Precalculus Notes

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11-Newton

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0 Introduction

Welcome to Precalculus Notes

1 Conic Sections

Definition (Conic Section)**.** A *conic* is the curve obtained as the intersection of a plane, called the cutting plane, with the surface of a double right circular cone.

Definition (Circle)**.** The *circle* is obtained when the cutting plane is perpendicular to the axis of symmetry of the cone and parallel to the plane of the generating *circle* of the cone.

Definition (Parabola)**.** If the cutting plane is parallel to exactly one generating line of the cone, then the conic is unbounded and is called a *parabola*

Definition (Ellipse)**.** *Ellipses* arise when the intersection of the cone and plane is a closed curve. The plane that intersects the curve is neither parallel or perpendicular to the axis of symmetry.

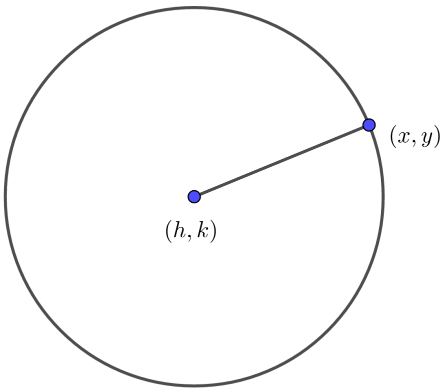
Definition (Hyperbola)**.** The plane that intersects the cone is parallel to the axis of symmetry of the cone, cutting both halves of the cone, producing two separate unbounded curves.

1.1 Circle

Definition (Circle)**.** A circle with center (h, k) and radius $r > 0$ is the set of all points (x, y) in the plane whose distance to (h, k) is r.

$$r = \sqrt{(x-h)^2 + (y-k)^2}$$

By squaring both sides of this equation, we get an equivalent equation (since $r > 0$) which gives us the



Definition (Standard Equation of a Circle)**.** The equation of a circle with center (h, k) and radius $r > 0$ is

$$(x-h)^2 + (y-k)^2 = r^2$$

Definition (General Equation of a Circle)**.**

$$x^2 + y^2 + Dx + Ey + F = 0$$

For some constants D, E, and F.

There are 3 cases that might happen to the radius r.

- $r^2 > 0 \to$ Circle
- $r^2 = 0 \to$ Degenerate Circle or Point Circle
- $r^2 < 0 \to \emptyset$

Definition (Distance from Point to Line)**.** Distance from a point (x_1, y_1) to a line $Ax + By + C = 0$.

$$d = \frac{\sqrt{Ax_1 + By_1 + C}}{\sqrt{A^2 + B^2}}$$

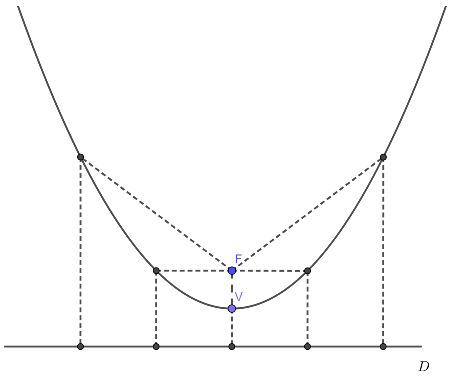
1.2 Parabola

Definition (Parabola)**.** Let F be a point in the plane and ℓ be a line not containing F. A *parabola* is the set of all points equidistant from F and ℓ. The point F is called the *focus* of the parabola and the line ℓ is called the *directrix* of the *parabola*.

Definition (Standard Equation of a Vertical Parabola)**.** The equation with Vertex $V:(h,k)$ and focal length c is

$$(x-h)^2 = \pm 4c(y-k)$$

Opening either upward or downward.



Definition (Standard Equation of a Horizontal Parabola)**.** The equation with Vertex $V:(h,k)$ and focal length c is

$$(y-h)^2 = \pm 4c(x-k)$$

Opening either to the right or left.

Parts of Parabola

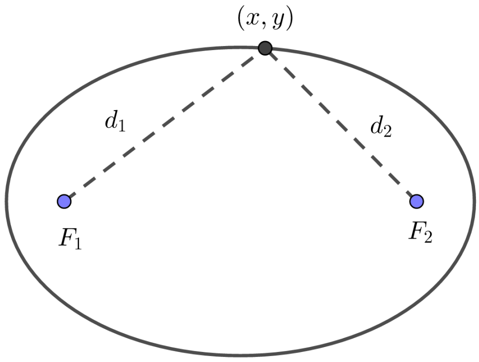
- *Vertex* $\rightarrow$ The point midway between the focus and the directrix.
- *Focus* $\rightarrow$ Fixed point that is c units above or below the vertex. (right or left if Horizontal Parabola)
- *Directrix* $\rightarrow$ Fixed line that is c units above or below the vertex. (right or left if Horizontal Parabola)
- *Axis of Symettry* $\rightarrow$ The line that divides the parabola into two parts which are mirror images of each other.
- *Latus Rectum* $\rightarrow$ The line segment through the focus perpendicular to the axis of symmetry and whose length is $4c$ called the *focal diameter*.

1.3 Ellipse

Definition (Ellipse)**.** Given two distinct points F_1 and F_2 in the plane and a fixed distance d, an *ellipse* is the set of all points (x, y) in the plane such that the sum of each of the distances from F_1 and F_2 to (x, y) is d. The points F_1 and F_2 are called the *foci* of the *ellipse*.

Definition (Standard Equation of a Horizontal Ellipse)**.** The standard equation of an ellipse with Center $C : (h, k)$ with the *major axis* of length $2a$ along the x-axis and a *minor axis* of length $2b$ along the y-axis is

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$



From the definition of an ellipse, defining the foci $F_1 : (-c, 0)$ and $F_2 : (c, 0)$ and $V : (a, 0)$, lying all on the same line, we can conclude the following:

$$\begin{aligned} \text{distance from } (-c,0) \text{ to } (a,0) + \text{distance from } (c,0) \text{ to } (a,0) &= d \\ (a-c) + (a+c) &= d \\ 2a &= d \end{aligned}$$

Define the following, the foci are $F_1 : (-c, 0)$ and $F_2 : (c, 0)$, and letting a point on the ellipse be a covertex $P : (b, 0)$, from the definition of an ellipse, we can conclude the following:

$$\begin{aligned} \text{distance from } (-c,0) \text{ to } (0,b) + \text{distance from } (c,0) \text{ to } (0,b) &= d \\ \sqrt{(-c-0)^2 + (0-b)^2} + \sqrt{(c-0)^2 + (0-b)^2} &= 2a \\ \sqrt{c^2+b^2} + \sqrt{c^2+b^2} &= 2a \\ 2\sqrt{c^2+b^2} &= 2a \\ \sqrt{c^2+b^2} &= a \end{aligned}$$

From this we get $a^2 = b^2 + c^2$, or $c^2 = a^2 - b^2$.

Definition (Standard Equation of a Vertical Ellipse)**.** The standard equation of an ellipse with Center $C : (h, k)$ with the *major axis* of length $2a$ along the y-axis and a *minor axis* of length $2b$ along the x-axis is

$$\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1$$

Parts of Ellipse

- *Center* $\rightarrow$ The intersection of the major and minor axis. The midpoint of the line segment joining the foci.
- *Foci* $\rightarrow$ Each focus is c units away from the center. For any points on the ellipse, the sum of its distances from the focis is $2a$.
- *Vertices* $\rightarrow$ Points on the ellipse that are collinear with the center and foci. Each vertex is a units away from the center. Endpoints of the *major axis*.
- *Covertices* $\rightarrow$ Points on the ellipse that are b units away from the center. Endpoints of the *minor axis*.
- *Directrices* $\rightarrow$ Each *directrix* is parallel to the *minor axis* and is located outside the curve.

$$\textit{Horizontal Ellipse}\colon\ x = \pm\frac{a^2}{c} \qquad \textit{Vertical Ellipse}\colon\ y = \pm\frac{a^2}{c}$$

- *Latera Recta* $\rightarrow$ *Latus Rectume* of an ellipse is a line segment perpendicular to the major axis through any of the foci and whose endpoints lie on the ellipse.

$$\text{Length of Latus Rectum: } \frac{2b^2}{a}$$

- *Major Axis* $\rightarrow$ The line segment joining the vertices and contains the foci.

$$\text{Length of Major Axis: } 2a$$

- *Minor Axis* $\rightarrow$ The line segment joining the covertices.

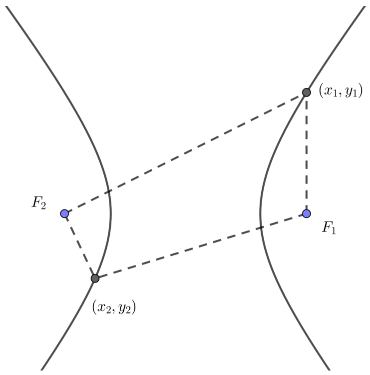
$$\text{Length of Minor Axis: } 2b$$

- *Eccentricity* $\rightarrow$ denoted by e is the ratio

$$e = \frac{\text{Distance from center to foci}}{\text{Distance from center to vertex}} = \frac{c}{a}$$

1.4 Hyperbola

Definition. Given two distinct points F_1 and F_2 in the plane and a fixed distance d, a hyperbola is the set of all points (x, y) in the plane such that the absolute value of the difference of each of the distances from F_1 and F_2 to (x, y) is d. The points F_1 and F_2 are called the foci of the hyperbola.



Since $V : (a, 0)$ is on the hyperbola, it must satisfy the definition. That is, the distance from $(-c, 0)$ to $(a, 0)$ minus the distance from $(c, 0)$ to $(a, 0)$ must equal the fixed distance d. Since all these points lie on the same line, we get

$$\begin{aligned} |\text{distance from } (-c,0) \text{ to } (a,0) - \text{distance from } (c,0) \text{ to } (a,0)| &= d \\ |(a+c) - (c-a)| &= d \\ |2a| &= d \\ 2a &= d \end{aligned}$$

We also have the following equation

$$c^2 = a^2 + b^2$$

Definition (Standard Equation of a Horizontal Hyperbola)**.** For positive numbers a and b, the equation of a horizontal hyperbola with center (h, k) is:

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

A hyperbola with branches that opens to the left and right.

If the roles of x and y were interchanged, then the hyperbola's branches would open upwards and downwards and we would get a *Vertical Hyperbola*

Definition (Standard Equation of a Vertical Hyperbola)**.** For positive numbers a and b, the equation of a horizontal hyperbola with center (h, k) is:

$$\frac{(y-h)^2}{a^2} + \frac{(x-k)^2}{b^2} = 1$$

Parts of Hyperbola

- *Transverse Axis* $\rightarrow$ The equivalent of the major axis of an ellipse. It contains the foci, center, and vertices as its endpoints.
- *Conjugate Axis* $\rightarrow$ The equivalent of the minor axis of an ellipse. The line segment which is a perpendicular bisector of the transverse axis.
- *Latera Recta* $\rightarrow$ A line segment that contains the foci and perpendicular to the transverse axis.

$$\text{Length of Latus a Rectum: } \frac{2b^2}{a}$$

- *Directrices* $\rightarrow$ A directrix is parallel to the conjugate axis.

$$\text{Distance from center: } \frac{a^2}{c}$$

- *Asymptotes* $\rightarrow$ Two lines passing through the center with each branch of the hyperbola approaching the asymptotes as $|x| \rightarrow \infty$.

$$\text{Equation of Asymptotes: } y = \pm\frac{b}{a}x$$

1.5 Applications of Parabola

Example. The cable of a suspension bridge hangs in the shape of a parabola. The towers supporting the cable are 400 ft apart and 150 ft high. If the cable, at its lowest, is 30 ft above the bridge at its midpoint, how high is the cable 50 ft away (horizontally) from either tower?

Proof. We may write it with the equation $(x-0)^2 = a(y-30)$; since we dont need the focal distance,we use the simpler variable a in place of $4c$. Since the towers are 150 ft high and 400 ft apart, we deduce that $(200, 150)$ is a point on the parabola.

$$x^2 = a(y-30)$$

$$(200)^2 = a(150-30)$$

$$a = \frac{200^2}{120} = \frac{1000}{3}$$

The parabola has equation $x^2 = \frac{1000}{3}(y-30)$ or equivalently, $y = 0.003x^2 + 30$. For the two points on the parabola 50 ft away from the towers, $x = 150$ or $x = -150$. If $x = 150$, then

$$y = 0.0003(150)^2 + 30 = 97.5.$$

Thus the cable is 97.5 ft high 50 ft away from the tower. □

Example. A satellite dish has a hape calleda paraboloid, where each cross-section is a parabola. Since radio signals (parallel to the axis) will bounce off the surface of the dish to the focus, the receiver should be placed at the focus. How far should the receiver be from the vertex, if the dish is 12 ft across and 4.5 ft deep at the vertex?

Proof. We take a cross-section of the satellite dish drawn on a rectungular coordinate system, with the vertex at the origin. From the problem, we deduce that $(6, 4.5)$ is a point on the parabola. We need the distance of the focus from the vertex, i.e, the value of c in $x^2 = 4cy$.

$$x^2 = 4cy$$

$$6^2 = 4c(4.5)$$

$$c = \frac{36}{4 \cdot 4.5} = 2$$

Thus, the receiver should be 2 ft away from the vertex. □

1.6 Applicatios of Hyperbola

Example. An explosion is recorded by two microphones that are 10km apart. the first microphone receives the sound 3 seconds before the second microphone. Assuming the sound travels at 1.2 km/s, determing the possible locations of the expolosion relative to the location of the microphone.

Proof. Let M_A be the first microphone and M_B the second microphone which are the foci the hyperbola. Let the explosion be at the point : $V : (a, 0)$. It is given that M_A received the sound 3 seconds before M_B given that sound travels at 1.2 km/s, we can solve for the distance between the two vertices.

$$3 \cdot 1.2 = 3.6\ km = 2a$$

$$a = 1.8$$

$$a^2 = 3.24$$

We can assume that the hyperbola is of the form

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

We know that the distance between M_A and M_B is 10 km which serves as the focal distance $2c$.

$$2c = 10$$

$$c = 5$$

$$c^2 = 25$$

Using $c^2 = a^2 + b^2$ we get

$$c^2 = a^2 + b^2$$

$$b^2 = c^2 - a^2$$

$$b^2 = 25 - 3.24$$

$$b^2 = 21.76$$

The equation of the hyperbola with microphones at each focus is

$$\frac{x^2}{3.24} - \frac{y^2}{21.76} = 1$$

If we assume that M_A resides at the right side, then the explosion occured on the right branch of the hyperbola, which is closer to M_A. □

2 Linear Systems of Equations

We will discuss Linear Algebra instead of basic Linear equations. Topics such as matrices, pivots, gaussian elimination, spans, linear combinations, vector spaces, and linear transformations

2.1 Linear Equations

Definition (Linear Equations)**.** A *linear equation* is an equation that may be put in the form

$$a_1x_1 + \cdots + a_nx_n + b = 0$$

where $x_1, \ldots, x_n$ are the variables, and $b, a_1, \ldots, a_n$ are the coefficients. (where often it is the case that $x_i \in \mathbb{R}$).

We can solve system of linear equations for example in $\mathbb{R}^2$ where there are only two variables.

Example. If we graph the two lines

$$3x + 2y = 4$$

$$6x + 47 = 5$$

we can see that they are parallel and do not intersect, so that this system of linear equations has no solution.

Example. If we graph the two lines

$$3x + 2y = 5$$

$$x + y = 2$$

it is easy to see that the two lines are not parallel and intersect at the point $(1, 1)$, so that this system of two linear equations has exactly one solution.

Example. If we graph the two lines

$$3x + 2y = 5$$

$$6x + 4y = 10$$

It is easy to see that the two lines overlap completely, so that this system of two linear equations has infinitely many solutions.

In general, we shall study a system of m linear equations of the form

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= d_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= d_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= d_m \end{aligned}$$

with n variables $x_1, x_2, \cdots, x_n$. Here we may not be so lucky as to be able to see geometrically what is going on. We therefore need to study the problem from a more algebraic viewpoint.

We can omit the variables, then the system can be represented by an array of all the coefficients known as the *augmented matrix*.

$$\left(\begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & b_3 \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_4 \end{array}\right)$$

We can also write it as $A\mathbf{x} = \mathbf{b}$, a matrix A multiplied by the vector $\mathbf{x}$ is the vector $\mathbf{b}$, where

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}$$

and

$$\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

represents the coefficients and

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix}$$

represents the variables.

Example. The array

$$\left(\begin{array}{ccccc|c} 1 & 3 & 1 & 5 & 1 & 5 \\ 0 & 1 & 1 & 2 & 1 & 4 \\ 2 & 4 & 0 & 7 & 1 & 3 \end{array}\right)$$

represents the three linear equations

$$\begin{aligned} x_1 + 3x_2 + x_3 + 5x_4 + x_5 &= 5, \\ x_2 + x_3 + 2x_4 + x_5 &= 4, \\ 2x_1 + 4x_2 + +7x_4 + x_5 &= 3, \end{aligned}$$

with five variables x_1, x_2, x_3, x_4, x_5. We can also write

$$\begin{pmatrix} 1 & 3 & 1 & 5 & 1 \\ 0 & 1 & 1 & 2 & 1 \\ 2 & 4 & 0 & 7 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 5 \\ 4 \\ 3 \end{pmatrix}$$

2.2 Elementary Row Operations

Proposition (Elementary Row Operations)**.** There are three types of elementary matrices, which correspond to three types of row operations

1. **Row Switching** $\rightarrow$ A row within the matrix can be switched with another row.
$$R_i \leftrightarrow R_j$$
2. **Row Multiplication** $\rightarrow$ Each element in a row can be multiplied by a non-zero constant.
$$kR_i \rightarrow R_i, \text{ where } k \neq 0$$
3. **Row Addition** $\rightarrow$ A row can be replaced by the sum of that row and a multiple of another row.
$$R_i + kR_j \rightarrow R_i, \text{ where } i \neq j$$

Example. Consider again the system of linear equations

$$\begin{aligned} x_1 + 3x_2 + x_3 + 5x_4 + x_5 &= 5, \\ x_2 + x_3 + 2x_4 + x_5 &= 4, \\ 2x_1 + 4x_2 + +7x_4 + x_5 &= 3, \end{aligned}$$

represented by the array
$$\left(\begin{array}{ccccc|c} 1 & 3 & 1 & 5 & 1 & 5 \\ 0 & 1 & 1 & 2 & 1 & 4 \\ 2 & 4 & 0 & 7 & 1 & 3 \end{array}\right)$$

Proof. Let us now perform elementary row operations on the augmented matrix. We first label the rows as $R_1, R_2,$ *and* R_3.

Adding -2 times the first row to the third row.

$$-2R_1 + R_3 \rightarrow R_3 \qquad \left(\begin{array}{ccccc|c} 1 & 3 & 1 & 5 & 1 & 5 \\ 0 & 1 & 1 & 2 & 1 & 4 \\ 0 & -2 & -2 & -3 & -1 & -7 \end{array}\right)$$

From here, we add 2 times the second row to the third row to obtain

$$2R_2 + R_3 \rightarrow R_3 \qquad \left(\begin{array}{ccccc|c} 1 & 3 & 1 & 5 & 1 & 5 \\ 0 & 1 & 1 & 2 & 1 & 4 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{array}\right)$$

Next, we add -3 times the second row to the first row to obtain.

$$-3R_2 + R_1 \rightarrow R_3 \qquad \left(\begin{array}{ccccc|c} 1 & 0 & -2 & -1 & -2 & -7 \\ 0 & 1 & 1 & 2 & 1 & 4 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{array}\right)$$

Next, we add the third row to the first row to obtain

$$3R_3 + R_1 \rightarrow R_3 \qquad \left(\begin{array}{ccccc|c} 1 & 0 & -2 & 0 & -1 & -6 \\ 0 & 1 & 1 & 2 & 1 & 4 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{array}\right)$$

Finally, we add -2 times the third row to the second row to obtain

$$-2R_3 + R_2 \to R_3 \qquad \left(\begin{array}{ccccc|c} 1 & 0 & -2 & 0 & -1 & -6 \\ 0 & 1 & 1 & 0 & -1 & 2 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{array}\right)$$

We know that this augmented matrix is equivalent to the system of linear equations:

$$\begin{aligned} x_1 + -2x_3 + -x_5 &= -6, \\ x_2 + x_3 + -x_5 &= 2, \\ x_4 + x_5 &= 1, \end{aligned}$$

First of all, take the third equation

$$x_4 + x_5 = 1$$

If we let $x_5 = t$, then $x_4 = 1 - t$. Substituting to the second equation, we obtain

$$x_2 + x_3 = 2 + t$$

If we let $x_3 = s$, then $x_2 = 2 + t - s$. Substituting all these info into the first equation, we obtaion

$$x_1 = -6 + t + 2s$$

Hence

$$\mathbf{x} = (x_1, x_2, x_3, x_4, x_5) = (-6 + t + 2s, 2 + t - s, s, 1 - t, t)$$

is a solution of the system of linear equations for every $s, t \in \mathbb{R}$. $\square$

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