COMP1021 Mathematics for Computer Science
Linear Algebra (Part 2)
Practical - Week 12

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Instructions: Work on these problems in the practical sessions for the week specified. First try them on your own. If you're stuck, try discussing things with others. If you get the answer, still discuss with others to see if maybe you missed something. If you run into major roadblocks, ask the demonstrators for hints.

Solutions will be posted on Learn Ultra at the end of the week. Make sure you're all set with the solutions and understand them before the next practical.

Purpose of this practical: This practical is about LU decomposition and all the ideas surrounding it. The goal of this practical is to get you familiar with decomposing a matrix into its LU factorization and subsequently using that form to solve a system of linear equations. This practical also gets you familiar with which matrices can be decomposed into LU form and which cannot. And what to do for matrices that can't be decomposed into LU form.

1. Consider the following matrices A, B, C, D, and F:

$$A = \begin{pmatrix} 2 & 6 & -8 \\ 0 & 5 & 3 \\ 4 & 7 & 9 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 6 & -8 \\ 0 & 5 & 3 \\ 0 & -5 & 25 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 3 & -4 \\ 0 & 5 & 3 \\ 4 & 7 & 9 \end{pmatrix}$$

$$D = \begin{pmatrix} 2 & 6 & -8 \\ 0 & 0 & 28 \\ 0 & -5 & 25 \end{pmatrix}, \quad F = \begin{pmatrix} 4 & 7 & 9 \\ 0 & 5 & 3 \\ 1 & 3 & -4 \end{pmatrix}$$

For each of the following equations, find an elementary matrix E that satisfies the equation:

(a) $EA = B$ (b) $EB = A$ (c) $EA = C$ (d) $EC = A$

(e) $EB = D$ (f) $ED = B$ (g) $EC = F$ (h) $EF = C$

2. Consider a matrix A that can be decomposed into the LU form. If U is the result of elementary matrices E_1, E_2, $\ldots$, E_k being applied to A, then what is L equal to in terms of these elementary matrices and why?

3. Solve the linear system $A\mathbf{x} = \mathbf{b}$ by using the LU method, where

$$A = \begin{pmatrix} 3 & -6 & -3 \\ 2 & 0 & 6 \\ -4 & 7 & 4 \end{pmatrix} \text{ and } \mathbf{b} = \begin{pmatrix} -3 \\ -22 \\ 3 \end{pmatrix}.$$

4. We talked in class about why using LU decomposition is preferable to Gaussian elimination when you must solve multiple systems of linear equations using the same linear mapping. To really drive that home, actually calculate the number of operations it would take to use Gaussian elimination to solve a system of equations using an $n \times n$ linear mapping. Let's assume that that each of the following constitute a single operation: (i) multiplication of two numbers (ii) division of two numbers (iii) addition of two numbers (iv) subtraction of two numbers. **A loose (but not too loose) upper bound on the number of operations is fine.** (Why is this fine? Because what we truly care about is the asymptotic time to utilize this method.) Subsequently, calculate how many operations it would take to compute the LU decomposition of the same $n \times n$ linear mapping, and then calculate how many operations it would take to solve a linear system using the computed LU decomposition.

5. Prove that the matrix $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ has no LU-decomposition

6. Recall the theorem we saw in class: Let A be a square matrix and let U be its (non-reduced) row echelon form, obtained by Gaussian elimination. If A and U are as above and **no row exchanges** were performed while obtaining U from A, then A can be factored $A = LU$, where L is lower triangular.

   Let's build some insight into why this is true.

   (a) Consider the elementary matrix E corresponding to the elementary row operation $R_3 = R_3 - 2R_1$. For this elementary matrix, what is its inverse? Find the inverses for different types of elementary matrices to see if you notice a pattern.

   (b) Consider the product of two lower triangular matrices. Can you say something about whether the product is also lower triangular or not?

(c) Recall that applying Gaussian elimination to a matrix A to get an upper triangular matrix U corresponds to left multiplying A by a sequence of elementary matrix operations, say E_1 to E_k. Using this and what you've seen so far, can you argue about why the theorem is true.

(d) With regards to the theorem, what exactly is the problem when one of elementary matrices corresponds to a row exchange?

7. Implement the LU decomposition algorithm in python.