MCS Calculus
Practical Exercises 5
(Week 13)

Epiphany Term 2025

If you wish, try typesetting your answers with LaTeX. LaTeXis a pretty useful tool for writing scientific papers and reports with plenty of mathematical notation. A not so short introduction to it can be found here.

1. Find the points on the surface $xy + z^2 = 4$ that are closest to the origin $(0, 0, 0)$.

2. Find the maximum and minimum values of $f(x, y) = x^2 + x + 2y^2$ on the unit circle.

3. Find the maximum and minimum values of $f(x, y) = x^2 - xy + y^2$ on the quarter circle $x^2 + y^2 = 1, x, y \geq 0$.

4. Find the maximum and minimum values of $f(x, y) = x^2 + y^2$ on the curve $g(x, y) = x^2 - 2x + y^2 - 4y = 0$.

5. Assume that among all rectangular (3D) boxes with fixed surface area of 20 square metres, there is a box of largest possible volume. Find its dimensions.

6. Design a 1 litre cylindrical metal container (with a lid) using the minimum possible amount of metal.
Hint: what is the function that gives the volume of a cylinder with respect to the height of the cylinder and the radius of its base? What is the function that gives its total surface?

7. Consider the geometric series $a + ar + ar^2 + ar^3 + ...$ with initial term a and common ratio r. So $S_n = \sum_{m=0}^{n} ar^m$. By considering the difference $S_n - rS_{n-1}$ prove that
$$S_n = a\left(\frac{1 - r^{n+1}}{1 - r}\right).$$
If $r < 1$ deduce that
$$\sum_{m=0}^{\infty} ar^m = \frac{a}{1 - r}.$$

8. Determine whether the following series converge and, if so, their value.

   (a) $2 + 1 + \frac{2}{4} + \frac{1}{5} + \frac{2}{4^2} + \frac{1}{5^2} + \frac{2}{4^3} + \frac{1}{5^3} + ...$

   (b) $2 - 1 + \frac{2}{4} - \frac{1}{5} + \frac{2}{4^2} - \frac{1}{5^2} + \frac{2}{4^3} - \frac{1}{5^3} + ...$