Terence Tao Analysis I Exercises

Duncan Bandojo

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Chapter Title

Contents

0 Introduction

This are important notes and solutions to some of the exercises in *Terence Tao Analysis I*. Most of the chapters are skipped and only the essential ones are made.

Needed definitions that are used in the book and exercises are added.

0.1 Functions

Definition (Axioms of Equality)**.** a relation linking two objects x, y of the same type T. How equlity is defined depends on the class T of objects under consideration. We require equality to obey the four *axioms of equality.*

- (Reflexive Axiom). Given any object x, we have $x = x$.
- (Symmetry Axioms). Given any two objects x and y of the same type, if $x = y$, then $y = x$.
- (Transitive Axiom). Given any three objects x, y, z of the same type, if $x = y$ and $y = z$, then $x = z$.
- (Substitution Axiom). Given any two objects x and y of the same type, if $x = y$, then $f(x) = f(y)$ for all functions or operations f.

Definition (Equality of Functions)**.** Functions $f : X \to Y$ and $g : X \to Y$ with same domain and range are equal, if and only if $f(x) = g(x)$ for all $x \in X$.

Exercise (3.3.1)**.** Show that the definition of the equality of functions is a reflexive, symmetric, and transitive. Also verify the substitution proeprty: if $f, \tilde{f} : X \to Y$ and $g, \tilde{g} : Y \to Z$ are functions such that $f = \tilde{f}$ and $g = \tilde{g}$, then $g \circ f = g \circ \tilde{f}$.

Proof. We seperate each property and make use of the *Axioms of Equality.*

- (Reflexive). Let $f : X \to Y$ be a function and $x \in X$. Then there exists $f(x) \in Y$, and by the *reflexive axiom* of equality

  $$f(x) = f(x).$$

  Hence, $f = f$, since x is an arbitrary element of X.
- (Symmetric). Let $f : X \to Y$and $g : X \to Y$ be functions and $x \in X$ be arbitrary. Assume for $f(x), g(x) \in Y$ that

  $$f(x) = g(x),$$

  by the *symmetry axiom* of equality of objects of the same type,

  $$g(x) = f(x).$$

  Therefore, $f = g$ implies $g = f$.
- (Transitivity). Let f, g, h be functions with domain X and range Y. Assume that $f = g$ and $g = h$, so that for all $x \in X$,

  $$f(x) = g(x) \quad \text{and} \quad g(x) = h(x),$$

  where $f(x), g(x), h(x) \in Y$. By the *trasitivity axiom* of equality,

  $$f(x) = h(x).$$

  Since x is arbitrary, $f = h$.

- (Substitution). Let $f, \tilde{f} : X \to Y$ and $g, \tilde{g} : Y \to Z$ be functions such that $f = \tilde{f}$ and $g = \tilde{g}$. Hence, $f(x) = \tilde{f}(x)$ for all $x \in X$ and $g(y) = \tilde{g}(y)$ for all $y \in Y$.

  To show that $g \circ f = \tilde{g} \circ \tilde{f}$, we must show that $(g \circ f)(x) = (\tilde{g} \circ \tilde{f})(x)$ for all $x \in X$. The two functions already have the same domain and range.

  We make use of the *subsitution axiom* of equality, on the elements $f(x), \tilde{f}(x) \in Y$, and notice that $g, \tilde{g}$ are functions.

  $$f(x) = \tilde{f}(x) \quad \Longrightarrow \quad g(f(x)) = g(\tilde{f}(x)),$$

  and it follows that

  $$g(\tilde{f}(x)) = \tilde{g}(\tilde{f}(x)).$$

  Making use of the *axioms of equality*,

  $$(g \circ f)(x) = g(f(x)) = g(\tilde{f}(x)) = \tilde{g}(\tilde{f}(x)) = (\tilde{g} \circ \tilde{f})(x).$$

  Which holds for all $x \in X$, hence, $g \circ f = \tilde{g} \circ \tilde{f}$.

□

Exercise (3.3.2)**.** Let $f : X \to Y$ and $g : Y \to Z$ be functions. Show that if f and g are both injective, then so is $g \circ f$; similarly, show that if f and g are both surjective, then so is $g \circ f$.

Proof. There are two proofs we need to do:

- (Injective). Let $x, x' \in X$ be arbitrary. Assume that f and g are both injective. Since g is injective and $f(x), f(x') \in Y$,

  $$g(f(x)) = g(f(x')) \implies f(x) = f(x').$$

  We also know that f is injective and $x, x' \in X$ so that

  $$f(x) = f(x') \implies x = x'.$$

  Therefore,

  $$g(f(x)) = g(f(x')) \implies x = x',$$

  Hence, $g \circ f$ is injective.

- (Surjective). Assume f and g are surjective. By the definition of surjection,

  $$\forall z \in Z, \exists y \in Y \ st. \ g(y) = z,$$

  and

  $$\forall y \in Y \exists x \in X \ st. \ f(x) = y.$$

  Therefore, for every $z \in Z$, we can choose an $x \in X$ such that

  $$z = g(y) = g(f(x)) = (g \circ f)(x).$$

  Hence, $g \circ f$ is surjective.

□

Exercise (3.3.3)**.** When is the *empty function* injective? surjective? bijective? (The empty function $f : \emptyset \to Y$ is a function with the empty set as its domain).

Proof. A function f is injective if for arbitrary x, x' in its domain,

$$x \neq x' \implies f(x) \neq f(x').$$

Notice that there exists no element in the empty set, hence the *empty function* is always injective no matter the range is.

A function is surjective if for every y in its range, there exists x in its domain such that

$$f(x) = y.$$

Notice that if the range is nonempty, then there exists an element in the range wherein $f(x) = y$ is not satisfied since the empty set has no element. Therefore, if the range is empty, then the *empty function* is surjective vacuously.

The *empty function* is bijective if the range is empty. □

Exercise (3.3.4)**.** Let $f : X \to Y, \tilde{f} : X \to Y, g : Y \to Z$,and $\tilde{g} : y \to Z$ be functions. Show that if $g \circ f = g \circ \tilde{f}$ and g is injective, then $f = \tilde{f}$. Is the same statement true if g is not injective? Show that if $g \circ f = \tilde{g} \circ f$ and f is injective, then $g = \tilde{g}$. Is the same statement true if f is not surjective?

Proof. □