MCS Calculus
Practical Exercises 6
(Week 15)

Epiphany Term 2025

Make sure you have completed all exercises from the previous Calculus practical. If you wish, try typesetting your answers with LaTeX.

1. Consider the harmonic series $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+....$ By grouping 1 then 1 then 2 then 4 then 8 etc terms, obtain a underestimate for S_m where $m = 1+2^k-1$. Deduce that the series diverges.

2. Use the comparison test to determine if the following series converge.

   (a) $1+(\frac{2}{3})+(\frac{2}{3})^4+(\frac{2}{3})^9+...+(\frac{2}{3})^{n^2}+...$

   (b) $\frac{3}{4}+\frac{4}{7}+\frac{5}{12}+...+\frac{n+2}{n^2+3}+...$

   (c) $1+\frac{1}{3^2}+\frac{1}{5^3}+\frac{1}{7^4}+...+\frac{1}{(2n-1)^n}+...$

   (d) $\frac{1}{3-1}+\frac{1}{3^2-2}+\frac{1}{3^3-3}+...+\frac{1}{3^n-n}+...$

3. Use the ratio test to determine if the following series converge. If the ratio test fails, identify another way of testing for convergence.

   (a) $\frac{2}{1}+\frac{2.5}{1.5}+\frac{2.5.8}{1.5.9}+...+\frac{2.5.8...(3n-1)}{1.5.9...(4n-3)}+...$

   (b) $\frac{1}{\sqrt{3}}+\frac{3}{3}+\frac{5}{(\sqrt{3})^3}+...+\frac{2n-1}{(\sqrt{3})^n}+...$

   (c) $\frac{2}{5}+\frac{5}{14}+\frac{10}{29}+...+\frac{n^2+1}{3n^2+2}+...$

   (d) $\frac{1}{10}+\frac{2!}{10^2}+\frac{3!}{10^3}+...+\frac{n!}{10^n}+...$

4. Consider the series
$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \left(\frac{nk}{3n+1}\right)^n$$
   where $k \in \mathbb{R}$ is some constant.

   (a) Determine the value of $\lim_{n\to\infty} \sqrt[n]{a_n}$.

   (b) Let $R_m = \sum_{n=m+1}^{\infty} a_n$. By comparison with a suitable geometric series, show that

    i. if $k < 3$ then $\lim_{m\to\infty} R_m = 0$, whereas
    ii. if $k > 3$ then $\lim_{m\to\infty} R_m \to \infty$.

   (c) For what values of k can you conclude that $\sum_{n=1}^{\infty} a_n$ converges or diverges and for what values of k can you reach no conclusion?

5. After tackling the question above, recall 'The nth root test' and have a look at some examples on Math24.

6. Determine whether the following series converge.

   (a) $\sum_{n=2}^{\infty}(-1)^n \frac{\sqrt{n}}{\ln n}$
   (b) $\sum_{n=2}^{\infty}(-1)^n \frac{\ln n}{\sqrt{n}}$