MCS Calculus
Practical Exercises 5
(Week 13)

Epiphany Term 2025

If you wish, try typesetting your answers with LaTeX. LaTeXis a pretty useful tool for writing scientific papers and reports with plenty of mathematical notation. A not so short introduction to it can be found here.

1. Find the points on the surface $xy+z^2 = 4$ that are closest to the origin $(0, 0, 0)$.

Answer: We are tasked with minimising the distance from the origin, i.e. the distance $\sqrt{x^2 + y^2 + z^2}$, subject to the constraint $g(x, y, z) = xy + z^2 - 4 = 0$. To simplify our calculations, we can equivalently minimise the function:

$$f(x, y, z) = x^2 + y^2 + z^2,$$

subject to the constraint $g(x, y, z) = xy + z^2 - 4 = 0$.

We use the method of Lagrange multipliers. The Lagrangian is:

$$\mathcal{L}(x, y, z, \lambda) = f(x, y, z) - \lambda g(x, y, z) = x^2 + y^2 + z^2 - \lambda(xy + z^2 - 4).$$

Taking partial derivatives with respect to x, y, z, and λ, we get:

$$\begin{aligned}
\mathcal{L}_x &= \frac{\partial \mathcal{L}}{\partial x} = 2x - \lambda y, \\
\mathcal{L}_y &= \frac{\partial \mathcal{L}}{\partial y} = 2y - \lambda x, \\
\mathcal{L}_z &= \frac{\partial \mathcal{L}}{\partial z} = 2z - 2\lambda z, \\
\mathcal{L}_\lambda &= \frac{\partial \mathcal{L}}{\partial \lambda} = -(xy + z^2 - 4).
\end{aligned}$$

Setting these derivatives equal to 0, we obtain the system of equations:

$$\left\{ \begin{array}{rcl} 2x &=& \lambda y, \\ 2y &=& \lambda x, \\ 2z &=& 2\lambda z, \\ xy + z^2 - 4 &=& 0. \end{array} \right\}$$

Notice that the system above is exactly the same as the system of equations you get when setting:

$$\left\{ \begin{array}{rl} \nabla f = & \lambda \nabla g \\ xy + z^2 - 4 = & 0. \end{array} \right\} \Leftrightarrow \left\{ \begin{array}{rl} \text{Lagrange equations:} & f_x = \lambda g_x \Leftrightarrow 2x = \lambda y \\ & f_y = \lambda g_y \Leftrightarrow 2y = \lambda x \\ & f_z = \lambda g_z \Leftrightarrow 2z = 2\lambda z \\ \text{Constraint:} & xy + z^2 - 4 = 0. \end{array} \right\}$$

Let us now solve the above system of equations. From the third equation, we have that $\lambda = 1$ or $z = 0$. We distinguish cases.

Case 1: If $\lambda = 1$, substituting into the two first equations, we can see that it must be that $x = y = 0$. Now substituting into the last equation, we have:

$$z^2 = 4 \implies z = \pm 2.$$

So, the critical points in this case are $(x, y, z) = (0, 0, 2)$ and $(0, 0, -2)$.

Case 2: If $z = 0$, the constraint becomes: $xy = 4$ and so it must be that both x and y are non-zero and, in particular, $x = \frac{4}{y}$.

Substituting into the first and second equations, we can solve both equations for λ and set the results equal to each other; we then get that $y = \pm 2$.

Now substituting back into the equation $x = \frac{4}{y}$ that we got above, we have that $x = \pm 2$.

Therefore, the critical points in this case are $(x, y, z) = (2, 2, 0)$ and $(-2, -2, 0)$.

Conclusion:

We have four candidates for points on the surface $xy + z^2 = 4$ that are closest to the origin. We evaluate f on each of them and we have:

$$f(0, 0, 2) = f(0, 0, -2) = 4$$

and

$$f(2, 2, 0) = f(-2, -2, 0) = 8.$$

So, the points closest to the origin are: $(0, 0, 2)$ and $(0, 0, -2)$.

2. Find the maximum and minimum values of $f(x,y) = x^2 + x + 2y^2$ on the unit circle.

Answer: The objective function is $f(x,y)$. The constraint is $g(x,y) = x^2 + y^2 - 1 = 0$.

So, the Lagrangian is: $\mathcal{L}(x,y,\lambda) = f(x,y) - \lambda g(x,y) = x^2 + x + 2y^2 - \lambda(x^2 + y^2 - 1)$.

The system of equations we need to solve is given by setting the partial derivatives of $\mathcal{L}(x,y,\lambda)$ with respect to x, y, and λ equal to zero:

$$\left\{ \begin{array}{c} \mathcal{L}_x = 0 \\ \mathcal{L}_y = 0 \\ \mathcal{L}_\lambda = 0 \end{array} \right\},$$

or equivalently:

$$\left\{ \begin{array}{rrl} \text{Lagrange equations:} & f_x = \lambda g_x \Leftrightarrow & 2x + 1 = \lambda 2x \\ & f_y = \lambda g_y \Leftrightarrow & 4y = \lambda 2y \\ \text{Constraint:} & & x^2 + y^2 = 1. \end{array} \right\}$$

The second equation implies that $y = 0$ or $\lambda = 2$.

Case 1: $y = 0$. Then from the last equation in the system we get: $x = \pm 1$.

Case 2: $\lambda = 2$. Then from the first equation we get: $x = 1/2$. Now, substituting in the last equation, we have that it must be: $y = \pm\sqrt{3}/2$.

So, the critical points (and candidates for constrained extrema) are:
$(1/2, \sqrt{3}/2), (1/2, -\sqrt{3}/2), (1,0)$, and $(-1,0)$.

We calculate the value of f on those and based on that we conclude on their nature:
$f(1/2, \pm\sqrt{3}/2) = 9/4$ (maximum).
$f(1,0) = 2$ (neither min. nor max).
$f(-1,0) = 0$ (minimum).

3. Find the maximum and minimum values of $f(x, y) = x^2 - xy + y^2$ on the quarter circle $x^2 + y^2 = 1, x, y \geq 0$.

Answer: The objective function is $f(x, y)$. The constraint is $g(x, y) = x^2 + y^2 - 1 = 0$ with x and y non-negative. The maximum and minimum values of f will occur where $\nabla f = \lambda \nabla g$ or at endpoints of the quarter circle.

The Lagrangian is: $\mathcal{L}(x, y, \lambda) = f(x, y) - \lambda g(x, y) = x^2 - xy + y^2 - \lambda(x^2 + y^2 - 1)$.

The system of equations we need to solve is given by setting the partial derivatives of $\mathcal{L}(x, y, \lambda)$ with respect to x, y, and λ equal to zero:

$$\left\{ \begin{array}{l} \mathcal{L}_x = 0 \\ \mathcal{L}_y = 0 \\ \mathcal{L}_\lambda = 0 \end{array} \right\},$$

or equivalently:

$$\left\{ \begin{array}{rcc} \text{Lagrange equations:} & f_x = \lambda g_x \Leftrightarrow & 2x - y = \lambda 2x \\ & f_y = \lambda g_y \Leftrightarrow & -x + 2y = \lambda 2y \\ \text{Constraint:} & & x^2 + y^2 = 1 \end{array} \right\}$$

We distinguish cases:

Case 1: $x, y \neq 0$. Then, we can solve the first two equations for λ (divide by $2x$ and $2y$, respectively) and set the results equal to each other. This way, we get:

$$\frac{2x - y}{2x} = \frac{-x + 2y}{2y} \Rightarrow x^2 = y^2.$$

Because we're constrained to $x^2 + y^2 = 1$ with x and y non-negative, we conclude that $x = y = 1/\sqrt{2}$.

Case 2: $x = 0$. Then, by the first equation, it must also be that $y = 0$ but we now have a contradiction with the third equation. So, no solution in this case.

Case 3: $y = 0$. Then, by the second equation, it must also be that $x = 0$ but we now have a contradiction with the third equation. So, no solution in this case.

In summary, the extreme points of f will be at $(1/\sqrt{2}, 1/\sqrt{2})$, $(1, 0)$, or $(0, 1)$. By evaluating f on these points, we find that $f(1/\sqrt{2}, 1/\sqrt{2}) = 1/2$ is the minimum value of f on this quarter circle and $f(1, 0) = f(0, 1) = 1$ is the maximum.

4. Find the maximum and minimum values of $f(x, y) = x^2 + y^2$ on the curve $g(x, y) = x^2 - 2x + y^2 - 4y = 0$.

Answer: The objective function is $f(x, y)$. The constraint is $g(x, y) = 0$. The maximum and minimum values of f will occur where $\nabla f = \lambda \nabla g$ and $g(x, y) = 0$, so the system of equations to solve is:

$$\left\{ \begin{array}{rrl} \text{Lagrange equations:} & f_x = \lambda g_x \Leftrightarrow & 2x = 2\lambda x - 2\lambda \;\; (1) \\ & f_y = \lambda g_y \Leftrightarrow & 2y = 2\lambda y - 4\lambda \;\; (2) \\ \text{Constraint:} & & x^2 - 2x + y^2 - 4y = 0 \end{array} \right\}$$

Subtracting 1/2 of the second equation from the first equation, i.e. $(1) - \frac{1}{2} \cdot (2)$, we get:

$$2x - y = 2\lambda x - \lambda y \Leftrightarrow 2x - y = \lambda(2x - y) \Leftrightarrow (2x - y)(\lambda - 1) = 0,$$

i.e., we get that $\lambda = 1$ or $2x = y$. We distinguish cases.

Case 1: $\lambda = 1$. We can see that there is no solution (e.g. try to substitute in the first equation).

Case 2: $2x = y$. Now, the third equation becomes:

$$x^2 - 2x + 4x^2 - 8x = 0 \Leftrightarrow 5x^2 - 10x = 0 \Leftrightarrow x(5x - 10) = 0,$$

i.e. $x = 0$ or $5x - 10 = 0 \Leftrightarrow x = 2$.

For $x = 0$, we get $y = 2x = 0$; for $x = 2$, we get $y = 2x = 4$. So, the two candidates for constrained extrema are $(0, 0)$ and $(2, 4)$.

The value of f at those two points is $f(0, 0) = 0$ and $f(2, 4) = 20$, respectively.

Therefore, there is a minimum of f at $(0, 0)$ and a maximum of f at $(2, 4)$ on the given curve $g(x, y) = 0$.

5. Assume that among all rectangular (3D) boxes with fixed surface area of 20 square metres, there is a box of largest possible volume. Find its dimensions.

Answer: Let $x, y, z > 0$ be the lengths of the sides of the box we are looking for. The volume of the box is given by the function $f(x, y, z) = xyz$ and its surface area is given by $2xy + 2yz + 2zx$, which we are constrained to keep equal to 20.

Using the method of Lagrange multipliers, the objective function is $f(x, y, z)$ and the constraint is $g(x, y) = xy + yz + zx - 10$ with x, y, z all greater than zero.

The Lagrangian is: $\mathcal{L}(x, y, z, \lambda) = f(x, y, z) - \lambda g(x, y, z) = xyz - \lambda(xy + yz + zx - 10)$.

The system of equations we need to solve is given by setting the partial derivatives of $\mathcal{L}(x, y, z, \lambda)$ with respect to x, y, z, and λ equal to zero:

$$\left\{ \begin{array}{l} \mathcal{L}_x = 0 \\ \mathcal{L}_y = 0 \\ \mathcal{L}_z = 0 \\ \mathcal{L}_\lambda = 0 \end{array} \right\},$$

or equivalently:

$$\left\{ \begin{array}{rrr} \text{Lagrange equations:} & f_x = \lambda g_x \Leftrightarrow & yz = \lambda(y + z) \\ & f_y = \lambda g_y \Leftrightarrow & xz = \lambda(x + z) \\ & f_z = \lambda g_z \Leftrightarrow & xy = \lambda(x + y) \\ \text{Constraint:} & & xy + yz + zx = 10 \end{array} \right\}$$

Since $x, y, z > 0$ the only solution to the above system can be found as $x = y = z = \sqrt{10/3}$, which are the dimensions of the box with maximum volume in question.

6. Design a 1 litre cylindrical metal container (with a lid) using the minimum possible amount of metal.
Hint: what is the function that gives the volume of a cylinder with respect to the height of the cylinder and the radius of its base? What is the function that gives its total surface?

Answer: The volume of a cylinder is given by the function

$$V(r,h) = \pi r^2 h$$

and its total surface (bases + cylinder surface) is given by the function

$$S(r,h) = 2\pi r h + 2\pi r^2,$$

where r and h are the radius of the base and the height of the cylinder, respectively.

Therefore, we must minimise $S(r,h)$ under the constraint $g(r,h) = V(r,h) - 1 = 0$.

The Lagrangian is:

$$\mathcal{L}(r,h,\lambda) = f(r,h) - \lambda g(rh) = 2\pi r h + 2\pi r^2 - \lambda(\pi r^2 h - 1).$$

Using the method of Lagrange multipliers, we must solve the following system:

$$\left\{ \begin{array}{rrl} \text{Lagrange equations:} & S_r = \lambda g_r \Leftrightarrow & 2\pi h + 4\pi r = \lambda 2\pi h r \\ & S_h = \lambda g_h \Leftrightarrow & 2\pi r = \lambda \pi r^2 \\ \text{Constraint:} & & \pi r^2 h = 1 \end{array} \right\}$$

i.e.

$$\left\{ \begin{array}{rl} h + 2r = & \lambda h r \\ r(2 - \lambda r) = & 0 \\ \pi r^2 h = & 1. \end{array} \right\}$$

Since $r > 0$, the second equation gives $\lambda = \frac{2}{r}$. So, from the first equation we get $h + 2r = 2h \Leftrightarrow h = 2r$. We substitute in the third equation and get:

$$2\pi r^3 = 1 \Leftrightarrow r = \frac{1}{\sqrt[3]{2\pi}}.$$

Substituting into the equation $h = 2r$ we got above, we have:

$$h = \frac{2}{\sqrt[3]{2\pi}}.$$

So, the requested cylindrical container should have a base radius equal to $\frac{1}{\sqrt[3]{2\pi}}$ and height equal to $\frac{2}{\sqrt[3]{2\pi}}$.

7. Consider the geometric series $a + ar + ar^2 + ar^3 + ...$ with initial term a and common ratio r. So $S_n = \sum_{m=0}^{n} ar^m$. By considering the difference $S_n - rS_{n-1}$ prove that

$$S_n = a\left(\frac{1 - r^{n+1}}{1 - r}\right).$$

If $r < 1$ deduce that

$$\sum_{m=0}^{\infty} ar^m = \frac{a}{1 - r}.$$

Answer:

$$\begin{aligned}
S_n - rS_{n-1} &= (a + ar + ar^2 + ar^3 + ... + ar^n) - r(a + ar + ar^2 + ar^3 + ... + ar^{n-1}) \\
S_n - rS_{n-1} &= (a + ar + ar^2 + ar^3 + ... + ar^n) - (ar + ar^2 + ar^3 + ... + ar^n) \\
S_n - rS_{n-1} &= a
\end{aligned}$$

But $S_{n-1} = S_n - ar^n$ so we have

$$\begin{aligned}
S_n - rS_{n-1} &= a \\
S_n - r(S_n - ar^n) &= a \\
S_n(1 - r) + ar^{n+1} &= a \\
S_n(1 - r) &= a - ar^{n+1} \\
S_n &= \frac{a(1 - r^{n+1})}{1 - r}
\end{aligned}$$

Since $r < 1$, $r^{n+1} \to 0$ as $n \to \infty$, hence

$$S_n \to \frac{a}{1 - r}.$$

8. Determine whether the following series converge and, if so, their value.

   (a) $2 + 1 + \frac{2}{4} + \frac{1}{5} + \frac{2}{4^2} + \frac{1}{5^2} + \frac{2}{4^3} + \frac{1}{5^3} + ...$
   (b) $2 - 1 + \frac{2}{4} - \frac{1}{5} + \frac{2}{4^2} - \frac{1}{5^2} + \frac{2}{4^3} - \frac{1}{5^3} + ...$

Answer: (a) Let $A = \sum \frac{2}{4^n}$, $B \sum \frac{1}{5^n}$. By the previous question, $A = 8/3, B = 5/4$. Then taking $C = \sum \frac{2}{4^n} + \frac{1}{5^n}$ we can deduce $C = A + B = 47/12$.

(b) Let $A = \sum \frac{2}{4^n}$, $B \sum \frac{1}{5^n}$. By the previous question, $A = 8/3, B = 5/4$. Then taking $C = \sum \frac{2}{4^n} - \frac{1}{5^n}$ we can deduce $C = A - B = 17/12$.