Lecture 2: Paths, Cycles, Connectivity

Dr. Amitabh Trehan

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**Based on the slides of ADS-21/22 by Dr. George Mertzios*

Reminder from last lecture

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- A graph G is a pair $(V(G), E(G))$, where
  - $V(G)$ is a nonempty set of vertices (or nodes),
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- Paths, cycles, bipartite graphs, complete graphs, hypercubes

Contents for today's lecture

- Paths and directed paths;
- The shortest path problem;
- Connectivity and connected components;
- Eulerian and Hamiltonian cycles;
- Examples and exercises.

Walks, paths, cycles, and distances

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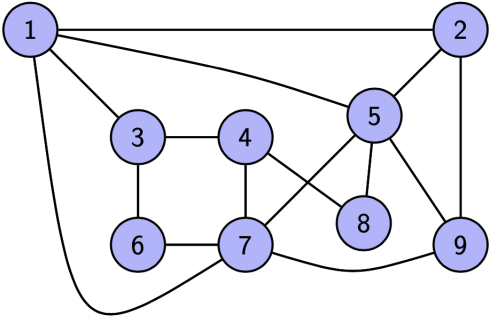
- The diameter of a graph is the largest distance between two vertices in it

Exercise



- Does this graph contain a path of length 7?
- Does it contain a cycle of length 7?
- What is the distance from 2 to 6?
- What is the diameter of this graph?
- Answer at *PollEV.com/amitabhtrehan005*

Exercise: Solutions

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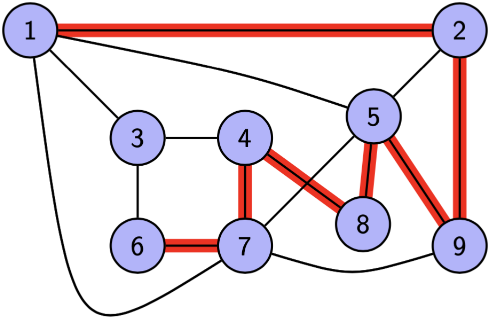

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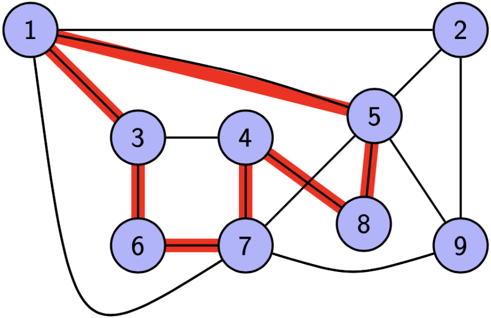

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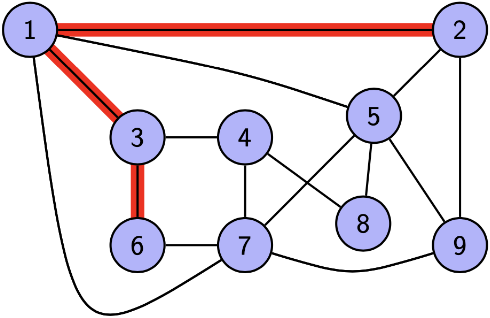

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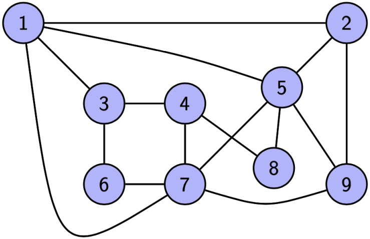

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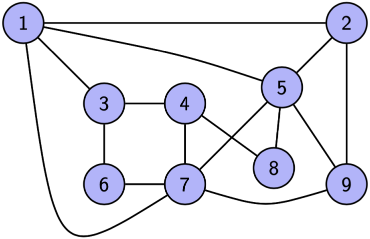

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  Proof: there is a DOMINATING SET {1,5,7} which is a TRIANGLE

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- This is known as the "small world phenomenon".
- There is a popular play (and a film) based on this, called "Six degrees of separation".

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- Btw, my Erdös number is 3. (Can you put this in plain words?)

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- About 90% of actors have a Bacon number (i.e. the distance is not ∞)

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- Btw, my Erdös-Bacon number is ∞, but I have a colleague of a colleague who has a co-author (Hubie Chen) with Erdös-Bacon number 5 (3+2)

Shortest-path problems

In a graph (possibly with edge weights, the problem of computing a path from a given vertex u ("source") to a given vertex v ("target") with the smallest total length (or weight) is known as the shortest-path problem.

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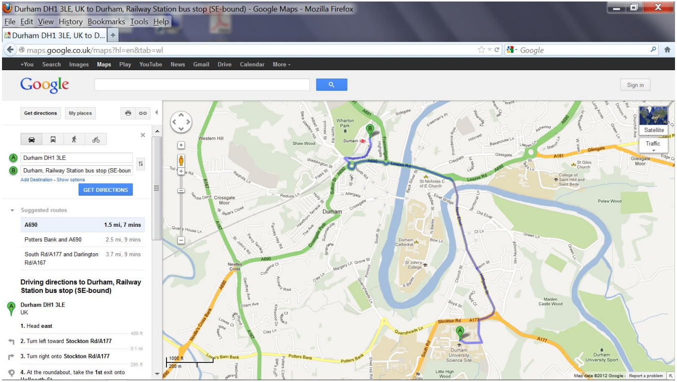

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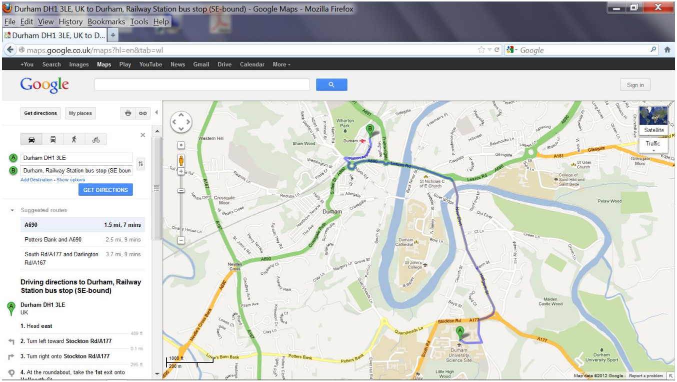

We will learn about algorithms for the (unweighted) problem in a few lectures.

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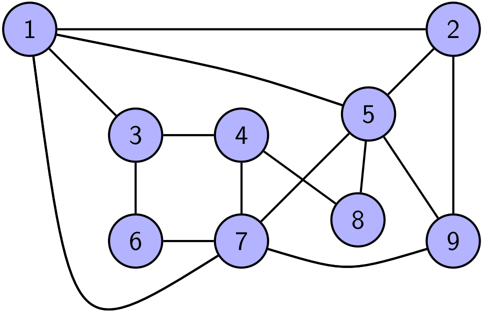
A connected component of G is a maximal connected subgraph of G.



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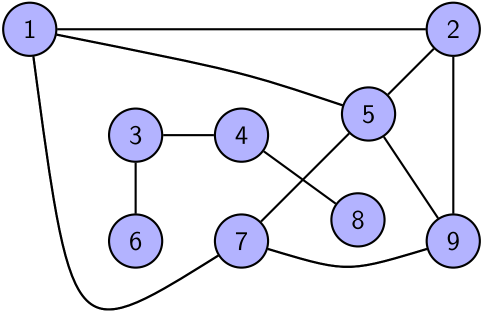

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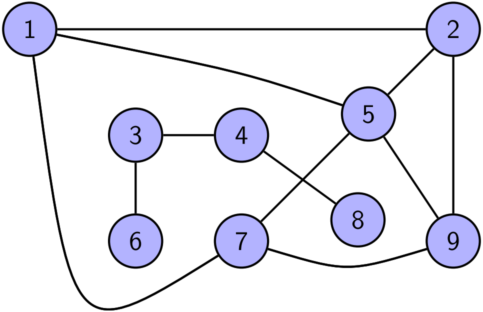

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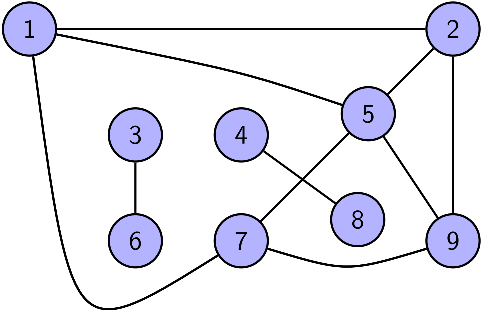

- How many connected components does this graph have?

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Induction step: Let $G = (V, E)$ with $|E| = m + 1$. Consider an arbitrary $e \in E$, and define $E' = E \setminus \{e\}$. By induction hypothesis: $G' = (V, E')$ has at least $|V| - |E'| = |V| - m$ connected components. Two cases:

1. With e, the number of connected components does not change.
2. With e, the number of connected components ???

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Corollary (useful in various algorithmic proofs)

If $G = (V, E)$ is connected then $|E| \geq |V| - 1$.

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- Similarly, $N(v)$ contains none of the neighbours of u nor u itself.
- Hence G contains at least $\frac{n-1}{2} + 1 + \frac{n-1}{2} + 1 = n + 1$ vertices, a contradiction.

Exercise

Exercise 1: Prove that if G is a graph on n vertices and $\delta(G) \geq (n-1)/2$ then G is connected.

- Proof by contradiction: Assume G is not connected, derive a contradiction.
- Take two vertices u and v in different connected components.
- We have that $deg(u) \geq (n-1)/2$ and $deg(v) \geq (n-1)/2$.
- The set $N(u)$ contains none of the neighbours of v nor v itself.
- Similarly, $N(v)$ contains none of the neighbours of u nor u itself.
- Hence G contains at least $\frac{n-1}{2} + 1 + \frac{n-1}{2} + 1 = n+1$ vertices, a contradiction.
- In fact, we even proved more: any two vertices in G are at distance at most 2 (so the diameter of G is at most 2).

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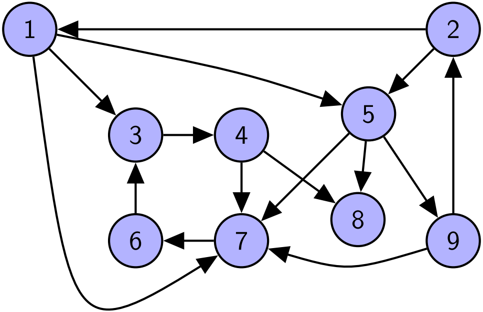
A directed graph G is called (weakly) connected if the graph obtained from G by forgetting directions is connected.
A directed graph is called strongly connected if any two distinct vertices are connected by directed paths in both directions.
A strongly connected component (or simply strong component) of a digraph G is a maximal strongly connected subgraph of G.



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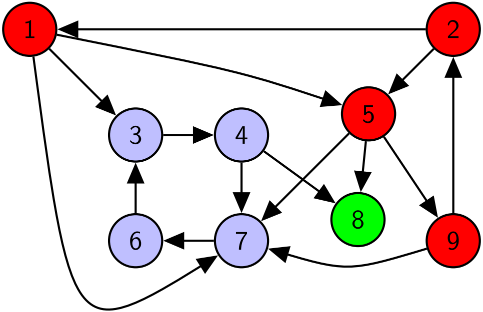

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Connectivity properties of commercially important graphs

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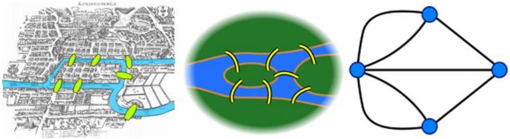

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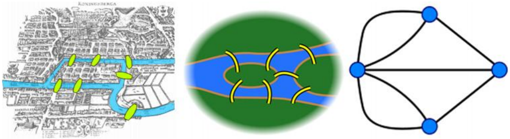

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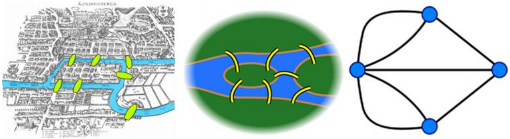

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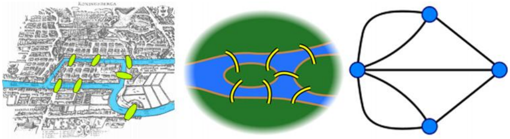

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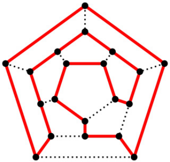

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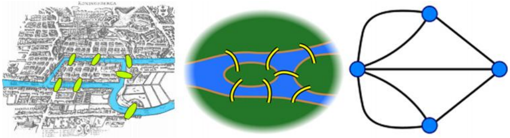

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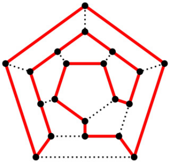

- Detecting one of these two types of circuits is easy, while detecting the other is not easy at all. Which is which?

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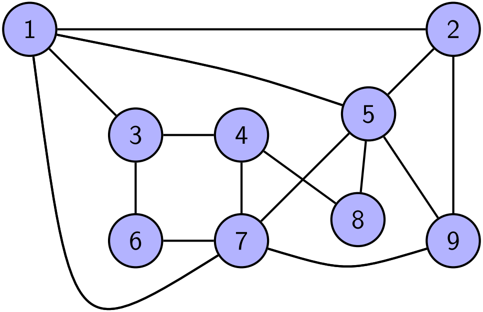
- Hence we have a circuit C. Delete it from G to obtain a smaller graph H in which all degrees are also even.
- By induction hypothesis, each connected component of H has an Eulerian circuit.
- Combine C and these circuits to obtain the required circuit for G.

Hamiltonian cycles



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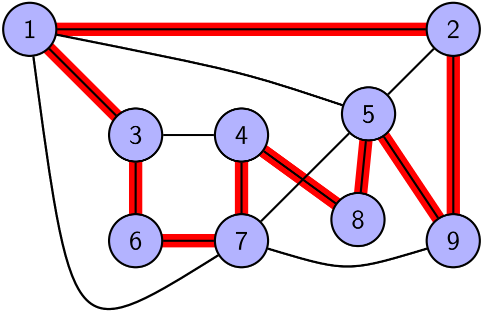

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- Detecting Eulerian circuits algorithmically is easy. (How?)
- Detecting Hamiltonian cycles is hard (NP-complete).

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You have just witnessed a Reduction! (from one problem to another).

Thank You!