MCS Calculus
Practical Exercises 8
(Week 19)

Epiphany Term 2025

If you wish, try typesetting your answers with LaTeX.

1. Let $f(x) = \lambda + \mu x$ for some positive real numbers λ and $\mu > 0$.

   Split the interval $[a, b]$ into n equal strips and determine m_i, the minimum value of $f(x)$ on the ith strip, and M_i, the maximum value of $f(x)$ on the ith strip. Then

   - determine lower and upper bounds for the area under $f(x)$,
   - confirm that they both converge to the same limit as $n \to \infty$,
   - deduce the value of $\int_a^b (\lambda + \mu x)dx$.

Answer: Let $\Delta_n = \frac{b-a}{n}$. Since $\mu > 0$, we have $m_i = \lambda + \mu(a + (i-1)\Delta_n)$ and $M_i = \lambda + \mu(a + i\Delta_n)$. Then

$$\sum_{i=1}^{n} m_i \Delta_n \leq Area \leq \sum_{i=1}^{n} M_i \Delta_n$$

$$\sum_{i=1}^{n} (\lambda + \mu a + \mu(i-1)\Delta_n)\Delta_n \leq Area \leq \sum_{i=1}^{n} (\lambda + \mu a + \mu i \Delta_n)\Delta_n$$

$$n(\lambda + \mu a)\tfrac{(b-a)}{n} + \mu\tfrac{(b-a)^2}{n^2}\sum_{i=1}^{n}(i-1) \leq Area \leq n(\lambda + \mu a)\tfrac{(b-a)}{n} + \mu\tfrac{(b-a)^2}{n^2}\sum_{i=1}^{n} i$$

$$(\lambda + \mu a)(b-a) + \mu\tfrac{(b-a)^2}{n^2}\tfrac{(n-1)n}{2} \leq Area \leq (\lambda + \mu a)(b-a) + \mu\tfrac{(b-a)^2}{n^2}\tfrac{n(n+1)}{2}$$

Both lower and upper bounds tends to $(\lambda + \mu a)(b-a) + \mu\frac{(b-a)^2}{2}$ as $n \to \infty$.

2. Evaluate the integral

$$I = \int_0^3 f(x)dx$$

given that $f(x) = \begin{cases} x & \text{if } 0 \leq x \leq 1 \\ 2x + 2 & \text{if } 1 \leq x \leq 2 \\ x - 1 & \text{if } 2 \leq x \leq 3 \end{cases}$

Answer: Discontinuities at $x1, 2$ so split up the integral:

$$I = \int_0^1 x dx + \int_1^2 2x + 2dx + \int_2^3 x - 1dx = \left[\tfrac{x^2}{2}\right]_0^1 + \left[x^2 + 2x\right]_1^2 + \left[\tfrac{x^2}{2} - x\right]_2^3$$

Thus

$$I = \tfrac{1}{2} + [8 - 3] + \tfrac{3}{2} = 7.$$

3. Consider the first mean value theorem for integrals. In each case below determine the value of the integral and a value ξ satisfying the theorem, or state why one does not exist.

   (a) $\int_{-1}^3 x^2 dx$

Answer: $\int_{-1}^3 x^2 dx = \left[\frac{x^3}{3}\right]_{-1}^3 = 9 - \frac{-1}{3} = 28/3$. So $4\xi^2 = 28/3, \xi = \sqrt{7/3}$.

   (b) $\int_1^5 f(x)dx$ given that $f(x) = \begin{cases} x+1 & \text{if } 1 \le x \le 3 \\ \frac{5-x}{2} & \text{if } 3 \le x \le 5 \end{cases}$

Answer: $I = \int_1^3 x + 1dx + \int_3^5 \frac{5-x}{2}dx = \left[\frac{x^2}{2} + x\right]_1^3 + \left[\frac{-x^2}{4} + \frac{5}{2}x\right]_3^5$
$= 6 + 1 = 7$. So $f(\xi) = 7/4$, and no such ξ exists. The theorem does not apply because f is discontinuous.

   (c) $\int_{-1}^2 \frac{\sqrt{|x|}}{x} dx$

Answer: $I = \lim_{\epsilon \to 0} \int_{-1}^{-\epsilon} \frac{-1}{\sqrt{-x}} dx + \lim_{\epsilon' \to 0} \int_{\epsilon'}^2 \frac{1}{\sqrt{x}} dx$
$= \lim_{\epsilon \to 0} \left[2\sqrt{-x}\right]_{-1}^{-\epsilon} + \lim_{\epsilon' \to 0} \left[2\sqrt{x}\right]_{\epsilon'}^2 = -2 + 2\sqrt{2} = 2(\sqrt{2} - 1)$. So we would need $\frac{\sqrt{|\xi|}}{\xi} = \frac{2}{3}(\sqrt{2} - 1) \approx 0.28$, i.e. $\xi = 13.1$ but this is not in the interval of integration. The theorem does not apply because there is a discontinuity in the range.

4. Find the following antiderivatives.

   (a) $\int \frac{1}{3} \cos(4x) dx$

Answer: $\frac{1}{12} \sin(4x) + C$.

   (b) $\int x^2 + 3\sin(x) + 1dx$

Answer: $\frac{1}{3}x^3 - 3\cos(x) + x + C$.

   (c) $\int 4^x + 2\cos(2x) + \frac{3}{x} dx$

Answer: Recall that $4^x = e^{x \ln 4}$. So $I = \frac{4^x}{\ln 4} + \sin(2x) + 3 \ln x + C$.

5. Using integration by substitution determine the antiderivatives

   (a) $\int x(3x^2 + 1)^5 dx$.

Answer: Sub $u = 3x^2 + 1$, so $u' = 6x$. Then $I = \int \frac{1}{6}(u(x))^5 u'(x)dx = \int \frac{1}{6}u^5 du = \frac{1}{36}u^6 + C = \frac{1}{36}(3x^2 + 1)^6 + C$.

(b) $\int \tan x dx$

Answer: Sub $u = \cos x$, so $u' = -\sin x$. Then
$I = \int -\frac{1}{u} du = -\ln|u| + C = -\ln(\cos x) + C$.

6. What is wrong here:

Given

$$\int_1^2 \frac{2\cos x + 3}{(3x + 2\sin x)^4} dx$$

I decide to make the substitution $u = 3x + 2\sin x, u' = 3 + 2\cos x$.

The integral is therefore $\int_1^2 \frac{2\cos x+3}{(3x+2\sin x)^4} dx = \int_1^2 \frac{1}{(3x+2\sin x)^4}(3 + 2\cos x)dx$ and substituting we get

$$= \int_1^2 \frac{1}{u^4}\frac{du}{dx}dx = \int_1^2 \frac{1}{u^4}du = \left[\frac{-1}{3u^3}\right]_1^2 = \frac{-1}{3 \cdot 2^3} - \frac{-1}{3 \cdot 1^3} = \frac{7}{24}.$$

But surely this can't be right: the integrand is less than $\frac{1}{100}$ over the whole range $[1, 2]$?

Answer: The range $[1, 2]$ is a range for the original dummy variable x. We must either: return to the original variable after finding the antiderivative and before evaluating the definite integral:

$$\left[\frac{-1}{3u^3}\right]_{x=1}^{x=2} = \left[\frac{-1}{3(3x + 2\sin x)^3}\right]_{x=1}^{x=2} = -(6 + 2\sin 2)^{-3} + (3 + 2\sin 1)^{-3}$$

$$= -0.00210 + 0.00974 = 0.00764$$

or determine a corresponding range of the substitute variable u:

$$\left[\frac{-1}{3u^3}\right]_{x=1}^{x=2} = \left[\frac{-1}{3(3x + 2\sin x)^3}\right]_{u=3+2\sin 1}^{u=6+2\sin 2} = -(6 + 2\sin 2)^{-3} + (3 + 2\sin 1)^{-3}$$

7. Evaluate the definite integral $\int_0^1 (2x + 5)\cosh(x^2 + 5x + 1)dx$

Answer: Use the substitution $u(x) = x^2 + 5x + 1$, then

$$I = \int_{x=0}^{x=1} \cosh u \; du = \int_{u=1}^{u=7} \cosh u \; du$$

$= \sinh 7 - \sinh 1$.

8. Use integration by parts to find the antiderivatives

(a) $I = \int e^{ax} \sin x \; dx$

Answer: Take $u = e^{ax}$ and $v' = \sin x$, then using integration by parts

$$I = uv - \int vu'dx = -e^{ax}\cos x - \int -\cos x.ae^{ax\ dx} = -e^{ax}\cos x + a\int e^{ax}\cos x\, dx$$

This looks like we have got nowhere! But wait: we can repeat, again setting $u = e^{ax}$ and $v' = \cos x$. Using the integration by parts formula on the remaining integral we get:

$$I = -e^{ax}\cos x + a\left[e^{ax}\sin x - \int \sin x.ae^{ax}\ dx\right]$$

$$I = -e^{ax}\cos x + ae^{ax}\sin x - a^2 I + C$$

so

$$(1 + a^2)I = e^{ax}(a\sin x - \cos x)$$

and

$$I = \frac{1}{(1 + a^2)}e^{ax}(a\sin x - \cos x)$$

(b) $I = \int \sin x \sinh x\ dx$

Answer: Take $u = \sin x$ and $v' = \sinh x$, then using integration by parts

$$I = uv - \int vu'dx = \sin x\cosh x - \int \cosh x\cos x\ dx$$

Again repeat. Take $u = \cos x$ and $v' = \cosh x$, then using integration by parts

$$I = \sin x\cosh x - \cos x\sinh x + \int \sinh x(-\sin x)\ dx$$

So

$$I = \frac{1}{2}(\sin x\cosh x - \cos x\sinh x).$$

(c) $\int \ln x\ dx$ [Hint: you can take $\ln x = \ln x \times 1$ and then set $v' = 1$.]

Answer: Take $u = \ln x$ and $v' = 1$, then using integration by parts

$$I = uv - \int vu'dx = \ln x.x - \int \tfrac{1}{x}x\ dx$$

$$I = x\ln x - x + C$$

(d) $\int (\ln x)^2\ dx$

Answer: Take $u = \ln^2 x$ and $v' = 1$, then using integration by parts

$$I = uv - \int vu'dx = \ln^2 x.x - \int x2\ln x.\tfrac{1}{x}\ dx$$

$$I = x\ln^2 x - 2\int \ln x\ dx$$

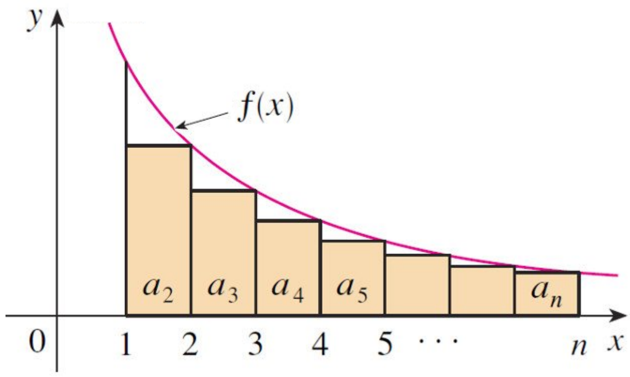
Using the previous part

$$I = x\ln^2 x - 2x\ln x + 2x + C$$

9. Consider a continuous, positive and decreasing function $f(x)$ on the interval $[1, \infty)$ and let $f(n) = a_n$.

   (a) Obtain a lower bound on the area under the curve of f on the interval $[1, n]$ (for some integer n) by splitting the interval into $n - 1$ strips / subintervals of width one and summing the areas of the rectangles in each strip where the height of each rectangle is the value of f at the right endpoint of the subinterval.

Answer: Note that $f(2) = a_2, \ldots, f(n) = a_n$.



Therefore, the lower bound we are looking for, namely a lower bound for $\int_1^n f(x)\, dx$, is $A = 1 \cdot f(2) + \ldots + 1 \cdot f(n) = a_2 + \ldots + a_n = \sum_{i=2}^n a_i$.

   (b) Suppose that $I = \int_1^\infty f(x)\, dx$ is convergent, i.e. has a real positive value. Considering the lower bound you derived above, give an upper bound for the partial sum of the series $\sum_{i=1}^\infty a_i$, $S_n = \sum_{i=1}^n a_i$, in terms of I.

Answer: Since f is positive, it must be that $\int_1^n f(x)\, dx < \int_1^\infty f(x)\, dx = I$ for any real integer n.

We already have from above that $\sum_{i=2}^n a_i$ is upper bounded by $\int_1^n f(x)\, dx$, therefore:

$$\sum_{i=2}^{n} a_i < \int_1^\infty f(x)\, dx = I.$$

So, we get the following upper bound for our partial sum:

$$S_n = \sum_{i=1}^{n} a_i < a_1 + \int_1^\infty f(x)\, dx \qquad (= a_1 + I).$$

   (c) Let U be the upper bound you derived above for the partial sum $\sum_{i=1}^n a_i$. Take the sequence of partial sums of the series, $s_m = \sum_{i=1}^m a_i$. Using the fact that these are now all upper bounded by U, prove that $\sum_{i=1}^\infty a_i$ converges.

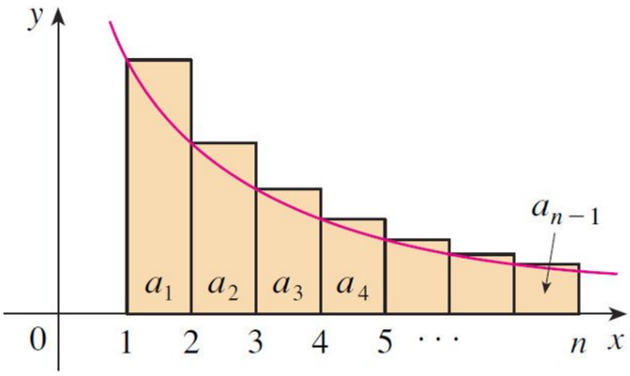
Answer: Because all the terms of our series are positive (since f is positive), we know that $s_m < s_m + a_{m+1} = \sum_{i=1}^m a_i + a_{m+1} = \sum_{i=1}^{m+1} a_i = s_{m+1}$.

So, the sequence $\{s_m\}_{m=1}^\infty$ is an increasing sequence which is upper bounded by U. Therefore, it is convergent and so is our series $\sum_{i=1}^\infty a_i$.

10. Consider a continuous, positive and decreasing function $f(x)$ on the interval $[1, \infty)$ and let $f(n) = a_n$. The previous exercise should have given you a proof of the integral test for convergence. Use a similar approach to prove the integral test for divergence, namely that if $\int_1^\infty f(x)\,dx$ is divergent, so is $\sum_{n=1}^\infty a_n$.

Note that the general integral test statement is for a series that starts at some arbitrary n=k, but here we look at the special case where $n = 1$. A similar proof can be given for arbitrary $n = k$.

Instead of giving a lower bound for $\int_1^n f(x)\,dx$, we will now give an upper bound by using the left endpoints of each subinterval / strip for the height of the rectangles as shown below:



The upper bound we get for $\int_1^n f(x)\,dx$, is $A' = 1 \cdot f(1) + \ldots + 1 \cdot f(n-1) = a_1 + \ldots + a_{n-1} = \sum_{i=1}^{n-1} a_i$.

We can also similarly deduce that:

$$s_{n-1} = \sum_{i=1}^{n-1} a_i = a_1 + \ldots + a_{n-1} > \int_1^{n-1} f(x)\,dx.$$

Since $\int_1^\infty f(x)\,dx$ is divergent and f is positive, it must be that $\int_1^n f(x)\,dx$ tends to infinity as n tends to infinity. Therefore, it must also be that $\int_1^{n-1} f(x)\,dx \to \infty$ as $n \to \infty$.

Since $s_{n-1} > \int_1^{n-1} f(x)\,dx$ it must be that $s_{n-1} \to \infty$ as $n \to \infty$, and therefore also $s_n \to \infty$ as $n \to \infty$. This tells us that the sequence of partial sums $\{s_m\}$ is divergent and therefore so is the series $\sum_{n=1}^\infty a_n$.

11. Use the integral test to determine if the following series converge.

(a) $\frac{1}{2\ln 2} + \frac{1}{3\ln 3} + \cdots + \frac{1}{n \ln n} + \cdots$

Answer: Consider the integral $I = \int_2^\infty \frac{1}{x \ln x} dx$.

Use the substitution $u(x) = \ln(x)$, then $I = \lim_{N\to\infty} \int_{x=2}^{x=N} \frac{1}{u} du = \lim_{N\to\infty} \ln u]_{x=2}^{x=N}$ $= \lim_{N\to\infty} \ln\ln x]_{x=2}^{x=N} = \ln\ln N - \ln\ln 2 \to \infty$. Hence series diverges.

(b) $\frac{1}{2(\ln 2)^2} + \frac{1}{3(\ln 3)^2} + \cdots + \frac{1}{n(\ln n)^2} + \cdots$

Answer: Consider the integral $I = \int_2^{\infty} \frac{1}{x(\ln x)^2} dx$.

Use the substitution $u(x) = \ln(x)$, then $I = \lim_{N\to\infty} \int_{x=2}^{x=N} \frac{1}{u^2} du = \lim_{N\to\infty} -\frac{1}{u}\Big]_{x=2}^{x=N}$
$= \lim_{N\to\infty} \frac{-1}{\ln x}\Big]_{x=2}^{x=N} = \frac{-1}{\ln N} - \frac{-1}{\ln 2} \to \frac{1}{\ln 2}$. Hence series converges.