 0, i.e.:

$$\nabla_u f(\boldsymbol{x_0}) = 0 \Leftrightarrow \nabla f . u = 0$$

Recall that the directional derivative of f in the direction of a vector u gives the gradient of the slope if you were to move in the u direction through point $f(\boldsymbol{x_0})$ and is defined as:

$$\nabla_v f(\boldsymbol{x_0}) = \lim_{h \to 0} \frac{f(\boldsymbol{x_0} + h\boldsymbol{v}) \; - f(\boldsymbol{x_0})}{h}$$

That means that any such vector u is perpendicular to the gradient ∇f, i.e. $\nabla f \perp$ level set of g. However, we know that $\nabla g \perp$ level set of g. So, it must be:

$$\nabla f \; // \; \nabla g$$

Another example

Optimise $f(x,y) = xy + 1$ subject to the constraint $g(\boldsymbol{x}) = x^2 + y^2 - 1 = 0$.

1. Define the Lagrangian $L(x,y,\lambda) = xy + 1 - \lambda x^2 - \lambda y^2 + \lambda$
2. Solve:

$$\begin{Bmatrix} f_x = \lambda g_x \\ f_y = \lambda g_y \\ g = 0 \end{Bmatrix} \Rightarrow \begin{Bmatrix} y = 2\lambda x \\ x = 2\lambda y \\ x^2 + y^2 = 1 \end{Bmatrix} \Rightarrow \begin{Bmatrix} y = 2\lambda x \\ x = 4\lambda^2 x \\ x^2 + y^2 = 1 \end{Bmatrix} \Rightarrow \begin{Bmatrix} y = 2\lambda x \\ x(1 - 4\lambda^2) = 0 \\ x^2 + y^2 = 1 \end{Bmatrix}$$

- For $x = 0$, we also get from the 1st equation that $y = 0$ but this contradicts the 3rd equation.
- $4\lambda^2 = 1$ gives $\lambda = \pm\frac{1}{2}$.
  - For $\lambda = \frac{1}{2}$, we get from the 1st equation that $y = x$, so from the 3rd equation we get $x = \pm\frac{1}{\sqrt{2}}$ and the extrema are found at $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ and $(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$; *(maxima)*
  - for $\lambda = -\frac{1}{2}$, we get from the 1st equation that $y = -x$, so from the 3rd equation we get $x = \pm\frac{1}{\sqrt{2}}$ and the extrema are found at $(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$ and $\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$. *(minima)*

Minimise $f(x, y) = x^2 + y^2$ subject to $g(x, y) = x + y - 2 = 0$. I.e., solve the system of equations: $\nabla f(x, y) = \lambda \nabla g(x, y)$ and $g(x, y) = 0$.

- There is no constrained minimum.
- The minimum is at (0,0).
- The minimum is at (1,1).
- The minimum is at (2,0).
- There is a constrained minimum but it is none of the above.

Minimise $f(x,y) = x^2 + y^2$ subject to $g(x,y) = x + y - 2 = 0$. I.e., solve the system of equations: $\nabla f(x,y) = \lambda \nabla g(x,y)$ and $g(x,y) = 0$.

There is no constrained minimum. 0%

The minimum is at (0,0). 0%

The minimum is at (1,1). 0%

The minimum is at (2,0). 0%

There is a constrained minimum but it is none of the above. 0%

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Minimise $f(x,y) = x^2 + y^2$ subject to $g(x,y) = x + y - 2 = 0$. I.e., solve the system of equations: $\nabla f(x,y) = \lambda \nabla g(x,y)$ and $g(x,y) = 0$.

There is no constrained minimum.

0%

The minimum is at (0,0).

0%

The minimum is at (1,1).

0%

The minimum is at (2,0).

0%

There is a constrained minimum but it is none of the above.

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More advanced example
Surface minimising pyramid

We want to build a pyramid with a given fixed triangular base B of area A and a given volume V so as to minimise the total surface area A'.

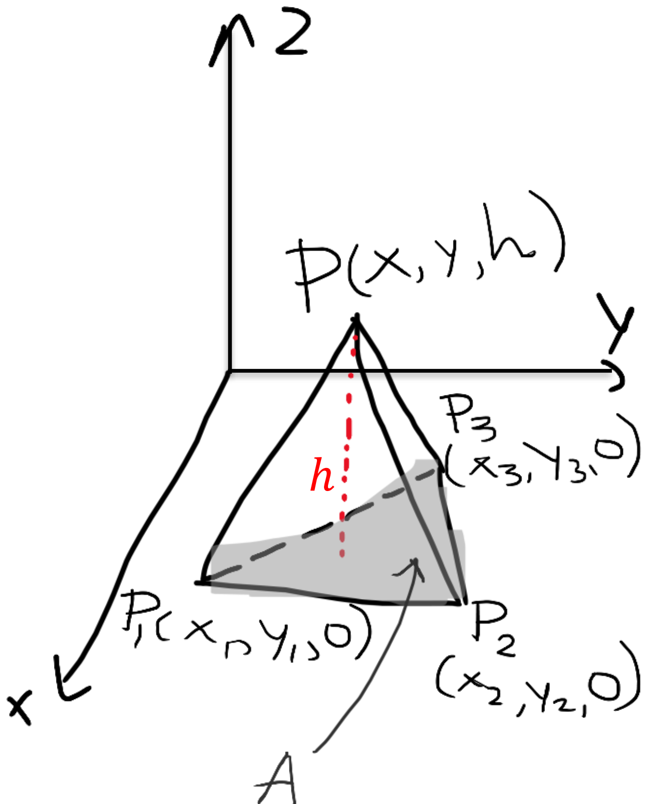
1. Notice that since A and V are fixed, the only thing that we can 'adjust' is where to place the top point of the pyramid.

2. Recall $V = \frac{1}{3}.A.h$

3. Since V and A are fixed, so is h.

4. We, thus, must choose where to place the top point of the pyramid, P, given that its height from the base is h (i.e. its z coordinate is fixed).

5. We can't solve if we just express the area we wish to minimize as a function of x, y, $A'(x, y)$.



More advanced example

Surface minimising pyramid

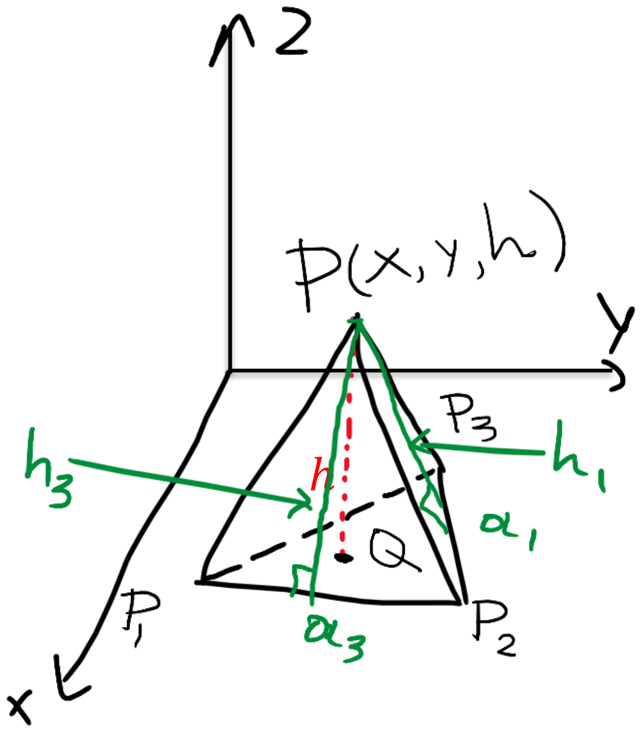
We want to build a pyramid with a given fixed triangular base B of area A and a given volume V so as to minimise the total surface area.

6. Let us consider the heights of the side triangles (faces) of the pyramid.

7. Then, A' will be the sum of three terms, namely

$$A' = \sum_{i=1}^{3} \frac{1}{2} \cdot a_i \cdot h_i$$

8. The heights $h_i, i = 1,2,3,$ are distances in the space; however we can reduce them to distances on the plane if we consider the point $Q(x, y, 0)$



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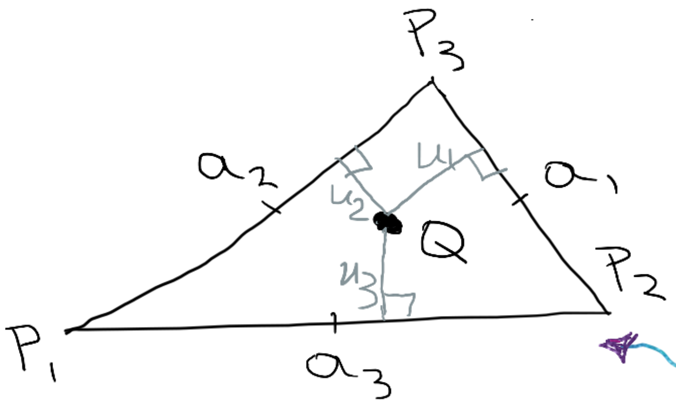
More advanced example
Surface minimising pyramid

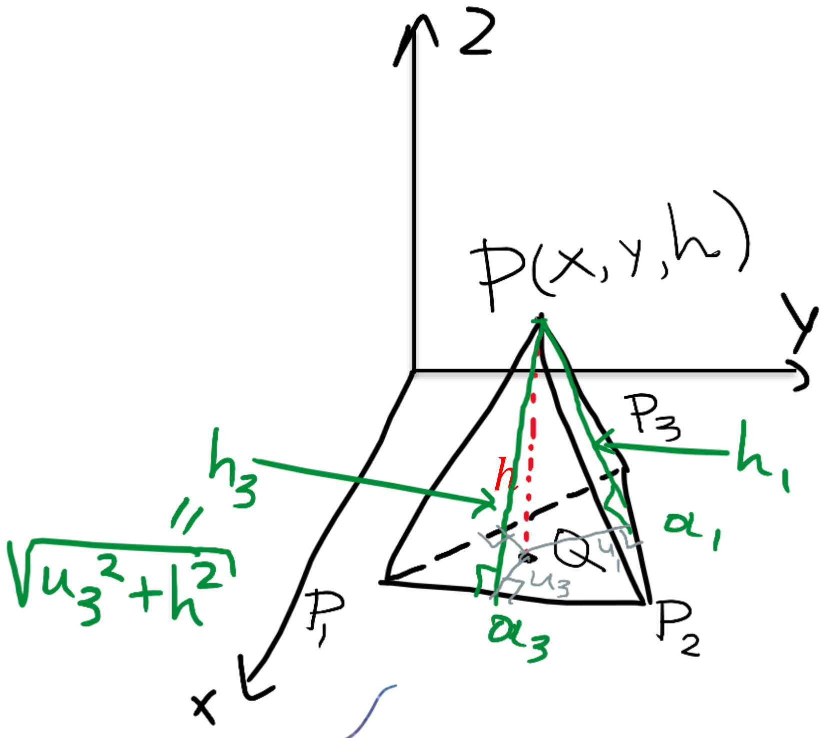
We want to build a pyramid with a given fixed triangular base B of area A and a given volume V so as to minimise the total surface area.

9. Let us call u_1, u_2, u_3 the distances from Q to the sides of the base triangle

10. Then $h_i = \sqrt{u_i^2 + h^2},\ i = 1,2,3$

11. Therefore, $A' = \sum_{i=1}^{3} \frac{1}{2} \cdot a_i \cdot \sqrt{u_i^2 + h^2} = f(u_1, u_2, u_3)$





More advanced example

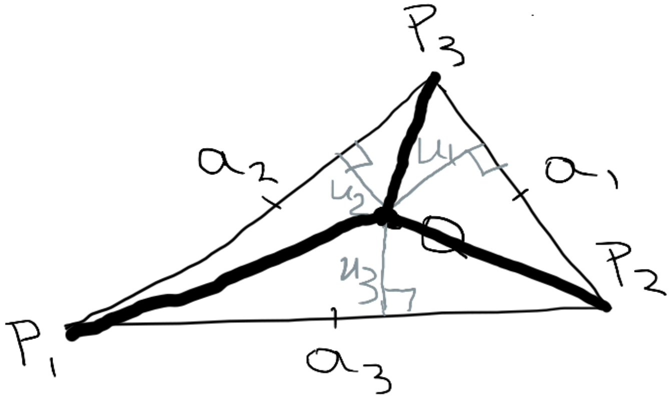
Surface minimising pyramid

We want to build a pyramid with a given fixed triangular base B of area A and a given volume V so as to minimise the total surface area.

12. So, we have the function f that we wish to minimise but we don't have a relation between the u_i's **(yet!)**

13. Let us split the base triangle into three triangles, as seen below, by connecting Q to the P_i's.

14. Then the area of the base is $A = \frac{1}{2}a_1u_1 + \frac{1}{2}a_2u_2 + \frac{1}{2}a_3u_3 = g(u_1, u_2, u_3)$



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More advanced example

Surface minimising pyramid

We want to build a pyramid with a given fixed triangular base B of area A and a given volume V so as to minimise the total surface area.

12. Let us solve the Langrange multiplier equations $\nabla f = \lambda \nabla g$, where

$$f(u_1, u_2, u_3) = \sum_{i=1}^{3} \frac{1}{2} \cdot a_i \cdot \sqrt{u_i^2 + h^2} \quad \& \quad g(u_1, u_2, u_3) = \sum_{i=1}^{3} \frac{1}{2} \cdot a_i \cdot u_i$$

$$\frac{\vartheta f}{\vartheta u_1} = \lambda \frac{\vartheta g}{\vartheta u_1} \Rightarrow \frac{1}{2} a_1 \frac{u_1}{\sqrt{u_1^2 + h^2}} = \lambda \frac{1}{2} a_1 \Rightarrow \lambda = \frac{u_1}{\sqrt{u_1^2 + h^2}}$$

$$\frac{\vartheta f}{\vartheta u_2} = \lambda \frac{\vartheta g}{\vartheta u_2} \Rightarrow \frac{1}{2} a_2 \frac{u_2}{\sqrt{u_2^2 + h^2}} = \lambda \frac{1}{2} a_2 \Rightarrow \lambda = \frac{u_2}{\sqrt{u_2^2 + h^2}}$$

$$\frac{\vartheta f}{\vartheta u_3} = \lambda \frac{\vartheta g}{\vartheta u_3} \Rightarrow \frac{1}{2} a_3 \frac{u_3}{\sqrt{u_3^2 + h^2}} = \lambda \frac{1}{2} a_3 \Rightarrow \lambda = \frac{u_3}{\sqrt{u_3^2 + h^2}}$$

13. Therefore $\frac{u_1}{\sqrt{u_1^2+h^2}} = \frac{u_2}{\sqrt{u_2^2+h^2}} = \frac{u_3}{\sqrt{u_3^2+h^2}} \Rightarrow u_1 = u_2 = u_3 \Rightarrow$

Q should be the incentre of the base triangle!

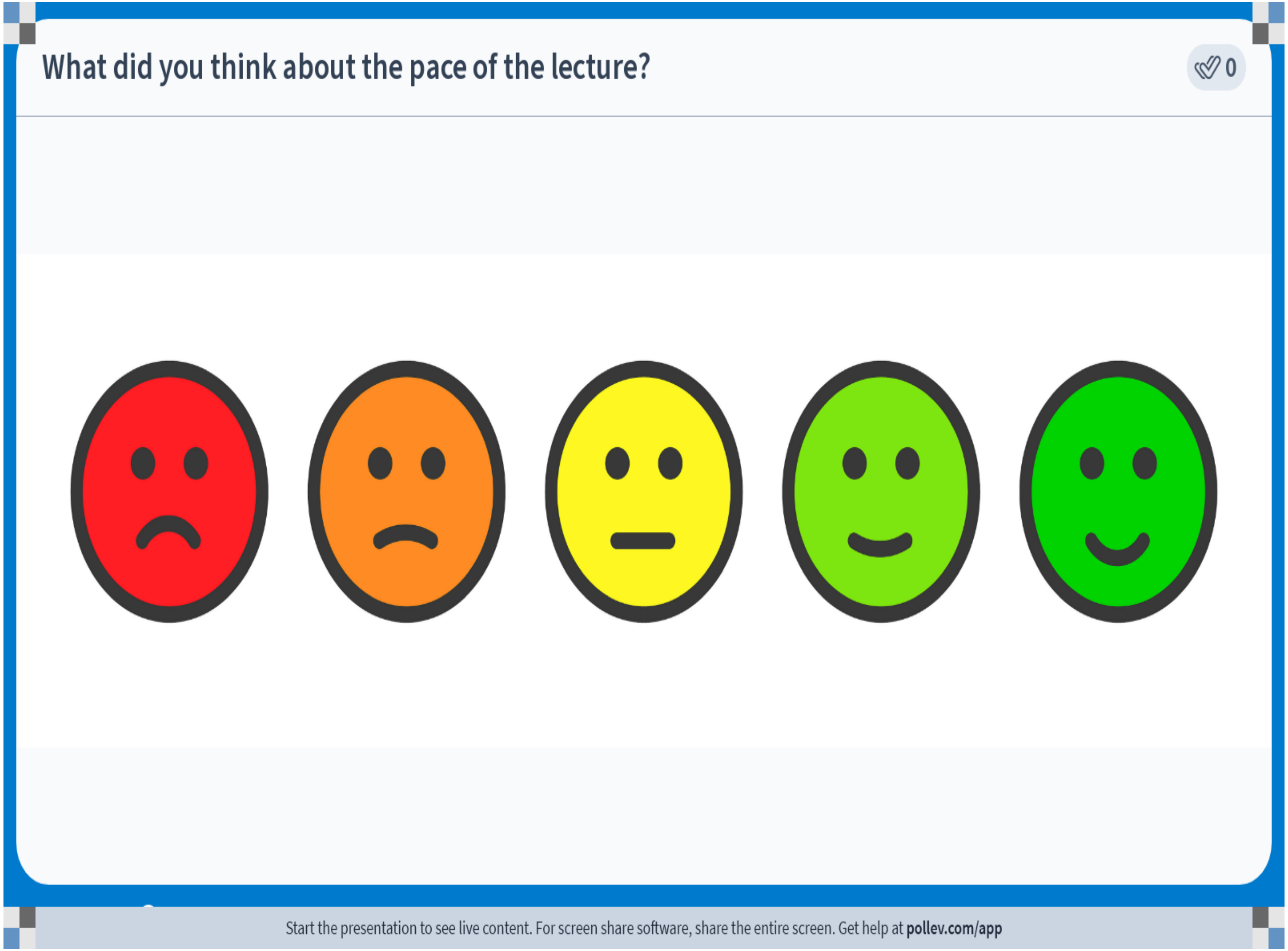
Highlights – what we learnt

Method of Lagrange multipliers:

- Used when we wish to min/max a function f but the variables are related via a function g
- We solve the system of equations $\nabla f(\boldsymbol{x}) = \lambda \nabla g(\boldsymbol{x})$ and $g(\boldsymbol{x}) = 0$ for the unknowns $\boldsymbol{x}, \lambda$.
  - I.e. we look for stationary points of the **Lagrangian**:

$$L(\boldsymbol{x}, \lambda) = f(\boldsymbol{x}) - \lambda g(\boldsymbol{x})$$

- The constrained extrema are found among those solutions.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