# Mathematics for Computer Science
# Linear Algebra (Part 2)
# QR Decomposition

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Thanks to Andrei Krokhin and William Moses Jr for use of some slides.

# Outline

# Roadmap for Lectures 5, 6, & 7

- **End Goal:** Application - linear regression.
- **Using:** QR decomposition.
- Requires knowledge of Gram-Schmidt Process.
- Requires knowledge of some basics: Inner product spaces.

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Now we recap last lecture & look at what we'll cover today.

# Last Lecture

- An inner product space is a (real) vector space $V$ equipped with an inner product — a function that associates to each pair $\mathbf{u}, \mathbf{v} \in V$ a real number denoted $\langle \mathbf{u}, \mathbf{v} \rangle \in \mathbb{R}$. This function must satisfy four axioms:
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  A vector $\mathbf{v}$ with $||\mathbf{v}|| = 1$ is called a unit vector. Each non-zero vector can be normalised (scaled to become a unit vector): $\mathbf{v} \mapsto \frac{1}{||\mathbf{v}||}\mathbf{v}$.

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- Vectors $\mathbf{u}$ and $\mathbf{v}$ are called orthogonal if $\langle \mathbf{u}, \mathbf{v} \rangle = 0$.
- For a subspace $W$ of an inner product space $V$, can define the orthogonal complement

$$W^{\perp} = \{\mathbf{x} \in V \mid \langle \mathbf{u}, \mathbf{x} \rangle = 0 \text{ for all } \mathbf{u} \in W\}$$

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- A vector space can be equipped with different inner products — the notions of norm and orthogonality depend on the choice of inner product.

# Today's Lecture

- Orthogonal and orthonormal bases in an inner product space
- Constructing such bases - the Gram-Schmidt process

# Outline

# Orthogonal and Orthonormal Sets of Vectors

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Example in $\mathbb{R}^3$ (with the Euclidean inner product): Let

$$\mathbf{v}_1 = (0,1,0),\ \mathbf{v}_2 = (1,0,1),\ \mathbf{v}_3 = (1,0,-1).$$

The set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is orthogonal, since $\langle \mathbf{v}_1, \mathbf{v}_2 \rangle = \langle \mathbf{v}_1, \mathbf{v}_3 \rangle = \langle \mathbf{v}_2, \mathbf{v}_3 \rangle = 0$.

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The norms of the vectors are:

$$||\mathbf{v}_1|| = 1,\ ||\mathbf{v}_2|| = \sqrt{2},\ ||\mathbf{v}_3|| = \sqrt{2}.$$

By normalising (i.e., setting $\mathbf{q}_i = \frac{1}{||\mathbf{v}_i||}\mathbf{v}_i$), we get an orthonormal set $\{\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3\}$.

$$\mathbf{q}_1 = (0, 1, 0),\ \mathbf{q}_2 = (\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}),\ \mathbf{q}_3 = (\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}).$$

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**Theorem**

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**Previous example:** For dot product, consider $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ where
$\mathbf{v}_1 = (0, 1, 0)$, $\mathbf{v}_2 = (1, 0, 1)$, $\mathbf{v}_3 = (1, 0, -1)$.

# Orthogonal and Orthonormal Bases

An orthogonal (resp. orthonormal) basis in an inner product space is a basis, which is an orthogonal (resp. orthonormal) set. For example,

$\{\mathbf{v}_1 = (0, 1, 0), \mathbf{v}_2 = (\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}), \mathbf{v}_3 = (\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}})\}$ is an orthonormal basis in $\mathbb{R}^3$.

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## Theorem

*If $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is an orthogonal basis in inner product space $V$ then for any $\mathbf{u}$*

$$\mathbf{u} = \frac{\langle \mathbf{u}, \mathbf{v}_1 \rangle}{||\mathbf{v}_1||^2}\mathbf{v}_1 + \frac{\langle \mathbf{u}, \mathbf{v}_2 \rangle}{||\mathbf{v}_2||^2}\mathbf{v}_2 + \ldots + \frac{\langle \mathbf{u}, \mathbf{v}_n \rangle}{||\mathbf{v}_n||^2}\mathbf{v}_n.$$

*Moreover, if $S$ is an orthonormal basis in $V$ then*

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# Orthogonal and Orthonormal Bases (contd.)

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## Proof.

If $\mathbf{u} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_n\mathbf{v}_n$ then, since $\langle \mathbf{v}_j, \mathbf{v}_i \rangle = 0$ for all $j \neq i$, we have that, for each $i$, $\langle \mathbf{u}, \mathbf{v}_i \rangle = c_i \langle \mathbf{v}_i, \mathbf{v}_i \rangle = c_i ||\mathbf{v}_i||^2$, so $c_i = \frac{\langle \mathbf{u}, \mathbf{v}_i \rangle}{||\mathbf{v}_i||^2}$, as required. $\square$

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# Orthogonal Projections

### Theorem (Projection Theorem)

*If $W$ is a subspace in a finite-dimensional inner product space $V$ then every vector $\mathbf{u} \in V$ can be uniquely expressed as $\mathbf{u} = \mathbf{w}_1 + \mathbf{w}_2$ where $\mathbf{w}_1 \in W$ and $\mathbf{w}_2 \in W^{\perp}$.*

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If $\mathbf{u}, \mathbf{w}_1$ and $\mathbf{w}_2$ are as above then $\mathbf{w}_1$ is the orthogonal projection of $\mathbf{u}$ onto $W$.
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Moreover, if $\{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$ is an orthonormal basis in $W$ then

$$\mathrm{proj}_W \mathbf{u} = \langle \mathbf{u}, \mathbf{v}_1 \rangle \mathbf{v}_1 + \langle \mathbf{u}, \mathbf{v}_2 \rangle \mathbf{v}_2 + \ldots + \langle \mathbf{u}, \mathbf{v}_r \rangle \mathbf{v}_r.$$

# Example 16.1

Consider $\mathbb{R}^3$ with the Euclidean inner product. Let $W$ be the subspace formed by *span*$(\mathbf{v}_1, \mathbf{v}_2)$ where $\mathbf{v}_1 = (0, 1, 0)$ and $\mathbf{v}_2 = (-4, 0, 3)$. Express $\mathbf{u} = (1, 1, 1)$ in the form $\mathbf{u} = \mathbf{w}_1 + \mathbf{w}_2$, where $\mathbf{w}_1 \in W$ and $\mathbf{w}_2 \in W^{\perp}$.

**Step 1.** $\mathbf{w}_1 = \mathrm{proj}_W \mathbf{u} = \frac{\langle \mathbf{u}, \mathbf{v}_1 \rangle}{||\mathbf{v}_1||^2}\mathbf{v}_1 + \frac{\langle \mathbf{u}, \mathbf{v}_2 \rangle}{||\mathbf{v}_2||^2}\mathbf{v}_2 = \frac{1}{1}(0, 1, 0) + \frac{-1}{25}(-4, 0, 3) = (\frac{4}{25}, 1, \frac{-3}{25})$

**Step 2.** $\mathbf{w}_2 = \mathrm{proj}_{W^\perp} \mathbf{u} = \mathbf{u} - \mathbf{w}_1 = (1, 1, 1) - (\frac{4}{25}, 1, \frac{-3}{25}) = (\frac{21}{25}, 0, \frac{28}{25})$

**Step 3.** $\mathbf{u} = (\frac{4}{25}, 1, \frac{-3}{25}) + (\frac{21}{25}, 0, \frac{28}{25})$

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- Let $W$ be a subspace of $V$ and let $\{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ be a basis of $W$.
  Consider the subspaces $W_r = span(\mathbf{u}_1, \ldots, \mathbf{u}_r)$, $r = 1, \ldots, n$, in $W$.
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- The Gram-Schmidt process inductively constructs orthogonal bases for the subspaces $W_i$, eventually constructing an orthogonal basis for $W_n = W$.
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**The Gram-Schmidt process:**

**Step 1.** Let $\mathbf{v}_1 = \mathbf{u}_1$. Clearly, $\{\mathbf{v}_1\}$ is an orthogonal basis for $W_1 = span(\mathbf{u}_1)$.

**Step $r$ $(2 \leq r \leq n)$.** Assuming that we have an orthogonal basis $\{\mathbf{v}_1, \ldots, \mathbf{v}_{r-1}\}$ for $W_{r-1}$, add a vector $\mathbf{v}_r$ to it to get an orthogonal basis for $W_r$.

# The Gram-Schmidt (Orthogonalisation) Process

**Step** $r$**:** If $\{\mathbf{v}_1, \ldots, \mathbf{v}_{r-1}\}$ is an orthogonal basis for $W_{r-1} = span(\mathbf{u}_1, \ldots, \mathbf{u}_{r-1})$, find a vector $\mathbf{v}_r$ such that $\{\mathbf{v}_1, \ldots, \mathbf{v}_{r-1}, \mathbf{v}_r\}$ is an orthogonal basis for $W_r$.

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Apply the Projection theorem to $\mathbf{u}_r \in W_r$ and $W_{r-1}$ (as a subspace of $W_r$):

$$\mathbf{u}_r = \mathrm{proj}_{W_{r-1}} \mathbf{u}_r + \mathrm{proj}_{{W_{r-1}}^{\perp}} \mathbf{u}_r.$$

(Note that the orthogonal complement ${W_{r-1}}^{\perp}$ here is taken in $W_r$).

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Recall that, since $\{\mathbf{v}_1, \dots, \mathbf{v}_{r-1}\}$ is an orthogonal basis for $W_{r-1}$, we have

$$\mathrm{proj}_{W_{r-1}}\mathbf{u}_r = \frac{\langle \mathbf{u}_r, \mathbf{v}_1 \rangle}{||\mathbf{v}_1||^2}\mathbf{v}_1 + \frac{\langle \mathbf{u}_r, \mathbf{v}_2 \rangle}{||\mathbf{v}_2||^2}\mathbf{v}_2 + \dots + \frac{\langle \mathbf{u}_r, \mathbf{v}_{r-1} \rangle}{||\mathbf{v}_{r-1}||^2}\mathbf{v}_{r-1}.$$

# The Gram-Schmidt (Orthogonalisation) Process

**Step** $r$**:** If $\{\mathbf{v}_1, \ldots, \mathbf{v}_{r-1}\}$ is an orthogonal basis for $W_{r-1} = span(\mathbf{u}_1, \ldots, \mathbf{u}_{r-1})$, find a vector $\mathbf{v}_r$ such that $\{\mathbf{v}_1, \ldots, \mathbf{v}_{r-1}, \mathbf{v}_r\}$ is an orthogonal basis for $W_r$.

Apply the Projection theorem to $\mathbf{u}_r \in W_r$ and $W_{r-1}$ (as a subspace of $W_r$):

$$\mathbf{u}_r = \mathrm{proj}_{W_{r-1}}\mathbf{u}_r + \mathrm{proj}_{W_{r-1}^{\perp}}\mathbf{u}_r.$$

(Note that the orthogonal complement $W_{r-1}^{\perp}$ here is taken in $W_r$).

Recall that, since $\{\mathbf{v}_1, \ldots, \mathbf{v}_{r-1}\}$ is an orthogonal basis for $W_{r-1}$, we have

$$\mathrm{proj}_{W_{r-1}}\mathbf{u}_r = \frac{\langle \mathbf{u}_r, \mathbf{v}_1 \rangle}{||\mathbf{v}_1||^2}\mathbf{v}_1 + \frac{\langle \mathbf{u}_r, \mathbf{v}_2 \rangle}{||\mathbf{v}_2||^2}\mathbf{v}_2 + \ldots + \frac{\langle \mathbf{u}_r, \mathbf{v}_{r-1} \rangle}{||\mathbf{v}_{r-1}||^2}\mathbf{v}_{r-1}.$$

Set

$$\mathbf{v}_r = \mathrm{proj}_{W_{r-1}^{\perp}}\mathbf{u}_r = \mathbf{u}_r - \frac{\langle \mathbf{u}_r, \mathbf{v}_1 \rangle}{||\mathbf{v}_1||^2}\mathbf{v}_1 - \frac{\langle \mathbf{u}_r, \mathbf{v}_2 \rangle}{||\mathbf{v}_2||^2}\mathbf{v}_2 - \ldots - \frac{\langle \mathbf{u}_r, \mathbf{v}_{r-1} \rangle}{||\mathbf{v}_{r-1}||^2}\mathbf{v}_{r-1}.$$

# The Gram-Schmidt (Orthogonalisation) Process

**Step $r$:** If $\{\mathbf{v}_1, \ldots, \mathbf{v}_{r-1}\}$ is an orthogonal basis for $W_{r-1} = span(\mathbf{u}_1, \ldots, \mathbf{u}_{r-1})$, find a vector $\mathbf{v}_r$ such that $\{\mathbf{v}_1, \ldots, \mathbf{v}_{r-1}, \mathbf{v}_r\}$ is an orthogonal basis for $W_r$.

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(Note that the orthogonal complement $W_{r-1}^{\perp}$ here is taken in $W_r$).

Recall that, since $\{\mathbf{v}_1, \ldots, \mathbf{v}_{r-1}\}$ is an orthogonal basis for $W_{r-1}$, we have

$$\mathrm{proj}_{W_{r-1}}\mathbf{u}_r = \frac{\langle \mathbf{u}_r, \mathbf{v}_1\rangle}{||\mathbf{v}_1||^2}\mathbf{v}_1 + \frac{\langle \mathbf{u}_r, \mathbf{v}_2\rangle}{||\mathbf{v}_2||^2}\mathbf{v}_2 + \ldots + \frac{\langle \mathbf{u}_r, \mathbf{v}_{r-1}\rangle}{||\mathbf{v}_{r-1}||^2}\mathbf{v}_{r-1}.$$

Set

$$\mathbf{v}_r = \mathrm{proj}_{W_{r-1}^{\perp}}\mathbf{u}_r = \mathbf{u}_r - \frac{\langle \mathbf{u}_r, \mathbf{v}_1\rangle}{||\mathbf{v}_1||^2}\mathbf{v}_1 - \frac{\langle \mathbf{u}_r, \mathbf{v}_2\rangle}{||\mathbf{v}_2||^2}\mathbf{v}_2 - \ldots - \frac{\langle \mathbf{u}_r, \mathbf{v}_{r-1}\rangle}{||\mathbf{v}_{r-1}||^2}\mathbf{v}_{r-1}.$$

Since $\mathbf{v}_r \in W_{r-1}^{\perp}$, the set $\{\mathbf{v}_1, \ldots, \mathbf{v}_{r-1}, \mathbf{v}_r\}$ is orthogonal (and so linearly independent)

$W_r = span(\mathbf{u}_1, \ldots, \mathbf{u}_{r-1}, \mathbf{u}_r) = span(\mathbf{v}_1, \ldots, \mathbf{v}_{r-1}, \mathbf{u}_r) = span(\mathbf{v}_1, \ldots, \mathbf{v}_{r-1}, \mathbf{v}_r)$.

# The Gram-Schmidt Process: Summary

To convert a (linearly independent) set $S = \{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ into an orthogonal basis for *span*$(S)$, do the following:

Step 1. $\mathbf{v}_1 = \mathbf{u}_1$.

Step 2. $\mathbf{v}_2 = \mathbf{u}_2 - \mathrm{proj}_{W_1}\mathbf{u}_2 = \mathbf{u}_2 - \frac{\langle \mathbf{u}_2, \mathbf{v}_1 \rangle}{||\mathbf{v}_1||^2}\mathbf{v}_1$.

Step 3. $\mathbf{v}_3 = \mathbf{u}_3 - \mathrm{proj}_{W_2}\mathbf{u}_3 = \mathbf{u}_3 - \frac{\langle \mathbf{u}_3, \mathbf{v}_1 \rangle}{||\mathbf{v}_1||^2}\mathbf{v}_1 - \frac{\langle \mathbf{u}_3, \mathbf{v}_2 \rangle}{||\mathbf{v}_2||^2}\mathbf{v}_2$

Step 4. $\mathbf{v}_4 = \mathbf{u}_4 - \mathrm{proj}_{W_3}\mathbf{u}_4 = \mathbf{u}_4 - \frac{\langle \mathbf{u}_4, \mathbf{v}_1 \rangle}{||\mathbf{v}_1||^2}\mathbf{v}_1 - \frac{\langle \mathbf{u}_4, \mathbf{v}_2 \rangle}{||\mathbf{v}_2||^2}\mathbf{v}_2 - \frac{\langle \mathbf{u}_4, \mathbf{v}_3 \rangle}{||\mathbf{v}_3||^2}\mathbf{v}_3$

$\vdots$

(continue for $n$ steps)

Optional step. Normalise all vectors $\mathbf{v}_i$ if an orthonormal basis is needed.

# Example 16.2 Using the Gram-Schmidt process in $\mathbb{R}^3$

**Task:** Consider $\mathbb{R}^3$ with the Euclidean inner product. Find an orthonormal basis of $W = \mathit{span}(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3)$ where $\mathbf{u}_1 = (1,1,1), \mathbf{u}_2 = (0,1,1), \mathbf{u}_3 = (0,0,1)$.

**Step 1.** $\mathbf{v}_1 = \mathbf{u}_1 = (1,1,1)$.

**Step 2.** $\mathbf{v}_2 = \mathbf{u}_2 - \mathrm{proj}_{W_1}\mathbf{u}_2 = \mathbf{u}_2 - \frac{\langle \mathbf{u}_2, \mathbf{v}_1 \rangle}{||\mathbf{v}_1||^2}\mathbf{v}_1 =$

$(0,1,1) - \frac{2}{3}(1,1,1) = (-\frac{2}{3}, \frac{1}{3}, \frac{1}{3})$.

**Step 3.** $\mathbf{v}_3 = \mathbf{u}_3 - \mathrm{proj}_{W_2}\mathbf{u}_3 = \mathbf{u}_3 - \frac{\langle \mathbf{u}_3, \mathbf{v}_1 \rangle}{||\mathbf{v}_1||^2}\mathbf{v}_1 - \frac{\langle \mathbf{u}_3, \mathbf{v}_2 \rangle}{||\mathbf{v}_2||^2}\mathbf{v}_2 =$

$$(0,0,1) - \frac{1}{3}(1,1,1) - \frac{1/3}{2/3}(-\frac{2}{3}, \frac{1}{3}, \frac{1}{3}) = (0, -\frac{1}{2}, \frac{1}{2}).$$

$\{\mathbf{v}_1 = (1,1,1),\ \mathbf{v}_2 = (-\frac{2}{3}, \frac{1}{3}, \frac{1}{3}),\ \mathbf{v}_3 = (0, -\frac{1}{2}, \frac{1}{2})\}$ is an orthogonal basis for $W$.

Since $||\mathbf{v}_1|| = \sqrt{3},\ ||\mathbf{v}_2|| = \frac{\sqrt{6}}{3},\ ||\mathbf{v}_3|| = \frac{1}{\sqrt{2}}$, we have an orthonormal basis for $W$:

$\{\mathbf{q}_1 = \frac{1}{||\mathbf{v}_1||}\mathbf{v}_1 = (\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}),\ \mathbf{q}_2 = (-\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}),\ \mathbf{q}_3 = (0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})\}$

# Example 16.3 Using the Gram-Schmidt process in $C[-1,1]$

**Task:** Consider the space $C[-1,1]$. Find an orthogonal basis of $W = span(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3)$ where $\mathbf{u}_1 = 1, \mathbf{u}_2 = x, \mathbf{u}_3 = x^2$.

**Step 1.** $\mathbf{v}_1 = \mathbf{u}_1 = 1$.

**Step 2.** $\mathbf{v}_2 = \mathbf{u}_2 - \mathrm{proj}_{W_1}\mathbf{u}_2 = \mathbf{u}_2 - \frac{\langle \mathbf{u}_2, \mathbf{v}_1 \rangle}{||\mathbf{v}_1||^2}\mathbf{v}_1$. We have $\langle \mathbf{u}_2, \mathbf{v}_1 \rangle = \int_{-1}^{1} x \, dx = 0$ and $||\mathbf{v}_1||^2 = \langle \mathbf{v}_1, \mathbf{v}_1 \rangle = \int_{-1}^{1} 1 \, dx = 2$, so $\mathbf{v}_2 = \mathbf{u}_2 - 0\mathbf{v}_1 = x$.

**Step 3.** $\mathbf{v}_3 = \mathbf{u}_3 - \mathrm{proj}_{W_2}\mathbf{u}_3 = \mathbf{u}_3 - \frac{\langle \mathbf{u}_3, \mathbf{v}_1 \rangle}{||\mathbf{v}_1||^2}\mathbf{v}_1 - \frac{\langle \mathbf{u}_3, \mathbf{v}_2 \rangle}{||\mathbf{v}_2||^2}\mathbf{v}_2$. We have

$$\langle \mathbf{u}_3, \mathbf{v}_1 \rangle = \int_{-1}^{1} x^2 \, dx = \frac{2}{3} \quad \text{and} \quad \langle \mathbf{u}_3, \mathbf{v}_2 \rangle = \int_{-1}^{1} x^3 \, dx = 0.$$

Hence, $\mathbf{v}_3 = x^2 - \frac{2/3}{2}\mathbf{v}_1 - 0\mathbf{v}_2 = x^2 - \frac{1}{3}$.

So, $\{\mathbf{v}_1 = 1, \ \mathbf{v}_2 = x, \ \mathbf{v}_3 = x^2 - \frac{1}{3}\}$ is an orthogonal basis for $W$.

# Extending an Orthogonal Set to an Orthogonal Basis

## Theorem

*If $V$ is a finite-dimensional inner product space then*

1. *Any orthogonal set of vectors in $V$ can be extended to an orthogonal basis.*
2. *Any orthonormal set of vectors in $V$ can be extended to an orthonormal basis.*

## Proof.

Let $\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ be an orthogonal set in $V$ and $\{\mathbf{u}_{k+1}, \ldots, \mathbf{u}_{k+n}\}$ some basis in $V$.

- Apply Gram-Schmidt to the set $\{\mathbf{u}_1, \ldots, \mathbf{u}_k, \mathbf{u}_{k+1}, \ldots, \mathbf{u}_{k+n}\}$.
- Since $\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ is an orthogonal set, we will have $\mathbf{v}_i = \mathbf{u}_i$ for $1 \leq i \leq k$.
- If $\mathbf{v}_r = \mathbf{0}$ at any Step $r$ (with $r > k$), do not add it to the output set.

  (This happens iff $\mathbf{u}_r \in \mathit{span}(\mathbf{u}_1, \ldots, \mathbf{u}_{r-1})$, i.e. if $W_{r-1} = W_r$.)

□

# Extending an Orthogonal Set to an Orthogonal Basis (contd.)

**Theorem**

*If $V$ is a finite-dimensional inner product space then*

1. *Any orthogonal set of vectors in $V$ can be extended to an orthogonal basis.*
2. *Any orthonormal set of vectors in $V$ can be extended to an orthonormal basis.*

**Proof.**

- If $\mathbf{v}_r = \mathbf{0}$ at any Step $r$ (with $r > k$), do not add it to the output set.

  (This happens iff $\mathbf{u}_r \in span(\mathbf{u}_1, \ldots, \mathbf{u}_{r-1})$, i.e. if $W_{r-1} = W_r$.)

The final set will extend $\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$, it will be orthogonal (and hence linearly independent), and its span will be $span(\mathbf{u}_1, \ldots, \mathbf{u}_k, \mathbf{u}_{k+1}, \ldots, \mathbf{u}_{k+n}) = V$.

For item (2), normalise all vectors in the final set. $\square$

# Outline

# Example exam question

(a) Give the definition of an orthonormal set of vectors. **[2 Marks]**

(b) Let $S$ be the set of vectors $\left\{ \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 9 \\ 9 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix} \right\}$.

Use the Gram-Schmidt process to construct an orthonormal basis of $span(S)$.

**[6 Marks]**

Figure

# Wrapping Things Up

**What we learnt today:**

- Orthogonal and orthonormal bases in an inner product space
- Constructing such bases - the Gram-Schmidt process

**Next time:**

- QR decomposition and Least squares - solving inconsistent linear systems

# The End

# Supplementary Slide: Orthogonal Sets are Linearly Independent

**Theorem**

*If $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is an orthogonal set of non-zero vectors in an inner product space then $S$ is linearly independent.*

# Supplementary Slide: Orthogonal Sets are Linearly Independent

## Theorem

*If $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is an orthogonal set of non-zero vectors in an inner product space then $S$ is linearly independent.*

## Proof.

**Proof idea:** If $S$ was linearly dependent, then sum of products of scalars and vectors $= \mathbf{0}$. Show that this is the case only when all scalars are zero. So vectors are linearly independent.

# Supplementary Slide: Orthogonal Sets are Linearly Independent

## Theorem

*If $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is an orthogonal set of <u>non-zero</u> vectors in an inner product space then $S$ is linearly independent.*

## Proof.

**Proof idea:** If $S$ was linearly dependent, then sum of products of scalars and vectors $= \mathbf{0}$. Show that this is the case only when all scalars are zero. So vectors are linearly independent.
Assume that $k_1\mathbf{v}_1 + \ldots + k_n\mathbf{v}_n = \mathbf{0}$ and prove that $k_1 = \ldots = k_n = 0$.

Pick any $\mathbf{v}_i$ and take the product of both sides of the above equation with this $\mathbf{v}_i$:

$$\langle k_1\mathbf{v}_1 + \ldots + k_n\mathbf{v}_n, \mathbf{v}_i\rangle = \langle \mathbf{0}, \mathbf{v}_i\rangle = 0$$

# Supplementary Slide: Orthogonal Sets are Linearly Independent

## Theorem

*If $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is an orthogonal set of <u>non-zero</u> vectors in an inner product space then $S$ is linearly independent.*

## Proof.

**Proof idea:** If $S$ was linearly dependent, then sum of products of scalars and vectors $= \mathbf{0}$. Show that this is the case only when all scalars are zero. So vectors are linearly independent. Assume that $k_1\mathbf{v}_1 + \ldots + k_n\mathbf{v}_n = \mathbf{0}$ and prove that $k_1 = \ldots = k_n = 0$.

Pick any $\mathbf{v}_i$ and take the product of both sides of the above equation with this $\mathbf{v}_i$:

$$\langle k_1\mathbf{v}_1 + \ldots + k_n\mathbf{v}_n, \mathbf{v}_i\rangle = \langle \mathbf{0}, \mathbf{v}_i\rangle = 0$$

By using the linearity (i.e. additivity + homogeneity) of the inner product, as well as orthogonality of $S$ (i.e. $\langle \mathbf{v}_j, \mathbf{v}_i\rangle = 0$ for $j \neq i$), we get

$$\langle k_1\mathbf{v}_1 + \ldots + k_n\mathbf{v}_n, \mathbf{v}_i\rangle = k_1\langle \mathbf{v}_1, \mathbf{v}_i\rangle + \ldots + k_n\langle \mathbf{v}_n, \mathbf{v}_i\rangle = k_i\langle \mathbf{v_i}, \mathbf{v}_i\rangle.$$

Thus, $k_i\langle \mathbf{v_i}, \mathbf{v}_i\rangle = 0$, and, since $\mathbf{v}_i$ is non-zero, it follows that $k_i = 0$. □