Spivak Calculus Notes and Exercises

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Contents

0 Introduction

These are notes and selected exercises from Spivak Calculus. All the proofs given are my own proofs (unless stated otherwise) which is not assured for correctness and preciseness.

1 Prologue

1.1 Basic Properties of Numbers

Definition (Field Properties)**.** The following properties hold in $\mathbb{R}$

- P1 (Associative law for addition) $a + (b + c) = (a + b) + c.$
- P2 (Existence of an additive identity) $a + 0 = 0 + a = a.$
- P3 (Existence of additive inverse) $a + (-a) = (-a) + a = 0.$
- P4 (Commutative law for addition) $a + b = b + a.$
- P5 (Associative law for multiplication) $a \cdot (b \cdot c) = (a \cdot b) \cdot c.$
- P6 (Existence of multiplicative identity) $a \cdot 1 = 1 \cdot a = a; \quad 1 \neq 0.$
- P7 (Existence of multiplicative inverses) $a \cdot a^{-1} = a^{-1} \cdot a = 1, \text{ for } a \neq 0.$
- P8 (Commutative law for multiplication) $a \cdot b = b \cdot a.$
- P9 (Distributive law) $a \cdot (b + c) = a \cdot b + a \cdot c.$
- P10 (Trichotomy law) For every number a, one and only one of the following holds: (Denote P as the collection of positive numbers)
  - (i) $a = 0$,
  - (ii) a is in the collection P,
  - (iii) $-a$ is in the collection P.
- P11 (Closure under addition) If a and b are in P, then $a + b$ is in P.
- P12 (Closure under multiplication) If a and b are in P, then $a \cdot b$ is in P.

Definition (Absolute Value)**.** For any number a, we define the *absolute value* $|a|$ of a as follows:

$$|a| = \begin{cases} a, & a \geq 0 \\ -a, & a \leq 0 \end{cases}$$

Theorem (Triangle Inequality)**.** For all numbers a and b, we have

$$|a + b| \leq |a| + |b|$$

Proof. We make use of the fact that if both x and y are nonnegative, then $x^2 < y^2$ implies $x < y$.

$$\begin{aligned} |a + b|^2 &= a^2 + 2ab + b^2 \\ &= |a|^2 + 2ab + |b|^2 \\ &\leq |a|^2 + 2|a||b| + |b|^2 \\ &= (|a| + |b|)^2 \end{aligned}$$

Since $|a + b|$ and $(|a| + |b|)$ are both nonnegative, then

$$|a + b| \leq |a| + |b|.$$

$\square$

1.1.1 Exercises

Exercise (1). Prove the following:

(i) If $ax = a$ for some number $a \neq 0$, then $x = 1$.

Proof. Assume that $ax = a$ fro some number $a \neq 0$.

$$\begin{aligned} x = x \cdot 1 = x \cdot (a \cdot a^{-1}) &= ax \cdot (a^{-1}) \\ &= a \cdot (a^{-1}) \\ &= (a \cdot a^{-1}) \\ &= 1 \end{aligned}$$

□

(ii) $x^2 - y^2 = (x-y)(x+y)$.

Proof. Using the field axioms.

$$\begin{aligned} (x-y)(x+y) &= x \cdot (x+y) + (-y) \cdot (x+y) \\ &= (x^2 + xy) + ((-y) \cdot x + (-y) \cdot y) \\ &= x^2 + xy - xy - y^2 \\ &= x^2 - y^2 \end{aligned}$$

□

(iii) If $x^2 = y^2$, then $x = y$ or $x = -y$.

Proof. Assume that $x^2 = y^2$. We make use of (ii).

$$\begin{aligned} x^2 = y^2 &\Leftrightarrow x^2 - y^2 = 0 \\ &\Leftrightarrow (x-y)(x+y) = 0 \\ &\Rightarrow x = y \text{ or } x = -y. \end{aligned}$$

□

(iv) $x^3 - y^3 = (x-y)(x^2 + xy + y^2)$.

Proof. Using the field axioms

$$\begin{aligned} (x-y)(x^2+xy+y^2) &= x^2(x-y) + xy(x-y) + y^2(x-y) \\ &= (x^3 - x^2y) + (x^2y - xy^2) + (xy^2 - y^3) \\ &= x^3 + (x^2y - x^2y) + (xy^2 - xy^2) - y^3 \\ &= x^3 - y^3 \end{aligned}$$

□

(v) $x^n - y^n = (x-y)(x^{n-1} + x^{n-2}y + \cdots + xy^{n-2} + y^{n-1})$.

Proof. Using the field axioms

$$\begin{aligned}(x-y)(x^{n-1}+x^{n-2}y+\cdots+xy^{n-2}+y^{n-1}) &= x(x^{n-1}+x^{n-2}y+\cdots+xy^{n-2}+y^{n-1})\\ &\quad-[y(x^{n-1}+x^{n-2}y+\cdots+xy^{n-2}+y^{n-1})]\\ &= x^n+x^{n-1}y+\cdots+x^2y^{n-2}+xy^{n-1}\\ &\quad-[x^{n-1}y+x^{n-2}y^2+\cdots+xy^{n-1}+y^n]\\ &= x^n-y^n\end{aligned}$$

□

Alternative Proof. We make us of sigma notation

$$\begin{aligned}(x-y)\cdot\sum_{i=0}^{n-1}x^iy^{n-(i+1)} &= x\left(\sum_{i=0}^{n-1}x^iy^{n-(i+1)}\right)-\left[y\left(\sum_{i=0}^{n-1}x^iy^{n-(i+1)}\right)\right]\\ &= \sum_{i=0}^{n-1}x^{i+1}y^{n-(i+1)}-\left[\sum_{i=0}^{n-1}x^iy^{n-i}\right]\\ &= x^n+\sum_{i=0}^{n-2}x^{i+1}y^{n-(i+1)}-\left[\sum_{i=1}^{n-1}x^iy^{n-i}+y^n\right]\\ &= x^n+\sum_{i=0}^{n-2}x^{i+1}y^{n-(i+1)}-\left[\sum_{i=0}^{n-2}x^{i+1}y^{n-(i+1)}+y^n\right]\\ &= x^n-y^n+\sum_{i=0}^{n-2}\left[x^{i+1}y^{n-(i+1)}-(x^{i+1}y^{n-(i+1)})\right]\\ &= x^n-y^n+\sum_{i=0}^{n-2}0\\ &= x^n-y^n\end{aligned}$$

□

(vi) $x^3+y^3=(x+y)(x^2-xy+y^2)$.

Proof. Replace y by $-y$ in part (iv)

$$\begin{aligned}x^3-y^3=(x-y)(x^2+xy+y^2) &\Leftrightarrow x^3-(-y)^3=(x-(-y))(x^2+x(-y)+(-y)^2)\\ &\Leftrightarrow x^3+y^3=(x+y)(x^2-xy+y^2)\end{aligned}$$

□

Exercise (2). What is wrong with the following "proof"? Let $x=y$. Then

$$\begin{aligned}x^2&=xy,\\ x^2-y^2&=xy-y^2,\\ (x+y)(x-y)&=y(x-y),\\ x+y&=y,\\ 2y&=y,\\ 2&=1.\end{aligned}$$

Solution. For all $a \in \mathbb{R}$ we know that $a \cdot a^{-1} = 0$ with the assumption $a \neq 0$. The $4th$ step is contradictory on the given fact that $x = y$ which implies $x - y = 0$ and has no multiplicative inverse. $\square$

Exercise (3). Prove the following:

(i) $\dfrac{a}{b} = \dfrac{ac}{bc}$, if $b, c \neq 0$.

Proof. Using the field axioms

$$\begin{aligned} \frac{a}{b} = ab^{-1} &= (ab^{-1})(c \cdot c^{-1}) \\ &= (ac)(b^{-1}c^{-1}) \\ &= (ac)(bc)^{-1} \\ &= \frac{ac}{bc} \end{aligned}$$

$\square$

(ii) $\dfrac{a}{b} + \dfrac{c}{d} = \dfrac{ad + bc}{bd}$, if $b, d \neq 0$.

Proof. Using the field axioms

$$\begin{aligned} \frac{a}{b} + \frac{c}{d} = ab^{-1} + cd^{-1} &= (ab^{-1} + cd^{-1}) \cdot (bd)(bd)^{-1} \\ &= (ad(b \cdot b^{-1}) + bc(d \cdot d^{-1})) \cdot (bd)^{-1} \\ &= (ad + bc) \cdot (bd)^{-1} \\ &= \frac{ad + bc}{bd} \end{aligned}$$

$\square$

(iii) $(ab)^{-1} = a^{-1}b^{-1}$, if $a, b \neq 0$.

Proof. Using the field axioms

$$\begin{aligned} ab(a^{-1}b^{-1}) &= 1 \\ a^{-1}b^{-1} &= (ab)^{-1} \end{aligned}$$

$\square$

(iv) $\dfrac{a}{b} \cdot \dfrac{c}{d} = \dfrac{ac}{db}$ if $b, d \neq 0$.

Proof. Using the field axioms

$$\begin{aligned} \frac{a}{b} \cdot \frac{c}{d} &= (ab^{-1}) \cdot (cd^{-1}) \\ &= (ac) \cdot (d^{-1}b^{-1}) \\ &= (ac) \cdot (db)^{-1} \\ &= \frac{ac}{db} \end{aligned}$$

$\square$

(v) $\dfrac{a}{b} \Big/ \dfrac{c}{d} = \dfrac{ad}{bc}$, if $b, d \neq 0$.

Proof. Using the field axioms

$$\begin{aligned} \frac{a}{b} \Big/ \frac{c}{d} &= \frac{a}{b} \cdot \left(\frac{c}{d}\right)^{-1} \\ &= ab^{-1} \cdot (cd^{-1})^{-1} \\ &= ab^{-1} \cdot c^{-1}(d^{-1})^{-1} \\ &= ab^{-1} \cdot c^{-1}d \\ &= (ad) \cdot (b^{-1}c^{-1}) \\ &= (ad) \cdot (bc)^{-1} \\ &= \frac{ad}{bc} \end{aligned}$$

$\square$

(vi) If $b, d \neq 0$, then $\dfrac{a}{b} = \dfrac{c}{d}$ if and only if $ad = bc$. Also determine when $\dfrac{a}{b} = \dfrac{b}{a}$.

Proof. There are two cases to prove for the first part.

($\Rightarrow$) Let $b, d \neq 0$. Assume that $\dfrac{a}{b} = \dfrac{c}{d}$,

$$\begin{aligned}
\frac{a}{b} &= \frac{c}{d}, \\
ab^{-1} &= cd^{-1}, \\
(ab^{-1})(bd) &= (cd^{-1})(bd), \\
(ad)(b \cdot b^{-1}) &= (bc)(d \cot d^{-1}), \\
ad &= bc.
\end{aligned}$$

($\Leftarrow$) Let $b, d \neq 0$. Assume that $ad = bc$,

$$\begin{aligned}
ad &= bc, \\
(ad)(bd)^{-1} &= (bc)(bd)^{-1} \\
(ab^{-1})(d \cdot d^{-1}) &= (cd^{-1})(b \cdot b^{-1}) \\
ab^{-1} &= cd^{-1} \\
\frac{a}{b} &= \frac{c}{d}
\end{aligned}$$

$\square$

Proof. From Exercise 1 Part (*iii*) we make use of the fact, if $x^2 = y^2$ then $x = y$ or $x = -y$.

$$\begin{aligned}
\frac{a}{b} &= \frac{b}{a}, \\
ab^{-1} &= ba^{-1}, \\
(ab^{-1})(ab) &= (ba^{-1})(ab), \\
(a \cdot a)(b \cdot b^{-1}) &= (b \cdot b)(a \cdot a^{-1}), \\
a^2 &= b^2.
\end{aligned}$$

and so it must be that $a = b$ or $a = -b$. $\square$

Exercise (4). Find all numbers x for which

(i) $4 - x < 3 - 2x$.

Proof. Using the field axioms

$$\begin{aligned} 4 - x &< 3 - 2x \\ 4 - x + (2x - 4) &< 3 - 2x + (2x - 4) \\ x &< -1 \end{aligned}$$

□

(ii) $5 - x^2 < 8$.

Proof. Using the field axioms

$$\begin{aligned} 5 - x^2 + (x^2 - 5) &< 8 + (x^2 - 5) \\ x^2 + 3 &> 0 \end{aligned}$$

since $x^2 \geq 0$ for all $x \in \mathbb{R}$, then it must be that $x^2 + 3 > 0$ for all $x \in \mathbb{R}$. □

(iii) $5 - x^2 < -2$

Proof. Using the field axioms

$$\begin{aligned} 5 - x^2 &< -2 \\ x^2 &> 7 \\ |x| &> \sqrt{7} \\ x < -\sqrt{7} &\text{ or } x > \sqrt{7} \end{aligned}$$

□

(iv) $(x - 3)(x - 1) > 0$ (When is a product of two numbers positive?)

Proof. The product of two numbers is postivie if and only if the numbers are both positive or both negative. For all $a, b \in \mathbb{R}$, $ab > 0 \Leftrightarrow a > 0$ and $b > 0$, or $a < 0$ and $b < 0$.

Hence,

$$x - 3 > 0 \qquad \text{and} \qquad x - 1 > 0$$

so it must be that $x > 3$. Or

$$x - 3 < 0 \qquad \text{and} \qquad x - 1 > 0$$

and it must be that $x < 1$. That is $(x - 3)(x - 1) > 0$ if $x > 3$ or $x < 1$. □

(v) $x^2 - 2x + 2 > 0$.

Proof. Using the field axioms

$$\begin{aligned} x^2 - 2x + 2 &= (x^2 + 2x + 1) + 1 \\ &= (x-1)^2 + 1 \end{aligned}$$

for all $x \in \mathbb{R}$ notice that, $(x-1)^2 \geq 0$, so it must be that $(x-1)^2 + 1 > 0$. $\square$

(vi) $x^2 + x + 1 > 2$.

Proof. Using the field axioms

$$\begin{aligned} x^2 + x + 1 &> 2 \\ x^2 + x - 1 &> 0 \\ (x^2 + x + \frac{1}{4}) - \frac{5}{4} &> 0 \\ \left(x + \frac{1}{2}\right)^2 &> \frac{5}{4} \\ \left|x + \frac{1}{2}\right| &> \frac{\sqrt{5}}{2} \\ x + \frac{1}{2} > \frac{\sqrt{5}}{2} \text{ or } x + \frac{1}{2} &< -\frac{\sqrt{5}}{2} \end{aligned}$$

so it must be that

$$x > \frac{\sqrt{5}-1}{2} \qquad \text{or} \qquad x < \frac{-\sqrt{5}-1}{2}$$

$\square$

(vii) $x^2 - x + 10 > 16$.

Proof. Using the field axioms

$$\begin{aligned} x^2 - x + 10 &> 16 \\ x^2 - x - 6 &> 0 \\ (x-3)(x+2) &> 0 \end{aligned}$$

To assure that the product is positive, it must be that the two numbers are both positive or both negative. Hence,

$$x - 3 > 0 \qquad \text{and} \qquad x + 2 > 0$$

such that $x > 3$. Or

$$x - 3 < 0 \qquad \text{and} \qquad x + 2 < 0$$

such that $x < -2$. Therefore, $x^2 - x + 10 > 16$ if $x > 3$ or $x < -2$. $\square$

(viii) $x^2 + x + 1 > 0$.

Proof. Using the field axioms

$$\begin{aligned} x^2 + x + 1 &= (x^2 + x + \frac{1}{4}) + \frac{3}{4}, \\ &= \left(x + \frac{1}{2}\right)^2 + \frac{3}{4}. \end{aligned}$$

for all $x \in \mathbb{R}$, notice that $(x + \frac{1}{2})^2 \geq 0$, so it must be that $(x + \frac{1}{2})^2 + \frac{3}{4} > 0$ for all $x \in \mathbb{R}$. □

(ix) $(x - \pi)(x + 5)(x - 3) > 0$.

Proof. The expression $(x - \pi)(x + 5)(x - 3)$ can be rearranged as a product of two numbers, namely, $(x - \pi)\left[(x + 5)(x - 3)\right]$.

Notice, the product of two real numbers ab is greater than zero if a and b are both greater than zero, or both less than zero.

There are two cases:

- Let $(x - \pi) > 0$ so that $x > \pi$, and $(x + 5)(x - 3) > 0$ so that $x < -5$ or $x > 3$. Therefore it must be that $x > \pi$.
- Let $(x - \pi) < 0$ so that $x < \pi$, and $(x + 5)(x - 3) < 0$ so that $-5 < x < 3$. Therefore it must be that $-5 < x < 3$.

Therefore, $(x - \pi)(x + 5)(x - 3) > 0$ if $x > \pi$, or $-5 < x < 3$. □

(x) $(x - \sqrt[3]{2})(x - \sqrt{2}) > 0$.

Proof. Either both numbers are greater than zero or less than zero.

$$x > \sqrt[3]{2} \qquad \text{and} \qquad x > \sqrt{2}$$

so that $x > \sqrt{2}$. Or

$$x < \sqrt[3]{2} \qquad \text{and} \qquad x < \sqrt{2}$$

so that $x < \sqrt[3]{2}$.

Therefore, $(x - \sqrt[3]{2})(x - \sqrt{2}) > 0$ if $x > \sqrt{2}$ or $x < \sqrt[3]{2}$. □

(xi) $2^x < 8$.

Proof. We can rewrite it as

$$2^x < 2^3$$

Both have the same base, so it must be that the inequality is preserved on the exponents.

$$x < 3$$

so $2^x < 8$, whenever $x < 3$. □

(xii) $x + 3^x < 4$.

Proof. We first notice that $x + 3^x = 4$ if $x = 1$

$$\begin{aligned} x + 3^x &= (1) + 3^1 \\ &= 4 \end{aligned}$$

observe that $x + 3^x$ is always increasing as x increase, and decreasing as x decrease. Therefore $x + 3^x < 4$ if $x < 1$. □

(xiii) $\dfrac{1}{x} + \dfrac{1}{1-x} > 0.$

Proof. We can rewrite the expression as

$$\begin{aligned} \frac{1}{x} + \frac{1}{1-x} &= \frac{(1-x)+x}{x(1-x)} \\ &= \frac{1}{x(1-x)} \end{aligned}$$

Notice that $\frac{1}{x(1-x)} > 0$, whenever $x(1-x) > 0$. So it must be that x and $(1-x)$ are greater than zero

$$x > 0 \qquad \text{and} \qquad x < 1$$

or x and $(1-x)$ are both less than zero

$$x < 0 \qquad \text{and} \qquad x > 1$$

but there exists no x sucht that $x < 0$ and $x > 1$. Therefore, $\frac{1}{x} + \frac{1}{1-x} > 0$ if $x > 0$ and $x < 1$. □

(xiv) $\dfrac{x-1}{x+1} > 0.$

Proof. Either both $(x-1)$ and $(x+1)$ are greater than zero or both less than zero.

$$x > 1 \qquad \text{and} \qquad x > -1$$

so it must be that $x > 1$. Or

$$x < 1 \qquad \text{and} \qquad x < -1$$

so it must be that $x < -1$. □

Exercise (5). Prove the following:

(i) If $a < b$ and $c < d$, then $a + c < b + d$.

Proof. Assume that $a < b$ and $c < d$. Notice that $b - a > 0$ and $d - c > 0$, therefore their sum is also positive, namely,

$$(b - a) + (d - c) > 0$$

so that

$$a + c < b + d$$

□

(ii) If $a < b$, then $-b < -a$.

Proof. Assume that $a < b$. Therefore, $a - b < 0$. Notice that,

$$\begin{aligned} -(a - b) &< 0, \\ b - a &< 0, \\ -a &< -b. \end{aligned}$$

□

(iii) If $a < b$ and $c > d$, then $a - c < b - d$.

Proof. Assume that $a < b$ and $c > d$. Therefore, $a - b < 0$ and $c - d >$ so that $a - b < 0 < c - d$. Therefore,

$$\begin{aligned} a - b &< c - d, \\ a - c &< b - d. \end{aligned}$$

□

(iv) If $a < b$ and $c > 0$, then $ac < bc$.

Proof. Assume that $a < b$ and $c > 0$. Therefore, $b - a > 0$ and their product is positive,

$$\begin{aligned} (b - a) \cdot c &> 0, \\ bc - ac &> 0, \\ ac &< bc. \end{aligned}$$

□

(v) If $a < b$ and $c < 0$, then $ac > bc$.

Proof. Assume that $a < b$ and $c < 0$. Then $b - a > 0$ and $0 - c = -c > 0$, and their product must be positive,

$$\begin{aligned} (b - a) \cdot -c &> 0, \\ -bc + ac &> 0, \\ ac &> bc. \end{aligned}$$

□

(vi) If $a > 1$, then $a^2 > a$.

Proof. Assume that $a > 1$. Notice that $a - 1 > 0$ and $a > 0$, so their product is positive,

$$\begin{aligned} (a-1)\cdot a &> 0, \\ a^2 - a &> 0, \\ a^2 &> a. \end{aligned}$$

$\square$

(vii) If $0 < a < 1$, then $a^2 < a$.

Proof. Assume that $0 < a < 1$, therefore $a > 0$ and $1 - a > 0$ so that their product is also positive,

$$\begin{aligned} (1-a)\cdot a &> 0, \\ a - a^2 &> 0, \\ a^2 &< a. \end{aligned}$$

$\square$

(viii) If $0 \le a < b$ and $0 \le c < d$, then $ac < bd$.

Proof. Assume that $0 \le a < b$ and $0 \le c < d$. Notice that $bd > 0$, and if either one of a or c is equal to zero so that ac is zero, then

$$0 = ac < bd.$$

Otherwise, if a and c are greater than zero then ac is also greater than zero,

$$0 < ac < bc < bd.$$

$\square$

(ix) If $0 \le a < b$, then $a^2 < b^2$.

Proof. Assume that $0 \le a < b$. If $a = 0$, then $a^2 = 0$ so that

$$a^2 < b^2.$$

Suppose that $a > 0$. From our assumption, $a < b$ therefore

$$a^2 < ab < b^2.$$

$\square$

(x) If $a, b \ge 0$ and $a^2 < b^2$, then $a < b$.

Proof. Suppose for contradiction that $a, b \ge 0$ and $a^2 < b^2$, but $a \ge b$. So either $a = b$ or $a > b$.
If $a = b$, then $a^2 = b^2$, a contradiction. Now if $a > b \ge 0$, then

$$a^2 > ab > b^2$$

also a contradiction. Therefore it must be that $a < b$. $\square$

Exercise (6). Prove the following:

(i) Prove that if $0 \leq x < y$, then $x^n < y^n$, $n = 1, 2, 3, \ldots$

Proof. Assume that $0 \leq x < y$. From the previous problems, it must be that $x^2 < y^2$, and so on for all $n \in \mathbb{N}$. □

(ii) Prove that if $x < y$ and n is odd, then $x^n < y^n$.

Proof. Assume tha $x < y$ and n is odd. There are three cases:

- If $0 \leq x$, then by the previous exercise, it must be that

$$x^n < y^n.$$

- If $x < y \leq 0$, then it must be that $0 \leq -y < -x$. Therefore,

$$\begin{aligned} (-y)^n &< (-x)^n, \\ -y^n &< -x^n, \\ x^n &< y^n. \end{aligned}$$

- If $x < 0 \leq y$, then since n is odd, it must be that

$$x^n < 0 \leq y^n$$

so that $x^n < y^n$.

□

(iii) Prove that if $x^n = y^n$ and n is odd, then $x = y$.(

Proof. If it is not, then it must be that $x^n < y^n$ or $x^n > y^n$. □

(iv) Prove that if $x^n = y^n$ and n is even, then $x = y$ or $x = -y$.

Proof. We have three cases and we make use of (i):

- Let $x, y \geq 0$. If $x^n = y^n$, then $x = y$. Since from part (i), without loss of generality, $x \neq y$ implies that $x^n < y^n$ for all $n \in \mathbb{N}$.
- Let $x, y \leq 0$ so that $-x, -y \geq 0$. If $(-x)^n = (-y)^n$, then $-x = -y$ so that $x = y$ for all $n \in \mathbb{N}$.
- Let $x \leq 0, y \geq 0$ so that $-x, y \geq 0$. If $(-x)^n = y^n$, then $-x = y$ which is the same as $x = -y$, for all $n \in \mathbb{N}$.

We have now exhausted all the cases of x and y. Therefore, if $x^n = y^n$ where n is even, then $x = y$ or $x = -y$. □

Exercise (7). Prove that if $0 < a < b$, then

$$a < \sqrt{ab} < \frac{a+b}{2} < b.$$

Notice that the inequality $\sqrt{ab} \leq (a+b)/2$ holds for all $a, b \geq 0$. A generalization of this fact occurs in Exercise $2-22$.

Proof. Since $0 < a < b$, then we know that

$$a^2 = a \cdot a < ab < b \cdot b = b^2.$$

therefore $a < \sqrt{ab} < b$.

Notice also that

$$a = \frac{a+a}{2} < \frac{a+b}{2} < \frac{b+b}{2} = b.$$

so that $a < (a+b)/2$ and $b > (a+b)/2$.

Lastly, we must show that $\sqrt{ab} < (a+b)/2$. But since $a, b > 0$, we know that

$$\begin{aligned}
(a-b)^2 &> 0, \\
a^2 - 2ab + b^2 &> 0, \\
a^2 + 2ab + b^2 &> 4ab, \\
(a+b)^2 &> 4ab, \\
\frac{a+b}{2} &> \sqrt{ab}
\end{aligned}$$

for all a, b such that $0 < a < b$. □

Exercise (12). Prove the following:

(i) $|xy| = |x| \cdot |y|$.

Proof. We will prove this fact the same way we proved the *Triangle Inequality*, by using that fact that if $x^2 = y^2$ and x, y are nonnegative, then $x = y$. Notice

$$|xy|^2 = (xy)^2 = x^2y^2 = |x|^2 \cdot |y|^2 = (|x| \cdot |y|)^2.$$

Since $|xy|$ and $|x| \cdot |y|$ are always nonnegative, we find that

$$|xy| = |x| \cdot |y|.$$

□

(ii) $\left|\frac{1}{x}\right| = \frac{1}{|x|}$, if $x \neq 0$.

Proof. We make use of the previous exercise.

$$\left|\frac{1}{x}\right| \cdot |x| = \left|\frac{1}{x} \cdot x\right| = |1| = 1$$

therefore

$$\left|\frac{1}{x}\right| = \frac{1}{|x|}.$$

□

(iii) $\dfrac{|x|}{|y|} = \left|\dfrac{x}{y}\right|$, if $y \neq 0$.

Proof. It follows by using the result of the previous exercise,

$$|x^{-1}| = |x|^{-1}.$$

$\square$

(iv) $|x - y| \leq |x| + |y|$.

Proof. We make use of the *Triangle Inequality* and the fact that for all $x \in \mathbb{R}$, $|x| = |-x|$. Let $y_0 = -y$,

$$\begin{aligned} |x + y_0| &\leq |x| + |y_0|, \\ |x + (-y)| &\leq |x| + |-y|, \\ |x - y| &\leq |x| + |y|. \end{aligned}$$

$\square$

(v) $|x| - |y| \leq |x - y|$.

Proof. From the *Triangle Inequality*,

$$\begin{aligned} |x| &= |(x - y) + y| \\ &\leq |x - y| + |y| \end{aligned}$$

so that

$$|x| - |y| \leq |x - y|.$$

$\square$

(vi) $|(|x| - |y|)| \leq |x - y|$.

Proof. To show that $|(|x| - |y|)| \leq |x - y|$, we must show that $-|x - y| \leq |x| - |y| \leq |x - y|$.

- The second inequality follows from the previous exercise.
- The first inequality is the same as $|y| - |x| \leq |x - y|$, but this also follows from the previous exercises since $|y| - |x| \leq |y - x| = |x - y|$.

$\square$

(vii) $|x + y + z| \leq |x| + |y| + |z|$.

Proof. We just apply the *Triangle Inequality* multiple times.

$$\begin{aligned} |x + y + z| &\leq |x| + |y + z| \\ &\leq |x| + |y| + |z|. \end{aligned}$$

$\square$

Exercise (14). Prove the following

(i) Prove that $|a| = |-a|$.

Proof. If $a \geq 0$, then $-a \leq 0$ and its absolute value is

$$|-a| = -(-a) = a = |a|$$

Now, if $a \leq 0$, then $-a \geq 0$ so it follows from the previous one that $|a| = |-a|$. $\square$

(ii) Prove that $-b \leq a \leq b$ if and only if $|a| \leq b$. In particular, it follows that $-|a| \leq a \leq |a|$.

Proof. This is a biconditional, so we prove it in both directions.
Assume that $-b \leq a \leq b$. If $a \geq 0$, then

$$|a| = a \leq b.$$

If $a \leq 0$ and also notice that $-a \leq b$, then

$$|a| = -a \leq b.$$

For the converse, we assume that $|a| \leq b$ and it follows that $b \geq 0$. If $a \geq 0$, then

$$a = |a| \leq b.$$

and

$$-b \leq 0 \leq a.$$

Now, if $a \leq 0$, then $-a = |a| \leq b$ so that

$$-b \leq a.$$

and

$$a \leq 0 \leq b.$$

$\square$

(iii) Use this fact to give a new proof that $|a + b| \leq |a| + |b|$.

Proof. We make use of the fact that $-|a| \leq a \leq |a|$ and $-|b| \leq b \leq |b|$, and it follows that

$$-(|a| + |b|) \leq a + b \leq |a| + |b|,$$

so $|a + b| \leq |a| + |b|$. $\square$

Exercise (18). Prove the following

(i) Suppose that $b^2 - 4c \geq 0$. Show that the numbers

$$\frac{-b + \sqrt{b^2 - 4c}}{2}, \qquad \frac{-b - \sqrt{b^2 - 4c}}{2}$$

both satisfy the equation $x^2 + bx + c = 0$.

Proof. From the equation, we can find that

$$\begin{aligned} x^2 + bx + c &= 0, \\ \left(x + \frac{b}{2}\right)^2 &= -c + \frac{b^2}{4}, \\ \left|x + \frac{b}{2}\right| &= \frac{\sqrt{b^2 - 4c}}{|2|}, \\ x &= \frac{-b \pm \sqrt{b^2 - 4c}}{2}. \end{aligned}$$

$\square$

(ii) Suppose that $b^2 - 4c < 0$. Show there are no numbers x satisfying $x^2 + bx + c = 0$; in fact, $x^2 + bx + c > 0$ for all x.

Proof. We complete the square

$$\begin{aligned} x^2 + bx + c &= \left(x + \frac{b}{2}\right)^2 + c - \frac{b^2}{4} \\ &= \left(x + \frac{b}{2}\right)^2 - \left(\frac{b^2 - 4c}{4}\right) \end{aligned}$$

We know that squares are always nonnegative, and also we know that $b^2 - 4c < 0$ so the second term is always positive. Therefore $x^2 + bx + c$ is always positive for all x. $\square$

(iii) Use this fact to give another proof that if x and y are not both 0, then $x^2 + xy + y^2 > 0$.

Proof. From the previous exercise, we make $y^2 - 4y^2 < 0$ to assure that $x^2 + xy + y^2 > 0$ for all x, given that $y \neq 0$ (since $y^2 - 4y^2 = 0$ if $y = 0$). In the case that $y = 0$, $x^2 + xy + y^2 = x^2$ is still positive given that $x \neq 0$. $\square$

(iv) For which numbers α is it true that $x^2 + \alpha xy + y^2 > 0$ whenever x and y are not both 0.

Proof. From the previous exercise, $b^2 - 4c = (\alpha y)^2 - 4y^2 < 0$ will assure that $x^2 + \alpha xy + y^2 > 0$, given that both x and y are not 0.
We just solve for the values of α that will assure $(\alpha y)^2 - 4y^2$ is less than zero.

$$\begin{aligned}(\alpha y)^2 - 4y^2 &< 0,\\ \alpha^2 y^2 &< 4y^2,\\ \alpha^2 &< 4,\\ |\alpha| &< 2.\end{aligned}$$

so the values of α that will make $x^2 + \alpha xy + y^2 > 0$ are

$$-2 < \alpha < 2.$$

$\square$

(v) Find the smallest possible value of $x^2 + bx + c$ and of $ax^2 + bx + c$, for $a > 0$.

Proof. By completing the square, we can find the minimum value.

$$x^2 + bx + c = \left(x + \frac{b}{2}\right)^2 + c - \frac{b^2}{4},$$

and the minimum $c - \dfrac{b^2}{4}$ is achieved if $x = -\dfrac{b}{2}$.
For $ax^2 + bx + c$ where $a > 0$,

$$\begin{aligned}ax^2 + bx + c &= a\left(x^2 + \frac{b}{a}x\right) + c\\ &= a\left(x + \frac{b}{2a}\right)^2 + c - \frac{b^2}{4a},\end{aligned}$$

and the minimum $c - \dfrac{b^2}{4a}$ is achieved if $x = -\dfrac{b}{2a}$. $\square$

Exercise (20). Prove that if

$$|x - x_0| < \frac{\varepsilon}{2} \quad \text{and} \quad |y - y_0| < \frac{\varepsilon}{2},$$

then

$$\begin{aligned} |(x + y) - (x_0 + y_0)| < \varepsilon, \\ |(x - y) - (x_0 - y_0)| < \varepsilon. \end{aligned}$$

Proof. Assume the premise. Therefore, their sum would be

$$|x - x_0| + |y - y_0| < \varepsilon,$$

By the *Triange Inequality* we get

$$\begin{aligned} |(x + y) - (x_0 + y_0)| &= |(x - x_0) + (y - y_0)| \\ &\leq |x - x_0| + |y - y_0| \\ &< \varepsilon. \end{aligned}$$

Notice also that,

$$|x - x_0| + |y - y_0| = |x - x_0| + |y_0 - y| < \varepsilon,$$

and by the *Triangle Inequality*,

$$\begin{aligned} |(x - y) - (x_0 - y_0)| &= |(x - x_0) - (y_0 - y)| \\ &\leq |x - x_0| + |y_0 - y| \\ &< \varepsilon. \end{aligned}$$

□

1.2 Number of Various Sorts

Definition (Mathematical Induction)**.** Suppose that $P(x)$ means that the property P holds for the number x. Then the principle of mathematical induction states that $P(x)$ holds for all natural numbers x provided that

- $P(1)$ is true.
- Whenever $P(k)$ is true, $P(k+1)$ is true.

(Note: In the construction of the natural numbers from the *Peano Axioms*, induction is an axiom and 0 is an element of the naturals.)

Definition (Strong Induction)**.** We use a stronger hypothesis since in some cases it happens that in order to prove $P(k+1)$ we must assume not only $P(k)$, but also $P(l)$ for all natural numbers $l \leq k$. $P(x)$ holds for all natural numbers x provideded that

- $P(1)$ is true
- $P(k+1)$ is true, if $P(l)$ is true for $1 < l \leq k$

Definition (Recursive Definitions)**.** A *recursive definition* of a function defines values of the function for some inputs in terms of the values of the function for other inputs.

For example, the number $n!$ is defined as the product of all then natural numbers less then or equal to n:

$$n! = 1 \ \cdot 2 \cdot 3 \cdot \ldots \cdot (n-1) \cdot n.$$

More formally:

$$1! = 1 \tag{1}$$
$$n! = n \cdot (n-1)!. \tag{2}$$

Another example is the convenient notation regarding sums. Instead of writing

$$a_1 + a_1 + \cdots + a_n.$$

we will use the Greek letter Σ and write

$$\sum_{i=1}^{n} a_i.$$

The define $\sum\limits_{i=1}^{n} a_i$ precisely really requires a recursive definition:

$$\sum_{i=1}^{1} a_i = a_1, \tag{3}$$
$$\sum_{i=1}^{n} a_i = \sum_{i=1}^{n-1} a_i + a_n. \tag{4}$$

As an example where induction is used for its proof,

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

1.2.1 Exercises

Exercise (1). Prove the following formulas by induction.

(i) $1^2 + \cdots + n^2 = \dfrac{n(n+1)(2n+1)}{6}$.

Proof. We induct on n. For the base case $n = 1$, the equation is trivial.

$$\begin{aligned}1^2 &= \frac{1(1+1)(2(1)+1)}{6}\\ &= \frac{2(3)}{6}\\ &= 1\end{aligned}$$

Now we assume that the equation hold for arbitrary n, and we shall prove that it also hold for $n+1$.

$$\begin{aligned}1^2 + \cdots + n^2 + (n+1)^2 &= \sum_{i=1}^{n+1} i^2\\ &= \sum_{i=1}^{n} i^2 + (n+1)^2\\ &= \frac{n(n+1)(2n+1)}{6} + (n+1)^2\\ &= \frac{n(n+1)(2n+1) + 6(n+1)^2}{6}\\ &= \frac{(n+1)[2n^2 + n + 6n + 6]}{6}\\ &= \frac{(n+1)(n+2)(2(n+1)+1)}{6}.\end{aligned}$$

Hence, the equation hold for all natural numbers. □

(ii) $1^3 + \cdots + n^3 = (1 + \cdots + n)^2$.

Proof. We induct on n and prove the base case $n = 1$,

$$1^3 = (1)^2$$

which is trivial.
Let us now assume that the equation hold for arbitrary n. We must now show that it also hold for $n+1$.

$$\begin{aligned}(1 + \cdots + (n+1))^2 &= (n+1)(2(1 + \cdots + n) + (n+1)) + (1 + \cdots + n)^2\\ &= (1^3 + \cdots + n^3) + (n+1)(2(1 + \cdots + n) + (n+1))\\ &= (1^3 + \cdots + n^3) + (n+1)(2 \cdot \frac{n(n+1)}{2} + n + 1)\\ &= (1^3 + \cdots + n^3) + (n+1)((n+1)(n+1))\\ &= 1^3 + \cdots + (n+1)^3,\end{aligned}$$

hence, the equation holds for all natural numbers. □

Exercise (2). Find a formula for

(i) $\sum_{i=1}^{n}(2n-1) = 1+3+5+\cdots+(2n-1)$.

Proof. We can make use of the properties of the summation symbol and we can easily derive the formla. However, we will not make use of it.

$$\begin{aligned}
\sum_{i=1}^{n}(2n-1) &= 1+3+5+\cdots+(2n-1) \\
&= 1+2+3+\cdots+(2n) - 2(1+2+3+\cdots+n) \\
&= \frac{2n(2n+1)}{2} - 2\cdot\frac{n(n+1)}{2} \\
&= n((2n+1)-(n+1)) \\
&= n^2.
\end{aligned}$$

□

(ii) $\sum_{i=1}^{n}(2n-1)^2 = 1^2+3^2+5^2+\cdots+(2n-1)^2$.

Proof. We can make use of the properties of the summation symbol and we can easily derive the formla. However, we will not make use of it.

$$\begin{aligned}
\sum_{i=1}^{n}(2n-1)^2 &= 1^2+3^2+5^2+\cdots+(2n-1)^2 \\
&= 1^2+2^2+3^2+\cdots+(2n)^2 - 4(1^2+2^2+3^2+\cdots+n^2) \\
&= \frac{2n(2n+1)(4n+1)}{6} - 4\cdot\frac{n(n+1)(2n+1)}{6} \\
&= \frac{2n(2n+1)\,(4n+1-2(n+1))}{6} \\
&= \frac{n(2n+1)(2n-1)}{3}.
\end{aligned}$$

□

Exercise (3). If $0 \le k \le n$, the *"binomial coefficient"* $\binom{n}{k}$ is defined by

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n(n-1)\cdots(n-k+1)}{k!}, \text{if } k \neq 0, n$$

and a special case of the first formula if we define $0! = 1$,

$$\binom{n}{0} = \binom{n}{n} = 1,$$

and for $k < 0$ or $k > n$ we just define the binomial coefficient to be 0.

(i) Prove that

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}.$$

Proof. It is just a manipulation of the expressions and from the fact that,

$$\binom{n+1}{k} = \frac{(n+1)!}{(n+1-k)!k!}.$$

Now for the proof,

$$\begin{aligned}\binom{n}{k-1} + \binom{n}{k} &= \frac{n!}{(n+1-k)!(k-1)!} + \frac{n!}{(n-k)!k!} \\ &= \frac{n!k + n!(n+1-k)}{(n+1-k)!k!} \\ &= \frac{(n+1)!}{(n+1-k)!k!}.\end{aligned}$$

□

This relation gives rise to the following configuration, known as *"Pascal's Triangle"*—a number not on one of the sides is the sum of the two numbers above it; the binomial coefficient $\binom{n}{k}$ is the $(k+1)$st number in the $(n+1)$st row.

(ii) Notice that all the numbers in Pascal's Triangle are natural numbers. Use part (i) to prove by induction that $\binom{n}{k}$ is always a natural number.

Proof. We want to prove the assertion that for fixed n (Note: The case $k = 0$ is trivial),

$$\binom{n}{k} \text{is a natural number for all } k, 1 \le k \le n.$$

We prove it by induction on n. For the base case where $n = 1$, we only need to prove for $k = 1$.

$$\binom{1}{1} = 1.$$

For the inductive step, we assume that the assertion is true for arbitrary n and we will show that

$$\binom{n+1}{k} \text{ is a natural number for all } k, 1 \leq k \leq n+1.$$

The case where $k = 1$ and $k = n+1$ are trivial. We can now assume that for $2 \leq k \leq n$, it must be that $1 \leq k-1 < k \leq n$ and we can make use this with part (i)

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k},$$

since the sum of naturals is a natural, $\binom{n+1}{k}$ is natural. We have shown that $\binom{n+1}{k}$ is natural for all k, $1 \leq k \leq n+1$ if $\binom{n}{k}$ is natural for all k $1 \leq k \leq n$. Hence, the induction is complete. □

(iii) Give another proof that $\binom{n}{k}$ is natural number by showing that $\binom{n}{k}$ is the number of sets of exactly k integers each chosen from $1, \ldots, n$.

Proof. In choosing k objects from n, there are $n(n-1)\cdots(n-k+1)$ ways to choose a set with k elements, $\{k_1, \ldots, k_n\}$. However, for each set the k objects can be arranged in $k!$ ways, that is why we divide by $k!$ as the order of the kth element in a certain set with the same elements does not matter. Hence,

$$\binom{n}{k} = \frac{n(n-1)\ldots(n-k+1)}{k!} = \frac{n!}{(n-k)!k!},$$

is the number of sets of exactly k integers chosen from n and that is why $\binom{n}{k}$ is always a natural. □

(iv) Prove the *Binomial Theorem*: If a and b are any numbers and n is a natural number, then

$$\begin{aligned}(a+b)^n &= a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \cdots + \binom{n}{n-1}ab^{n-1} + b^n \\ &= \sum_{j=0}^{n}\binom{n}{j}a^{n-j}b^j.\end{aligned}$$

Proof. We prove by induction. For the base case, let $n = 1$ in which we get

$$(a+b)^1 = a + b = \binom{1}{0}a + \binom{1}{1}b = \sum_{j=0}^{1}\binom{1}{j}a^{1-j}b^j.$$

For the inductive step, we assume that the property holds for arbitrary $n = k$. We now show that it also holds for $n = k + 1$.

$$\begin{aligned}(a+b)^{k+1} &= (a+b)\sum_{j=0}^{k}\binom{k}{j}a^{k-j}b^j \\ &= \left(\sum_{j=0}^{k}\binom{k}{j}a^{(k+1)-j}b^j\right) + \left(\sum_{j=1}^{k+1}\binom{k}{j-1}a^{(k+1)-j}b^j\right) \\ &= \left(a^{k+1} + \sum_{j=1}^{k}\binom{k}{j}a^{(k+1)-j}b^j\right) + \left(b^{k+1} + \sum_{j=1}^{k}\binom{k}{j-1}a^{(k+1)-j}b^j\right) \\ &= a^{k+1} + b^{k+1} + \sum_{j=1}^{k}\binom{k}{j}a^{(k+1)-j}b^j + \binom{k}{j-1}a^{(k+1)-j}b^j \\ &= a^{k+1} + b^{k+1} + \sum_{j=1}^{k}(a^{(k+1)-j}b^j)\left(\binom{k}{j} + \binom{k}{j-1}\right) \\ &= a^{k+1} + b^{k+1} + \sum_{j=1}^{k}(a^{(k+1)-j}b^j)\binom{k+1}{j} \\ &= \sum_{j=0}^{k+1}(a^{(k+1)-j}b^j)\binom{k+1}{j}.\end{aligned}$$

□

Remark. To better understand it, an expression $(a^n + \cdots + b^n)$ have $n+1$ terms and if multiplied by $(a+b)$, the resulting expression will have $2n+2$ terms. The "ends" of the expression, a^{n+1} and b^{n+1} represent the two terms, while the $2n$ terms will be reduced to n terms. The first sum ${}^1a(a^n + \cdots b^n)$ will have its 2nd-$(n+1)$th term match the 1st-nth term of the second sum ${}^2b(a^n + \cdots b^n)$. The two sums have the same indexing of the terms (The coefficient is dependent on the ith index), hence the coefficient $\binom{n}{k}$ of these terms differ, since the kth term of the first sum, except for the "end" terms, will match to the $(k-1)$th term of the second term.

(v) Prove that

(a) $\sum\limits_{j=0}^{n} \binom{n}{j} = \binom{n}{0} + \cdots + \binom{n}{n} = 2^n$.

Proof. To be continued...

□

Exercise (4). Prove the following:

(i) Prove that
$$\sum_{k=0}^{l} \binom{n}{k}\binom{m}{l-k} = \binom{n+m}{l}.$$
Hint: Apply the binomial theorem to $(1+x)^n(1+x)^m$.

Proof. We apply the binomial theorem to get
$$\sum_{k=0}^{n} \binom{n}{k} x^k \sum_{j=0}^{m} \binom{m}{j} x^j = \sum_{l=0}^{n+m} \binom{n+m}{l} x^l.$$
The LHS and RHS are both polynomials, and notice that if polynomials are equal, then the coefficients of each term with the same degree is equal. We just find another form of the LHS that seperates the coefficient of each term so that we can equate the coefficient to the RHS.
$$\sum_{l=0}^{n+m} \left(\sum_{\substack{0\le k\le n \\ 0\le j\le m \\ k+j=l}} \binom{n}{k}\binom{m}{j} \right) x^l = \sum_{l=0}^{n+m} \binom{n+m}{l} x^l.$$
Then, $k+j=l$ and $j=l-k$, we find
$$\sum_{l=0}^{n+m} \left(\sum_{k=0}^{l} \binom{n}{k}\binom{m}{l-k} \right) x^l = \sum_{l=0}^{n+m} \binom{n+m}{l} x^l.$$
We can now compare the coefficients and we see that
$$\sum_{k=0}^{l} \binom{n}{k}\binom{m}{l-k} = \sum_{l=0}^{n+m} \binom{n+m}{l} x^l.$$

□

Remark. The proof/idea is taken from `https://math.stackexchange.com/a/3950826`, and further discussions can be found on `https://math.stackexchange.com/a/3950826`.

(ii) Prove that
$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}.$$

Exercise (8). Prove that every natural number is either even or odd.

Proof. We prove by induction. For $n=1$, it is trivial since 1 is odd. Now, we assume that for arbitrary $k \in N$, it is either even or odd.

- If k is even, so $k = 2c$ for some $c \in \mathbb{Z}^+$, then $k + 1 = 2c + 1$ which is odd.
- If k is odd, so $k = 2d + 1$ for some $d \in \mathbb{Z}^+$, then $k + 1 = 2(c + 2)$ and we know that $c + 2 \in \mathbb{Z}^+$, hence $k + 1$ is even.

We have shown that if k is either even or odd, then $k + 1$ is also even or odd. The induction is complete. $\square$