

Spivak Calculus Exercises

Duncan Bandojo

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Chapter Title
(Summary Information)

[6]

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0 Introduction

These are exercises from Spivak Calculus. All the proofs given are my own proofs (unless stated otherwise) which is not assured for correctness and preciseness.

1 Prologue

1.1 Numbers of Various Sorts

Definition (Field Properties). The following properties hold in \mathbb{R}

- | | |
|---|---|
| P1 (Associative law for addition) | $a + (b + c) = (a + b) + c.$ |
| P2 (Existence of an additive identity) | $a + 0 = 0 + a = a.$ |
| P3 (Existence of additive inverse) | $a + (-a) = (-a) + a = 0.$ |
| P4 (Commutative law for addition) | $a + b = b + a.$ |
| P5 (Associative law for multiplication) | $a \cdot (b \cdot c) = (a \cdot b) \cdot c.$ |
| P6 (Existence of multiplicative identity) | $a \cdot 1 = 1 \cdot a = a; \quad 1 \neq 0.$ |
| P7 (Existence of multiplicative inverses) | $a \cdot a^{-1} = a^{-1} \cdot a = 1, \text{ for } a \neq 0.$ |
| P8 (Commutative law for multiplication) | $a \cdot b = b \cdot a.$ |
| P9 (Distributive law) | $a \cdot (b + c) = a \cdot b + a \cdot c.$ |

P10 (Trichotomy law) For every number a , one and only one of the following holds:
(Denote P as the collection of positive numbers)

- (i) $a = 0$,
- (ii) a is in the collection P ,
- (iii) $-a$ is in the collection P .

P11 (Closure under addition) If a and b are in P , then $a + b$ is in P .

P12 (Closure under multiplication) If a and b are in P , then $a \cdot b$ is in P .

Theorem (Triangle Inequality). For all numbers a and b , we have

$$|a + b| \leq |a| + |b|$$

Exercise (1). Prove the following:

- (i) If $ax = a$ for some number $a \neq 0$, then $x = 1$.

Proof. Assume that $ax = a$ for some number $a \neq 0$.

$$\begin{aligned} x = x \cdot 1 &= x \cdot (a \cdot a^{-1}) = ax \cdot (a^{-1}) \\ &= a \cdot (a^{-1}) \\ &= (a \cdot a^{-1}) \\ &= 1 \end{aligned}$$

□

- (ii) $x^2 - y^2 = (x - y)(x + y).$

Proof. Using the field axioms.

$$\begin{aligned} (x - y)(x + y) &= x \cdot (x + y) + (-y) \cdot (x + y) \\ &= (x^2 + xy) + ((-y) \cdot x + (-y) \cdot y) \\ &= x^2 + xy - xy - y^2 \\ &= x^2 - y^2 \end{aligned}$$

□

(iii) If $x^2 = y^2$, then $x = y$ or $x = -y$.

Proof. Assume that $x^2 = y^2$. We make use of (ii).

$$\begin{aligned} x^2 = y^2 &\Leftrightarrow x^2 - y^2 = 0 \\ &\Leftrightarrow (x - y)(x + y) = 0 \\ &\Rightarrow x = y \text{ or } x = -y. \end{aligned}$$

□

(iv) $x^3 - y^3 = (x - y)(x^2 + xy + y^2)$.

Proof. Using the field axioms

$$\begin{aligned} (x - y)(x^2 + xy + y^2) &= x^2(x - y) + xy(x - y) + y^2(x - y) \\ &= (x^3 - x^2y) + (x^2y - xy^2) + (xy^2 - y^3) \\ &= x^3 + (x^2y - x^2y) + (xy^2 - xy^2) - y^3 \\ &= x^3 - y^3 \end{aligned}$$

□

(v) $x^n - y^n = (x - y)(x^{n-1} + x^{n-2}y + \cdots + xy^{n-2} + y^{n-1})$.

Proof. Using the field axioms

$$\begin{aligned} (x - y)(x^{n-1} + x^{n-2}y + \cdots + xy^{n-2} + y^{n-1}) &= x(x^{n-1} + x^{n-2}y + \cdots + xy^{n-2} + y^{n-1}) \\ &\quad - [y(x^{n-1} + x^{n-2}y + \cdots + xy^{n-2} + y^{n-1})] \\ &= x^n + x^{n-1}y + \cdots + x^2y^{n-2} + xy^{n-1} \\ &\quad - [x^{n-1}y + x^{n-2}y^2 + \cdots + xy^{n-1} + y^n] \\ &= x^n - y^n \end{aligned}$$

□

Alternative Proof. We make use of sigma notation

$$\begin{aligned} (x - y) \cdot \sum_{i=0}^{n-1} x^i y^{n-(i+1)} &= x \left(\sum_{i=0}^{n-1} x^i y^{n-(i+1)} \right) - \left[y \left(\sum_{i=0}^{n-1} x^i y^{n-(i+1)} \right) \right] \\ &= \sum_{i=0}^{n-1} x^{i+1} y^{n-(i+1)} - \left[\sum_{i=0}^{n-1} x^i y^{n-i} \right] \\ &= x^n + \sum_{i=0}^{n-2} x^{i+1} y^{n-(i+1)} - \left[\sum_{i=1}^{n-1} x^i y^{n-i} + y^n \right] \\ &= x^n + \sum_{i=0}^{n-2} x^{i+1} y^{n-(i+1)} - \left[\sum_{i=0}^{n-2} x^{i+1} y^{n-(i+1)} + y^n \right] \\ &= x^n - y^n + \sum_{i=0}^{n-2} [x^{i+1} y^{n-(i+1)} - (x^{i+1} y^{n-(i+1)})] \\ &= x^n - y^n + \sum_{i=0}^{n-2} 0 \\ &= x^n - y^n \end{aligned}$$

□

(vi) $x^3 + y^3 = (x + y)(x^2 - xy + y^2)$.

Proof. Replace y by $-y$ in part (iv)

$$\begin{aligned} x^3 - y^3 &= (x - y)(x^2 + xy + y^2) \Leftrightarrow x^3 - (-y)^3 = (x - (-y))(x^2 + x(-y) + (-y)^2) \\ &\Leftrightarrow x^3 + y^3 = (x + y)(x^2 - xy + y^2) \end{aligned}$$

□

Exercise (2). What is wrong with the following "proof"? Let $x = y$. Then

$$\begin{aligned} x^2 &= xy, \\ x^2 - y^2 &= xy - y^2, \\ (x + y)(x - y) &= y(x - y), \\ x + y &= y, \\ 2y &= y, \\ 2 &= 1. \end{aligned}$$

Solution. For all $a \in \mathbb{R}$ we know that $a \cdot a^{-1} = 1$ with the assumption $a \neq 0$. The 4th step is contradictory on the given fact that $x = y$ which implies $x - y = 0$ and has no multiplicative inverse. □

Exercise (3). Prove the following:

(i) $\frac{a}{b} = \frac{ac}{bc}$, if $b, c \neq 0$.

Proof. Using the field axioms

$$\begin{aligned} \frac{a}{b} &= ab^{-1} = (ab^{-1})(c \cdot c^{-1}) \\ &= (ac)(b^{-1}c^{-1}) \\ &= (ac)(bc)^{-1} \\ &= \frac{ac}{bc} \end{aligned}$$

□

(ii) $\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$, if $b, d \neq 0$.

Proof. Using the field axioms

$$\begin{aligned} \frac{a}{b} + \frac{c}{d} &= ab^{-1} + cd^{-1} = (ab^{-1} + cd^{-1}) \cdot (bd)(bd)^{-1} \\ &= (ad(b \cdot b^{-1}) + bc(d \cdot d^{-1})) \cdot (bd)^{-1} \\ &= (ad + bc) \cdot (bd)^{-1} \\ &= \frac{ad + bc}{bd} \end{aligned}$$

□

(iii) $(ab)^{-1} = a^{-1}b^{-1}$, if $a, b \neq 0$.

Proof. Using the field axioms

$$\begin{aligned} ab(a^{-1}b^{-1}) &= 1 \\ a^{-1}b^{-1} &= (ab)^{-1} \end{aligned}$$

□

(iv) $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{db}$ if $b, d \neq 0$.

Proof. Using the field axioms

$$\begin{aligned} \frac{a}{b} \cdot \frac{c}{d} &= (ab^{-1}) \cdot (cd^{-1}) \\ &= (ac) \cdot (d^{-1}b^{-1}) \\ &= (ac) \cdot (db)^{-1} \\ &= \frac{ac}{db} \end{aligned}$$

□

(v) $\frac{a}{b} \bigg/ \frac{c}{d} = \frac{ad}{bc}$, if $b, d \neq 0$.

Proof. Using the field axioms

$$\begin{aligned} \frac{a}{b} \bigg/ \frac{c}{d} &= \frac{a}{b} \cdot \left(\frac{c}{d}\right)^{-1} \\ &= ab^{-1} \cdot (cd^{-1})^{-1} \\ &= ab^{-1} \cdot c^{-1}(d^{-1})^{-1} \\ &= ab^{-1} \cdot c^{-1}d \\ &= (ad) \cdot (b^{-1}c^{-1}) \\ &= (ad) \cdot (bc)^{-1} \\ &= \frac{ad}{bc} \end{aligned}$$

□

(vi) If $b, d \neq 0$, then $\frac{a}{b} = \frac{c}{d}$ if and only if $ad = bc$. Also determine when $\frac{a}{b} = \frac{b}{a}$.

Proof. There are two cases to prove for the first part.

(\Rightarrow) Let $b, d \neq 0$. Assume that $\frac{a}{b} = \frac{c}{d}$,

$$\begin{aligned}\frac{a}{b} &= \frac{c}{d}, \\ ab^{-1} &= cd^{-1}, \\ (ab^{-1})(bd) &= (cd^{-1})(bd), \\ (ad)(b \cdot b^{-1}) &= (bc)(d \cot d^{-1}), \\ ad &= bc.\end{aligned}$$

(\Leftarrow) Let $b, d \neq 0$. Assume that $ad = bc$,

$$\begin{aligned}ad &= bc, \\ (ad)(bd)^{-1} &= (bc)(bd)^{-1} \\ (ab^{-1})(d \cdot d^{-1}) &= (cd^{-1})(b \cdot b^{-1}) \\ ab^{-1} &= cd^{-1} \\ \frac{a}{b} &= \frac{c}{d}\end{aligned}$$

□

Proof. From Exercise 1 Part (iii) we make use of the fact, if $x^2 = y^2$ then $x = y$ or $x = -y$.

$$\begin{aligned}\frac{a}{b} &= \frac{b}{a}, \\ ab^{-1} &= ba^{-1}, \\ (ab^{-1})(ab) &= (ba^{-1})(ab), \\ (a \cdot a)(b \cdot b^{-1}) &= (b \cdot b)(a \cdot a^{-1}), \\ a^2 &= b^2.\end{aligned}$$

and so it must be that $a = b$ or $a = -b$.

□

Exercise (4). Find all numbers x for which

(i) $4 - x < 3 - 2x$.

Proof. Using the field axioms

$$\begin{aligned} 4 - x &< 3 - 2x \\ 4 - x + (2x - 4) &< 3 - 2x + (2x - 4) \\ x &< -1 \end{aligned}$$

□

(ii) $5 - x^2 < 8$.

Proof. Using the field axioms

$$\begin{aligned} 5 - x^2 + (x^2 - 5) &< 8 + (x^2 - 5) \\ x^2 + 3 &> 0 \end{aligned}$$

since $x^2 \geq 0$ for all $x \in \mathbb{R}$, then it must be that $x^2 + 3 > 0$ for all $x \in \mathbb{R}$.

□

(iii) $5 - x^2 < -2$

Proof. Using the field axioms

$$\begin{aligned} 5 - x^2 &< -2 \\ x^2 &> 7 \\ |x| &> \sqrt{7} \\ x &< -\sqrt{7} \text{ or } x > \sqrt{7} \end{aligned}$$

□

(iv) $(x - 3)(x - 1) > 0$ (When is a product of two numbers positive?)

Proof. The product of two numbers is positive if and only if the numbers are both positive or both negative. For all $a, b \in \mathbb{R}$, $ab > 0 \Leftrightarrow a > 0$ and $b > 0$, or $a < 0$ and $b < 0$.

Hence,

$$x - 3 > 0 \quad \text{and} \quad x - 1 > 0$$

so it must be that $x > 3$. Or

$$x - 3 < 0 \quad \text{and} \quad x - 1 < 0$$

and it must be that $x < 1$. That is $(x - 3)(x - 1) > 0$ if $x > 3$ or $x < 1$.

□

(v) $x^2 - 2x + 2 > 0$.

Proof. Using the field axioms

$$\begin{aligned} x^2 - 2x + 2 &= (x^2 + 2x + 1) + 1 \\ &= (x - 1)^2 + 1 \end{aligned}$$

for all $x \in \mathbb{R}$ notice that, $(x - 1)^2 \geq 0$, so it must be that $(x - 1)^2 + 1 > 0$. \square

(vi) $x^2 + x + 1 > 2$.

Proof. Using the field axioms

$$\begin{aligned} x^2 + x + 1 &> 2 \\ x^2 + x - 1 &> 0 \\ (x^2 + x + \frac{1}{4}) - \frac{5}{4} &> 0 \\ \left(x + \frac{1}{2}\right)^2 &> \frac{5}{4} \\ \left|x + \frac{1}{2}\right| &> \frac{\sqrt{5}}{2} \\ x + \frac{1}{2} &> \frac{\sqrt{5}}{2} \text{ or } x + \frac{1}{2} < -\frac{\sqrt{5}}{2} \end{aligned}$$

so it must be that

$$x > \frac{\sqrt{5}-1}{2} \quad \text{or} \quad x < \frac{-\sqrt{5}-1}{2}$$

\square

(vii) $x^2 - x + 10 > 16$.

Proof. Using the field axioms

$$\begin{aligned} x^2 - x + 10 &> 16 \\ x^2 - x - 6 &> 0 \\ (x - 3)(x + 2) &> 0 \end{aligned}$$

To assure that the product is positive, it must be that the two numbers are both positive or both negative. Hence,

$$x - 3 > 0 \quad \text{and} \quad x + 2 > 0$$

such that $x > 3$. Or

$$x - 3 < 0 \quad \text{and} \quad x + 2 < 0$$

such that $x < -2$. Therefore, $x^2 - x + 10 > 16$ if $x > 3$ or $x < -2$. \square

(viii) $x^2 + x + 1 > 0$.

Proof. Using the field axioms

$$\begin{aligned} x^2 + x + 1 &= \left(x^2 + x + \frac{1}{4}\right) + \frac{3}{4}, \\ &= \left(x + \frac{1}{2}\right)^2 + \frac{3}{4}. \end{aligned}$$

for all $x \in \mathbb{R}$, notice that $(x + \frac{1}{2})^2 \geq 0$, so it must be that $(x + \frac{1}{2})^2 + \frac{3}{4} > 0$ for all $x \in \mathbb{R}$. \square

(ix) $(x - \pi)(x + 5)(x - 3) > 0$.

Proof. The expression $(x - \pi)(x + 5)(x - 3)$ can be rearranged as a product of two numbers, namely, $(x - \pi)[(x + 5)(x - 3)]$.

Notice, the product of two real numbers ab is greater than zero if a and b are both greater than zero, or both less than zero.

There are two cases:

- Let $(x - \pi) > 0$ so that $x > \pi$, and $(x + 5)(x - 3) > 0$ so that $x < -5$ or $x > 3$. Therefore it must be that $x > \pi$.
- Let $(x - \pi) < 0$ so that $x < \pi$, and $(x + 5)(x - 3) < 0$ so that $-5 < x < 3$. Therefore it must be that $-5 < x < 3$.

Therefore, $(x - \pi)(x + 5)(x - 3) > 0$ if $x > \pi$, or $-5 < x < 3$. \square

(x) $(x - \sqrt[3]{2})(x - \sqrt{2}) > 0$.

Proof. Either both numbers are greater than zero or less than zero.

$$x > \sqrt[3]{2} \quad \text{and} \quad x > \sqrt{2}$$

so that $x > \sqrt{2}$. Or

$$x < \sqrt[3]{2} \quad \text{and} \quad x < \sqrt{2}$$

so that $x < \sqrt[3]{2}$.

Therefore, $(x - \sqrt[3]{2})(x - \sqrt{2}) > 0$ if $x > \sqrt{2}$ or $x < \sqrt[3]{2}$. \square

(xi) $2^x < 8$.

Proof. We can rewrite it as

$$2^x < 2^3$$

Both have the same base, so it must be that the inequality is preserved on the exponents.

$$x < 3$$

so $2^x < 8$, whenever $x < 3$. \square

(xii) $x + 3^x < 4$.

Proof. We first notice that $x + 3^x = 4$ if $x = 1$

$$\begin{aligned} x + 3^x &= (1) + 3^1 \\ &= 4 \end{aligned}$$

observe that $x + 3^x$ is always increasing as x increase, and decreasing as x decrease. Therefore $x + 3^x < 4$ if $x < 1$. \square

$$(xiii) \quad \frac{1}{x} + \frac{1}{1-x} > 0.$$

Proof. We can rewrite the expression as

$$\begin{aligned} \frac{1}{x} + \frac{1}{1-x} &= \frac{(1-x) + x}{x(1-x)} \\ &= \frac{1}{x(1-x)} \end{aligned}$$

Notice that $\frac{1}{x(1-x)} > 0$, whenever $x(1-x) > 0$. So it must be that x and $(1-x)$ are greater than zero

$$x > 0 \quad \text{and} \quad x < 1$$

or x and $(1-x)$ are both less than zero

$$x < 0 \quad \text{and} \quad x > 1$$

but there exists no x such that $x < 0$ and $x > 1$. Therefore, $\frac{1}{x} + \frac{1}{1-x} > 0$ if $x > 0$ and $x < 1$. \square

$$(xiv) \quad \frac{x-1}{x+1} > 0.$$

Proof. Either both $(x-1)$ and $(x+1)$ are greater than zero or both less than zero.

$$x > 1 \quad \text{and} \quad x > -1$$

so it must be that $x > 1$. Or

$$x < 1 \quad \text{and} \quad x < -1$$

so it must be that $x < -1$. \square

Exercise (5). Prove the following:

- (i) If $a < b$ and $c < d$, then $a + c < b + d$.

Proof. Assume that $a < b$ and $c < d$. Notice that $b - a > 0$ and $d - c > 0$, therefore their sum is also positive, namely,

$$(b - a) + (d - c) > 0$$

so that

$$a + c < b + d$$

□

- (ii) If $a < b$, then $-b < -a$.

Proof. Assume that $a < b$. Therefore, $a - b < 0$. Notice that,

$$\begin{aligned} -(a - b) &< 0, \\ b - a &< 0, \\ -a &< -b. \end{aligned}$$

□

- (iii) If $a < b$ and $c > d$, then $a - c < b - d$.

Proof. Assume that $a < b$ and $c > d$. Therefore, $a - b < 0$ and $c - d > 0$ so that $a - b < 0 < c - d$. Therefore,

$$\begin{aligned} a - b &< c - d, \\ a - c &< b - d. \end{aligned}$$

□

- (iv) If $a < b$ and $c > 0$, then $ac < bc$.

Proof. Assume that $a < b$ and $c > 0$. Therefore, $b - a > 0$ and their product is positive,

$$\begin{aligned} (b - a) \cdot c &> 0, \\ bc - ac &> 0, \\ ac &< bc. \end{aligned}$$

□

- (v) If $a < b$ and $c < 0$, then $ac > bc$.

Proof. Assume that $a < b$ and $c < 0$. Then $b - a > 0$ and $0 - c = -c > 0$, and their product must be positive,

$$\begin{aligned} (b - a) \cdot -c &> 0, \\ -bc + ac &> 0, \\ ac &> bc. \end{aligned}$$

□

(vi) If $a > 1$, then $a^2 > a$.

Proof. Assume that $a > 1$. Notice that $a - 1 > 0$ and $a > 0$, so their product is positive,

$$\begin{aligned}(a - 1) \cdot a &> 0, \\ a^2 - a &> 0, \\ a^2 &> a.\end{aligned}$$

□

(vii) If $0 < a < 1$, then $a^2 < a$.

Proof. Assume that $0 < a < 1$, therefore $a > 0$ and $1 - a > 0$ so that their product is also positive,

$$\begin{aligned}(1 - a) \cdot a &> 0, \\ a - a^2 &> 0, \\ a^2 &< a.\end{aligned}$$

□

(viii) If $0 \leq a < b$ and $0 \leq c < d$, then $ac < bd$.

Proof. Assume that $0 \leq a < b$ and $0 \leq c < d$. Notice that $bd > 0$, and if either one of a or c is equal to zero so that ac is zero, then

$$0 = ac < bd.$$

Otherwise, if a and c are greater than zero then ac is also greater than zero,

$$0 < ac < bc < bd.$$

□

(ix) If $0 \leq a < b$, then $a^2 < b^2$.

Proof. Assume that $0 \leq a < b$. If $a = 0$, then $a^2 = 0$ so that

$$a^2 < b^2.$$

Suppose that $a > 0$. From our assumption, $a < b$ therefore

$$a^2 < ab < b^2.$$

□

(x) If $a, b \geq 0$ and $a^2 < b^2$, then $a < b$.

Proof. Suppose for contradiction that $a, b \geq 0$ and $a^2 < b^2$, but $a \geq b$. So either $a = b$ or $a > b$.

If $a = b$, then $a^2 = b^2$, a contradiction. Now if $a > b \geq 0$, then

$$a^2 > ab > b^2$$

also a contradiction. Therefore it must be that $a < b$.

□