Maths for Computer Science
Calculus

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Power Series

Contents for today's lecture

- Reminder of tests for series convergence / divergence
- Intro to power series:
  - Definition
  - Radius of convergence
  - Differentiation (and integration) of power series
- Maclaurin series
- Taylor series

Comparison test

Convergence:

Let $\sum_{n=1}^{\infty} b_n$ be a convergent series of positive terms.

Let $\sum_{n=1}^{\infty} a_n$ be a series and N be an integer such that

$$|a_n| \leq b_n \text{ for all } n > N$$

then $\sum_{n=1}^{\infty} a_n$ is an absolutely convergent series.

Divergence:

Let $\sum_{n=1}^{\infty} b_n$ be a divergent series of positive terms.

Let $\sum_{n=1}^{\infty} a_n$ be a series and N be an integer such that

$$0 \leq b_n \leq a_n \text{ for all } n > N$$

then $\sum_{n=1}^{\infty} a_n$ is a divergent series.

The Ratio test

Let $\sum_{n=1}^{\infty} a_n$ be a series where $a_n \neq 0$ and such that $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L$. Then

a) If $\mathrm{L} < 1$, $\sum_{n=1}^{\infty} a_n$ is an absolutely convergent series.

b) If $\mathrm{L} > 1$, $\sum_{n=1}^{\infty} a_n$ is a divergent series.

c) If $\mathrm{L} = 1$, the test fails.

The n^{th} root test

Consider the series $\sum_{n=1}^{\infty} a_n$ such that $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L$. Then

a) If $\mathrm{L} < 1$, $\sum_{n=1}^{\infty} a_n$ is an absolutely convergent series.

b) If $\mathrm{L} > 1$, $\sum_{n=1}^{\infty} a_n$ is a divergent series.

c) If $\mathrm{L} = 1$, the test fails.

Alternating series test

The series $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$ converges if

- $a_n > 0$, and
- $a_{n+1} \leq a_n$ for all n, and
- $\lim_{n\to\infty} a_n = 0$.

Power series

A **power series** is a series involving a variable x and a constant x_0 of the form:

$$\sum_{n=0}^{\infty} a_n(x - x_0)^n = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + a_3(x - x_0)^3 + \cdots$$

At any specific value of x the power series becomes a normal series and we can use the tests we have discussed to determine convergence.

One of the key uses of series is **power series**. A Power Series looks like a normal series with an added **variable $\boldsymbol{x}$** in it. In the general case, we 'centre' x around a point x_0, i.e. x_0 is a constant value.

Power series

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For example, earlier you saw the power series for e^x given as:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots$$

Radius of convergence

For an arbitrary power series $\sum_{n=0}^{\infty} a_n (x - x_0)^n$, we can use the ratio test:

Ratio test:

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Except now we must remember to include $(x - x_0)^n$ in our ratio.

Radius of convergence

For an arbitrary power series $\sum_{n=0}^{\infty} a_n(x - x_0)^n$, we can use the ratio test:

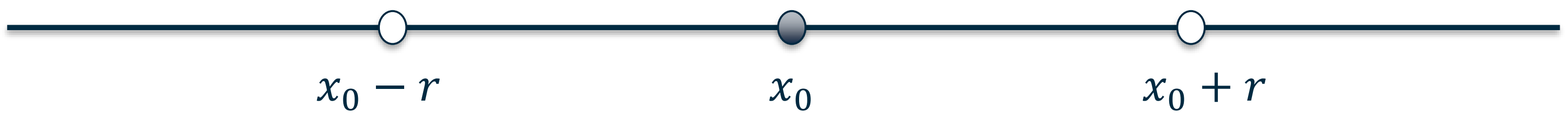
If $\lim\limits_{n\to\infty} \left|\frac{a_{n+1}(x-x_0)^{n+1}}{a_n(x-x_0)^n}\right| = L < 1$ we have convergence; if $L > 1$, divergence.

But for a specific value of x, $|x - x_0|$ is just a number, so:

if $\lim\limits_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < \frac{1}{|x-x_0|}$ we have absolute convergence; if $\lim\limits_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| > \frac{1}{|x-x_0|}$: divergence.

Equivalently, we call $\lim\limits_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| = r$ the **radius of convergence**, and:

- if $|x - x_0| < r$ we have absolute convergence,
- if $|x - x_0| > r$ we have divergence.



Radius of convergence

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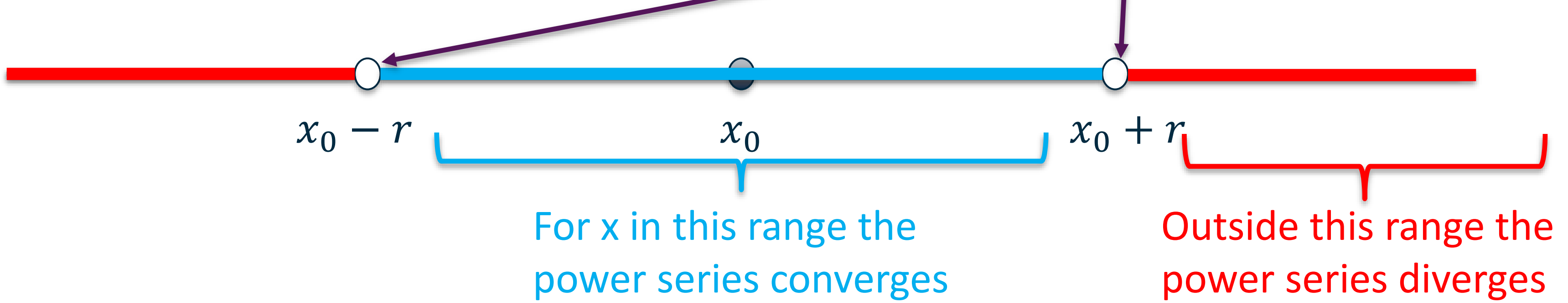
- if $|x-x_0| < r$ we have absolute convergence,
- if $|x-x_0| > r$ we have divergence.

At $x_0 \pm r$ we cannot tell with the ratio test.



The series $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$ converges for all $x \in \mathbb{R}$.

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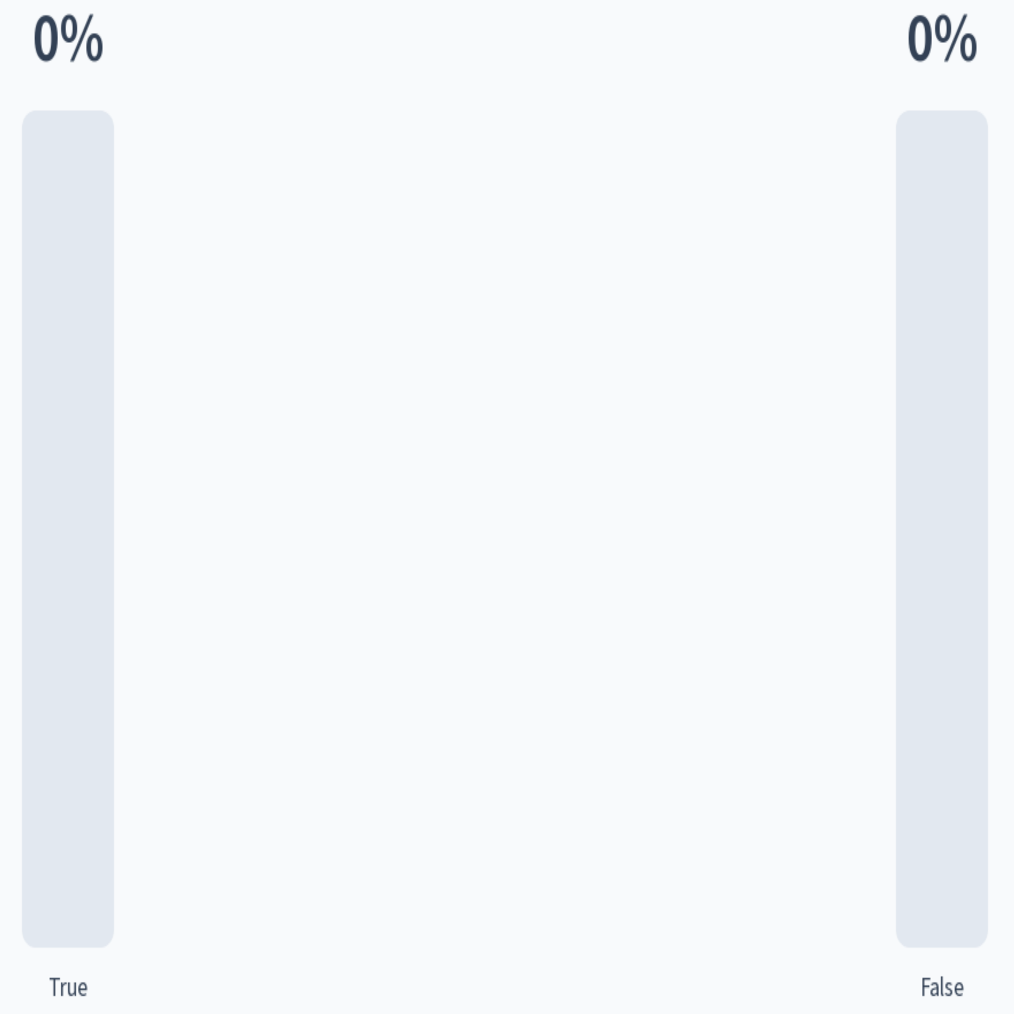

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Example 1: e^x

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots$$

Let us first evaluate r:

$$r = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| =$$

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Example 2: $\ln(x+1)$

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \qquad [= \ln(x+1)]$$

First we evaluate the limit $\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = \frac{1/n}{1/(n+1)} = \frac{n+1}{n} \to 1$ as $n \to \infty$.

So the radius of convergence is $r = 1$:

- For $|x| < 1$, $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^n}{n}$ converges absolutely
- For $|x| > 1$ it diverges.
- *What about $x = 1$?*
  The terms of $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}$ are decreasing in value and tending to zero, so it is convergent by the alternating series test.
- *What about $x = -1$?*
  $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{(-1)^n}{n} = -\sum_{n=1}^{\infty}\frac{1}{n} \to -\infty$ is divergent (harmonic series).

Example 3

$$\sum_{n=0}^{\infty} (nx)^n = 1 + x + (2x)^2 + \cdots$$

Differentiation (and integration) of power series

Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ be a power series with radius of convergence $r > 0$.

Then within the interval $(-r, r)$:

- $f(x)$ is continuous
- $f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$ (term by term differentiation)
- $\int_0^x f(t)dt = \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}$ (term by term integration).

Both series $f'(x)$ and $\int_0^x f(t)dt$ have the same radius of convergence as $f(x)$.

Power series from functions

If we have a **function** and would like to **determine a power series for it**, how – *and when!* – can we do it?

“Why?”, you ask.

Because once we have that, the term-by-term integration, and term-by-term differentiation are very easy, and everything works … like a polynomial!

Power series from functions

Suppose we start with some function $f(x)$ and we wish to determine a power series $\sum_{n=0}^{\infty} a_n x^n$ such that $f(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$.

First observe that by setting $x = 0$, we must have $a_0 = f(0)$.

Now assume that we have been able to create a power series for f and let its radius of convergence be $r > 0$. Differentiate once for $-r < x < r$:

$$f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \cdots,$$

i.e. $f'(0) = a_1$

Differentiating again for $-r < x < r$:

$$f''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = 2a_2 + 3 \cdot 2\, a_3 x + 4 \cdot 3\, a_4 x^2 + 5 \cdot 4\, a_5 x^3 + \cdots,$$

i.e. $f''(0) = 2a_2$.

Proceeding systematically:

$f^{(m)}(x) = \sum_{n=m}^{\infty} n(n-1) \ldots (n-m+1) a_n x^{n-m}$, therefore $f^{(m)}(0) = m!\, a_m$.

Power series from functions: Maclaurin series

Suppose we start with some function $f(x)$ and we wish to determine a power series $\sum_{n=0}^{\infty} a_n x^n$ such that $f(x) = \sum_{n=0}^{\infty} a_n x^n$.

Putting these values in, we see that if such a power series exists then

$$f(x) = f(0) + f'(0)x + f''(0)\frac{x^2}{2!} + f^{(3)}(0)\frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} f^{(n)}(0)\frac{x^n}{n!}$$

This is called the **Maclaurin series for f**.

To use a Maclaurin series for some function f we must have that:

- f can be differentiated an infinite number of times ($f \in \mathrm{C}^{\infty}$);
- the power series converges.

Maclaurin series

- Maclaurin entered the University of Glasgow at 11 y.o. as a child prodigy at the time, and was later elected professor of mathematics at 19 y.o.
- One of his many contributions was related to the characterisation of maxima, minima, and points of inflection for infinitely differentiable functions, for which he made extensive use of what is now known as the Maclaurin series
- The series was known before to Newton and Gregory, and in special cases to Madhava of Sangamagrama in 14th century India
- The Maclaurin series is in fact a special case of the *Taylor series*

Colin Maclaurin (1698-1746)

Power series from functions: Taylor series

Taylor series is the same; all we do is recentre away from 0 on to some arbitrary point x_0.

Let us *recentre* our power series on x_0.

We do that by defining a new function $g(x) = f(x + x_0)$.

Then $g(0) = f(x_0)$; also, $g'(x) = f'(x + x_0)$, so $g'(0) = f'(x_0)$.

Repeatedly differentiating $g^{(n)}(x) = f^{(n)}(x + x_0)$, so $g^{(n)}(0) = f^{(n)}(x_0)$.

So we build the Maclaurin series for g, which gives us:

$$f(x + x_0) = g(x) = \sum_{n=0}^{\infty} g^{(n)}(0) \frac{x^n}{n!} = \sum_{n=0}^{\infty} f^{(n)}(x_0) \frac{x^n}{n!}$$

Or

$$f(x) = g(x - x_0) = \sum_{n=0}^{\infty} g^{(n)}(0) \frac{(x - x_0)^n}{n!} = \sum_{n=0}^{\infty} f^{(n)}(x_0) \frac{(x - x_0)^n}{n!}$$

$$= f(x_0) + f'(x_0)(x - x_0) + f''(x_0) \frac{(x - x_0)^2}{2!} + f^{(3)}(x_0) \frac{(x - x_0)^3}{3!} + \cdots.$$

This form is called the **Taylor series expansion of f**.

Example 5

$$f(x) = ln(x+1)$$

Differentiating:

$$f'(x) = \frac{1}{x+1}, f''(x) = -\frac{1}{(x+1)^2}, f'''(x) = \frac{2}{(x+1)^3}, \dots, f^{(n)}(x) = (-1)^{n+1}\frac{(n-1)!}{(x+1)^n}$$

At $x_0 = 0$:

$$f(0) = 0, \qquad f'(0) = 1, \qquad \dots \qquad , \qquad f^{(n)}(0) = (-1)^{n+1}(n-1)!$$

So the Maclaurin series for f is

$$\ln(x+1) = f(0) + \sum_{n=1}^{\infty} f^{(n)}(0)\frac{x^n}{n!} = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^n}{n},$$

which has radius of convergence $r = 1$ and converges on (-1,1].

Example 6

$$f(x) = x\ln(x)$$

Differentiating:

$$f'(x) = 1 + \ln x\,, f''(x) = \frac{1}{x}, f'''(x) = -\frac{1}{x^2}, \dots, f^{(n)}(x) = (-1)^n \frac{(n-2)!}{x^{n-1}}$$

Note: at 0 these are undefined! So we can't make a Maclaurin series.

But we can make the Taylor series at $x_0 = 1$:

$$f(1) = 0, \qquad f'(1) = 1, \qquad f''(1) = 1, \dots, \qquad f^{(n)}(1) = (-1)^n (n-2)!$$

So the Taylor series for f is:

$$\begin{aligned} x\ln(x) &= (x-1) + \frac{(x-1)^2}{1\cdot 2} - \frac{(x-1)^3}{2\cdot 3} + \frac{(x-1)^4}{3\cdot 4} - \frac{(x-1)^5}{4\cdot 5} + \cdots \\ &= (x-1) + \sum_{n=2}^{\infty} \frac{(-1)^n}{n(n-1)} (x-1)^n, \end{aligned}$$

which has radius of convergence $r = 1$ and converges on [0,2].

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What we learnt today

- A **power series** is a series of the form:

$$\sum_{n=0}^{\infty} a_n (x - x_0)^n$$

- The **radius of convergence** of a power series is $r = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right|$ and the series converges in $(x_0 - r, x_0 + r)$; also, potentially at distance r from x_0
- We may **differentiate** (and integrate) a power series term by term on the interior of the domain of convergence
  - The resulting series have the same radius of convergence as the original
- The **Maclaurin** series for f is $f(x) = \sum_{n=0}^{\infty} f^{(n)}(0)\frac{x^n}{n!}$, provided $f \in C^{\infty}$ and the power series converges.
- Similarly, the **Taylor** series for f is $f(x) = \sum_{n=0}^{\infty} f^{(n)}(x_0)\frac{(x-x_0)^n}{n!}$