MCS Calculus
Practical Exercises 8
(Week 19)

Epiphany Term 2025

If you wish, try typesetting your answers with LaTeX.

1. Let $f(x) = \lambda + \mu x$ for some positive real numbers λ and $\mu > 0$.

   Split the interval $[a, b]$ into n equal strips and determine m_i, the minimum value of $f(x)$ on the ith strip, and M_i, the maximum value of $f(x)$ on the ith strip. Then

   - determine lower and upper bounds for the area under $f(x)$,
   - confirm that they both converge to the same limit as $n \to \infty$,
   - deduce the value of $\int_a^b (\lambda + \mu x)dx$.

2. Evaluate the integral
$$I = \int_0^3 f(x)dx$$
   given that $f(x) = \begin{cases} x & \text{if } 0 \le x \le 1 \\ 2x + 2 & \text{if } 1 \le x \le 2 \\ x - 1 & \text{if } 2 \le x \le 3 \end{cases}$

3. Consider the first mean value theorem for integrals. In each case below determine the value of the integral and a value ξ satisfying the theorem, or state why one does not exist.

   (a) $\int_{-1}^3 x^2 dx$

   (b) $\int_1^5 f(x)dx$ given that $f(x) = \begin{cases} x + 1 & \text{if } 1 \le x \le 3 \\ \frac{5-x}{2} & \text{if } 3 \le x \le 5 \end{cases}$

   (c) $\int_{-1}^2 \frac{\sqrt{|x|}}{x} dx$

4. Find the following antiderivatives.

   (a) $\int \frac{1}{3} \cos(4x)dx$

(b) $\int x^2 + 3\sin(x) + 1dx$

(c) $\int 4^x + 2\cos(2x) + \frac{3}{x}dx$

5. Using integration by substitution determine the antiderivatives

   (a) $\int x(3x^2+1)^5 dx$.

   (b) $\int \tan x dx$

6. What is wrong here:

   Given

$$\int_1^2 \frac{2\cos x + 3}{(3x + 2\sin x)^4} dx$$

   I decide to make the substitution $u = 3x + 2\sin x, u' = 3 + 2\cos x$.

   The integral is therefore $\int_1^2 \frac{2\cos x+3}{(3x+2\sin x)^4} dx = \int_1^2 \frac{1}{(3x+2\sin x)^4}(3 + 2\cos x)dx$ and substituting we get

$$= \int_1^2 \frac{1}{u^4} \frac{du}{dx} dx = \int_1^2 \frac{1}{u^4} du = \left[\frac{-1}{3u^3}\right]_1^2 = \frac{-1}{3 \cdot 2^3} - \frac{-1}{3 \cdot 1^3} = \frac{7}{24}.$$

   But surely this can't be right: the integrand is less than $\frac{1}{100}$ over the whole range $[1, 2]$?

7. Evaluate the definite integral $\int_0^1 (2x + 5)\cosh(x^2 + 5x + 1)dx$

8. Use integration by parts to find the antiderivatives

   (a) $I = \int e^{ax} \sin x \; dx$

   (b) $I = \int \sin x \sinh x \; dx$

   (c) $\int \ln x \; dx$ [Hint: you can take $\ln x = \ln x \times 1$ and then set $v' = 1$.]

   (d) $\int (\ln x)^2 \; dx$

9. Consider a continuous, positive and decreasing function $f(x)$ on the interval $[1, \infty)$ and let $f(n) = a_n$.

   (a) Obtain a lower bound on the area under the curve of f on the interval $[1, n]$ (for some integer n) by splitting the interval into $n - 1$ strips / subintervals of width one and summing the areas of the rectangles in each strip where the height of each rectangle is the value of f at the right endpoint of the subinterval.

   (b) Suppose that $I = \int_1^\infty f(x) \; dx$ is convergent, i.e. has a real positive value. Considering the lower bound you derived above, give an upper bound for the partial sum of the series $\sum_{i=1}^\infty a_i$, $S_n = \sum_{i=1}^n a_i$, in terms of I.

   (c) Let U be the upper bound you derived above for the partial sum $\sum_{i=1}^n a_i$. Take the sequence of partial sums of the series, $s_m = \sum_{i=1}^m a_i$. Using the fact that these are now all upper bounded by U, prove that $\sum_{i=1}^\infty a_i$ converges.

10. Consider a continuous, positive and decreasing function $f(x)$ on the interval $[1, \infty)$ and let $f(n) = a_n$. The previous exercise should have given you a proof of the integral test for convergence. Use a similar approach to prove the integral test for divergence, namely that if $\int_1^\infty f(x)\,dx$ is divergent, so is $\sum_{n=1}^\infty a_n$.

11. Use the integral test to determine if the following series converge.

    (a) $\frac{1}{2\ln 2} + \frac{1}{3\ln 3} + \cdots + \frac{1}{n\ln n} + \cdots$

    (b) $\frac{1}{2(\ln 2)^2} + \frac{1}{3(\ln 3)^2} + \cdots + \frac{1}{n(\ln n)^2} + \cdots$