Spivak Calculus Notes and Exercises

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Contents

0 Introduction

These are notes and exercises from Spivak Calculus. All the proofs given are my own proofs (unless stated otherwise) which is not assured for correctness and preciseness.

1 Prologue

1.1 Numbers of Various Sorts

Definition (Field Properties)**.** The following properties hold in $\mathbb{R}$

- P1 (Associative law for addition) $a + (b + c) = (a + b) + c.$
- P2 (Existence of an additive identity) $a + 0 = 0 + a = a.$
- P3 (Existence of additive inverse) $a + (-a) = (-a) + a = 0.$
- P4 (Commutative law for addition) $a + b = b + a.$
- P5 (Associative law for multiplication) $a \cdot (b \cdot c) = (a \cdot b) \cdot c.$
- P6 (Existence of multiplicative identity) $a \cdot 1 = 1 \cdot a = a; \quad 1 \neq 0.$
- P7 (Existence of multiplicative inverses) $a \cdot a^{-1} = a^{-1} \cdot a = 1, \text{ for } a \neq 0.$
- P8 (Commutative law for multiplication) $a \cdot b = b \cdot a.$
- P9 (Distributive law) $a \cdot (b + c) = a \cdot b + a \cdot c.$
- P10 (Trichotomy law) For every number a, one and only one of the following holds: (Denote P as the collection of positive numbers)
  - (i) $a = 0$,
  - (ii) a is in the collection P,
  - (iii) $-a$ is in the collection P.
- P11 (Closure under addition) If a and b are in P, then $a + b$ is in P.
- P12 (Closure under multiplication) If a and b are in P, then $a \cdot b$ is in P.

Definition (Absolute Value)**.** For any number a, we define the *absolute value* $|a|$ of a as follows:

$$|a| = \begin{cases} a, & a \geq 0 \\ -a, & a \leq 0 \end{cases}$$

Theorem (Triangle Inequality)**.** For all numbers a and b, we have

$$|a + b| \leq |a| + |b|$$

Proof. We make use of the fact that if both x and y are nonnegative, then $x^2 < y^2$ implies $x < y$.

$$\begin{aligned} |a + b|^2 &= a^2 + 2ab + b^2 \\ &= |a|^2 + 2ab + |b|^2 \\ &\leq |a|^2 + 2\,|a|\,|b| + |b|^2 \\ &= (|a| + |b|)^2 \end{aligned}$$

Since $|a + b|$ and $(|a| + |b|)$ are both nonnegative, then

$$|a + b| \leq |a| + |b|\,.$$

$\square$

1.1.1 Exercises

Exercise (1). Prove the following:

(i) If $ax = a$ for some number $a \neq 0$, then $x = 1$.

Proof. Assume that $ax = a$ fro some number $a \neq 0$.

$$\begin{aligned} x = x \cdot 1 = x \cdot (a \cdot a^{-1}) &= ax \cdot (a^{-1}) \\ &= a \cdot (a^{-1}) \\ &= (a \cdot a^{-1}) \\ &= 1 \end{aligned}$$

□

(ii) $x^2 - y^2 = (x - y)(x + y)$.

Proof. Using the field axioms.

$$\begin{aligned} (x-y)(x+y) &= x \cdot (x+y) + (-y) \cdot (x+y) \\ &= (x^2 + xy) + ((-y) \cdot x + (-y) \cdot y) \\ &= x^2 + xy - xy - y^2 \\ &= x^2 - y^2 \end{aligned}$$

□

(iii) If $x^2 = y^2$, then $x = y$ or $x = -y$.

Proof. Assume that $x^2 = y^2$. We make use of (ii).

$$\begin{aligned} x^2 = y^2 &\Leftrightarrow x^2 - y^2 = 0 \\ &\Leftrightarrow (x-y)(x+y) = 0 \\ &\Rightarrow x = y \text{ or } x = -y. \end{aligned}$$

□

(iv) $x^3 - y^3 = (x - y)(x^2 + xy + y^2)$.

Proof. Using the field axioms

$$\begin{aligned} (x-y)(x^2+xy+y^2) &= x^2(x-y) + xy(x-y) + y^2(x-y) \\ &= (x^3 - x^2y) + (x^2y - xy^2) + (xy^2 - y^3) \\ &= x^3 + (x^2y - x^2y) + (xy^2 - xy^2) - y^3 \\ &= x^3 - y^3 \end{aligned}$$

□

(v) $x^n - y^n = (x - y)(x^{n-1} + x^{n-2}y + \cdots + xy^{n-2} + y^{n-1})$.

Proof. Using the field axioms

$$\begin{aligned}(x-y)(x^{n-1}+x^{n-2}y+\cdots+xy^{n-2}+y^{n-1}) &= x(x^{n-1}+x^{n-2}y+\cdots+xy^{n-2}+y^{n-1})\\ &\quad -[y(x^{n-1}+x^{n-2}y+\cdots+xy^{n-2}+y^{n-1})]\\ &= x^n+x^{n-1}y+\cdots+x^2y^{n-2}+xy^{n-1}\\ &\quad -[x^{n-1}y+x^{n-2}y^2+\cdots+xy^{n-1}+y^n]\\ &= x^n-y^n\end{aligned}$$

□

Alternative Proof. We make us of sigma notation

$$\begin{aligned}(x-y)\cdot\sum_{i=0}^{n-1}x^iy^{n-(i+1)} &= x\left(\sum_{i=0}^{n-1}x^iy^{n-(i+1)}\right)-\left[y\left(\sum_{i=0}^{n-1}x^iy^{n-(i+1)}\right)\right]\\ &= \sum_{i=0}^{n-1}x^{i+1}y^{n-(i+1)}-\left[\sum_{i=0}^{n-1}x^iy^{n-i}\right]\\ &= x^n+\sum_{i=0}^{n-2}x^{i+1}y^{n-(i+1)}-\left[\sum_{i=1}^{n-1}x^iy^{n-i}+y^n\right]\\ &= x^n+\sum_{i=0}^{n-2}x^{i+1}y^{n-(i+1)}-\left[\sum_{i=0}^{n-2}x^{i+1}y^{n-(i+1)}+y^n\right]\\ &= x^n-y^n+\sum_{i=0}^{n-2}\left[x^{i+1}y^{n-(i+1)}-(x^{i+1}y^{n-(i+1)})\right]\\ &= x^n-y^n+\sum_{i=0}^{n-2}0\\ &= x^n-y^n\end{aligned}$$

□

(vi) $x^3+y^3=(x+y)(x^2-xy+y^2)$.

Proof. Replace y by $-y$ in part (iv)

$$\begin{aligned}x^3-y^3=(x-y)(x^2+xy+y^2) &\Leftrightarrow x^3-(-y)^3=(x-(-y))(x^2+x(-y)+(-y)^2)\\ &\Leftrightarrow x^3+y^3=(x+y)(x^2-xy+y^2)\end{aligned}$$

□

Exercise (2). What is wrong with the following "proof"? Let $x=y$. Then

$$\begin{aligned}x^2&=xy,\\ x^2-y^2&=xy-y^2,\\ (x+y)(x-y)&=y(x-y),\\ x+y&=y,\\ 2y&=y,\\ 2&=1.\end{aligned}$$

Solution. For all $a \in \mathbb{R}$ we know that $a \cdot a^{-1} = 0$ with the assumption $a \neq 0$. The $4th$ step is contradictory on the given fact that $x = y$ which implies $x - y = 0$ and has no multiplicative inverse. $\square$

Exercise (3). Prove the following:

(i) $\dfrac{a}{b} = \dfrac{ac}{bc}$, if $b, c \neq 0$.

Proof. Using the field axioms

$$\begin{aligned} \frac{a}{b} = ab^{-1} &= (ab^{-1})(c \cdot c^{-1}) \\ &= (ac)(b^{-1}c^{-1}) \\ &= (ac)(bc)^{-1} \\ &= \frac{ac}{bc} \end{aligned}$$

$\square$

(ii) $\dfrac{a}{b} + \dfrac{c}{d} = \dfrac{ad + bc}{bd}$, if $b, d \neq 0$.

Proof. Using the field axioms

$$\begin{aligned} \frac{a}{b} + \frac{c}{d} = ab^{-1} + cd^{-1} &= (ab^{-1} + cd^{-1}) \cdot (bd)(bd)^{-1} \\ &= (ad(b \cdot b^{-1}) + bc(d \cdot d^{-1})) \cdot (bd)^{-1} \\ &= (ad + bc) \cdot (bd)^{-1} \\ &= \frac{ad + bc}{bd} \end{aligned}$$

$\square$

(iii) $(ab)^{-1} = a^{-1}b^{-1}$, if $a, b \neq 0$.

Proof. Using the field axioms

$$\begin{aligned} ab(a^{-1}b^{-1}) &= 1 \\ a^{-1}b^{-1} &= (ab)^{-1} \end{aligned}$$

□

(iv) $\dfrac{a}{b} \cdot \dfrac{c}{d} = \dfrac{ac}{db}$ if $b, d \neq 0$.

Proof. Using the field axioms

$$\begin{aligned} \frac{a}{b} \cdot \frac{c}{d} &= (ab^{-1}) \cdot (cd^{-1}) \\ &= (ac) \cdot (d^{-1}b^{-1}) \\ &= (ac) \cdot (db)^{-1} \\ &= \frac{ac}{db} \end{aligned}$$

□

(v) $\dfrac{a}{b} \Big/ \dfrac{c}{d} = \dfrac{ad}{bc}$, if $b, d \neq 0$.

Proof. Using the field axioms

$$\begin{aligned} \frac{a}{b} \Big/ \frac{c}{d} &= \frac{a}{b} \cdot \left(\frac{c}{d}\right)^{-1} \\ &= ab^{-1} \cdot (cd^{-1})^{-1} \\ &= ab^{-1} \cdot c^{-1}(d^{-1})^{-1} \\ &= ab^{-1} \cdot c^{-1}d \\ &= (ad) \cdot (b^{-1}c^{-1}) \\ &= (ad) \cdot (bc)^{-1} \\ &= \frac{ad}{bc} \end{aligned}$$

□

(vi) If $b, d \neq 0$, then $\dfrac{a}{b} = \dfrac{c}{d}$ if and only if $ad = bc$. Also determine when $\dfrac{a}{b} = \dfrac{b}{a}$.

Proof. There are two cases to prove for the first part.

($\Rightarrow$) Let $b, d \neq 0$. Assume that $\dfrac{a}{b} = \dfrac{c}{d}$,

$$\begin{aligned} \frac{a}{b} &= \frac{c}{d}, \\ ab^{-1} &= cd^{-1}, \\ (ab^{-1})(bd) &= (cd^{-1})(bd), \\ (ad)(b \cdot b^{-1}) &= (bc)(d \cot d^{-1}), \\ ad &= bc. \end{aligned}$$

($\Leftarrow$) Let $b, d \neq 0$. Assume that $ad = bc$,

$$\begin{aligned} ad &= bc, \\ (ad)(bd)^{-1} &= (bc)(bd)^{-1} \\ (ab^{-1})(d \cdot d^{-1}) &= (cd^{-1})(b \cdot b^{-1}) \\ ab^{-1} &= cd^{-1} \\ \frac{a}{b} &= \frac{c}{d} \end{aligned}$$

□

Proof. From Exercise 1 Part (*iii*) we make use of the fact, if $x^2 = y^2$ then $x = y$ or $x = -y$.

$$\begin{aligned} \frac{a}{b} &= \frac{b}{a}, \\ ab^{-1} &= ba^{-1}, \\ (ab^{-1})(ab) &= (ba^{-1})(ab), \\ (a \cdot a)(b \cdot b^{-1}) &= (b \cdot b)(a \cdot a^{-1}), \\ a^2 &= b^2. \end{aligned}$$

and so it must be that $a = b$ or $a = -b$. □

Exercise (4). Find all numbers x for which

(i) $4 - x < 3 - 2x$.

Proof. Using the field axioms

$$\begin{aligned} 4 - x &< 3 - 2x \\ 4 - x + (2x - 4) &< 3 - 2x + (2x - 4) \\ x &< -1 \end{aligned}$$

□

(ii) $5 - x^2 < 8$.

Proof. Using the field axioms

$$\begin{aligned} 5 - x^2 + (x^2 - 5) &< 8 + (x^2 - 5) \\ x^2 + 3 &> 0 \end{aligned}$$

since $x^2 \geq 0$ for all $x \in \mathbb{R}$, then it must be that $x^2 + 3 > 0$ for all $x \in \mathbb{R}$. □

(iii) $5 - x^2 < -2$

Proof. Using the field axioms

$$\begin{aligned} 5 - x^2 &< -2 \\ x^2 &> 7 \\ |x| &> \sqrt{7} \\ x < -\sqrt{7} \text{ or } x &> \sqrt{7} \end{aligned}$$

□

(iv) $(x - 3)(x - 1) > 0$ (When is a product of two numbers positive?)

Proof. The product of two numbers is postivie if and only if the numbers are both positive or both negative. For all $a, b \in \mathbb{R}$, $ab > 0 \Leftrightarrow a > 0$ and $b > 0$, or $a < 0$ and $b < 0$.

Hence,

$$x - 3 > 0 \qquad \text{and} \qquad x - 1 > 0$$

so it must be that $x > 3$. Or

$$x - 3 < 0 \qquad \text{and} \qquad x - 1 > 0$$

and it must be that $x < 1$. That is $(x - 3)(x - 1) > 0$ if $x > 3$ or $x < 1$. □

(v) $x^2 - 2x + 2 > 0$.

Proof. Using the field axioms

$$\begin{aligned} x^2 - 2x + 2 &= (x^2 + 2x + 1) + 1 \\ &= (x-1)^2 + 1 \end{aligned}$$

for all $x \in \mathbb{R}$ notice that, $(x-1)^2 \geq 0$, so it must be that $(x-1)^2 + 1 > 0$. $\square$

(vi) $x^2 + x + 1 > 2$.

Proof. Using the field axioms

$$\begin{aligned} x^2 + x + 1 &> 2 \\ x^2 + x - 1 &> 0 \\ (x^2 + x + \frac{1}{4}) - \frac{5}{4} &> 0 \\ \left(x + \frac{1}{2}\right)^2 &> \frac{5}{4} \\ \left|x + \frac{1}{2}\right| &> \frac{\sqrt{5}}{2} \\ x + \frac{1}{2} > \frac{\sqrt{5}}{2} \text{ or } x + \frac{1}{2} &< -\frac{\sqrt{5}}{2} \end{aligned}$$

so it must be that

$$x > \frac{\sqrt{5}-1}{2} \qquad \text{or} \qquad x < \frac{-\sqrt{5}-1}{2}$$

$\square$

(vii) $x^2 - x + 10 > 16$.

Proof. Using the field axioms

$$\begin{aligned} x^2 - x + 10 &> 16 \\ x^2 - x - 6 &> 0 \\ (x-3)(x+2) &> 0 \end{aligned}$$

To assure that the product is positive, it must be that the two numbers are both positive or both negative. Hence,

$$x - 3 > 0 \qquad \text{and} \qquad x + 2 > 0$$

such that $x > 3$. Or

$$x - 3 < 0 \qquad \text{and} \qquad x + 2 < 0$$

such that $x < -2$. Therefore, $x^2 - x + 10 > 16$ if $x > 3$ or $x < -2$. $\square$

(viii) $x^2 + x + 1 > 0$.

Proof. Using the field axioms

$$
\begin{aligned}
x^2 + x + 1 &= (x^2 + x + \frac{1}{4}) + \frac{3}{4}, \\
&= \left(x + \frac{1}{2}\right)^2 + \frac{3}{4}.
\end{aligned}
$$

for all $x \in \mathbb{R}$, notice that $(x + \frac{1}{2})^2 \geq 0$, so it must be that $(x + \frac{1}{2})^2 + \frac{3}{4} > 0$ for all $x \in \mathbb{R}$. □

(ix) $(x - \pi)(x + 5)(x - 3) > 0$.

Proof. The expression $(x - \pi)(x + 5)(x - 3)$ can be rearranged as a product of two numbers, namely, $(x - \pi)\left[(x + 5)(x - 3)\right]$.
Notice, the product of two real numbers ab is greater than zero if a and b are both greater than zero, or both less than zero.
There are two cases:

- Let $(x - \pi) > 0$ so that $x > \pi$, and $(x + 5)(x - 3) > 0$ so that $x < -5$ or $x > 3$. Therefore it must be that $x > \pi$.
- Let $(x - \pi) < 0$ so that $x < \pi$, and $(x + 5)(x - 3) < 0$ so that $-5 < x < 3$. Therefore it must be that $-5 < x < 3$.

Therefore, $(x - \pi)(x + 5)(x - 3) > 0$ if $x > \pi$, or $-5 < x < 3$. □

(x) $(x - \sqrt[3]{2})(x - \sqrt{2}) > 0$.

Proof. Either both numbers are greater than zero or less than zero.

$$x > \sqrt[3]{2} \qquad \text{and} \qquad x > \sqrt{2}$$

so that $x > \sqrt{2}$. Or

$$x < \sqrt[3]{2} \qquad \text{and} \qquad x < \sqrt{2}$$

so that $x < \sqrt[3]{2}$.
Therefore, $(x - \sqrt[3]{2})(x - \sqrt{2}) > 0$ if $x > \sqrt{2}$ or $x < \sqrt[3]{2}$. □

(xi) $2^x < 8$.

Proof. We can rewrite it as

$$2^x < 2^3$$

Both have the same base, so it must be that the inequality is preserved on the exponents.

$$x < 3$$

so $2^x < 8$, whenever $x < 3$. □

(xii) $x + 3^x < 4$.

Proof. We first notice that $x + 3^x = 4$ if $x = 1$

$$
\begin{aligned}
x + 3^x &= (1) + 3^1 \\
&= 4
\end{aligned}
$$

observe that $x + 3^x$ is always increasing as x increase, and decreasing as x decrease. Therefore $x + 3^x < 4$ if $x < 1$. □

(xiii) $\dfrac{1}{x} + \dfrac{1}{1-x} > 0.$

Proof. We can rewrite the expression as

$$\begin{aligned}\frac{1}{x} + \frac{1}{1-x} &= \frac{(1-x)+x}{x(1-x)} \\ &= \frac{1}{x(1-x)}\end{aligned}$$

Notice that $\frac{1}{x(1-x)} > 0$, whenever $x(1-x) > 0$. So it must be that x and $(1-x)$ are greater than zero

$$x > 0 \qquad \text{and} \qquad x < 1$$

or x and $(1-x)$ are both less than zero

$$x < 0 \qquad \text{and} \qquad x > 1$$

but there exists no x sucht that $x < 0$ and $x > 1$. Therefore, $\frac{1}{x} + \frac{1}{1-x} > 0$ if $x > 0$ and $x < 1$. □

(xiv) $\dfrac{x-1}{x+1} > 0.$

Proof. Either both $(x-1)$ and $(x+1)$ are greater than zero or both less than zero.

$$x > 1 \qquad \text{and} \qquad x > -1$$

so it must be that $x > 1$. Or

$$x < 1 \qquad \text{and} \qquad x < -1$$

so it must be that $x < -1$. □

Exercise (5). Prove the following:

(i) If $a < b$ and $c < d$, then $a + c < b + d$.

Proof. Assume that $a < b$ and $c < d$. Notice that $b - a > 0$ and $d - c > 0$, therefore their sum is also positive, namely,

$$(b - a) + (d - c) > 0$$

so that

$$a + c < b + d$$

□

(ii) If $a < b$, then $-b < -a$.

Proof. Assume that $a < b$. Therefore, $a - b < 0$. Notice that,

$$\begin{aligned} -(a - b) &< 0, \\ b - a &< 0, \\ -a &< -b. \end{aligned}$$

□

(iii) If $a < b$ and $c > d$, then $a - c < b - d$.

Proof. Assume that $a < b$ and $c > d$. Therefore, $a - b < 0$ and $c - d >$ so that $a - b < 0 < c - d$. Therefore,

$$\begin{aligned} a - b &< c - d, \\ a - c &< b - d. \end{aligned}$$

□

(iv) If $a < b$ and $c > 0$, then $ac < bc$.

Proof. Assume that $a < b$ and $c > 0$. Therefore, $b - a > 0$ and their product is positive,

$$\begin{aligned} (b - a) \cdot c &> 0, \\ bc - ac &> 0, \\ ac &< bc. \end{aligned}$$

□

(v) If $a < b$ and $c < 0$, then $ac > bc$.

Proof. Assume that $a < b$ and $c < 0$. Then $b - a > 0$ and $0 - c = -c > 0$, and their product must be positive,

$$\begin{aligned} (b - a) \cdot -c &> 0, \\ -bc + ac &> 0, \\ ac &> bc. \end{aligned}$$

□

(vi) If $a > 1$, then $a^2 > a$.

Proof. Assume that $a > 1$. Notice that $a - 1 > 0$ and $a > 0$, so their product is positive,

$$\begin{aligned}(a-1) \cdot a &> 0, \\ a^2 - a &> 0, \\ a^2 &> a.\end{aligned}$$

□

(vii) If $0 < a < 1$, then $a^2 < a$.

Proof. Assume that $0 < a < 1$, therefore $a > 0$ and $1 - a > 0$ so that their product is also positive,

$$\begin{aligned}(1-a) \cdot a &> 0, \\ a - a^2 &> 0, \\ a^2 &< a.\end{aligned}$$

□

(viii) If $0 \leq a < b$ and $0 \leq c < d$, then $ac < bd$.

Proof. Assume that $0 \leq a < b$ and $0 \leq c < d$. Notice that $bd > 0$, and if either one of a or c is equal to zero so that ac is zero, then

$$0 = ac < bd.$$

Otherwise, if a and c are greater than zero then ac is also greater than zero,

$$0 < ac < bc < bd.$$

□

(ix) If $0 \leq a < b$, then $a^2 < b^2$.

Proof. Assume that $0 \leq a < b$. If $a = 0$, then $a^2 = 0$ so that

$$a^2 < b^2.$$

Suppose that $a > 0$. From our assumption, $a < b$ therefore

$$a^2 < ab < b^2.$$

□

(x) If $a, b \geq 0$ and $a^2 < b^2$, then $a < b$.

Proof. Suppose for contradiction that $a, b \geq 0$ and $a^2 < b^2$, but $a \geq b$. So either $a = b$ or $a > b$.
If $a = b$, then $a^2 = b^2$, a contradiction. Now if $a > b \geq 0$, then

$$a^2 > ab > b^2$$

also a contradiction. Therefore it must be that $a < b$. □

Exercise (6). Prove the following:

(i) Prove that if $0 \le x < y$, then $x^n < y^n$, $n = 1, 2, 3, \ldots$

Proof. Assume that $0 \le x < y$. From the previous problems, it must be that $x^2 < y^2$, and so on for all $n \in \mathbb{N}$. □

(ii) Prove that if $x < y$ and n is odd, then $x^n < y^n$.

Proof. Assume tha $x < y$ and n is odd. There are three cases:

- If $0 \le x$, then by the previous exercise, it must be that

$$x^n < y^n.$$

- If $x < y \le 0$, then it must be that $0 \le -y < -x$. Therefore,

$$\begin{aligned} (-y)^n &< (-x)^n, \\ -y^n &< -x^n, \\ x^n &< y^n. \end{aligned}$$

- If $x < 0 \le y$, then since n is odd, it must be that

$$x^n < 0 \le y^n$$

so that $x^n < y^n$.

□

(iii) Prove that if $x^n = y^n$ and n is odd, then $x = y$.

Proof. If it is not, then it must be that $x^n < y^n$ or $x^n > y^n$. □

(iv) Prove that if $x^n = y^n$ and n is even, then $x = y$ or $x = -y$.

Proof. □