Durham University

Examination Paper

Examination Session:	Year:	Exam Code:
May/June	2023	COMP1021-WE01

Mathematics for Computer Science

Release Date/Time	11/05/2023 09:30
Latest Submission Date/Time	11/05/2023 12:30
Format of Exam	Restricted window exam
Duration:	2 hours
Word/Page Limit:	
Additional Material provided:	
Expected form of Submission	A SINGLE PDF file submitted to Gradescope
Submission method	Gradescope

Instructions to Candidates:	Answer ALL questions

Section A Linear Algebra (Dr. William K. Moses Jr.)

Question 1

(a) Suppose you're working at a video game company and you've encoded the position data for many sprites (small avatars players can choose to represent themselves) for your game into vectors, three vectors per sprite (let's just pretend this is possible). In the course of the game, you need to update the positions of the sprites after performing one of a small number of actions, where each action moves a sprite to a new position and is encoded by a matrix that is applied to the sprite's current position to get the new position of the sprite after performing the action. If you want to speed up the process of applying these actions to the various sprites, give one decomposition from {LU decomposition, eigendecomposition, SVD, QR decomposition} that would aid you, assuming that all of them exist for each action. Give a very brief description of how it would aid you. **[4 Marks]**

(b) Consider the video game company from the previous question. Now, instead of quickly applying some action to many sprites, you want to instead apply the same action to the same sprite many times in succession to move it to some final position. If you want to speed up the process of calculating the final position of the sprite, give one decomposition from {LU decomposition, eigendecomposition, SVD, QR decomposition} that would aid you, assuming that all of them exist for each action. Give a very brief description of how it would aid you. **[4 Marks]**

(c) Suppose you've collected data from a tribe of dogs that plots the amount of food a dog consumes per year against their age. You've collected this data from hundreds of dogs for various ages up to age 8. You now want to predict how much food a dog will consume per year when they are 9 years old based on the previous data. Give one decomposition from {LU decomposition, eigendecomposition, SVD, QR decomposition} that would aid you. Give a very brief description of how it would aid you. **[4 Marks]**

(d) You have a high resolution picture that you want to compress. Give one decomposition from {LU decomposition, eigendecomposition, SVD, QR decomposition} that would aid you. Give a very brief description of how it would aid you. **[4 Marks]**

continued

Question 2

(a) Consider the matrix $A = \begin{pmatrix} 10^{-80} & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & 2 \end{pmatrix}$

Find the LU decomposition of A such that the diagonal elements of U are all 1, i.e., U is of the form $A = \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}$ where a, b, c are some numbers. Give an exact expression for each value throughout your computations (do not use decimal approximations; fractions are allowed). **[10 Marks]**

(b) Consider the matrix $A = \begin{pmatrix} 3 & 0 & 5 \\ 1 & 5 & 0 \\ 0 & 0 & 5 \end{pmatrix}$

i. Find its eigenvalues **[2 Marks]**, the algebraic multiplicity of each eigenvalue **[2 Marks]**, and a basis for each eigenspace **[4 Marks]**.

ii. For each of LU decomposition, eigendecomposition, SVD, and QR decomposition, does the matrix satisfy the requirements for the given decomposition to be performed? Give reasoning for each. **[16 Marks]**

Section B Calculus
(Dr Eleni Akrida)

Question 3

(a) Let $a_k > 0$ for all $k \in \mathbb{N}$. If the series $\sum_{k=1}^{\infty} a_k$ converges, show that both series $\sum_{k=1}^{\infty} a_k^2$ and $\sum_{k=1}^{\infty} \frac{a_k}{a_k+1}$ also converge. **[6 Marks]**

(b) Determine the values of $x \in \mathbb{R}$ for which the series below converges:

$$\sum_{k=1}^{\infty} \frac{1}{1+x^k}.$$

Show all your working. **[9 Marks]**

(c) Calculate the Hessian matrix for the below function $f : \mathbb{R} \to \mathbb{R}$.

$$f(x, y, z) = x^2 + y^2 + z^2 + xy + yz + y.$$

Show all your working. **[5 Marks]**

(d) Use the solution you produced in (c) to determine the location and nature of the stationary points of f. Show all your working. **[5 Marks]**

Question 4

(a) Let $f, g : [a, b] \to \mathbb{R}$ be continuous functions such that

$$\int_a^b f(x)dx = \int_a^b g(x)dx.$$

Show that there exists $x_0 \in [a, b]$ such that $f(x_0) = g(x_0)$.
[Note: You may consider it a given that the image of a continuous function on a closed interval is closed.]
[Hint: Consider the function $h(x) = f(x) - g(x)$.] **[15 Marks]**

(b) Show that for all $p \in \mathbb{R}$, the integral $\int_0^{\infty} x^p dx$ is not finite. **[10 Marks]**

END OF PAPER