Maths for Computer Science
Calculus

Dr Eleni Akrida

Taylor's Theorem

Contents for today's lecture

- Approximating functions using polynomials
- Deriving a Taylor series from a function:
  - The series is infinite
  - It is a precise equal to $f(x)$
- Taylor's theorem:
  - Capping the Taylor series expansion at n terms but still making the sum a precise equal to $f(x)$
  - Applications

Recall: Quadratic approximation from function

Suppose we start with some function $f(x)$ and we wish to determine a quadratic form such that near x_0:

$$f(x) \approx a_0 + a_1(x - x_0) + a_2(x - x_0)^2.$$

What do we mean by $\approx$?

Let's at least demand that f and our approximation have

- the same value at x_0
- the same slope (derivative) at x_0
- the same curvature (2nd derivative) at x_0

You could define your approximation based on something else, e.g. to be such that the sum of squared errors over some region of interest is minimum between f and your polynomial approximation, but then that would lead to a different optimisation problem.

Recall: Quadratic approximation from function

Suppose we start with some function $f(x)$ and we wish to determine a quadratic form such that near x_0:

$$f(x) \approx a_0 + a_1(x - x_0) + a_2(x - x_0)^2.$$

First observe that by setting $x = x_0$, we must have: $f(x_0) = a_0$.

Now differentiate once and set $x = x_0$; we must have: $f'(x_0) = a_1$.

Differentiating again: $f''(x_0) = 2a_2 \Rightarrow a_2 = \frac{f''(x_0)}{2}$.

Putting all this together, it must be:

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2}(x - x_0)^2.$$

Durham University

Recall: Taylor series from function

Suppose we start with some function $f(x)$ and we wish to determine a power series $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ such that $f(x) = \sum_{n=0}^{\infty} a_n(x-x_0)^n$.

Now we ask for exact equality.

As before, by setting $x = x_0$, we must have $a_0 = f(x_0)$.

Assuming the series has some radius of convergence $r > 0$, differentiate once for $-r < x < r$:

$$f'(x) = \sum_{n=1}^{\infty} n a_n (x-x_0)^{n-1} = a_1 + 2a_2(x-x_0) + 3a_3(x-x_0)^2 + \cdots, \qquad i.e. \quad f'(x_0) = a_1.$$

Differentiating again for $-r < x < r$:

$$f''(x) = \sum_{n=2}^{\infty} n(n-1) a_n (x-x_0)^{n-2} = 2a_2 + 6a_3(x-x_0) + \cdots, \qquad i.e. \quad f''(x_0) = 2a_2.$$

Recall: Taylor series from function

Suppose we start with some function $f(x)$ and we wish to determine a power series $\sum_{n=0}^{\infty} a_n (x - x_0)^n$ such that $f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$.

Proceeding systematically:

$f^{(m)}(x) = \sum_{n=m}^{\infty} n(n-1) \dots (n-m+1) a_n (x - x_0)^{n-m}$, therefore $f^{(m)}(x_0) = m!\, a_m$.

So putting it together, **if** f is equal to a power series then the power series must be:

$$f(x) = \sum_{n=0}^{\infty} f^{(n)}(x_0) \frac{(x - x_0)^n}{n!} .$$

This form is called the **Taylor series expansion of f**.

(Recall the Maclaurin expansion is just the special case when $x_0 = 0$.)

Durham University

Taylor's Theorem

When we came up with the Maclaurin / Taylor series, we said that ***if*** a series exists that is equal to f, **then** it must have these coefficients. This is not quite the same as proving that the series does indeed converge to the correct value.

So, when is a Maclaurin / Taylor series really equal to the function with which it is associated?

Let $f(x)$ be infinitely differentiable and have the Taylor series representation:

$$f(x) = \sum_{n=0}^{\infty} f^{(n)}(x_0)\frac{(x-x_0)^n}{n!}.$$

Let $P_{n-1}(x) = \sum_{i=0}^{n-1} f^{(i)}(x_0)\frac{(x-x_0)^i}{i!}$ be the sum of the first n terms of the series which terminates at the power $(x-x_0)^{n-1}$.

Then, a necessary and sufficient condition for the Taylor series to converge to $f(x)$ is obviously that:

$$\lim_{n\to\infty} |f(x) - P_{n-1}(x)| = 0\,.$$

Taylor's Theorem

This suggests that to establish convergence, we must examine the behaviour of the remainder of the series after n terms.

Suppose a function f is n times differentiable on $[a, x]$, then for then there is some $\xi \in (a, x)$ such that

remainder term $R_n(x)$

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \cdots + \frac{(x-a)^{n-1}}{(n-1)!}f^{(n-1)}(a) + \boxed{\frac{(x-a)^n}{n!}f^{(n)}(\xi)}$$

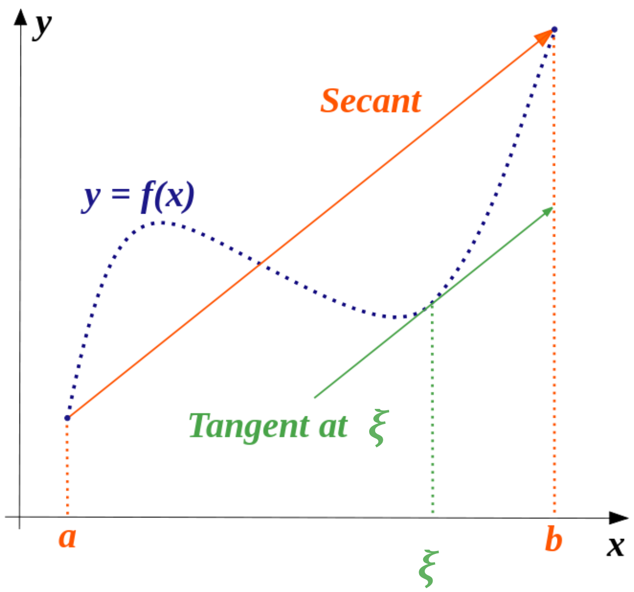
So, a necessary and sufficient condition for the Taylor series to converge to $f(x)$ is that:

$$\lim_{n\to\infty}|f(x) - P_{n-1}(x)| = 0 \Leftrightarrow \lim_{n\to\infty}|P_{n-1}(x) + R_n(x) - P_{n-1}(x)| = 0$$
$$\Leftrightarrow \lim_{n\to\infty}|R_n(x)| = 0$$

Taylor's Theorem

Suppose a function f is n times differentiable on $[a, x]$, then there exists some $\xi \in (a, x)$ such that

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \cdots + \frac{(x-a)^{n-1}}{(n-1)!}f^{(n-1)}(a) + \frac{(x-a)^n}{n!}f^{(n)}(\xi)$$



This expansion should seem a little familiar in style to the Mean Value Theorem: "the derivative at ξ is equal to the '*average derivative*' in the interval".

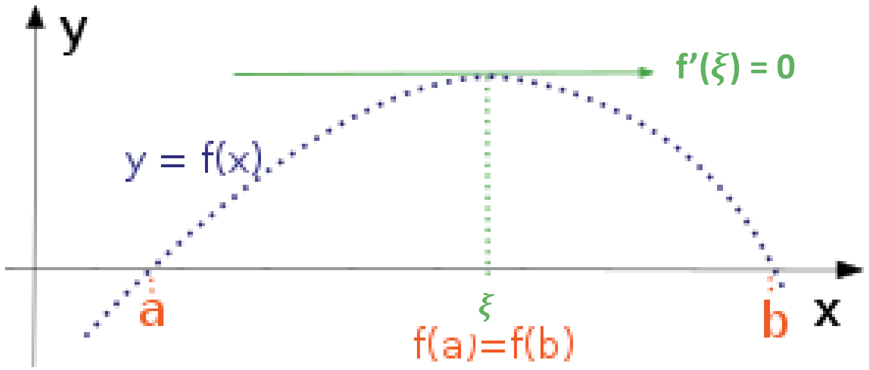
Recall the **Mean Value theorem**:
For any function that is continuous on $[a, b]$ and differentiable on (a, b) there exists some $\xi \in (a, b)$ such that the secant joining the endpoints of the interval $[a, b]$ is parallel to the tangent at ξ.

Durham University

Taylor’s Theorem

Suppose a function f is n times differentiable on $[a, x]$, then there exists some $\xi \in (a, x)$ such that

$$f(x) = f(a) + (x - a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \cdots + \frac{(x-a)^{n-1}}{(n-1)!}f^{(n-1)}(a) + \frac{(x-a)^n}{n!}f^{(n)}(\xi)$$



We will prove Taylor’s theorem similarly to the mean value theorem, using…

Rolle’s theorem:
If a real-valued function f is continuous on $[a, b]$, differentiable on (a, b), and $f(a) = f(b)$, then there exists $\xi \in (a, b)$ such that $f'(\xi) = 0$.

Durham University

Proof of Taylor's Theorem

Suppose a function f is n times differentiable on $[a, x]$.

Let k be a constant defined so that:

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \cdots + \frac{(x-a)^{n-1}}{(n-1)!}f^{(n-1)}(a) + \frac{(x-a)^n}{n!}k$$

Proof of Taylor's Theorem

The idea: design a suitable function F where we can apply Rolle's theorem.

We will then find ξ s.t. $F'(\xi) = 0$ and show that this implies $f^{(n)}(\xi) = k$, thus proving the theorem.

Suppose a function f is n times differentiable on $[a, x]$.

Let k be a constant defined so that:

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \cdots + \frac{(x-a)^{n-1}}{(n-1)!}f^{(n-1)}(a) + \frac{(x-a)^n}{n!}k$$

Proof of Taylor's Theorem

The idea: design a suitable function F where we can apply Rolle's theorem.

We will then find ξ s.t. $F'(\xi) = 0$ and show that this implies $f^{(n)}(\xi) = k$, thus proving the theorem.

Suppose a function f is n times differentiable on $[a, x]$.

Let k be a constant defined so that:

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \cdots + \frac{(x-a)^{n-1}}{(n-1)!}f^{(n-1)}(a) + \frac{(x-a)^n}{n!}k$$

Define the function $F(y) = f(x) - f(y) - (x-y)f'(y) - \cdots - \frac{(x-y)^{n-1}}{(n-1)!}f^{(n-1)}(y) - \frac{(x-y)^n}{n!}k$

Then $F(x) = 0$, and also

$$F(a) = f(x) - \left(f(a) + (x-a)f'(a) + \cdots + \frac{(x-a)^{n-1}}{(n-1)!}f^{(n-1)}(a) + \frac{(x-a)^n}{n!}k\right) = f(x) - f(x) = 0.$$

So we can apply Rolle's theorem to F on $[a, x]$, i.e. there is some $\xi \in (a, x)$ such that $F'(\xi) = 0$.

Proof of Taylor's Theorem

We have: $F(y) = f(x) - f(y) - (x-y)f'(y) - \frac{(x-y)^2}{2!}f''(y) - \cdots - \frac{(x-y)^{n-1}}{(n-1)!}f^{(n-1)}(y) - \frac{(x-y)^n}{n!}k.$

So, $F'(y) =$

$$= 0 - f'(y) + f'(y) - (x-y)f''(y) + (x-y)f''(y) - \cdots - \frac{(x-y)^{n-1}}{(n-1)!}f^{(n)}(y) + \frac{(x-y)^{n-1}}{(n-1)!}k,$$

$$\Rightarrow F'(y) = \frac{(x-y)^{n-1}}{(n-1)!}\left(k - f^{(n)}(y)\right).$$

Therefore:

$$\left.\begin{array}{llll} F'(\xi) = 0 & \Rightarrow & \frac{(x-\xi)^{n-1}}{(n-1)!}\left(k - f^{(n)}(\xi)\right) = 0 \\ \text{But} & \xi \in (a,x) & \Rightarrow \quad (x-\xi) \neq 0 \end{array}\right\} \Rightarrow f^{(n)}(\xi) = k,$$

which implies:

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \cdots + \frac{(x-a)^{n-1}}{(n-1)!}f^{(n-1)}(a) + \frac{(x-a)^n}{n!}f^{(n)}(\xi). \quad \square$$

Example: $\sin x$

Let $f(x) = \sin x$.

Then $f(0) = 0$,

$f'(0) = \cos(0) = 1$

$f''(0) = -\sin(0) = 0$

$f^{(3)}(0) = -\cos(0) = -1$

$f^{(2k)}(0) = 0$

$f^{(2k+1)}(0) = (-1)^k$

Since $f \in \mathrm{C}^\infty$, for any x, n there is some $\xi \in (0, x)$ such that

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots + \frac{x^n}{n!} \sin^{(n)} \xi$$

Since $|\sin^{(n)} \xi\,| \leq 1$ we have $\lim_{n\to\infty} \frac{x^n}{n!} \sin^{(n)} \xi = 0$, so the series converges to $\sin x$.

What is the radius of convergence of the Maclaurin series for $f(x) = \sin x$?

0

Infinite; the series converges for all $x \in \mathbb{R}$.

0; the series converges only at $x = 0$.

1; the series converges only in $[-1, 1)$.

None of the above

What is the radius of convergence of the Maclaurin series for $f(x) = \sin x$?

0

Infinite; the series converges for all $x \in \mathbb{R}$.

0%

0; the series converges only at $x = 0$.

0%

1; the series converges only in $[-1, 1)$.

0%

None of the above

0%

What is the radius of convergence of the Maclaurin series for $f(x) = \sin x$?

0

Infinite; the series converges for all $x \in \mathbb{R}$.

0%

0; the series converges only at $x = 0$.

0%

1; the series converges only in $[-1, 1)$.

0%

None of the above

0%

Example: sin x

$f(x) = \sin(x)$

$g(x) = x$

$h(x) = x - \frac{x^3}{6}$

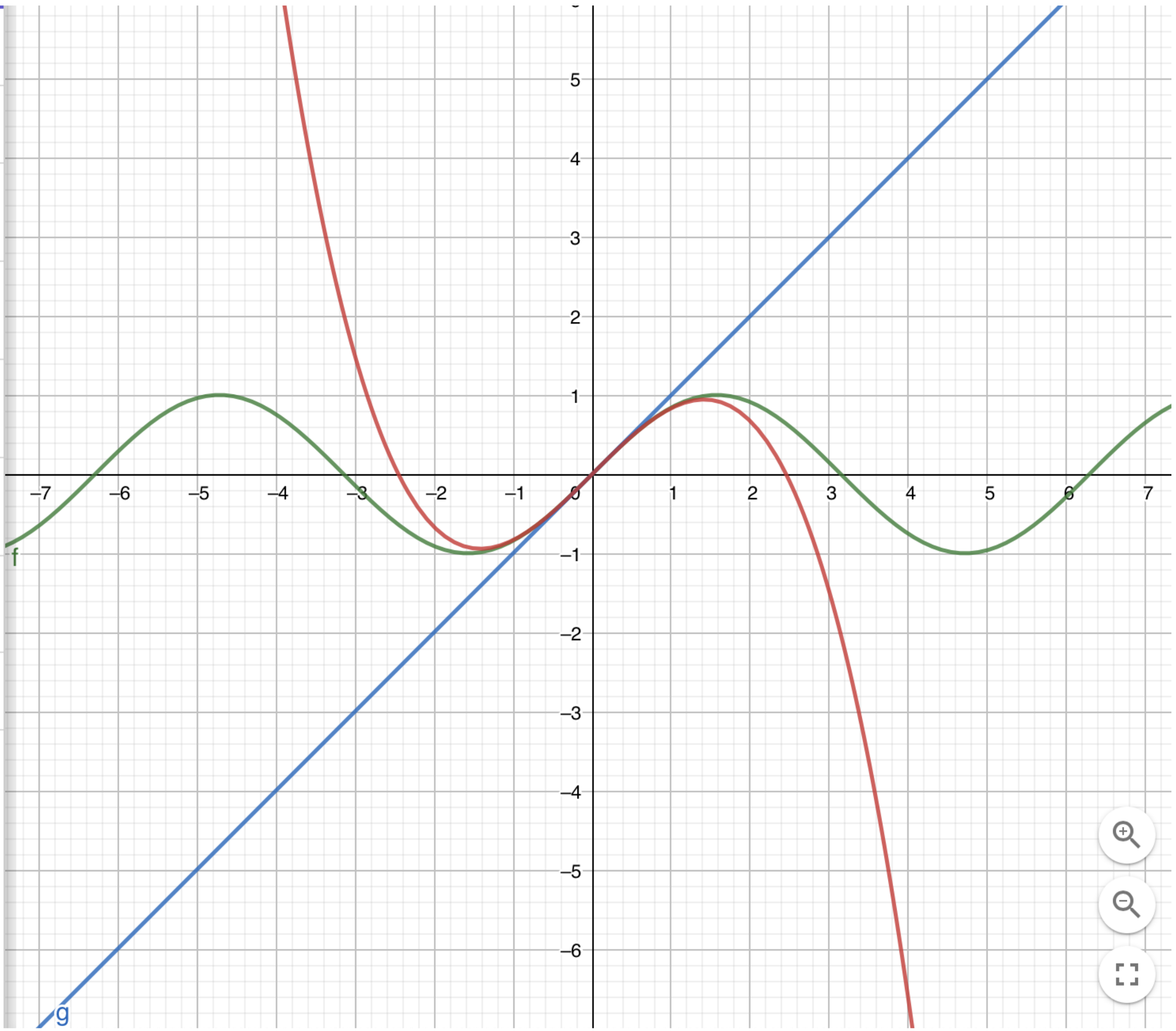
$p(x) = x - \frac{x^3}{6} + \frac{x^5}{5!}$

$q(x) = x - \frac{x^3}{6} + \frac{x^5}{5!} - \frac{x^7}{7!}$

$r(x) = x - \frac{x^3}{6} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!}$

$s(x) = x - \frac{x^3}{6} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!}$

+ Input…



Example: $\sin x$

$f(x) = \sin(x)$

$g(x) = x$

$h(x) = x - \frac{x^3}{6}$

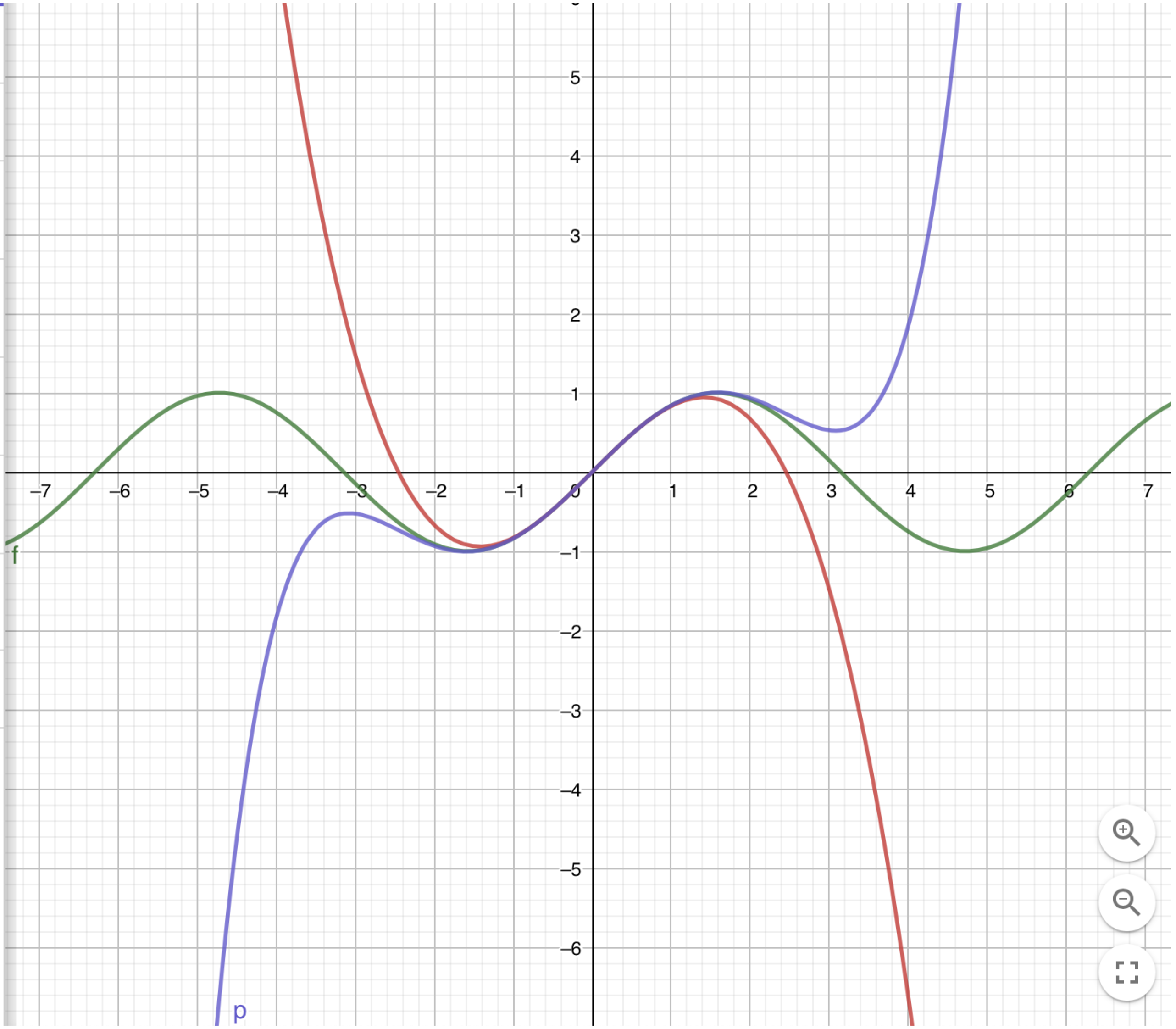
$p(x) = x - \frac{x^3}{6} + \frac{x^5}{5!}$

$q(x) = x - \frac{x^3}{6} + \frac{x^5}{5!} - \frac{x^7}{7!}$

$r(x) = x - \frac{x^3}{6} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!}$

$s(x) = x - \frac{x^3}{6} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!}$

Input...



Example: $\sin x$

$$f(x) = \sin(x)$$

$$g(x) = x$$

$$h(x) = x - \frac{x^3}{6}$$

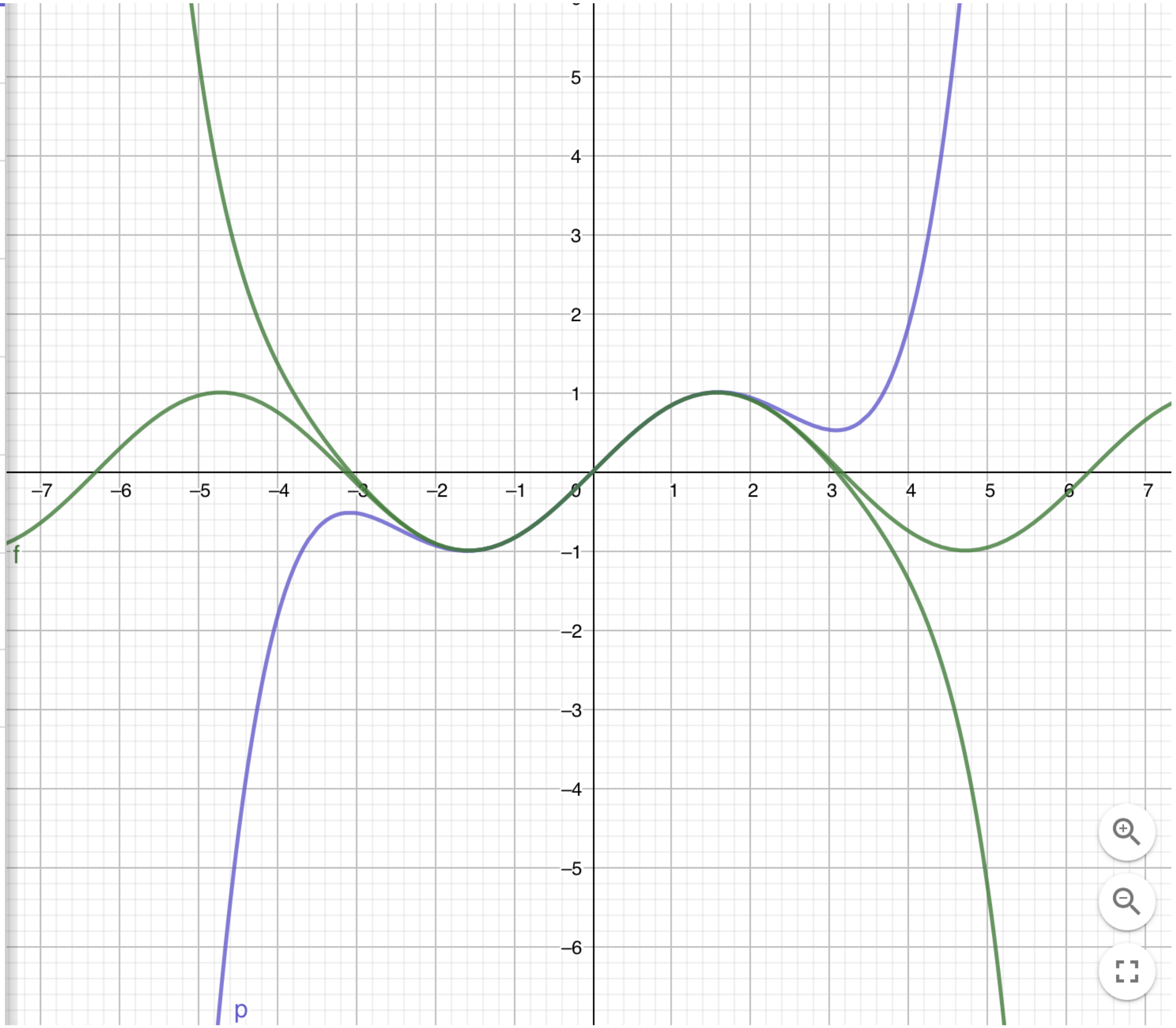
$$p(x) = x - \frac{x^3}{6} + \frac{x^5}{5!}$$

$$q(x) = x - \frac{x^3}{6} + \frac{x^5}{5!} - \frac{x^7}{7!}$$

$$r(x) = x - \frac{x^3}{6} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!}$$

$$s(x) = x - \frac{x^3}{6} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!}$$

Input…



Example: $\sin x$

$$f(x) = \sin(x)$$

$$g(x) = x$$

$$h(x) = x - \frac{x^3}{6}$$

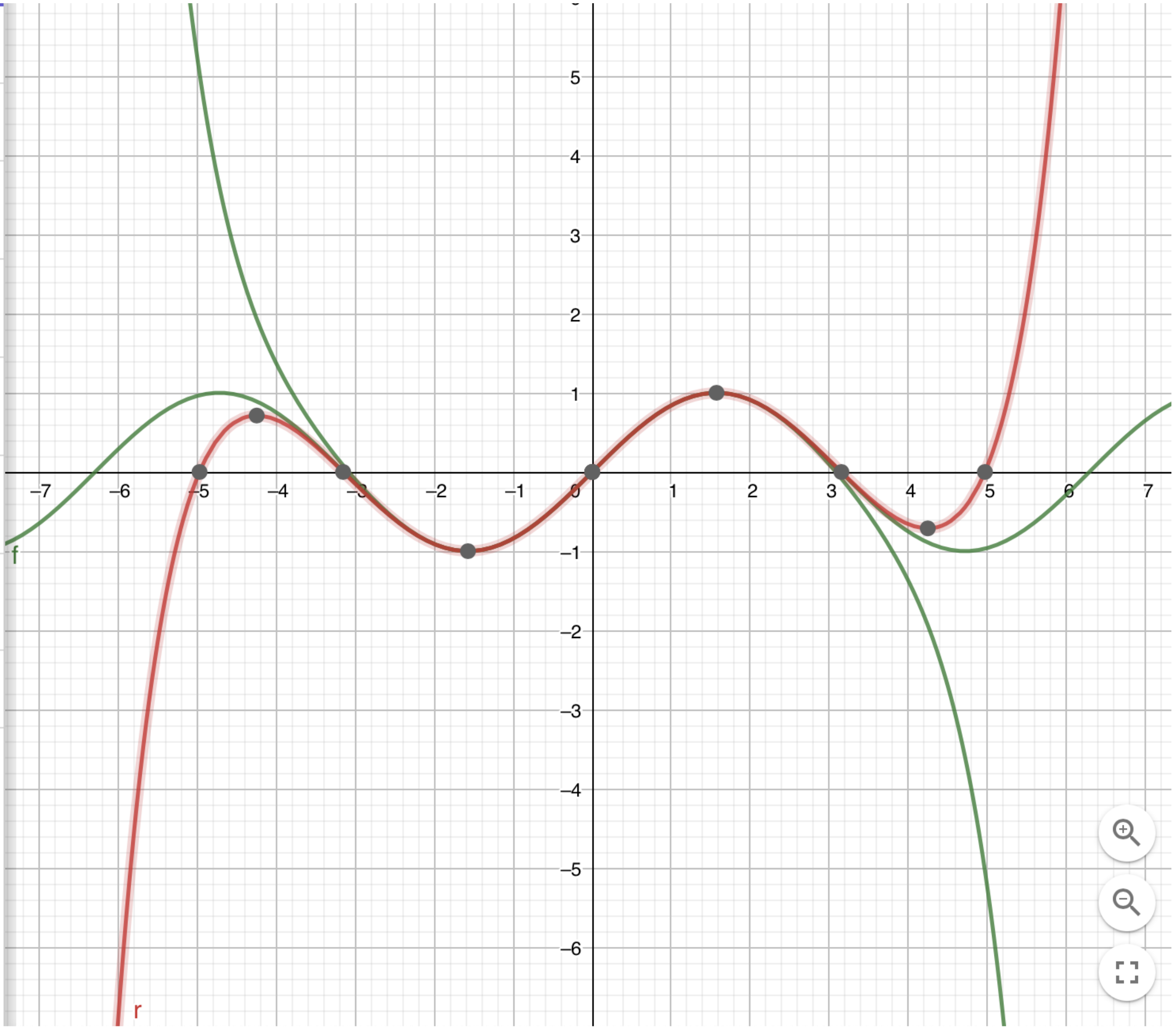
$$p(x) = x - \frac{x^3}{6} + \frac{x^5}{5!}$$

$$q(x) = x - \frac{x^3}{6} + \frac{x^5}{5!} - \frac{x^7}{7!}$$

$$r(x) = x - \frac{x^3}{6} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!}$$

$$s(x) = x - \frac{x^3}{6} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!}$$

+ Input…



Example: $\sin x$

$f(x) = \sin(x)$

$g(x) = x$

$h(x) = x - \frac{x^3}{6}$

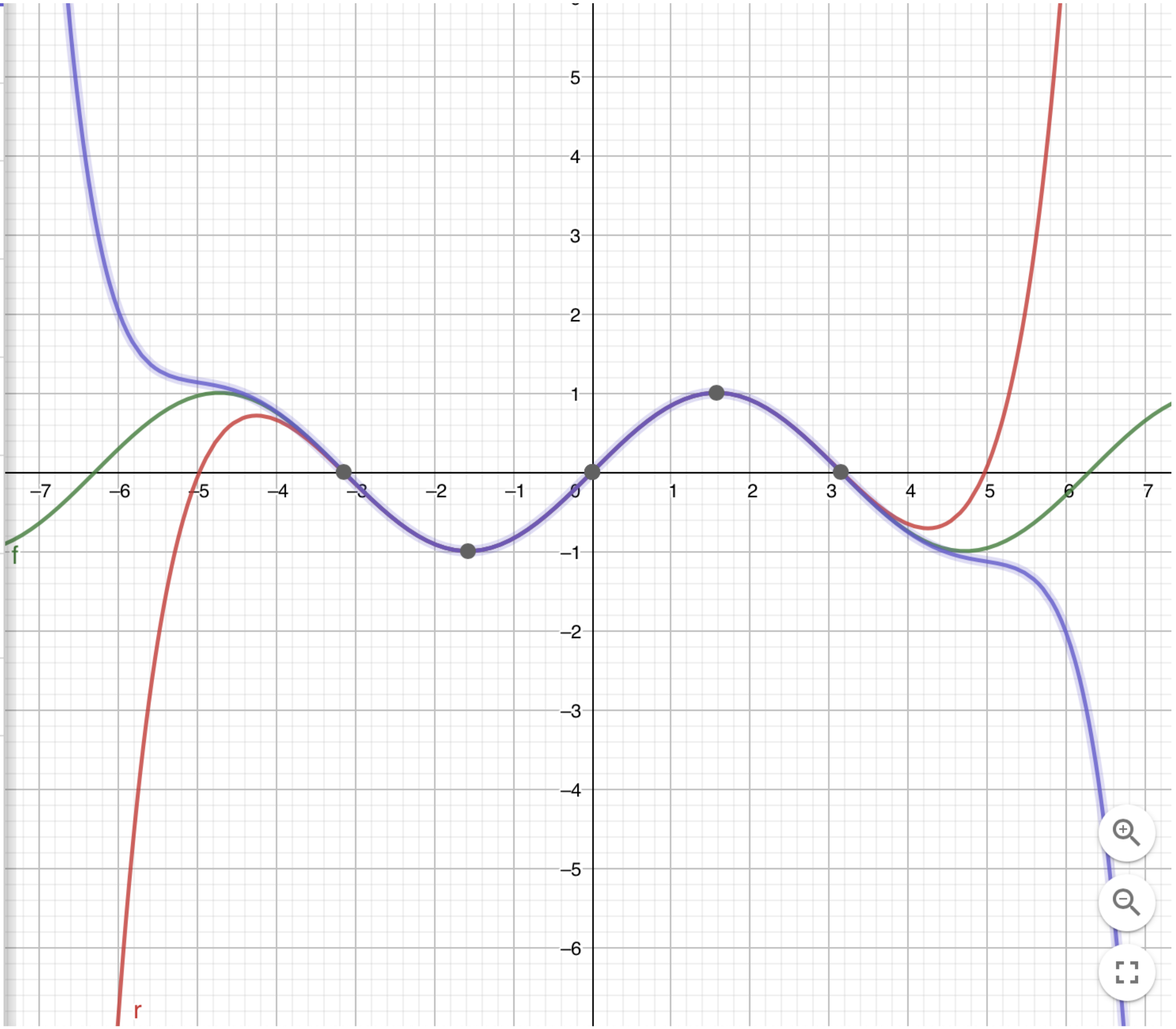
$p(x) = x - \frac{x^3}{6} + \frac{x^5}{5!}$

$q(x) = x - \frac{x^3}{6} + \frac{x^5}{5!} - \frac{x^7}{7!}$

$r(x) = x - \frac{x^3}{6} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!}$

$s(x) = x - \frac{x^3}{6} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!}$

Input...



Example: $\cos x$

Let $f(x) = \cos x$.

Then $f(0) = 1$,

$f'(0) = -\sin(0) = 0$

$f''(0) = -\cos(0) = -1$

$f^{(2k)}(0) = (-1)^k$

$f^{(2k+1)}(0) = 0$

Which of the following would be the Maclaurin series expansion for $f(x) = \cos x$?

$1 + \frac{x}{2} + \frac{x}{4} + \frac{x}{6} + \ldots + \frac{x}{2k} + \ldots$

$1 + \frac{x^2}{2} + \frac{x^4}{4} + \frac{x^6}{6} + \ldots + \frac{x^{2k}}{2k} + \ldots$

$1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots + (-1)^k \cdot \frac{x^{2k}}{(2k)!} + \ldots$

$x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \ldots + \frac{x^{2k+1}}{(2k+1)!} + \ldots$

None of the above

Which of the following would be the Maclaurin series expansion for $f(x) = \cos x$?

$1 + \frac{x}{2} + \frac{x}{4} + \frac{x}{6} + \ldots + \frac{x}{2k} + \ldots$

0%

$1 + \frac{x^2}{2} + \frac{x^4}{4} + \frac{x^6}{6} + \ldots + \frac{x^{2k}}{2k} + \ldots$

0%

$1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots + (-1)^k \cdot \frac{x^{2k}}{(2k)!} + \ldots$

0%

$x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \ldots + \frac{x^{2k+1}}{(2k+1)!} + \ldots$

0%

None of the above

0%

Which of the following would be the Maclaurin series expansion for $f(x) = \cos x$?

$1 + \frac{x}{2} + \frac{x}{4} + \frac{x}{6} + \ldots + \frac{x}{2k} + \ldots$

0%

$1 + \frac{x^2}{2} + \frac{x^4}{4} + \frac{x^6}{6} + \ldots + \frac{x^{2k}}{2k} + \ldots$

0%

$1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots + (-1)^k \cdot \frac{x^{2k}}{(2k)!} + \ldots$

0%

$x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \ldots + \frac{x^{2k+1}}{(2k+1)!} + \ldots$

0%

None of the above

0%

Example: $\cos x$

Let $f(x) = \cos x$.

Then $f(0) = 1$,

$f'(0) = -\sin(0) = 0$

$f''(0) = -\cos(0) = -1$

$f^{(2k)}(0) = (-1)^k$

$f^{(2k+1)}(0) = 0$

Since $f \in C^\infty$, for any x,

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots + (-1)^k \frac{x^{2k}}{(2k)!} + \cdots$$

Example: $\cos x, e^x$, etc.

We have just seen that:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots + (-1)^k \frac{x^{2k+1}}{(2k+1)!} + \cdots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots + (-1)^k \frac{x^{2k}}{(2k)!} + \cdots$$

Recall that the derivatives of sinh and cosh don't get the negatives, so:

Durham University

Example: $\cos x, e^x$, etc.

$(\sinh x)' = \cosh x$ and $(\cosh x)' = \sinh x$.

That means that when you take their Maclaurin series expansion, you get the same sums as for sinh and cosh, but with no negative terms.

We have just seen that:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots + (-1)^k \frac{x^{2k+1}}{(2k+1)!} + \cdots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots + (-1)^k \frac{x^{2k}}{(2k)!} + \cdots$$

Recall that the derivatives of sinh and cosh don't get the negatives, so:

Example: $\cos x, e^x$, etc.

$(\sinh x)' = \cosh x$ and $(\cosh x)' = \sinh x$.

That means that when you take their Maclaurin series expansion, you get the same sums as for sinh and cosh, but with no negative terms.

We have just seen that:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots + (-1)^k \frac{x^{2k+1}}{(2k+1)!} + \cdots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots + (-1)^k \frac{x^{2k}}{(2k)!} + \cdots$$

Recall that the derivatives of sinh and cosh don't get the negatives, so:

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} \ldots + \frac{x^{2k+1}}{(2k+1)!} + \cdots,$$

and

$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} \ldots + \frac{x^{2k}}{2k!} + \cdots.$$

Recall: $\sinh x = \frac{e^x - e^{-x}}{2}$

Recall: $\cosh x = \frac{e^x + e^{-x}}{2}$

Again this confirms what we have previously seen, i.e. that:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots = \sinh x + \cosh x$$

Durham University

Example: complex $\cos x, e^x$, etc.

We can also take the same expansions for cos and sin for complex values of x:

Let's try setting $x = i\theta$:

$$\sin i\theta = i\theta - \frac{(i\theta)^3}{3!} + \frac{(i\theta)^5}{5!} - \frac{(i\theta)^7}{7!} + \frac{(i\theta)^9}{9!} - \cdots$$

$$= i\left(\theta + \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \frac{\theta^7}{7!} + \frac{\theta^9}{9!} + \cdots\right) = i \sinh \theta$$

$$\cos i\theta = 1 - \frac{(i\theta)^2}{2!} + \frac{(i\theta)^4}{4!} - \frac{(i\theta)^6}{6!} + \cdots = \cosh \theta$$

$$e^{i\theta} = 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} \ldots$$

$$= 1 + i\theta - \frac{\theta^2}{2!} - i\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + i\frac{\theta^5}{5!} \ldots$$

$$= \cos\theta + i\sin\theta$$

In particular: $e^{i\pi} = \cos\pi + i\sin\pi = -1 + i \cdot 0 = -1$ (Euler's identity)

Durham University

Application: extended L'Hôpital's Rule

Let f and g be n-times differentiable functions such that

- $f(a) = g(a) = 0$,
- $f^{(r)}(a) = g^{(r)}(a) = 0$ for $1 \leq r \leq n-1$,
- $f^{(n)}(a),\ g^{(n)}(a)$ are not zero.

Then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f^{(n)}(x)}{\lim_{x\to a} g^{(n)}(x)}$$

Proof:

By Taylor's theorem there exist $\xi_1, \xi_2 \in (a, a+h)$ such that

$$\frac{f(a+h)}{g(a+h)} = \frac{f(a) + hf'(a) + \cdots + \frac{h^n}{n!} f^{(n)}(\xi_1)}{g(a) + hg'(a) + \cdots + \frac{h^n}{n!} g^{(n)}(\xi_2)} = \frac{f^{(n)}(\xi_1)}{g^{(n)}(\xi_2)}$$

Now if $h \to 0$ then $\xi_1, \xi_2 \to a$, so

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{h\to 0} \frac{f(a+h)}{g(a+h)} = \frac{\lim_{\xi_1\to a} f^{(n)}(\xi_1)}{\lim_{\xi_2\to a} g^{(n)}(\xi_2)}.$$

Durham University

Application: classifying extrema

1. A necessary and sufficient condition for a suitably differentiable function $f(x)$ to have a local maximum at $x = a$ is that the first derivative $f^{(n)}(x)$ with a non-zero value at $x = a$ is of even order (i.e. n is even) and $f^{(n)}(a) < 0$.
2. A necessary and sufficient condition for a suitably differentiable function $f(x)$ to have a local minimum at $x = a$ is that the first derivative $f^{(n)}(x)$ with a non-zero value at $x = a$ is of even order (i.e. n is even) and $f^{(n)}(a) > 0$.
3. If the first derivative $f^{(n)}(x)$ with a non-zero value at $x = a$ is of odd order and $n > 1$, then f has a stationary point of inflection at $x = a$.

Proof:

Suppose that the first derivative of f with a non-zero value at $x = a$ is of order n, namely $f^{(n)}(a)$.

By Taylor's theorem there exists $\xi \in (a, a+h)$ such that

$$f(a+h) = f(a) + hf'(a) + \cdots + \frac{h^{n-1}}{(n-1)!} f^{(n-1)}(a) + \frac{h^n}{n!} f^{(n)}(\xi) \ = f(a) + \frac{h^n}{n!} f^{(n)}(\xi)$$

So, $f(a+h) - f(a) = \frac{h^n}{n!} f^{(n)}(\xi)$, and

- if n is even then by continuity of $f^{(n)}$, $\frac{h^n}{n!} f^{(n)}(\xi)$ has the sign of $f^{(n)}(a)$ for small enough h, and therefore so does $f(a+h) - f(a)$. That is:
  - if $f^{(n)}(a) < 0$, then $f(a+h) < f(a)$
  - If $f^{(n)}(a) > 0$, then $f(a+h) > f(a)$
- if n is odd then $f(a+h) - f(a) = \frac{h^n}{n!} f^{(n)}(\xi)$ has dependent on sign of h.

What we learnt today

- Approximating functions using polynomials
  - Intuition via graphical representation
- Deriving a Taylor series from a function:
  - $f(x) = \sum_{n=0}^{\infty} f^{(n)}(x_0)\frac{(x-x_0)^n}{n!}$
- Taylor's theorem:
  - If f is n times differentiable on $[a, x]$ then $\exists \xi \in (a, x)$:

$$f(x) = f(a) + (x-a)f'(a) + \cdots + \frac{(x-a)^{n-1}}{(n-1)!}f^{(n-1)}(a) + \frac{(x-a)^n}{n!}f^{(n)}(\xi)$$

Durham University