Mechanics Notes

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December 2, 2020

Vectors and Kinematics
(Summary Information) [6]

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0 Introduction

(Introduction Summary)

Notes on Introduction to Mechanics that is based on the text Kleppner and Kolenkow Mechanics and Cambridge Dynamics and Relativity Notes.

1 Vectors and Kinematics

1.1 Vector Addition

Definition (Vector Quantities)**.** Vector quantities have magnitude and direction. (Force, Displacement, Velocity, Acceleration)

Denoted ($\mathbf{A}$, $\mathbf{B}, \ldots$)

The magnitude of a vector $\mathbf{A}$, denoted by $|\mathbf{A}|$ or A, is a scalar quantity that is always positive.

Example. For a vector $\mathbf{B} = 10m$ in some direction, then $|\mathbf{B}| = 10m$

Definition (Vector Addition)**.** For the sum of the vectors $\mathbf{A}$ and $\mathbf{B}$ defined as $\mathbf{C}$.

$$\mathbf{A} + \mathbf{B} = \mathbf{C}$$

The sum is the diaganoal of the parallelogram formed by the $\mathbf{A}$ and $\mathbf{B}$.

1.2 Vector Components

Let $\mathbf{A}$ be a vector, we define $A_x, A_y,\ and\ A_z$ as the components parallel to their each respective axes. The components are not vector quantities.

The magnitude of $\mathbf{A}$ is

$$A = \sqrt{A_x^2 + A_y^2}$$

and the direction of $\mathbf{A}$ makes and angle

$$\theta = \arctan(\frac{A_y}{A_x})$$

The law for vector addition is

$$\mathbf{A} + \mathbf{B} = (A_x + B_x,\ A_y + B_y\ A_z + B_z)$$

1.3 Vector Multiplication

Definition (Scalar Product)**.** The *scalar product* is an operation that combines vectors to form a scalar, denoted as $\mathbf{A} \cdot \mathbf{B}$, called the *dot product* of $\mathbf{A}$ and $\mathbf{B}$. It is define by

$$\mathbf{A} \cdot \mathbf{B} = AB \cos\theta$$

where θ is the angle between $\mathbf{A}$ and $\mathbf{B}$ drawn tail-to-tail.

Because $B\cos\theta$ is the projection of $\mathbf{B}$ along the direction of $\mathbf{A}$, it follows that

$$\begin{aligned}\mathbf{A} \cdot \mathbf{B} &= A \text{ times the projection of } \mathbf{B} \text{ on } \mathbf{A}.\\ &= B \text{ times the projection of } \mathbf{A} \text{ on } \mathbf{B}.\end{aligned}$$

Hence, $\mathbf{A} \cdot \mathbf{A} = |\mathbf{A}|^2 = A^2$. Also, $\mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{A}$.

If either $\mathbf{A}$ or $\mathbf{B}$ is zero, their dot product is zero. However, because $\cos\frac{\pi}{2} = 0$ the dot product of perpendicular vectors is zero.

Example. The dot product is used on *Work*. The work W done on an object by a force F is defined to be the product of the length of the displacement d and the component of F along the direction of displacement. If the force is applied at an angle θ with respect to the displacement,

$$W = (F \cos\theta)d$$

. Which can be written as vectors

$$W = \mathbf{W} \cdot \mathbf{d}$$

.

Definition (Vector Product)**.** Two vectors $\mathbf{A}$ and $\mathbf{B}$ are combined to form another vector $\mathbf{C}$. The vector product is often called as *cross product*:

$$\mathbf{C} = \mathbf{A} \times \mathbf{B}$$

The magnitude is defined as

$$C = AB \sin\theta$$

where θ is the angle between $\mathbf{A}$ and $\mathbf{B}$ when drawn tail-to-tail.

To eliminate ambiguity, θ is always taken as the angle smaller than π. Even if neither vector is zero, their vector product is zero if $\theta = 0$ or π, and also if the vectors are parallel or anti parallel. It follows that

$$\mathbf{A} \times \mathbf{A} = 0$$

for any vector $\mathbf{A}$.

Two vectors $\mathbf{A}$ and $\mathbf{B}$ determine a plane. We define the cross product, $\mathbf{C}$ to be perpendicular to the plane of $\mathbf{A}$ and $\mathbf{A}$.

The convention is the *right-hand rule* and we can think of if as a right-hand screw, where $\mathbf{A} \times \mathbf{B}$ can be thought of as swinging $\mathbf{A}$ into $\mathbf{B}$, then $\mathbf{C}$, lies in the direction the screw advances. Hence,

$$\mathbf{B} \times \mathbf{A} \neq \mathbf{A} \times \mathbf{B}$$

.

Example. One application is the definition of *torque*. Let the torque vector τ be defined by

$$\tau = \mathbf{r} \times \mathbf{F}$$

where $\mathbf{r}$ is a vector from the axis about which the torque is evaluated to the point of application of the force $\mathbf{F}$. This definition is consistent with the familiar idea that torque is a measure of the ability of an applied force to produce a twist. Note that a large force directed parallel to $\mathbf{r}$ produces no twist; it merely pulls. Only $F \sin\theta$, the component of force perpendicular to $\mathbf{r}$, produces a torque.

When we push a gate open, we instinctively apply force in such a way as to make $\mathbf{F}$ closely perpendicular to $\mathbf{F}$, to maximize the torque. Because the torque increases as the lever arm gets larger, we push at the edge of the gate, as far from the hinge line as possible.

1.4 Base Vectors

Base vectors are orthogonal (mutually perpendicular) unit vectors, one for each dimension. In the Cartesian coordinate system of three dimensions, the base vectors lie along the x,y, and z axes. Denoted by $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$.

$$\begin{aligned} \mathbf{i}\cdot\mathbf{i} = \mathbf{j}\cdot\mathbf{j} = \mathbf{k}\cdot\mathbf{k} = 1 \\ \mathbf{i}\cdot\mathbf{j} = \mathbf{j}\cdot\mathbf{k} = \mathbf{k}\cdot\mathbf{i} = 0 \\ \mathbf{i}\times\mathbf{j} = \mathbf{k} \\ \mathbf{j}\times\mathbf{k} = \mathbf{i} \\ \mathbf{k}\times\mathbf{i} = \mathbf{j} \end{aligned}$$

Hence, we can write any vector in terms of its components and base vectors:

$$\mathbf{A} = A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k}$$

To find the component of a vector in any direction, take the dot product with a unit vector in that direction.

$$A_z = \mathbf{A}\cdot\mathbf{k}$$

1.5 Position vector r and Displacement

The components of $\mathbf{r}$ are the coordinates of the point referred to the particular coordinate axes.

The three numbers (x, y, z) do not represent components of a vector, they only specify a single point. The position of an arbitrary point P at (x, y, z) is written as:

$$\mathbf{r} = (x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

The displacement vector $\mathbf{S}$ from point (x_1, y_1, z_1) to (x_2, y_2, z_2) is *true vector* and is not dependent on the coordinate system.

Let $\mathbf{r}$ and $\mathbf{r}'$ indicate the same position drawn in different coordinate systems. If $\mathbf{R}$ is the vector from the origin of the unprimed coordinate system to the origin of the primed coordinate system, we have $\mathbf{r} = \mathbf{R} + \mathbf{r}'$.

$$\begin{aligned} \mathbf{S} &= \mathbf{r_2} - \mathbf{r_1} \\ &= (\mathbf{R} + \mathbf{r'_2} - (\mathbf{R} + \mathbf{r'_1})) \\ &= \mathbf{r'_2} - \mathbf{r'_1} \end{aligned}$$

hence, it is independent of the coordinate systems of the initil and final position.

1.6 Velocitiy and Aceeleration

1.6.1 Motion in One Dimension

The *average velocity* $\bar{v}$ of the point between two times t_1 and t_2 is defined by

$$\bar{v} = \frac{x(t_2) - x(t_1)}{t_2 - t_1}$$

. The *instantaneous velocity* v is the limit of the average velocity:

$$v = \lim_{\Delta t \to 0} \frac{x(t + \Delta t) - x(t)}{\Delta t}$$

which as we know as the derivative, hence, we write:

$$v = \frac{dx}{dt}$$

or as

$$v = \dot{x}$$

The *instantaneous acceleration* a is

$$\begin{aligned} a &= \lim_{\Delta t \to 0} \frac{v(t + \Delta t) - v(t)}{\Delta t} \\ &= \frac{dv}{dt} = \dot{v}. \end{aligned}$$

Using $v = dx/dt$,

$$a = \frac{d^2x}{dt^2} = \ddot{x}$$

Here d^2x/dt^2 is called the second derivative of x with respect to t.

1.6.2 Motion in Several Dimensions

The instantaneous position of the particle at time t_1 is

$$\mathbf{r}(t_1) = (x(t_1), y(t_1))$$

or

$$\mathbf{r}(t_1) = (x_1, y_2)$$

The displacement of the particel between time t_1 and t_2 is

$$\mathbf{r}(t_2) - \mathbf{r}(t_1) = (x_2 - x_1, y_2 - y_1)$$

.

The displacement of the particel during the interval Δt is

$$\Delta\mathbf{r} = \mathbf{r}(t + \Delta t) - \mathbf{r}(t)$$

. This vector equation is equivalent to two scalar equations

$$\begin{aligned} \Delta x &= x(t + \Delta t) - x(t) \\ \Delta y &= y(t + \Delta t) - y(t). \end{aligned}$$

The velocity $\mathbf{v}$ of the particle as it moves along the path is

$$\begin{aligned}\mathbf{v} &= \lim \Delta t \to 0 \frac{\Delta \mathbf{r}}{\Delta t} \\ &= \frac{d\mathbf{r}}{dt},\end{aligned}$$

which is equivalent to the two scalar equations

$$V_x = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = \frac{dx}{dt}$$

$$V_y = \lim_{\Delta t \to 0} \frac{\Delta y}{\Delta t} = \frac{dy}{dt}.$$

We can also start with the definition $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, and differentiate:

$$\frac{d\mathbf{r}}{dt} = \frac{d(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})}{dt}$$

Where we can treat the Cartesian base vectors as constants:

$$\frac{d\mathbf{r}}{dt} = \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} + \frac{dz}{dt}\mathbf{k}$$

Similarly, acceleration $\mathbf{a}$ is defined by:

$$\begin{aligned}\mathbf{a} &= \frac{d\mathbf{v}}{dt} = \frac{dV_x}{dt}\mathbf{i} + \frac{dV_y}{dt}\mathbf{j} + \frac{dV_z}{dt}\mathbf{k} \\ &= \frac{d^2\mathbf{r}}{dt}.\end{aligned}$$

1.7 Formal Solution to Kinematical Equations

If the acceleration is a known function of time, the velocity can be found by

$$\frac{d\mathbf{v}(t)}{dt} = \mathbf{a}(t)$$

integration with respect to time. Writing the vector as

$$\frac{dv_x}{dt}\mathbf{i} + \frac{dv_y}{dt}\mathbf{j} + \frac{dv_z}{dt}\mathbf{k} = a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k}.$$

We can seperate the corresponding components

$$\frac{dv_x}{dt} = a_x, \qquad \frac{dv_y}{dt} = a_y, \qquad \frac{dv_z}{dt} = a_z$$

If we know the velocity at time t_0, then we can integrate with respect to time to find the velocity at later time t_1:

$$\int_{t_0}^{t_1} \frac{dv_x}{dt}\mathrm{d}t = \int_{t_0}^{t_1} a_x \mathrm{d}t,$$

$$v_x(t_1) - v_x(t_0) = \int_{t_0}^{t_1} a_x(t)\mathrm{d}t,$$

$$v_x(t_1) = v_x(t_0) + \int_{t_0}^{t_1} a_x(t)\mathrm{d}t$$

Treating the y and z velocity components similarly, we have

$$\mathbf{v}(t_1) = \mathbf{v}(t_0) + \int_{t_0}^{t_1} \mathbf{a}(t)\mathrm{d}t$$

To express the velocity at an arbitrary time t we write

$$\mathbf{v}(t) = \mathbf{v}_0 + \int_{t_0}^{t} \mathbf{a}(t')\mathrm{d}t'.$$

Position is found by second integration. Starting with

$$\frac{d\mathbf{r}(t)}{dt} = \mathbf{v}(t),$$

by the same argument before, we get

$$\mathbf{r}(t) = \mathbf{r}_0 + \int_{t_0}^{t} \mathbf{v}(t')\mathrm{d}t'$$

In *uniform acceleration*. If we take $\mathbf{a} =$ constant and $t_0 = 0$, we get

$$\mathbf{v}(t) = \mathbf{v}_0 + \mathbf{a}t$$

$$\mathbf{r}(t) = \mathbf{r}_0 + \int_{0}^{t} (\mathbf{v}_0 + \mathbf{a}t')\mathrm{d}t',$$

or

$$\mathbf{r}(t) = \mathbf{r}_0 + \mathbf{v}_0 t + \frac{1}{2}\mathbf{a}t^2.$$

1.8 Polar Coordinates

1.9 Time Derivatives

2 Newtonian Mechanics

2.1 Newton's Laws

Definition (Particle)**.** A *particle* is an object of insignificant size. This means that if you want to say what a particle looks like at a given time, the only information you have to specify is its position.

To describe the position of a particle we need a *reference frame.* This is a choice of origin, together with a set of axes which, for now, we pick to be Cartesian. With respect to this frame, the position of a particle is specified by a vector $\mathbf{x}$. Since a particle moves, the position depends on time, resulting in a *trajectory* of the particle obatined by

$$\mathbf{x} = \mathbf{x}(t)$$

- **N1** Left alone, a particle moves with constant velocity.

  Intertial frames exist.

- **N2** The acceleration (or, more precisely, the rate of change of momentum) of a particle is proportional to the force acting upon it.

  $$\frac{d}{dt}(m\dot{\mathbf{x}}) = \mathbf{F}(\mathbf{x}, \dot{\mathbf{x}})$$

  where the momentum is

  $$\mathbf{p} \equiv m\dot{\mathbf{x}}$$

- **N3** Every action has an equal and opposite reaction.