Spivak Calculus Notes and Exercises

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Contents

0 Introduction

These are notes and selected exercises from Spivak Calculus. All the proofs given are my own proofs (unless stated otherwise) which is not assured for correctness and preciseness.

1 Prologue

1.1 Basic Properties of Numbers

Definition (Field Properties)**.** The following properties hold in $\mathbb{R}$

- P1 (Associative law for addition) $a + (b + c) = (a + b) + c.$
- P2 (Existence of an additive identity) $a + 0 = 0 + a = a.$
- P3 (Existence of additive inverse) $a + (-a) = (-a) + a = 0.$
- P4 (Commutative law for addition) $a + b = b + a.$
- P5 (Associative law for multiplication) $a \cdot (b \cdot c) = (a \cdot b) \cdot c.$
- P6 (Existence of multiplicative identity) $a \cdot 1 = 1 \cdot a = a; \quad 1 \neq 0.$
- P7 (Existence of multiplicative inverses) $a \cdot a^{-1} = a^{-1} \cdot a = 1,$ for $a \neq 0.$
- P8 (Commutative law for multiplication) $a \cdot b = b \cdot a.$
- P9 (Distributive law) $a \cdot (b + c) = a \cdot b + a \cdot c.$
- P10 (Trichotomy law) For every number a, one and only one of the following holds: (Denote P as the collection of positive numbers)
  - (i) $a = 0$,
  - (ii) a is in the collection P,
  - (iii) $-a$ is in the collection P.
- P11 (Closure under addition) If a and b are in P, then $a + b$ is in P.
- P12 (Closure under multiplication) If a and b are in P, then $a \cdot b$ is in P.

Definition (Absolute Value)**.** For any number a, we define the *absolute value* $|a|$ of a as follows:

$$|a| = \begin{cases} a, & a \geq 0 \\ -a, & a \leq 0 \end{cases}$$

Theorem (Triangle Inequality)**.** For all numbers a and b, we have

$$|a + b| \leq |a| + |b|$$

Proof. We make use of the fact that if both x and y are nonnegative, then $x^2 < y^2$ implies $x < y$.

$$\begin{aligned} |a + b|^2 &= a^2 + 2ab + b^2 \\ &= |a|^2 + 2ab + |b|^2 \\ &\leq |a|^2 + 2\,|a|\,|b| + |b|^2 \\ &= (|a| + |b|)^2 \end{aligned}$$

Since $|a + b|$ and $(|a| + |b|)$ are both nonnegative, then

$$|a + b| \leq |a| + |b|\,.$$

$\square$

1.1.1 Exercises

Exercise (1). Prove the following:

(i) If $ax = a$ for some number $a \neq 0$, then $x = 1$.

Proof. Assume that $ax = a$ fro some number $a \neq 0$.

$$\begin{aligned} x = x \cdot 1 = x \cdot (a \cdot a^{-1}) &= ax \cdot (a^{-1}) \\ &= a \cdot (a^{-1}) \\ &= (a \cdot a^{-1}) \\ &= 1 \end{aligned}$$

□

(ii) $x^2 - y^2 = (x - y)(x + y)$.

Proof. Using the field axioms.

$$\begin{aligned} (x-y)(x+y) &= x \cdot (x+y) + (-y) \cdot (x+y) \\ &= (x^2 + xy) + ((-y) \cdot x + (-y) \cdot y) \\ &= x^2 + xy - xy - y^2 \\ &= x^2 - y^2 \end{aligned}$$

□

(iii) If $x^2 = y^2$, then $x = y$ or $x = -y$.

Proof. Assume that $x^2 = y^2$. We make use of (ii).

$$\begin{aligned} x^2 = y^2 &\Leftrightarrow x^2 - y^2 = 0 \\ &\Leftrightarrow (x-y)(x+y) = 0 \\ &\Rightarrow x = y \text{ or } x = -y. \end{aligned}$$

□

(iv) $x^3 - y^3 = (x - y)(x^2 + xy + y^2)$.

Proof. Using the field axioms

$$\begin{aligned} (x-y)(x^2+xy+y^2) &= x^2(x-y) + xy(x-y) + y^2(x-y) \\ &= (x^3 - x^2y) + (x^2y - xy^2) + (xy^2 - y^3) \\ &= x^3 + (x^2y - x^2y) + (xy^2 - xy^2) - y^3 \\ &= x^3 - y^3 \end{aligned}$$

□

(v) $x^n - y^n = (x - y)(x^{n-1} + x^{n-2}y + \cdots + xy^{n-2} + y^{n-1})$.

Proof. Using the field axioms

$$
\begin{aligned}
(x-y)(x^{n-1}+x^{n-2}y+\cdots+xy^{n-2}+y^{n-1}) &= x(x^{n-1}+x^{n-2}y+\cdots+xy^{n-2}+y^{n-1}) \\
&\quad -[y(x^{n-1}+x^{n-2}y+\cdots+xy^{n-2}+y^{n-1})] \\
&= x^n+x^{n-1}y+\cdots+x^2y^{n-2}+xy^{n-1} \\
&\quad -[x^{n-1}y+x^{n-2}y^2+\cdots+xy^{n-1}+y^n] \\
&= x^n-y^n
\end{aligned}
$$

□

Alternative Proof. We make us of sigma notation

$$
\begin{aligned}
(x-y)\cdot\sum_{i=0}^{n-1}x^iy^{n-(i+1)} &= x\left(\sum_{i=0}^{n-1}x^iy^{n-(i+1)}\right)-\left[y\left(\sum_{i=0}^{n-1}x^iy^{n-(i+1)}\right)\right] \\
&= \sum_{i=0}^{n-1}x^{i+1}y^{n-(i+1)}-\left[\sum_{i=0}^{n-1}x^iy^{n-i}\right] \\
&= x^n+\sum_{i=0}^{n-2}x^{i+1}y^{n-(i+1)}-\left[\sum_{i=1}^{n-1}x^iy^{n-i}+y^n\right] \\
&= x^n+\sum_{i=0}^{n-2}x^{i+1}y^{n-(i+1)}-\left[\sum_{i=0}^{n-2}x^{i+1}y^{n-(i+1)}+y^n\right] \\
&= x^n-y^n+\sum_{i=0}^{n-2}\left[x^{i+1}y^{n-(i+1)}-(x^{i+1}y^{n-(i+1)})\right] \\
&= x^n-y^n+\sum_{i=0}^{n-2}0 \\
&= x^n-y^n
\end{aligned}
$$

□

(vi) $x^3+y^3=(x+y)(x^2-xy+y^2)$.

Proof. Replace y by $-y$ in part (iv)

$$
\begin{aligned}
x^3-y^3=(x-y)(x^2+xy+y^2) &\Leftrightarrow x^3-(-y)^3=(x-(-y))(x^2+x(-y)+(-y)^2) \\
&\Leftrightarrow x^3+y^3=(x+y)(x^2-xy+y^2)
\end{aligned}
$$

□

Exercise (2). What is wrong with the following "proof"? Let $x=y$. Then

$$
\begin{aligned}
x^2 &= xy, \\
x^2-y^2 &= xy-y^2, \\
(x+y)(x-y) &= y(x-y), \\
x+y &= y, \\
2y &= y, \\
2 &= 1.
\end{aligned}
$$

Solution. For all $a \in \mathbb{R}$ we know that $a \cdot a^{-1} = 0$ with the assumption $a \neq 0$. The $4th$ step is contradictory on the given fact that $x = y$ which implies $x - y = 0$ and has no multiplicative inverse. □

Exercise (3). Prove the following:

(i) $\dfrac{a}{b} = \dfrac{ac}{bc}$, if $b, c \neq 0$.

Proof. Using the field axioms

$$\begin{aligned}\frac{a}{b} = ab^{-1} &= (ab^{-1})(c \cdot c^{-1}) \\ &= (ac)(b^{-1}c^{-1}) \\ &= (ac)(bc)^{-1} \\ &= \frac{ac}{bc}\end{aligned}$$

□

(ii) $\dfrac{a}{b} + \dfrac{c}{d} = \dfrac{ad + bc}{bd}$, if $b, d \neq 0$.

Proof. Using the field axioms

$$\begin{aligned}\frac{a}{b} + \frac{c}{d} = ab^{-1} + cd^{-1} &= (ab^{-1} + cd^{-1}) \cdot (bd)(bd)^{-1} \\ &= (ad(b \cdot b^{-1}) + bc(d \cdot d^{-1})) \cdot (bd)^{-1} \\ &= (ad + bc) \cdot (bd)^{-1} \\ &= \frac{ad + bc}{bd}\end{aligned}$$

□

(iii) $(ab)^{-1} = a^{-1}b^{-1}$, if $a, b \neq 0$.

Proof. Using the field axioms

$$\begin{aligned} ab(a^{-1}b^{-1}) &= 1 \\ a^{-1}b^{-1} &= (ab)^{-1} \end{aligned}$$

$\square$

(iv) $\dfrac{a}{b} \cdot \dfrac{c}{d} = \dfrac{ac}{db}$ if $b, d \neq 0$.

Proof. Using the field axioms

$$\begin{aligned} \frac{a}{b} \cdot \frac{c}{d} &= (ab^{-1}) \cdot (cd^{-1}) \\ &= (ac) \cdot (d^{-1}b^{-1}) \\ &= (ac) \cdot (db)^{-1} \\ &= \frac{ac}{db} \end{aligned}$$

$\square$

(v) $\dfrac{a}{b} \Big/ \dfrac{c}{d} = \dfrac{ad}{bc}$, if $b, d \neq 0$.

Proof. Using the field axioms

$$\begin{aligned} \frac{a}{b} \Big/ \frac{c}{d} &= \frac{a}{b} \cdot \left(\frac{c}{d}\right)^{-1} \\ &= ab^{-1} \cdot (cd^{-1})^{-1} \\ &= ab^{-1} \cdot c^{-1}(d^{-1})^{-1} \\ &= ab^{-1} \cdot c^{-1}d \\ &= (ad) \cdot (b^{-1}c^{-1}) \\ &= (ad) \cdot (bc)^{-1} \\ &= \frac{ad}{bc} \end{aligned}$$

$\square$

(vi) If $b, d \neq 0$, then $\dfrac{a}{b} = \dfrac{c}{d}$ if and only if $ad = bc$. Also determine when $\dfrac{a}{b} = \dfrac{b}{a}$.

Proof. There are two cases to prove for the first part.

($\Rightarrow$) Let $b, d \neq 0$. Assume that $\dfrac{a}{b} = \dfrac{c}{d}$,

$$\begin{aligned} \frac{a}{b} &= \frac{c}{d}, \\ ab^{-1} &= cd^{-1}, \\ (ab^{-1})(bd) &= (cd^{-1})(bd), \\ (ad)(b \cdot b^{-1}) &= (bc)(d \cot d^{-1}), \\ ad &= bc. \end{aligned}$$

($\Leftarrow$) Let $b, d \neq 0$. Assume that $ad = bc$,

$$\begin{aligned} ad &= bc, \\ (ad)(bd)^{-1} &= (bc)(bd)^{-1} \\ (ab^{-1})(d \cdot d^{-1}) &= (cd^{-1})(b \cdot b^{-1}) \\ ab^{-1} &= cd^{-1} \\ \frac{a}{b} &= \frac{c}{d} \end{aligned}$$

□

Proof. From Exercise 1 Part (*iii*) we make use of the fact, if $x^2 = y^2$ then $x = y$ or $x = -y$.

$$\begin{aligned} \frac{a}{b} &= \frac{b}{a}, \\ ab^{-1} &= ba^{-1}, \\ (ab^{-1})(ab) &= (ba^{-1})(ab), \\ (a \cdot a)(b \cdot b^{-1}) &= (b \cdot b)(a \cdot a^{-1}), \\ a^2 &= b^2. \end{aligned}$$

and so it must be that $a = b$ or $a = -b$. □

Exercise (4). Find all numbers x for which

(i) $4 - x < 3 - 2x$.

Proof. Using the field axioms

$$\begin{aligned} 4 - x &< 3 - 2x \\ 4 - x + (2x - 4) &< 3 - 2x + (2x - 4) \\ x &< -1 \end{aligned}$$

$\square$

(ii) $5 - x^2 < 8$.

Proof. Using the field axioms

$$\begin{aligned} 5 - x^2 + (x^2 - 5) &< 8 + (x^2 - 5) \\ x^2 + 3 &> 0 \end{aligned}$$

since $x^2 \geq 0$ for all $x \in \mathbb{R}$, then it must be that $x^2 + 3 > 0$ for all $x \in \mathbb{R}$. $\square$

(iii) $5 - x^2 < -2$

Proof. Using the field axioms

$$\begin{aligned} 5 - x^2 &< -2 \\ x^2 &> 7 \\ |x| &> \sqrt{7} \\ x < -\sqrt{7} \text{ or } x &> \sqrt{7} \end{aligned}$$

$\square$

(iv) $(x - 3)(x - 1) > 0$ (When is a product of two numbers positive?)

Proof. The product of two numbers is postivie if and only if the numbers are both positive or both negative. For all $a, b \in \mathbb{R}$, $ab > 0 \Leftrightarrow a > 0$ and $b > 0$, or $a < 0$ and $b < 0$.

Hence,

$$x - 3 > 0 \qquad \text{and} \qquad x - 1 > 0$$

so it must be that $x > 3$. Or

$$x - 3 < 0 \qquad \text{and} \qquad x - 1 > 0$$

and it must be that $x < 1$. That is $(x - 3)(x - 1) > 0$ if $x > 3$ or $x < 1$. $\square$

(v) $x^2 - 2x + 2 > 0$.

Proof. Using the field axioms

$$\begin{aligned} x^2 - 2x + 2 &= (x^2 + 2x + 1) + 1 \\ &= (x-1)^2 + 1 \end{aligned}$$

for all $x \in \mathbb{R}$ notice that, $(x-1)^2 \geq 0$, so it must be that $(x-1)^2 + 1 > 0$. □

(vi) $x^2 + x + 1 > 2$.

Proof. Using the field axioms

$$\begin{aligned} x^2 + x + 1 &> 2 \\ x^2 + x - 1 &> 0 \\ (x^2 + x + \frac{1}{4}) - \frac{5}{4} &> 0 \\ \left(x + \frac{1}{2}\right)^2 &> \frac{5}{4} \\ \left|x + \frac{1}{2}\right| &> \frac{\sqrt{5}}{2} \\ x + \frac{1}{2} > \frac{\sqrt{5}}{2} \text{ or } x + \frac{1}{2} &< -\frac{\sqrt{5}}{2} \end{aligned}$$

so it must be that

$$x > \frac{\sqrt{5}-1}{2} \qquad \text{or} \qquad x < \frac{-\sqrt{5}-1}{2}$$

□

(vii) $x^2 - x + 10 > 16$.

Proof. Using the field axioms

$$\begin{aligned} x^2 - x + 10 &> 16 \\ x^2 - x - 6 &> 0 \\ (x-3)(x+2) &> 0 \end{aligned}$$

To assure that the product is positive, it must be that the two numbers are both positive or both negative. Hence,

$$x - 3 > 0 \qquad \text{and} \qquad x + 2 > 0$$

such that $x > 3$. Or

$$x - 3 < 0 \qquad \text{and} \qquad x + 2 < 0$$

such that $x < -2$. Therefore, $x^2 - x + 10 > 16$ if $x > 3$ or $x < -2$. □

(viii) $x^2 + x + 1 > 0$.

Proof. Using the field axioms

$$\begin{aligned} x^2 + x + 1 &= (x^2 + x + \frac{1}{4}) + \frac{3}{4}, \\ &= \left(x + \frac{1}{2}\right)^2 + \frac{3}{4}. \end{aligned}$$

for all $x \in \mathbb{R}$, notice that $(x + \frac{1}{2})^2 \geq 0$, so it must be that $(x + \frac{1}{2})^2 + \frac{3}{4} > 0$ for all $x \in \mathbb{R}$. $\square$

(ix) $(x - \pi)(x + 5)(x - 3) > 0$.

Proof. The expression $(x - \pi)(x + 5)(x - 3)$ can be rearranged as a product of two numbers, namely, $(x - \pi)\left[(x + 5)(x - 3)\right]$.

Notice, the product of two real numbers ab is greater than zero if a and b are both greater than zero, or both less than zero.

There are two cases:

- Let $(x - \pi) > 0$ so that $x > \pi$, and $(x + 5)(x - 3) > 0$ so that $x < -5$ or $x > 3$. Therefore it must be that $x > \pi$.
- Let $(x - \pi) < 0$ so that $x < \pi$, and $(x + 5)(x - 3) < 0$ so that $-5 < x < 3$. Therefore it must be that $-5 < x < 3$.

Therefore, $(x - \pi)(x + 5)(x - 3) > 0$ if $x > \pi$, or $-5 < x < 3$. $\square$

(x) $(x - \sqrt[3]{2})(x - \sqrt{2}) > 0$.

Proof. Either both numbers are greater than zero or less than zero.

$$x > \sqrt[3]{2} \qquad \text{and} \qquad x > \sqrt{2}$$

so that $x > \sqrt{2}$. Or

$$x < \sqrt[3]{2} \qquad \text{and} \qquad x < \sqrt{2}$$

so that $x < \sqrt[3]{2}$.

Therefore, $(x - \sqrt[3]{2})(x - \sqrt{2}) > 0$ if $x > \sqrt{2}$ or $x < \sqrt[3]{2}$. $\square$

(xi) $2^x < 8$.

Proof. We can rewrite it as

$$2^x < 2^3$$

Both have the same base, so it must be that the inequality is preserved on the exponents.

$$x < 3$$

so $2^x < 8$, whenever $x < 3$. $\square$

(xii) $x + 3^x < 4$.

Proof. We first notice that $x + 3^x = 4$ if $x = 1$

$$\begin{aligned} x + 3^x &= (1) + 3^1 \\ &= 4 \end{aligned}$$

observe that $x + 3^x$ is always increasing as x increase, and decreasing as x decrease. Therefore $x + 3^x < 4$ if $x < 1$. $\square$

(xiii) $\dfrac{1}{x} + \dfrac{1}{1-x} > 0$.

Proof. We can rewrite the expression as

$$\begin{aligned}\frac{1}{x} + \frac{1}{1-x} &= \frac{(1-x)+x}{x(1-x)} \\ &= \frac{1}{x(1-x)}\end{aligned}$$

Notice that $\frac{1}{x(1-x)} > 0$, whenever $x(1-x) > 0$. So it must be that x and $(1-x)$ are greater than zero

$$x > 0 \qquad \text{and} \qquad x < 1$$

or x and $(1-x)$ are both less than zero

$$x < 0 \qquad \text{and} \qquad x > 1$$

but there exists no x sucht that $x < 0$ and $x > 1$. Therefore, $\frac{1}{x} + \frac{1}{1-x} > 0$ if $x > 0$ and $x < 1$. □

(xiv) $\dfrac{x-1}{x+1} > 0$.

Proof. Either both $(x-1)$ and $(x+1)$ are greater than zero or both less than zero.

$$x > 1 \qquad \text{and} \qquad x > -1$$

so it must be that $x > 1$. Or

$$x < 1 \qquad \text{and} \qquad x < -1$$

so it must be that $x < -1$. □

Exercise (5). Prove the following:

(i) If $a < b$ and $c < d$, then $a + c < b + d$.

Proof. Assume that $a < b$ and $c < d$. Notice that $b - a > 0$ and $d - c > 0$, therefore their sum is also positive, namely,

$$(b - a) + (d - c) > 0$$

so that

$$a + c < b + d$$

□

(ii) If $a < b$, then $-b < -a$.

Proof. Assume that $a < b$. Therefore, $a - b < 0$. Notice that,

$$\begin{aligned} -(a - b) &< 0, \\ b - a &< 0, \\ -a &< -b. \end{aligned}$$

□

(iii) If $a < b$ and $c > d$, then $a - c < b - d$.

Proof. Assume that $a < b$ and $c > d$. Therefore, $a - b < 0$ and $c - d >$ so that $a - b < 0 < c - d$. Therefore,

$$\begin{aligned} a - b &< c - d, \\ a - c &< b - d. \end{aligned}$$

□

(iv) If $a < b$ and $c > 0$, then $ac < bc$.

Proof. Assume that $a < b$ and $c > 0$. Therefore, $b - a > 0$ and their product is positive,

$$\begin{aligned} (b - a) \cdot c &> 0, \\ bc - ac &> 0, \\ ac &< bc. \end{aligned}$$

□

(v) If $a < b$ and $c < 0$, then $ac > bc$.

Proof. Assume that $a < b$ and $c < 0$. Then $b - a > 0$ and $0 - c = -c > 0$, and their product must be positive,

$$\begin{aligned} (b - a) \cdot -c &> 0, \\ -bc + ac &> 0, \\ ac &> bc. \end{aligned}$$

□

(vi) If $a > 1$, then $a^2 > a$.

Proof. Assume that $a > 1$. Notice that $a - 1 > 0$ and $a > 0$, so their product is positive,

$$\begin{aligned}(a-1) \cdot a &> 0,\\ a^2 - a &> 0,\\ a^2 &> a.\end{aligned}$$

□

(vii) If $0 < a < 1$, then $a^2 < a$.

Proof. Assume that $0 < a < 1$, therefore $a > 0$ and $1 - a > 0$ so that their product is also positive,

$$\begin{aligned}(1-a) \cdot a &> 0,\\ a - a^2 &> 0,\\ a^2 &< a.\end{aligned}$$

□

(viii) If $0 \leq a < b$ and $0 \leq c < d$, then $ac < bd$.

Proof. Assume that $0 \leq a < b$ and $0 \leq c < d$. Notice that $bd > 0$, and if either one of a or c is equal to zero so that ac is zero, then

$$0 = ac < bd.$$

Otherwise, if a and c are greater than zero then ac is also greater than zero,

$$0 < ac < bc < bd.$$

□

(ix) If $0 \leq a < b$, then $a^2 < b^2$.

Proof. Assume that $0 \leq a < b$. If $a = 0$, then $a^2 = 0$ so that

$$a^2 < b^2.$$

Suppose that $a > 0$. From our assumption, $a < b$ therefore

$$a^2 < ab < b^2.$$

□

(x) If $a, b \geq 0$ and $a^2 < b^2$, then $a < b$.

Proof. Suppose for contradiction that $a, b \geq 0$ and $a^2 < b^2$, but $a \geq b$. So either $a = b$ or $a > b$.
If $a = b$, then $a^2 = b^2$, a contradiction. Now if $a > b \geq 0$, then

$$a^2 > ab > b^2$$

also a contradiction. Therefore it must be that $a < b$. □

Exercise (6). Prove the following:

(i) Prove that if $0 \leq x < y$, then $x^n < y^n$, $n = 1, 2, 3, \ldots$

Proof. Assume that $0 \leq x < y$. From the previous problems, it must be that $x^2 < y^2$, and so on for all $n \in \mathbb{N}$. □

(ii) Prove that if $x < y$ and n is odd, then $x^n < y^n$.

Proof. Assume tha $x < y$ and n is odd. There are three cases:

– If $0 \leq x$, then by the previous exercise, it must be that

$$x^n < y^n.$$

– If $x < y \leq 0$, then it must be that $0 \leq -y < -x$. Therefore,

$$\begin{aligned} (-y)^n &< (-x)^n, \\ -y^n &< -x^n, \\ x^n &< y^n. \end{aligned}$$

– If $x < 0 \leq y$, then since n is odd, it must be that

$$x^n < 0 \leq y^n$$

so that $x^n < y^n$.

□

(iii) Prove that if $x^n = y^n$ and n is odd, then $x = y$.(

Proof. If it is not, then it must be that $x^n < y^n$ or $x^n > y^n$. □

(iv) Prove that if $x^n = y^n$ and n is even, then $x = y$ or $x = -y$.

Proof. We have three cases and we make use of (i):

– Let $x, y \geq 0$. If $x^n = y^n$, then $x = y$. Since from part (i), without loss of generality, $x \neq y$ implies that $x^n < y^n$ for all $n \in \mathbb{N}$.

– Let $x, y \leq 0$ so that $-x, -y \geq 0$. If $(-x)^n = (-y)^n$, then $-x = -y$ so that $x = y$ for all $n \in \mathbb{N}$.

– Let $x \leq 0, y \geq 0$ so that $-x, y \geq 0$. If $(-x)^n = y^n$, then $-x = y$ which is the same as $x = -y$, for all $n \in \mathbb{N}$.

We have now exhausted all the cases of x and y. Therefore, if $x^n = y^n$ where n is even, then $x = y$ or $x = -y$. □

Exercise (7). Prove that if $0 < a < b$, then

$$a < \sqrt{ab} < \frac{a+b}{2} < b.$$

Notice that the inequality $\sqrt{ab} \leq (a+b)/2$ holds for all $a, b \geq 0$. A generalization of this fact occurs in Exercise $2 - 22$.

Proof. Since $0 < a < b$, then we know that

$$a^2 = a \cdot a < ab < b \cdot b = b^2.$$

therefore $a < \sqrt{ab} < b$.

Notice also that

$$a = \frac{a+a}{2} < \frac{a+b}{2} < \frac{b+b}{2} = b.$$

so that $a < (a+b)/2$ and $b > (a+b)/2$.

Lastly, we must show that $\sqrt{ab} < (a+b)/2$. But since $a, b > 0$, we know that

$$\begin{aligned}
(a-b)^2 &> 0, \\
a^2 - 2ab + b^2 &> 0, \\
a^2 + 2ab + b^2 &> 4ab, \\
(a+b)^2 &> 4ab, \\
\frac{a+b}{2} &> \sqrt{ab}
\end{aligned}$$

for all a, b such that $0 < a < b$. ∎

Exercise (12). Prove the following:

(i) $|xy| = |x| \cdot |y|$.

Proof. We will prove this fact the same way we proved the *Triangle Inequality*, by using that fact that if $x^2 = y^2$ and x, y are nonnegative, then $x = y$. Notice

$$|xy|^2 = (xy)^2 = x^2y^2 = |x|^2 \cdot |y|^2 = (|x| \cdot |y|)^2.$$

Since $|xy|$ and $|x| \cdot |y|$ are always nonnegative, we find that

$$|xy| = |x| \cdot |y|.$$

∎

(ii) $\left|\frac{1}{x}\right| = \frac{1}{|x|}$, if $x \neq 0$.

Proof. We make use of the previous exercise.

$$\left|\frac{1}{x}\right| \cdot |x| = \left|\frac{1}{x} \cdot x\right| = |1| = 1$$

therefore

$$\left|\frac{1}{x}\right| = \frac{1}{|x|}.$$

∎

(iii) $\dfrac{|x|}{|y|} = \left|\dfrac{x}{y}\right|$, if $y \neq 0$.

Proof. It follows by using the result of the previous exercise,

$$|x^{-1}| = |x|^{-1}.$$

$\square$

(iv) $|x - y| \leq |x| + |y|$.

Proof. We make use of the *Triangle Inequality* and the fact that for all $x \in \mathbb{R}$, $|x| = |-x|$. Let $y_0 = -y$,

$$\begin{aligned} |x + y_0| &\leq |x| + |y_0|, \\ |x + (-y)| &\leq |x| + |-y|, \\ |x - y| &\leq |x| + |y|. \end{aligned}$$

$\square$

(v) $|x| - |y| \leq |x - y|$.

Proof. From the *Triangle Inequality*,

$$\begin{aligned} |x| &= |(x - y) + y| \\ &\leq |x - y| + |y| \end{aligned}$$

so that

$$|x| - |y| \leq |x - y|.$$

$\square$

(vi) $|(|x| - |y|)| \leq |x - y|$.

Proof. To show that $|(|x| - |y|)| \leq |x - y|$, we must show that $-|x - y| \leq |x| - |y| \leq |x - y|$.

- The second inequality follows from the previous exercise.
- The first inequality is the same as $|y| - |x| \leq |x - y|$, but this also follows from the previous exercises since $|y| - |x| \leq |y - x| = |x - y|$.

$\square$

(vii) $|x + y + z| \leq |x| + |y| + |z|$.

Proof. We just apply the *Triangle Inequality* multiple times.

$$\begin{aligned} |x + y + z| &\leq |x| + |y + z| \\ &\leq |x| + |y| + |z|. \end{aligned}$$

$\square$

Exercise (14). Prove the following

(i) Prove that $|a| = |-a|$.

Proof. If $a \geq 0$, then $-a \leq 0$ and its absolute value is

$$|-a| = -(-a) = a = |a|$$

Now, if $a \leq 0$, then $-a \geq 0$ so it follows from the previous one that $|a| = |-a|$. $\square$

(ii) Prove that $-b \leq a \leq b$ if and only if $|a| \leq b$. In particular, it follows that $-|a| \leq a \leq |a|$.

Proof. This is a biconditional, so we prove it in both directions.
Assume that $-b \leq a \leq b$. If $a \geq 0$, then

$$|a| = a \leq b.$$

If $a \leq 0$ and also notice that $-a \leq b$, then

$$|a| = -a \leq b.$$

For the converse, we assume that $|a| \leq b$ and it follows that $b \geq 0$. If $a \geq 0$, then

$$a = |a| \leq b.$$

and

$$-b \leq 0 \leq a.$$

Now, if $a \leq 0$, then $-a = |a| \leq b$ so that

$$-b \leq a.$$

and

$$a \leq 0 \leq b.$$

$\square$

(iii) Use this fact to give a new proof that $|a + b| \leq |a| + |b|$.

Proof. We make use of the fact that $-|a| \leq a \leq |a|$ and $-|b| \leq b \leq |b|$, and it follows that

$$-(|a| + |b|) \leq a + b \leq |a| + |b|,$$

so $|a + b| \leq |a| + |b|$. $\square$

Exercise (18). Prove the following

(i) Suppose that $b^2 - 4c \geq 0$. Show that the numbers

$$\frac{-b + \sqrt{b^2 - 4c}}{2}, \qquad \frac{-b - \sqrt{b^2 - 4c}}{2}$$

both satisfy the equation $x^2 + bx + c = 0$.

Proof. From the equation, we can find that

$$\begin{aligned} x^2 + bx + c &= 0, \\ \left(x + \frac{b}{2}\right)^2 &= -c + \frac{b^2}{4}, \\ \left|x + \frac{b}{2}\right| &= \frac{\sqrt{b^2 - 4c}}{|2|}, \\ x &= \frac{-b \pm \sqrt{b^2 - 4c}}{2}. \end{aligned}$$

$\square$

(ii) Suppose that $b^2 - 4c < 0$. Show there are no numbers x satisfying $x^2 + bx + c = 0$; in fact, $x^2 + bx + c > 0$ for all x.

Proof. We complete the square

$$\begin{aligned} x^2 + bx + c &= \left(x + \frac{b}{2}\right)^2 + c - \frac{b^2}{4} \\ &= \left(x + \frac{b}{2}\right)^2 - \left(\frac{b^2 - 4c}{4}\right) \end{aligned}$$

We know that squares are always nonnegative, and also we know that $b^2 - 4c < 0$ so the second term is always positive. Therefore $x^2 + bx + c$ is always positive for all x. $\square$

(iii) Use this fact to give another proof that if x and y are not both 0, then $x^2 + xy + y^2 > 0$.

Proof. From the previous exercise, we make $y^2 - 4y^2 < 0$ to assure that $x^2 + xy + y^2 > 0$ for all x, given that $y \neq 0$ (since $y^2 - 4y^2 = 0$ if $y = 0$). In the case that $y = 0$, $x^2 + xy + y^2 = x^2$ is still positive given that $x \neq 0$. $\square$

(iv) For which numbers α is it true that $x^2 + \alpha xy + y^2 > 0$ whenever x and y are not both 0.

Proof. From the previous exercise, $b^2 - 4c = (\alpha y)^2 - 4y^2 < 0$ will assure that $x^2 + \alpha xy + y^2 > 0$, given that both x and y are not 0.
We just solve for the values of α that will assure $(\alpha y)^2 - 4y^2$ is less than zero.

$$\begin{aligned}(\alpha y)^2 - 4y^2 &< 0,\\ \alpha^2 y^2 &< 4y^2,\\ \alpha^2 &< 4,\\ |\alpha| &< 2.\end{aligned}$$

so the values of α that will make $x^2 + \alpha xy + y^2 > 0$ are

$$-2 < \alpha < 2.$$

□

(v) Find the smallest possible value of $x^2 + bx + c$ and of $ax^2 + bx + c$, for $a > 0$.

Proof. By completing the square, we can find the minimum value.

$$x^2 + bx + c = \left(x + \frac{b}{2}\right)^2 + c - \frac{b^2}{4},$$

and the minimum $c - \frac{b^2}{4}$ is achieved if $x = -\frac{b}{2}$.
For $ax^2 + bx + c$ where $a > 0$,

$$\begin{aligned}ax^2 + bx + c &= a\left(x^2 + \frac{b}{a}x\right) + c\\ &= a\left(x + \frac{b}{2a}\right)^2 + c - \frac{b^2}{4a},\end{aligned}$$

and the minimum $c - \frac{b^2}{4a}$ is achieved if $x = -\frac{b}{2a}$. □

Exercise (20). Prove that if

$$|x - x_0| < \frac{\varepsilon}{2} \quad \text{and} \quad |y - y_0| < \frac{\varepsilon}{2},$$

then

$$\begin{aligned} |(x + y) - (x_0 + y_0)| < \varepsilon, \\ |(x - y) - (x_0 - y_0)| < \varepsilon. \end{aligned}$$

Proof. Assume the premise. Therefore, their sum would be

$$|x - x_0| + |y - y_0| < \varepsilon,$$

By the *Triange Inequality* we get

$$\begin{aligned} |(x + y) - (x_0 + y_0)| &= |(x - x_0) + (y - y_0)| \\ &\leq |x - x_0| + |y - y_0| \\ &< \varepsilon. \end{aligned}$$

Notice also that,

$$|x - x_0| + |y - y_0| = |x - x_0| + |y_0 - y| < \varepsilon,$$

and by the *Triangle Inequality*,

$$\begin{aligned} |(x - y) - (x_0 - y_0)| &= |(x - x_0) - (y_0 - y)| \\ &\leq |x - x_0| + |y_0 - y| \\ &< \varepsilon. \end{aligned}$$

□

1.2 Number of Various Sorts

Definition (Mathematical Induction)**.** Suppose that $P(x)$ means that the property P holds for the number x. Then the principle of mathematical induction states that $P(x)$ holds for all natural numbers x provided that

– $P(1)$ is true.

– Whenever $P(k)$ is true, $P(k+1)$ is true.

(Note: In the construction of the natural numbers from the *Peano Axioms*, induction is an axiom and 0 is an element of the naturals.)

Definition (Strong Induction)**.** We use a stronger hypothesis since in some cases it happens that in order to prove $P(k+1)$ we must assume not only $P(k)$, but also $P(l)$ for all natural numbers $l \leq k$. $P(x)$ holds for all natural numbers x provideded that

– $P(1)$ is true

– $P(k+1)$ is true, if $P(l)$ is true for $1 < l \leq k$

Definition (Recursive Definitions)**.** A *recursive definition* of a function defines values of the function for some inputs in terms of the values of the function for other inputs.

For example, the number $n!$ is defined as the product of all then natural numbers less then or equal to n:

$$n! = 1 \ \cdot 2 \cdot 3 \cdot \ldots \cdot (n-1) \cdot n.$$

More formally:

$$1! = 1 \tag{1}$$
$$n! = n \cdot (n-1)!. \tag{2}$$

Another example is the convenient notation regarding sums. Instead of writing

$$a_1 + a_1 + \cdots + a_n.$$

we will use the Greek letter Σ and write

$$\sum_{i=1}^{n} a_i.$$

The define $\sum\limits_{i=1}^{n} a_i$ precisely really requires a recursive definition:

$$\sum_{i=1}^{1} a_i = a_1, \tag{3}$$
$$\sum_{i=1}^{n} a_i = \sum_{i=1}^{n-1} a_i + a_n. \tag{4}$$

As an example where induction is used for its proof,

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

1.2.1 Exercises

Exercise (1). Prove the following formulas by induction.

(i) $1^2 + \cdots + n^2 = \dfrac{n(n+1)(2n+1)}{6}$.

Proof. We induct on n. For the base case $n = 1$, the equation is trivial.

$$\begin{aligned} 1^2 &= \frac{1(1+1)(2(1)+1)}{6} \\ &= \frac{2(3)}{6} \\ &= 1 \end{aligned}$$

Now we assume that the equation hold for arbitrary n, and we shall prove that it also hold for $n+1$.

$$\begin{aligned} 1^2 + \cdots + n^2 + (n+1)^2 &= \sum_{i=1}^{n+1} i^2 \\ &= \sum_{i=1}^{n} i^2 + (n+1)^2 \\ &= \frac{n(n+1)(2n+1)}{6} + (n+1)^2 \\ &= \frac{n(n+1)(2n+1) + 6(n+1)^2}{6} \\ &= \frac{(n+1)[2n^2 + n + 6n + 6]}{6} \\ &= \frac{(n+1)(n+2)(2(n+1)+1)}{6}. \end{aligned}$$

Hence, the equation hold for all natural numbers. □

(ii) $1^3 + \cdots + n^3 = (1 + \cdots + n)^2$.

Proof. We induct on n and prove the base case $n = 1$,

$$1^3 = (1)^2$$

which is trivial.

Let us now assume that the equation hold for arbitrary n. We must now show that it also hold for $n+1$.

$$\begin{aligned} (1 + \cdots + (n+1))^2 &= (n+1)(2(1 + \cdots + n) + (n+1)) + (1 + \cdots + n)^2 \\ &= (1^3 + \cdots + n^3) + (n+1)(2(1 + \cdots + n) + (n+1)) \\ &= (1^3 + \cdots + n^3) + (n+1)(2 \cdot \frac{n(n+1)}{2} + n + 1) \\ &= (1^3 + \cdots + n^3) + (n+1)((n+1)(n+1)) \\ &= 1^3 + \cdots + (n+1)^3, \end{aligned}$$

hence, the equation holds for all natural numbers. □

Exercise (2). Find a formula for

(i) $\sum\limits_{i=1}^{n}(2n-1) = 1+3+5+\cdots+(2n-1)$.

Proof. We can make use of the properties of the summation symbol and we can easily derive the formla. However, we will not make use of it.

$$\begin{aligned}
\sum_{i=1}^{n}(2n-1) &= 1+3+5+\cdots+(2n-1) \\
&= 1+2+3+\cdots+(2n)-2(1+2+3+\cdots+n) \\
&= \frac{2n(2n+1)}{2} - 2\cdot\frac{n(n+1)}{2} \\
&= n((2n+1)-(n+1)) \\
&= n^2.
\end{aligned}$$

$\square$

(ii) $\sum\limits_{i=1}^{n}(2n-1)^2 = 1^2+3^2+5^2+\cdots+(2n-1)^2$.

Proof. We can make use of the properties of the summation symbol and we can easily derive the formla. However, we will not make use of it.

$$\begin{aligned}
\sum_{i=1}^{n}(2n-1)^2 &= 1^2+3^2+5^2+\cdots+(2n-1)^2 \\
&= 1^2+2^2+3^2+\cdots+(2n)^2-4(1^2+2^2+3^2+\cdots+n^2) \\
&= \frac{2n(2n+1)(4n+1)}{6} - 4\cdot\frac{n(n+1)(2n+1)}{6} \\
&= \frac{2n(2n+1)\left(4n+1-2(n+1)\right)}{6} \\
&= \frac{n(2n+1)(2n-1)}{3}.
\end{aligned}$$

$\square$

Exercise (3). If $0 \leq k \leq n$, the *"binomial coefficient"* $\binom{n}{k}$ is defined by

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n(n-1)\cdots(n-k+1)}{k!}, \text{if } k \neq 0, n$$

and a special case of the first formula if we define $0! = 1$,

$$\binom{n}{0} = \binom{n}{n} = 1,$$

and for $k < 0$ or $k > n$ we just define the binomial coefficient to be 0.

(i) Prove that

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}.$$

Proof. It is just a manipulation of the expressions and from the fact that,

$$\binom{n+1}{k} = \frac{(n+1)!}{(n+1-k)!k!}.$$

Now for the proof,

$$\begin{aligned}\binom{n}{k-1} + \binom{n}{k} &= \frac{n!}{(n+1-k)!(k-1)!} + \frac{n!}{(n-k)!k!} \\ &= \frac{n!k + n!(n+1-k)}{(n+1-k)!k!} \\ &= \frac{(n+1)!}{(n+1-k)!k!}.\end{aligned}$$

□

This relation gives rise to the following configuration, known as *"Pascal's Triangle"*—a number not on one of the sides is the sum of the two numbers above it; the binomial coefficient $\binom{n}{k}$ is the $(k+1)$st number in the $(n+1)$st row.

(ii) Notice that all the numbers in Pascal's Triangle are natural numbers. Use part (i) to prove by induction that $\binom{n}{k}$ is always a natural number.

Proof. We want to prove the assertion that for fixed n (Note: The case $k = 0$ is trivial),

$$\binom{n}{k} \text{is a natural number for all } k, 1 \leq k \leq n.$$

We prove it by induction on n. For the base case where $n = 1$, we only need to prove for $k = 1$.

$$\binom{1}{1} = 1.$$

For the inductive step, we assume that the assertion is true for arbitrary n and we will show that

$$\binom{n+1}{k} \text{ is a natural number for all } k, 1 \leq k \leq n+1.$$

The case where $k = 1$ and $k = n+1$ are trivial. We can now assume that for $2 \leq k \leq n$, it must be that $1 \leq k-1 < k \leq n$ and we can make use this with part (i)

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k},$$

since the sum of naturals is a natural, $\binom{n+1}{k}$ is natural. We have shown that $\binom{n+1}{k}$ is natural for all k, $1 \leq k \leq n+1$ if $\binom{n}{k}$ is natural for all k $1 \leq k \leq n$. Hence, the induction is complete. □

(iii) Give another proof that $\binom{n}{k}$ is natural number by showing that $\binom{n}{k}$ is the number of sets of exactly k integers each chosen from $1, \ldots, n$.

Proof. In choosing k objects from n, there are $n(n-1)\cdots(n-k+1)$ ways to choose a set with k elements, $\{k_1, \ldots, k_n\}$. However, for each set the k objects can be arranged in $k!$ ways, that is why we divide by $k!$ as the order of the kth element in a certain set with the same elements does not matter. Hence,

$$\binom{n}{k} = \frac{n(n-1)\ldots(n-k+1)}{k!} = \frac{n!}{(n-k)!k!},$$

is the number of sets of exactly k integers chosen from n and that is why $\binom{n}{k}$ is always a natural. □

(iv) Prove the *"binomial theorem"*: If a and b are any numbers and n is a natural number, then

$$(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \cdots + \binom{n}{n-1}ab^{n-1} + b^n$$
$$= \sum_{j=0}^{n} \binom{n}{j} a^{n-j}b^j.$$

Proof. To be continued... □