Maths for Computer Science
Calculus

Dr Eleni Akrida

Fourier Series

Contents for today's lecture

Fourier series for piecewise continuous functions

Recall the Taylor Series

If $f(x)$ is infinitely differentiable, its Taylor series representation centred at x_0 is:

$$f(x) = \sum_{n=0}^{\infty} f^{(n)}(x_0) \frac{(x - x_0)^n}{n!}$$

and this converges to $f(x)$ on $(x_0 - r, x_0 + r)$ where r is the radius of convergence.

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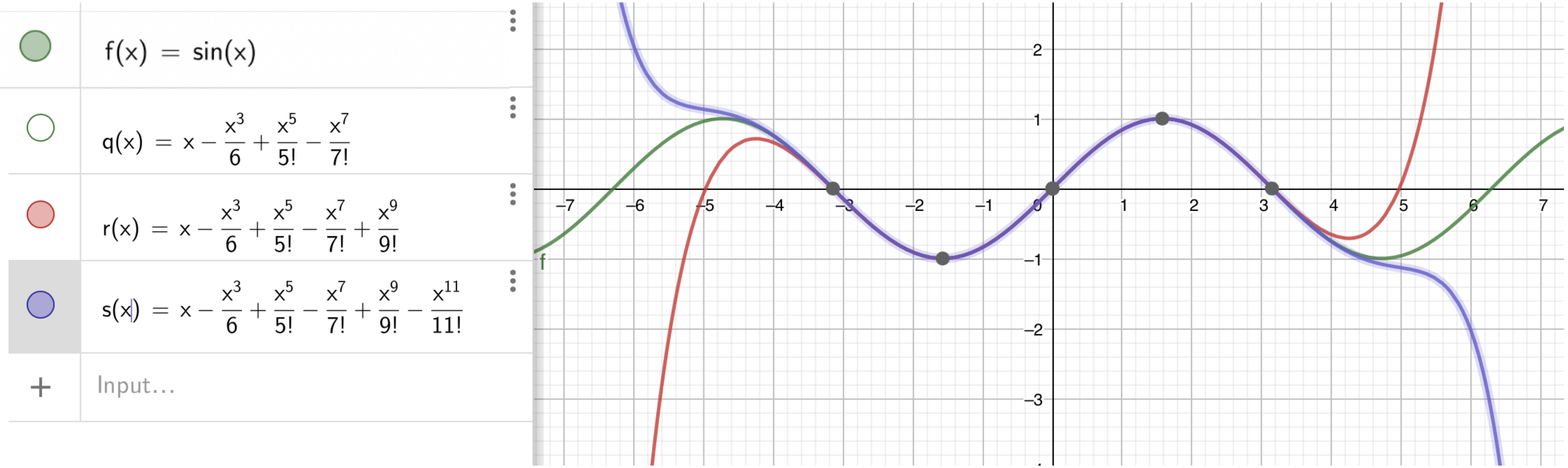

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- There is nothing wrong with our derivation of the Taylor series, but it does require that the derivatives of all orders of f exist at the point $x = x_0$.

Taylor series limitations

In many cases Taylor series is an appropriate representation and there is no need for a different kind of expansion. However, we need to also find ways to deal with functions where another type of series is preferable, easier to derive, or even required.

- It only works for many-times differentiable functions (infinitely differentiable for full Taylor series).

- No luck with functions like:

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A new series proposal!

- We will propose a new series expansion that uses **trigonometric functions** (example of a *trigonometric series*).

- We will show that we can express a function (that satisfies some constraints) as an infinite sum of sines and cosines such that the series converges to the function value at (almost) any point.

  - Where it does not converge exactly, it converges to a “sensible” value.

These series are easy to analyse because trigonometric functions are well-understood, and their derivatives fall into simple patterns.

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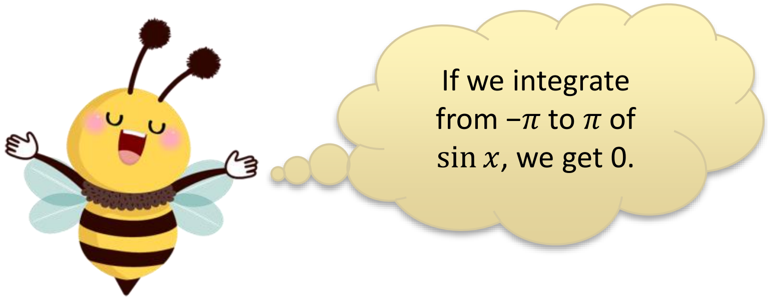
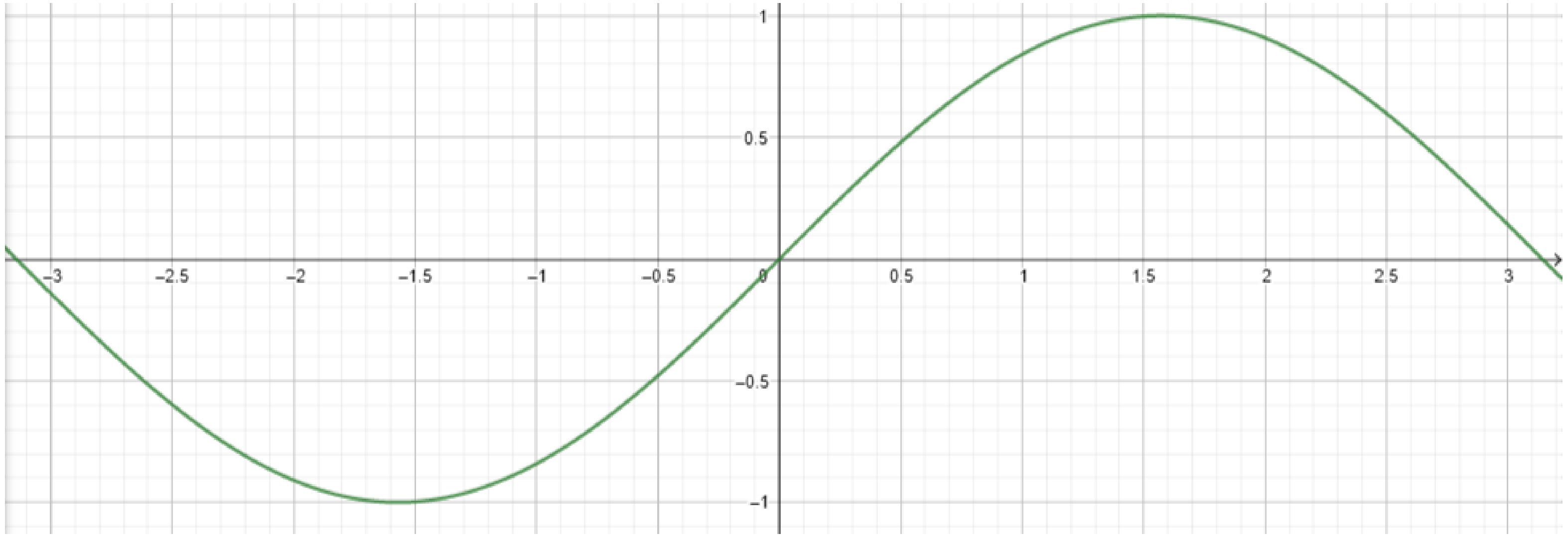
Some key facts



First note:

$$\int_{-\pi}^{\pi} \cos mx \, dx = \int_{-\pi}^{\pi} \sin nx \, dx = 0, \qquad \forall\, m, n \in \mathbb{Z}, \qquad m \neq 0,$$

since both are periodic functions with a period being an exact divisor of 2π.



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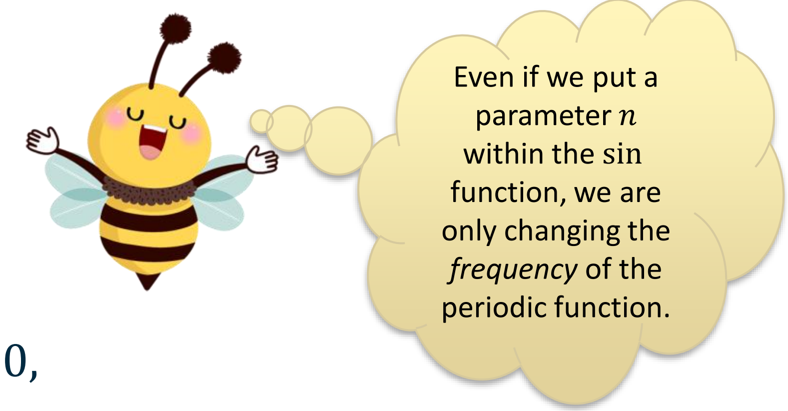

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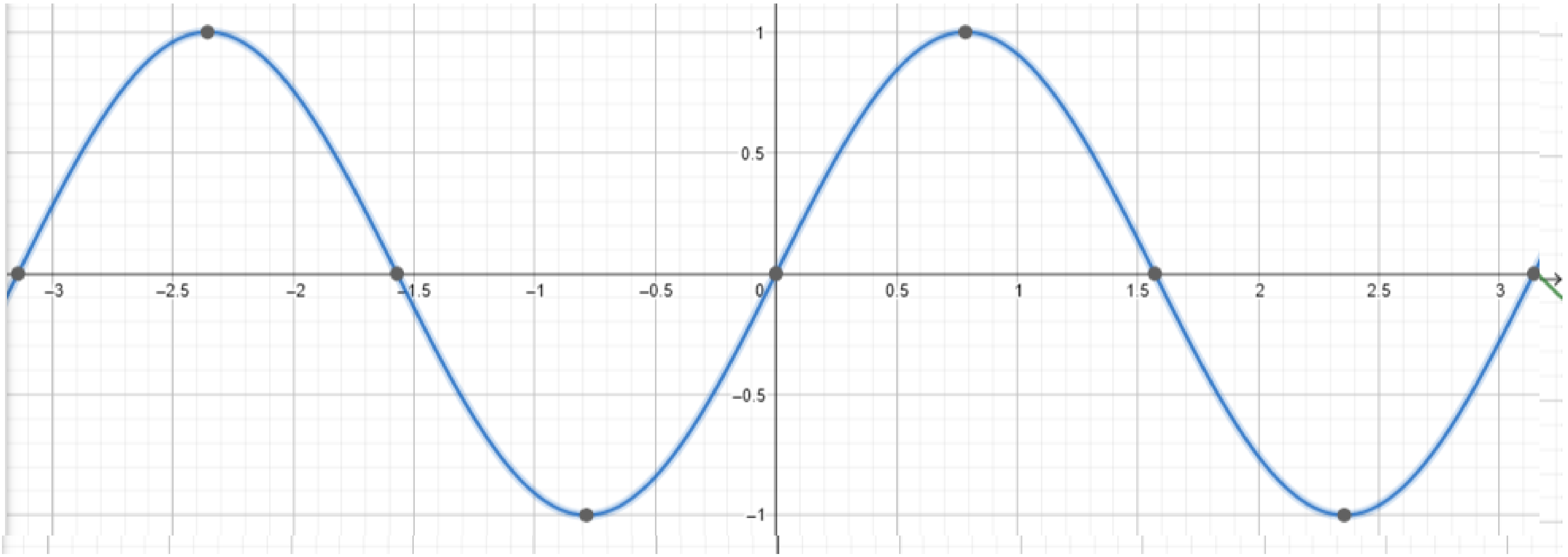

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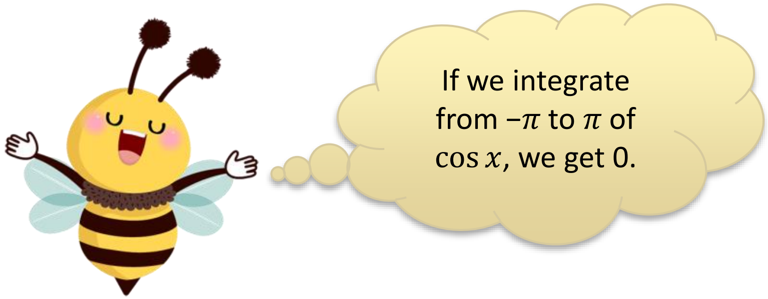

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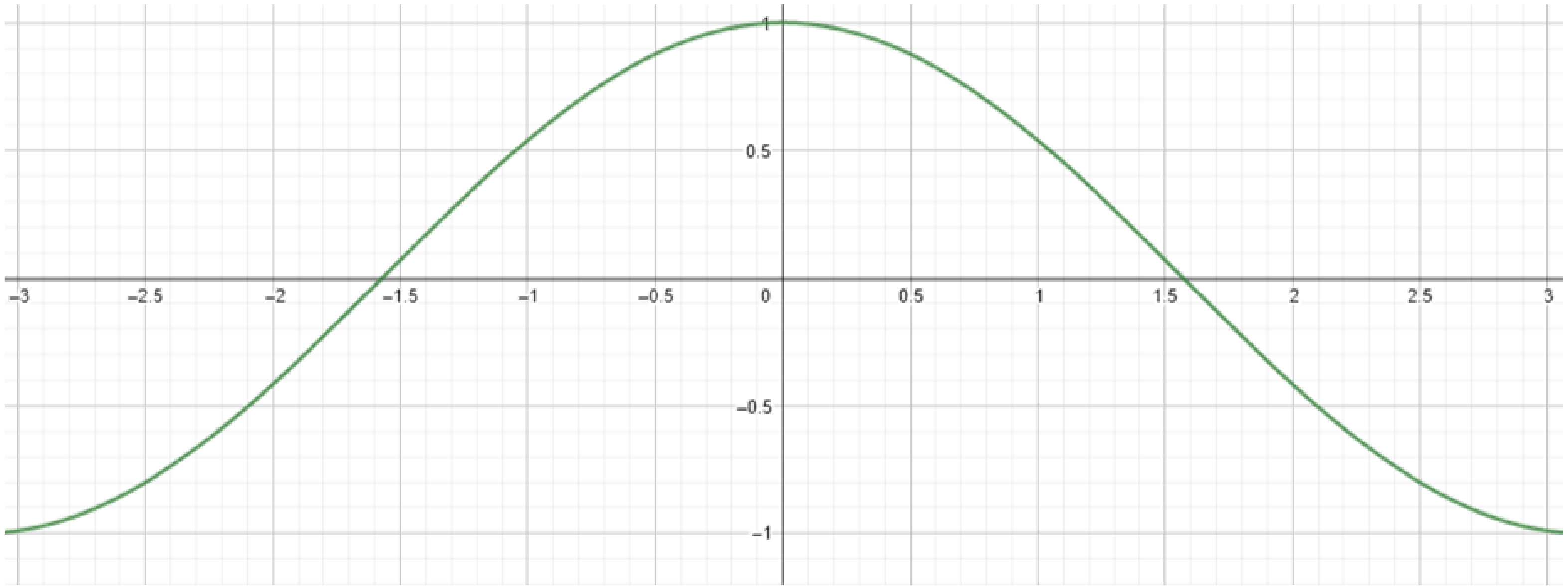

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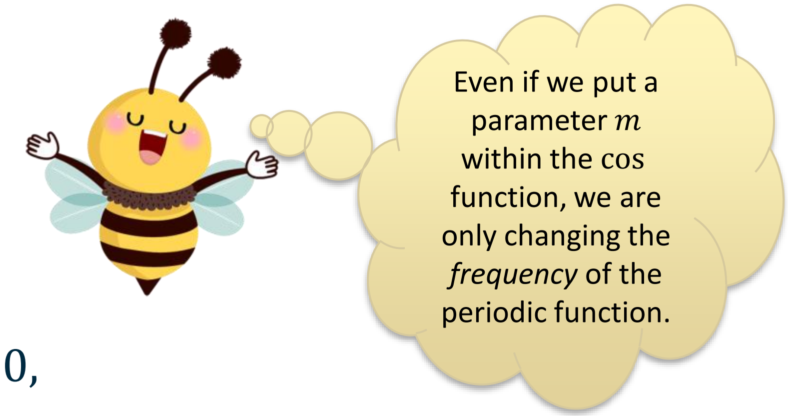

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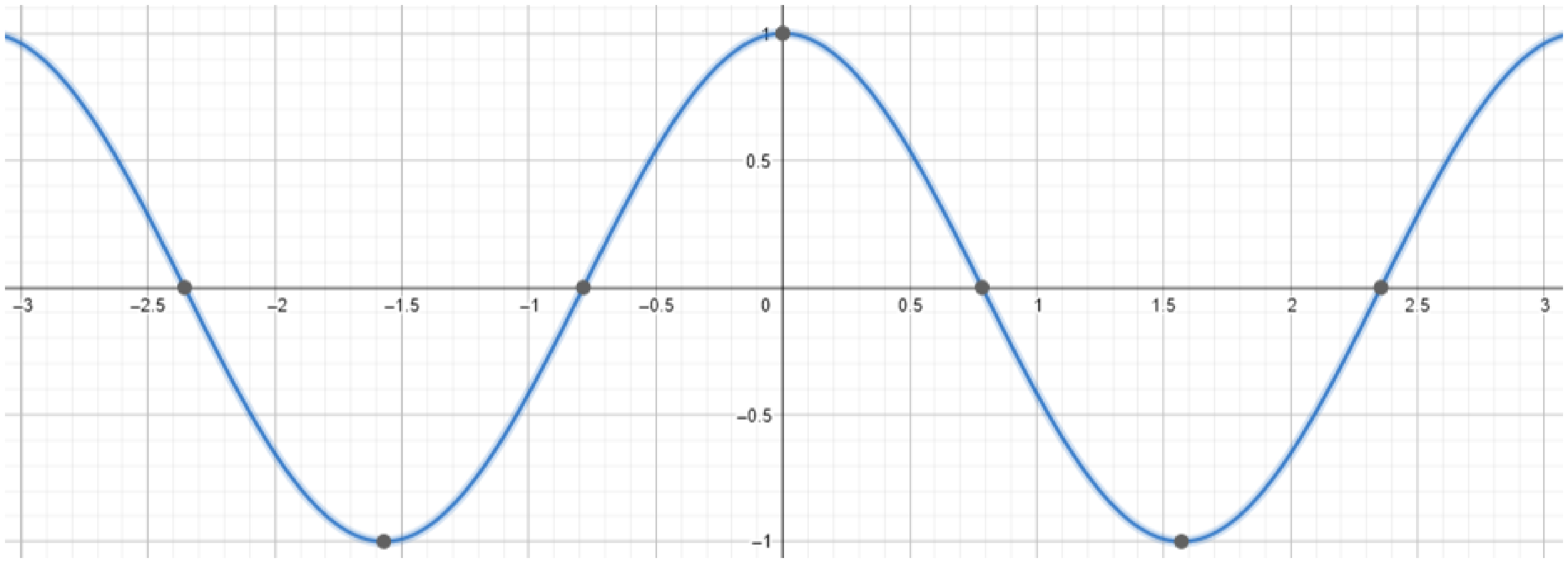
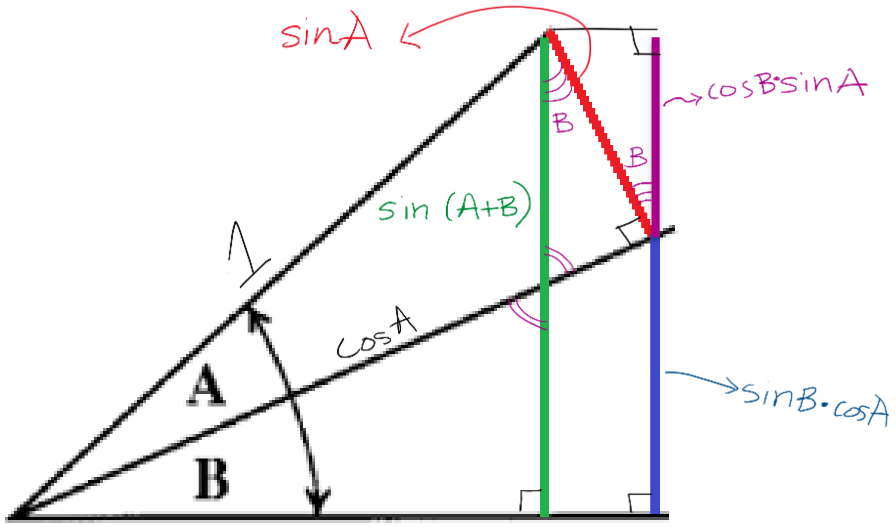


Some key facts

Recall (or look up) the trigonometric identities:

$$\sin(A + B) = \sin A \cos B + \cos A \sin B$$



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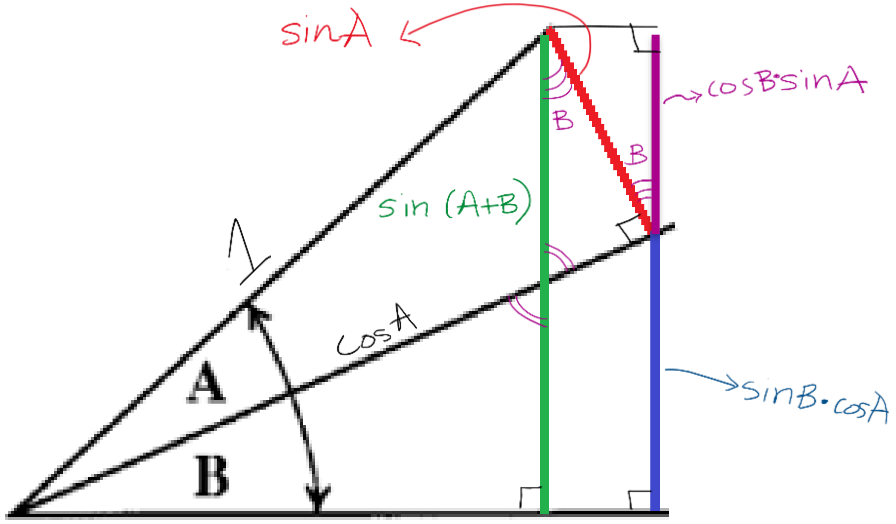
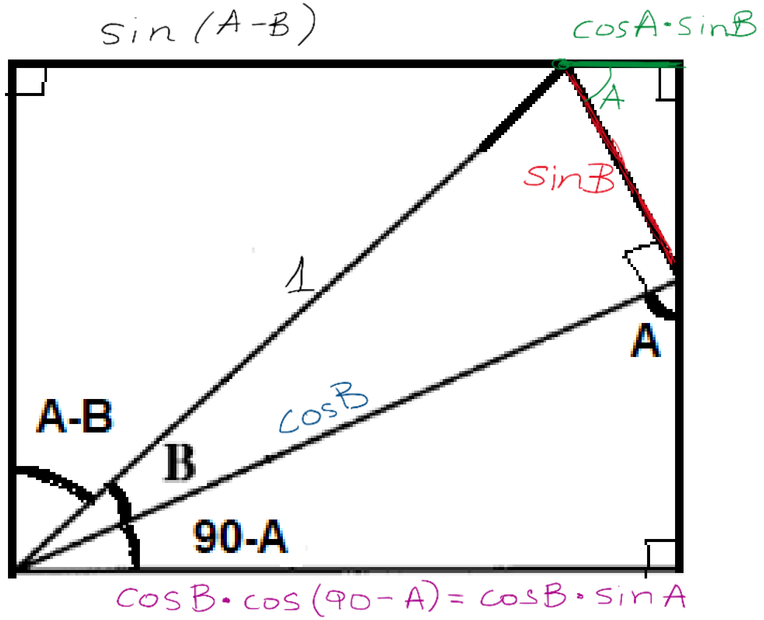


and

$$\sin(A-B) = \sin A \cos B - \cos A \sin B$$



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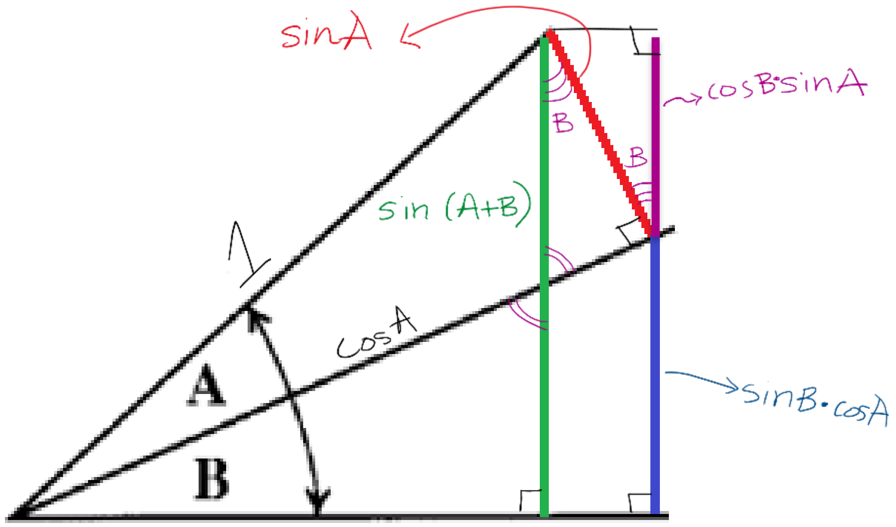

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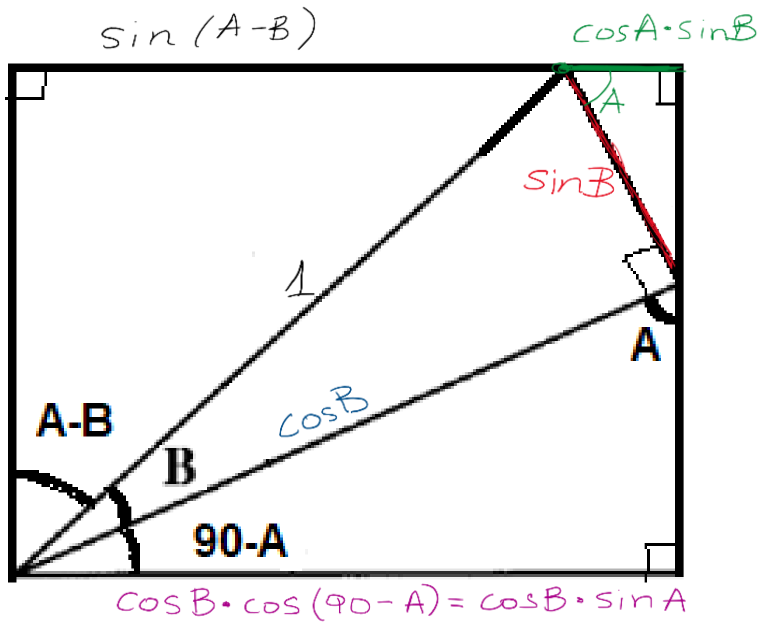

So we get the identity:

$$\sin A\cos B = \frac{1}{2}(\sin(A-B) + \sin(A+B))$$

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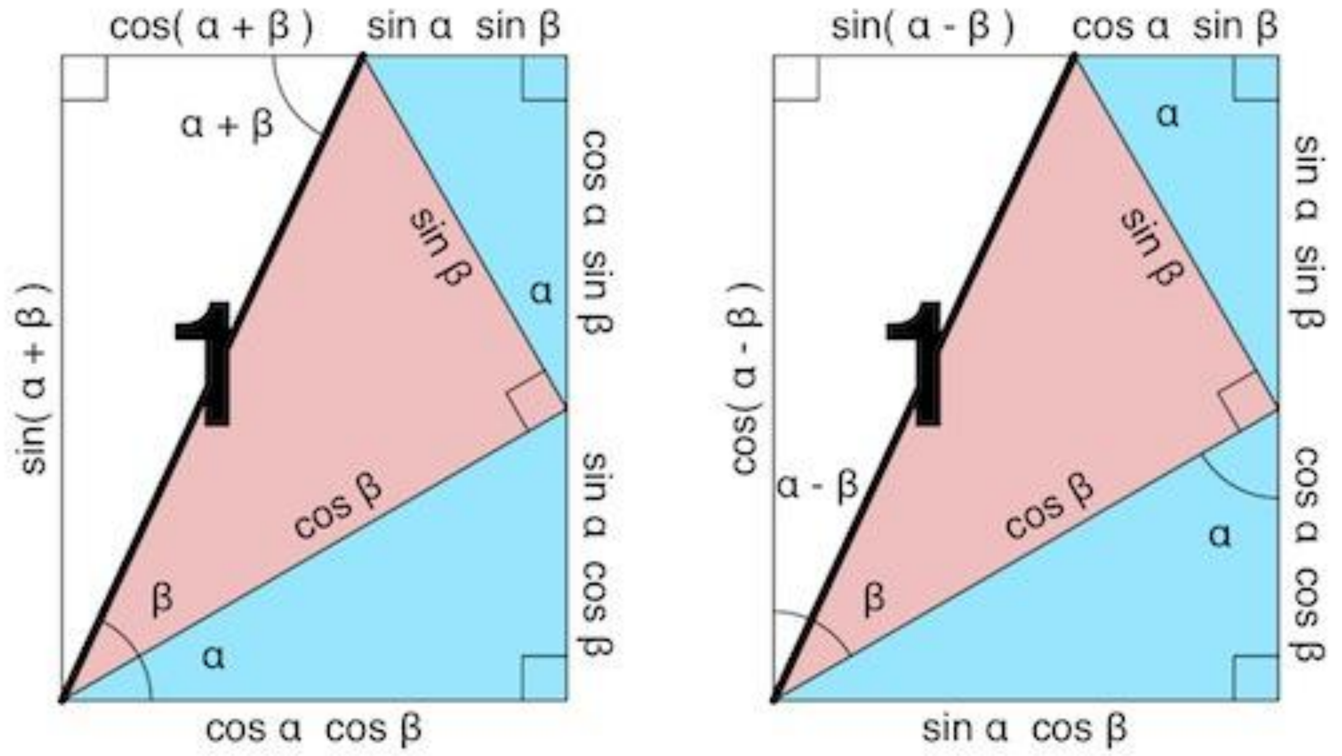
Some key facts



$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

$\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$

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$$\sin A \cos B = \frac{1}{2}(\sin(A-B) + \sin(A+B))$$

$$\sin A \sin B = \frac{1}{2}(\cos(A-B) - \cos(A+B))$$

$$\cos A \cos B = \frac{1}{2}(\cos(A-B) + \cos(A+B))$$

Therefore,

$$\int_{-\pi}^{\pi} \sin nx \cos mx \, dx = \frac{1}{2}\int_{-\pi}^{\pi} \sin(n-m)x \, dx + \frac{1}{2}\int_{-\pi}^{\pi} \sin(n+m)\, x \, dx$$

$$= 0, \qquad\qquad \forall n, m$$

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Similarly:

$$\int_{-\pi}^{\pi} \sin nx \sin mx \, dx = \frac{1}{2}\int_{-\pi}^{\pi} \cos\big((n-m)x\big)\, dx - \frac{1}{2}\int_{-\pi}^{\pi} \cos((n+m)x)\, dx$$

$$= 0, \quad \forall\, m \neq \pm n \ \text{ or } \ m = n = 0$$
$$= \pi, \quad if\ m = n \neq 0$$
$$= -\pi, \quad if\ m = -n \neq 0$$

and

$$\int_{-\pi}^{\pi} \cos nx \cos mx \, dx = \frac{1}{2}\int_{-\pi}^{\pi} \cos\big((n-m)x\big)\, dx + \frac{1}{2}\int_{-\pi}^{\pi} \cos((n+m)x)\, dx$$

$$= 0, \quad \forall\, m \neq \pm n$$
$$= \pi, \quad if\ m = \pm n \neq 0$$
$$= 2\pi, \quad if\ m = n = 0.$$

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A new series proposal (properly now)

For a function $f(x)$, $-\pi < x < \pi$, we will construct a series for f of the form

$$f(x) \approx \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$$

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Terms of higher and higher frequency

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$$f(x) \approx \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$$

The range $(-\pi, \pi)$ is no real restriction – we can always scale an arbitrary function f on a range (a, b) by setting

$$f(x) = g\left(\left(x - \frac{b+a}{2}\right)\frac{2\pi}{(b-a)}\right)$$

where g is a function on $(-\pi, \pi)$.

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The range $(-\pi, \pi)$ is no real restriction – we can always scale an arbitrary function f on a range $[a, b]$ by setting

$$f(x) = g\left(\left(x - \frac{b+a}{2}\right)\frac{2\pi}{(b-a)}\right)$$

'Move each point to the left' by $\frac{b+a}{2}$ and Scale by $\frac{2\pi}{b-a}$

where g is a function on $(-\pi, \pi)$.

Fourier coefficients

Suppose first that such a series exists.

If $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$, then integrating over the range $(-\pi, \pi)$ we get:

$$\int_{-\pi}^{\pi} f(x)\, dx = \int_{-\pi}^{\pi} \left(\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx \right) dx$$

$$= \frac{a_0}{2} \int_{-\pi}^{\pi} 1\, dx + \sum_{n=1}^{\infty} a_n \underbrace{\int_{-\pi}^{\pi} \cos nx\, dx}_{=0} + \sum_{n=1}^{\infty} b_n \underbrace{\int_{-\pi}^{\pi} \sin nx\, dx}_{=0}$$

$$= \pi a_0.$$

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Fourier coefficients

Suppose still that such a series exists.

If $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$, then, multiplying by $\cos mx, m > 0$ and integrating over the range $(-\pi, \pi)$ we get:

$$\int_{-\pi}^{\pi} f(x) \cos mx \; dx = \int_{-\pi}^{\pi} \left(\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx \right) \cos mx \, dx$$

$$= \frac{a_0}{2} \int_{-\pi}^{\pi} \cos mx \, dx + \sum_{n=1}^{\infty} a_n \int_{-\pi}^{\pi} \cos nx \cos mx \, dx + \sum_{n=1}^{\infty} b_n \int_{-\pi}^{\pi} \sin nx \cos mx \, dx$$

$$= \pi a_m.$$

Only one term is non-zero
when $n = m$

Fourier coefficients

Likewise, still assuming there is a series $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$, multiplying by this time by $\sin mx$ and integrating over the range:

$$\int_{-\pi}^{\pi} f(x) \sin mx \; dx = \int_{-\pi}^{\pi} \left(\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx \right) \sin mx \, dx$$

$$= \frac{a_0}{2} \int_{-\pi}^{\pi} \sin mx \, dx + \sum_{n=1}^{\infty} a_n \int_{-\pi}^{\pi} \cos nx \sin mx \, dx + \sum_{n=1}^{\infty} b_n \int_{-\pi}^{\pi} \sin nx \sin mx \, dx$$

$$= \pi b_m.$$

Only one term, only in this sum, is non-zero, when $n = m$

The Fourier Series

For a function $f(x)$ $-\pi < x < \pi$, the Fourier series is

$$f(x) \approx \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$$

where

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \; dx$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \; dx\,.$$

Theorem:

If f is piecewise continuous and $\int_{-\pi}^{\pi} [f(x)]^2 dx$ exists, then the Fourier series converges.

- For all points x at which f is continuous, the series converges to $f(x)$.
- For points y at which there is a jump discontinuity the series converges to

$$\frac{1}{2}\left(\lim_{x \to \mathrm{y}^-} f(x) + \lim_{x \to \mathrm{y}^+} f(x)\right)$$

Example $y = x$

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$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \; dx \, .$$

See MS whiteboard during / after lecture

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Example $y = x^2$

For a function $f(x)$, $-\pi < x < \pi$, the Fourier series is

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Visualisation

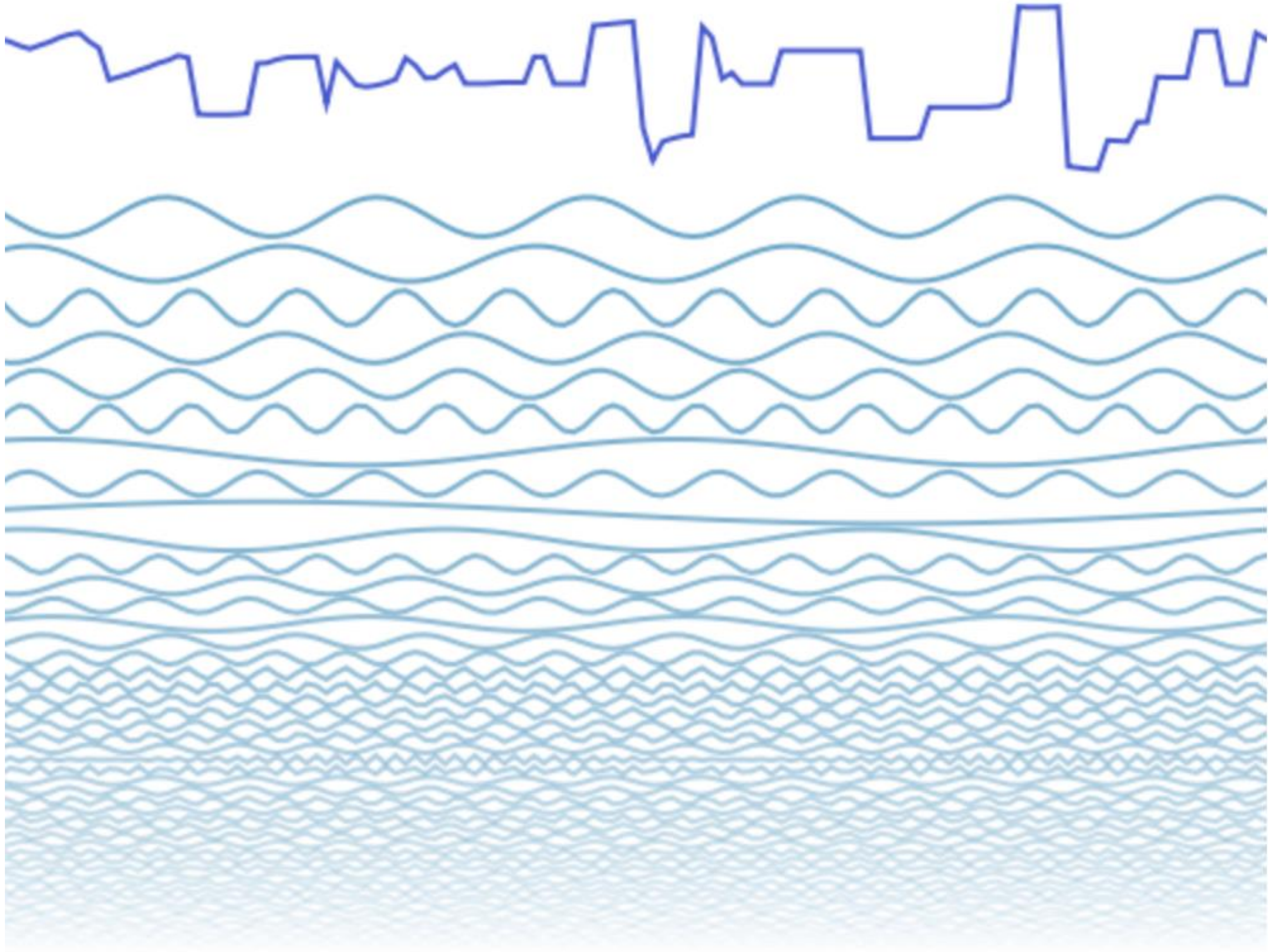

Image from
http://www.jezzamon.com/fourier/

Visit the site (strongly recommended) for some beautiful and interactive examples.

What we learnt today

Fourier series:

For a function $f(x)$ $-\pi < x < \pi$, the Fourier series is

$$f(x) \approx \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx\,, \qquad \text{where} \qquad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx \;dx\,,$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx \;dx\,.$$

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