Spivak Calculus Exercises

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0 Introduction

These are exercises from Spivak Calculus. All the proofs given are my own proofs (unless stated otherwise) which is not assured for correctness and preciseness.

1 Prologue

1.1 Numbers of Various Sorts

Definition (Field Properties)**.** The following properties hold in $\mathbb{R}$

- P1 (Associative law for addition) $a + (b + c) = (a + b) + c.$
- P2 (Existence of an additive identity) $a + 0 = 0 + a = a.$
- P3 (Existence of additive inverse) $a + (-a) = (-a) + a = 0.$
- P4 (Commutative law for addition) $a + b = b + a.$
- P5 (Associative law for multiplication) $a \cdot (b \cdot c) = (a \cdot b) \cdot c.$
- P6 (Existence of multiplicative identity) $a \cdot 1 = 1 \cdot a = a; \quad 1 \neq 0.$
- P7 (Existence of multiplicative inverses) $a \cdot a^{-1} = a^{-1} \cdot a = 1, \text{ for } a \neq 0.$
- P8 (Commutative law for multiplication) $a \cdot b = b \cdot a.$
- P9 (Distributive law) $a \cdot (b + c) = a \cdot b + a \cdot c.$
- P10 (Trichotomy law) For every number a, one and only one of the following holds: (Denote P as the collection of positive numbers)
  - (i) $a = 0$,
  - (ii) a is in the collection P,
  - (iii) $-a$ is in the collection P.
- P11 (Closure under addition) If a and b are in P, then $a + b$ is in P.
- P12 (Closure under multiplication) If a and b are in P, then $a \cdot b$ is in P.

Theorem (Triangle Inequality)**.** For all numbers a and b, we have

$$|a + b| \leq |a| + |b|$$

Exercise (1). Prove the following:

(i) If $ax = a$ for some number $a \neq 0$, then $x = 1$.

Proof. Assume that $ax = a$ fro some number $a \neq 0$.

$$\begin{aligned} x = x \cdot 1 = x \cdot (a \cdot a^{-1}) &= ax \cdot (a^{-1}) \\ &= a \cdot (a^{-1}) \\ &= (a \cdot a^{-1}) \\ &= 1 \end{aligned}$$

□

(ii) $x^2 - y^2 = (x - y)(x + y)$.

Proof. Using the field axioms.

$$\begin{aligned} (x - y)(x + y) &= x \cdot (x + y) + (-y) \cdot (x + y) \\ &= (x^2 + xy) + ((-y) \cdot x + (-y) \cdot y) \\ &= x^2 + xy - xy - y^2 \\ &= x^2 - y^2 \end{aligned}$$

□

(iii) If $x^2 = y^2$, then $x = y$ or $x = -y$.

Proof. Assume that $x^2 = y^2$. We make use of (ii).

$$\begin{aligned} x^2 = y^2 &\Leftrightarrow x^2 - y^2 = 0 \\ &\Leftrightarrow (x-y)(x+y) = 0 \\ &\Rightarrow x = y \text{ or } x = -y. \end{aligned}$$

□

(iv) $x^3 - y^3 = (x-y)(x^2 + xy + y^2)$.

Proof. Using the field axioms

$$\begin{aligned} (x-y)(x^2+xy+y^2) &= x^2(x-y) + xy(x-y) + y^2(x-y) \\ &= (x^3 - x^2y) + (x^2y - xy^2) + (xy^2 - y^3) \\ &= x^3 + (x^2y - x^2y) + (xy^2 - xy^2) - y^3 \\ &= x^3 - y^3 \end{aligned}$$

□

(v) $x^n - y^n = (x-y)(x^{n-1} + x^{n-2}y + \cdots + xy^{n-2} + y^{n-1})$.

Proof. Using the field axioms

$$\begin{aligned} (x-y)(x^{n-1} + x^{n-2}y + \cdots + xy^{n-2} + y^{n-1}) &= x(x^{n-1} + x^{n-2}y + \cdots + xy^{n-2} + y^{n-1}) \\ &\quad - [y(x^{n-1} + x^{n-2}y + \cdots + xy^{n-2} + y^{n-1})] \\ &= x^n + x^{n-1}y + \cdots + x^2y^{n-2} + xy^{n-1} \\ &\quad - [x^{n-1}y + x^{n-2}y^2 + \cdots + xy^{n-1} + y^n] \\ &= x^n - y^n \end{aligned}$$

□

Alternative Proof. We make us of sigma notation

$$\begin{aligned} (x-y) \cdot \sum_{i=0}^{n-1} x^i y^{n-(i+1)} &= x\left(\sum_{i=0}^{n-1} x^i y^{n-(i+1)}\right) - \left[y\left(\sum_{i=0}^{n-1} x^i y^{n-(i+1)}\right)\right] \\ &= \sum_{i=0}^{n-1} x^{i+1} y^{n-(i+1)} - \left[\sum_{i=0}^{n-1} x^i y^{n-i}\right] \\ &= x^n + \sum_{i=0}^{n-2} x^{i+1} y^{n-(i+1)} - \left[\sum_{i=1}^{n-1} x^i y^{n-i} + y^n\right] \\ &= x^n + \sum_{i=0}^{n-2} x^{i+1} y^{n-(i+1)} - \left[\sum_{i=0}^{n-2} x^{i+1} y^{n-(i+1)} + y^n\right] \\ &= x^n - y^n + \sum_{i=0}^{n-2} \left[x^{i+1} y^{n-(i+1)} - (x^{i+1} y^{n-(i+1)})\right] \\ &= x^n - y^n + \sum_{i=0}^{n-2} 0 \\ &= x^n - y^n \end{aligned}$$

□

(vi) $x^3 + y^3 = (x + y)(x^2 - xy + y^2)$.

Proof. Replace y by $-y$ in part (iv)

$$\begin{aligned} x^3 - y^3 = (x-y)(x^2+xy+y^2) &\Leftrightarrow x^3 - (-y)^3 = (x-(-y))(x^2 + x(-y) + (-y)^2) \\ &\Leftrightarrow x^3 + y^3 = (x+y)(x^2 - xy + y^2) \end{aligned}$$

□

Exercise (2). What is wrong with the following "proof"? Let $x = y$. Then

$$\begin{aligned} x^2 &= xy, \\ x^2 - y^2 &= xy - y^2, \\ (x+y)(x-y) &= y(x-y), \\ x + y &= y, \\ 2y &= y, \\ 2 &= 1. \end{aligned}$$

Solution. For all $a \in \mathbb{R}$ we know that $a \cdot a^{-1} = 0$ with the assumption $a \neq 0$. The $4th$ step is contradictory on the given fact that $x = y$ which implies $x - y = 0$ and has no multiplicative inverse. □

Exercise (3). Prove the following:

(i) $\dfrac{a}{b} = \dfrac{ac}{bc}$, if $b, c \neq 0$.

Proof. Using the field axioms

$$\begin{aligned} \frac{a}{b} = ab^{-1} &= (ab^{-1})(c \cdot c^{-1}) \\ &= (ac)(b^{-1}c^{-1}) \\ &= (ac)(bc)^{-1} \\ &= \frac{ac}{bc} \end{aligned}$$

□

(ii) $\dfrac{a}{b} + \dfrac{c}{d} = \dfrac{ad + bc}{bd}$, if $b, d \neq 0$.

Proof. Using the field axioms

$$\begin{aligned} \frac{a}{b} + \frac{c}{d} = ab^{-1} + cd^{-1} &= (ab^{-1} + cd^{-1}) \cdot (bd)(bd)^{-1} \\ &= (ad(b \cdot b^{-1}) + bc(d \cdot d^{-1})) \cdot (bd)^{-1} \\ &= (ad + bc) \cdot (bd)^{-1} \\ &= \frac{ad + bc}{bd} \end{aligned}$$

□

(iii) $(ab)^{-1} = a^{-1}b^{-1}$, if $a, b \neq 0$.

Proof. Using the field axioms

$$\begin{aligned} ab(a^{-1}b^{-1}) &= 1 \\ a^{-1}b^{-1} &= (ab)^{-1} \end{aligned}$$

$\square$

(iv) $\dfrac{a}{b} \cdot \dfrac{c}{d} = \dfrac{ac}{db}$ if $b, d \neq 0$.

Proof. Using the field axioms

$$\begin{aligned} \frac{a}{b} \cdot \frac{c}{d} &= (ab^{-1}) \cdot (cd^{-1}) \\ &= (ac) \cdot (d^{-1}b^{-1}) \\ &= (ac) \cdot (db)^{-1} \\ &= \frac{ac}{db} \end{aligned}$$

$\square$

(v) $\dfrac{a}{b} \Big/ \dfrac{c}{d} = \dfrac{ad}{bc}$, if $b, d \neq 0$.

Proof. Using the field axioms

$$\begin{aligned} \frac{a}{b} \Big/ \frac{c}{d} &= \frac{a}{b} \cdot \left(\frac{c}{d}\right)^{-1} \\ &= ab^{-1} \cdot (cd^{-1})^{-1} \\ &= ab^{-1} \cdot c^{-1}(d^{-1})^{-1} \\ &= ab^{-1} \cdot c^{-1}d \\ &= (ad) \cdot (b^{-1}c^{-1}) \\ &= (ad) \cdot (bc)^{-1} \\ &= \frac{ad}{bc} \end{aligned}$$

$\square$

(vi) If $b, d \neq 0$, then $\frac{a}{b} = \frac{c}{d}$ if and only if $ad = bc$. Also determine when $\frac{a}{b} = \frac{b}{a}$.

Proof. There are two cases to prove for the first part.

($\Rightarrow$) Let $b, d \neq 0$. Assume that $\frac{a}{b} = \frac{c}{d}$,

$$\begin{aligned}
\frac{a}{b} &= \frac{c}{d}, \\
ab^{-1} &= cd^{-1}, \\
(ab^{-1})(bd) &= (cd^{-1})(bd), \\
(ad)(b \cdot b^{-1}) &= (bc)(d \cot d^{-1}), \\
ad &= bc.
\end{aligned}$$

($\Leftarrow$) Let $b, d \neq 0$. Assume that $ad = bc$,

$$\begin{aligned}
ad &= bc, \\
(ad)(bd)^{-1} &= (bc)(bd)^{-1} \\
(ab^{-1})(d \cdot d^{-1}) &= (cd^{-1})(b \cdot b^{-1}) \\
ab^{-1} &= cd^{-1} \\
\frac{a}{b} &= \frac{c}{d}
\end{aligned}$$

□

Proof. From Exercise 1 Part (*iii*) we make use of the fact, if $x^2 = y^2$ then $x = y$ or $x = -y$.

$$\begin{aligned}
\frac{a}{b} &= \frac{b}{a}, \\
ab^{-1} &= ba^{-1}, \\
(ab^{-1})(ab) &= (ba^{-1})(ab), \\
(a \cdot a)(b \cdot b^{-1}) &= (b \cdot b)(a \cdot a^{-1}), \\
a^2 &= b^2.
\end{aligned}$$

and so it must be that $a = b$ or $a = -b$. □

Exercise (4). Find all numbers x for which

(i) $4 - x < 3 - 2x$.

Proof. Using the field axioms

$$4 - x < 3 - 2x$$
$$4 - x + (2x - 4) < 3 - 2x + (2x - 4)$$
$$x < -1$$

$\square$

(ii) $5 - x^2 < 8$.

Proof. Using the field axioms

$$5 - x^2 + (x^2 - 5) < 8 + (x^2 - 5)$$
$$x^2 + 3 > 0$$

since $x^2 \geq 0$ for all $x \in \mathbb{R}$, then it must be that $x^2 + 3 > 0$ for all $x \in \mathbb{R}$. $\square$

(iii) $5 - x^2 < -2$

Proof. Using the field axioms

$$5 - x^2 < -2$$
$$x^2 > 7$$
$$|x| > \sqrt{7}$$
$$x < -\sqrt{7} \text{ or } x > \sqrt{7}$$

$\square$

(iv) $(x - 3)(x - 1) > 0$ (When is a product of two numbers positive?)

Proof. The product of two numbers is postivie if and only if the numbers are both positive or both negative. For all $a, b \in \mathbb{R}$, $ab > 0 \Leftrightarrow a > 0$ and $b > 0$, or $a < 0$ and $b < 0$.
Hence,

$$x - 3 > 0 \qquad \text{and} \qquad x - 1 > 0$$

so it must be that $x > 3$. Or

$$x - 3 < 0 \qquad \text{and} \qquad x - 1 > 0$$

and it must be that $x < 1$. That is $(x - 3)(x - 1) > 0$ if $x > 3$ or $x < 1$. $\square$

(v) $x^2 - 2x + 2 > 0$.

Proof. Using the field axioms

$$\begin{aligned} x^2 - 2x + 2 &= (x^2 + 2x + 1) + 1 \\ &= (x - 1)^2 + 1 \end{aligned}$$

for all $x \in \mathbb{R}$ notice that, $(x-1)^2 \geq 0$, so it must be that $(x-1)^2 + 1 > 0$. $\square$

(vi) $x^2 + x + 1 > 2$.

Proof. Using the field axioms

$$\begin{gathered} x^2 + x + 1 > 2 \\ x^2 + x - 1 > 0 \\ (x^2 + x + \frac{1}{4}) - \frac{5}{4} > 0 \\ \left(x + \frac{1}{2}\right)^2 > \frac{5}{4} \\ \left|x + \frac{1}{2}\right| > \frac{\sqrt{5}}{2} \\ x + \frac{1}{2} > \frac{\sqrt{5}}{2} \text{ or } x + \frac{1}{2} < -\frac{\sqrt{5}}{2} \end{gathered}$$

so it must be that

$$x > \frac{\sqrt{5} - 1}{2} \qquad \text{or} \qquad x < \frac{-\sqrt{5} - 1}{2}$$

$\square$

(vii) $x^2 - x + 10 > 16$.

Proof. Using the field axioms

$$\begin{gathered} x^2 - x + 10 > 16 \\ x^2 - x - 6 > 0 \\ (x-3)(x+2) > 0 \end{gathered}$$

To assure that the product is positive, it must be that the two numbers are both positive or both negative. Hence,

$$x - 3 > 0 \qquad \text{and} \qquad x + 2 > 0$$

such that $x > 3$. Or

$$x - 3 < 0 \qquad \text{and} \qquad x + 2 < 0$$

such that $x < -2$. Therefore, $x^2 - x + 10 > 16$ if $x > 3$ or $x < -2$. $\square$

(viii) $x^2 + x + 1 > 0$.

Proof. Using the field axioms

$$\begin{aligned} x^2 + x + 1 &= (x^2 + x + \frac{1}{4}) + \frac{3}{4}, \\ &= \left(x + \frac{1}{2}\right)^2 + \frac{3}{4}. \end{aligned}$$

for all $x \in \mathbb{R}$, notice that $(x + \frac{1}{2})^2 \geq 0$, so it must be that $(x + \frac{1}{2})^2 + \frac{3}{4} > 0$ for all $x \in \mathbb{R}$. $\square$

(ix) $(x - \pi)(x + 5)(x - 3) > 0$.

Proof. The expression $(x - \pi)(x + 5)(x - 3)$ can be rearranged as a product of two numbers, namely, $(x - \pi)\left[(x + 5)(x - 3)\right]$.

Notice, the product of two real numbers ab is greater than zero if a and b are both greater than zero, or both less than zero.

There are two cases:

- Let $(x - \pi) > 0$ so that $x > \pi$, and $(x + 5)(x - 3) > 0$ so that $x < -5$ or $x > 3$. Therefore it must be that $x > \pi$.
- Let $(x - \pi) < 0$ so that $x < \pi$, and $(x + 5)(x - 3) < 0$ so that $-5 < x < 3$. Therefore it must be that $-5 < x < 3$.

Therefore, $(x - \pi)(x + 5)(x - 3) > 0$ if $x > \pi$, or $-5 < x < 3$. $\square$

(x) $(x - \sqrt[3]{2})(x - \sqrt{2}) > 0$.

Proof. Either both numbers are greater than zero or less than zero.

$$x > \sqrt[3]{2} \qquad \text{and} \qquad x > \sqrt{2}$$

so that $x > \sqrt{2}$. Or

$$x < \sqrt[3]{2} \qquad \text{and} \qquad x < \sqrt{2}$$

so that $x < \sqrt[3]{2}$.

Therefore, $(x - \sqrt[3]{2})(x - \sqrt{2}) > 0$ if $x > \sqrt{2}$ or $x < \sqrt[3]{2}$. $\square$

(xi) $2^x < 8$.

Proof. We can rewrite it as

$$2^x < 2^3$$

Both have the same base, so it must be that the inequality is preserved on the exponents.

$$x < 3$$

so $2^x < 8$, whenever $x < 3$. $\square$

(xii) $x + 3^x < 4$.

Proof. We first notice that $x + 3^x = 4$ if $x = 1$

$$\begin{aligned} x + 3^x &= (1) + 3^1 \\ &= 4 \end{aligned}$$

observe that $x + 3^x$ is always increasing as x increase, and decreasing as x decrease. Therefore $x + 3^x < 4$ if $x < 1$. $\square$

(xiii) $\dfrac{1}{x} + \dfrac{1}{1-x} > 0.$

Proof. We can rewrite the expression as

$$\begin{aligned}\frac{1}{x} + \frac{1}{1-x} &= \frac{(1-x)+x}{x(1-x)} \\ &= \frac{1}{x(1-x)}\end{aligned}$$

Notice that $\frac{1}{x(1-x)} > 0$, whenever $x(1-x) > 0$. So it must be that x and $(1-x)$ are greater than zero

$$x > 0 \qquad \text{and} \qquad x < 1$$

or x and $(1-x)$ are both less than zero

$$x < 0 \qquad \text{and} \qquad x > 1$$

but there exists no x sucht that $x < 0$ and $x > 1$. Therefore, $\frac{1}{x} + \frac{1}{1-x} > 0$ if $x > 0$ and $x < 1$. ☐

(xiv) $\dfrac{x-1}{x+1} > 0.$

Proof. Either both $(x-1)$ and $(x+1)$ are greater than zero or both less than zero.

$$x > 1 \qquad \text{and} \qquad x > -1$$

so it must be that $x > 1$. Or

$$x < 1 \qquad \text{and} \qquad x < -1$$

so it must be that $x < -1$. ☐

Exercise (5). Prove the following:

(i) If $a < b$ and $c < d$, then $a + c < b + d$.

Proof. $\square$

(ii) If $a < b$, then $-b < -a$.

Proof. $\square$

(iii) If $a < b$ and $c < d$, then $a - c < b - d$.

Proof. $\square$

(iv) If $a < b$ and $c > 0$, then $ac < bc$.

Proof. $\square$

(v) If $a < b$ and $c < 0$, then $ac > bc$.

Proof. $\square$

(vi) If $a > 1$, then $a^2 > a$.

Proof. $\square$

(vii) If $0 < a < 1$, then $a^2 < a$.

Proof. $\square$

(viii) If $0 \leq a < b$ and $0 \leq c < d$, then $ac < bd$.

Proof. $\square$

(ix) If $0 \leq a < b$, then $a^2 < b^2$.

Proof. $\square$

(x) If $a, b \geq 0$ and $a^2 < b^2$, then $a < b$.