# COMP1021 Mathematics for Computer Science
# Linear Algebra (Part 2)
# Practical - Week 16
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**Instructions:**

Work on these problems in the practical sessions for the week specified. First try them on your own. If you're stuck, try discussing things with others. If you get the answer, still discuss with others to see if maybe you missed something. If you run into major roadblocks, ask the demonstrators for hints.

Solutions will be posted on Learn Ultra at the end of the week. Make sure you're all set with the solutions and understand them before the next practical.

**Note:** If I don't mention some particular inner product when asking a question, assume it's the dot product.

**Purpose of this practical:** In this practical we review similarity of matrices, diagonilisation, eigendecomposition and inner product spaces.

1. Suppose a $3 \times 3$ upper triangular matrix has diagonal entries $1, 2$, and $7$. Can we say whether it is diagonalisable with the given information? Why or why not?

2. Find a matrix $A$ whose eigenvalues are 1 and 4 and whose corresponding eigenvectors are $(3, 1)$ and $(2, 1)$. Is this matrix unique?

3. We saw in class that a matrix $A$ of size $n \times n$ that has $n$ distinct eigenvalues will have an eigendecomposition. In other words, if all the eigenvalues of $A$ have algebraic multiplicity 1, then the eigendecomposition of $A$ exists. Can you say something about the relationship between the geometric multiplicity of eigevalues of $A$ and the existence of an eigendecomposition of $A$?

4. Prove the following properties about two similar matrices $A$ and $B$.
   (a) $A$ and $B$ have the same rank.
   (b) $A$ and $B$ have the same trace.
   (c) $A$ and $B$ have the same eigenvalues.

5. Show that for a diagonalisable $n \times n$ matrix $A$ with eigenvalues $\lambda_1, \ldots, \lambda_n$ counted with multiplicity, then $\det(A) = \prod_i \lambda_i$. Hint: consider the Eigendecomposition of $A$.

6. For each of the following expressions either prove that it defines an inner product on $\mathbb{R}^4$ or list all if the inner product axioms that fail to hold:
   (a) $\langle \mathbf{u}, \mathbf{b} \rangle = \mathbf{u}_1\mathbf{v}_1 - \mathbf{u}_2\mathbf{v}_2 + \mathbf{u}_3\mathbf{v}_3 - \mathbf{u}_4\mathbf{v}_4$
   (b) $\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u}_1^2\mathbf{v}_1^2 + \mathbf{u}_2^2\mathbf{v}_2^2 + \mathbf{u}_3^2\mathbf{v}_3^2 + \mathbf{u}_4^2\mathbf{v}_4^2$
   (c) $\langle \mathbf{u}, \mathbf{v} \rangle = 2\mathbf{u}_1 3\mathbf{v}_1 + 2\mathbf{u}_2 3\mathbf{v}_2 + 2\mathbf{u}_3 3\mathbf{v}_3 + 2\mathbf{u}_4 3\mathbf{v}_4$
   (d) $\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u}_1\mathbf{v}_4 + \mathbf{u}_2\mathbf{v}_3 + \mathbf{u}_3\mathbf{v}_2 + \mathbf{u}_4\mathbf{v}_1$

7. In each part, use the given inner product on $\mathbb{R}^2$ to find $||\mathbf{v}||$, where $\mathbf{v} = (-1, 3)$.
   (a) The Euclidean inner product.
   (b) The weighted Euclidean inner product with $w_1 = 3$ and $w_2 = 2$.
   (c) The inner product generated by the matrix
   $\begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix}$.

8. In each part, use the given inner product on $\mathbb{P}^2$ to find $\langle \mathbf{p}, \mathbf{q} \rangle$ where $\mathbf{p} = 1 - 2x^2$ and $\mathbf{q} = 1 - x + x^2$.
   (a) The standard inner product in $\mathbb{P}^2$.
   (b) The evaluation inner product with $x_0 = -1$, $x_1 = 0$ and $x_2 = 3$.
   (c) The inner product on $C[0, 1]$.

9. Consider $\mathbb{R}^4$ equipped with the weighted Euclidean inner product with following weights: $w_1 = 1, w_2 = 2, w_3 = 2, w_4 = 4$. Let $W = span(\mathbf{u}, \mathbf{v})$ where $\mathbf{u} = (1, 1, -1, 2)$ and $\mathbf{v} = (2, 2, 3, -1)$. Find a basis of $W^{\perp}$.

10. (Optional) Find all values of $a$, for which the following matrix $A$ is diagonalisable. For each such value, find an eigendecomposition of $A$.

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & a \\ 0 & 0 & 0 \end{pmatrix}.$$

11. (optional) (hard) Consider the Maclaurin series for $e^x$ as the definition of this function. By extending this definition to allow imaginary numbers, prove Euler's formula $e^{ix} = \cos(x) + i\sin(x)$, where $x \in \mathbb{R}$. Derive the equation $e^{i\pi} + 1 = 0$ connecting the five most important constants in mathematics.
(*Hint: The Maclaurin series* $\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$ *and* $\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$ *may be useful.*)

12. (optional) (hard) Prove the following properties of the orthogonal complement in $\mathbb{R}^n$ (with the dot product): If $W$ is a subspace of $\mathbb{R}^n$ then

    (a) $W^{\perp}$ is a subspace of $\mathbb{R}^n$.

    (b) $dim(W) + dim(W^{\perp}) = n$.

    (c) $(W^{\perp})^{\perp} = W$.

    Do your proofs generalise to an arbitrary (real) finite-dimensional inner product space?