MATH 121E

CALCULUS II

OBJECTIVES

- To use integration techniques to determine the integral of a function

TOPIC OUTLINE

Integration Techniques

- Integration by Parts
- Trigonometric Integrals
- Trigonometric Substitution
- Rational Functions
- Rationalizing Substitution

Integration by Parts

Integration by Parts

Integration by Parts Formula:

$\int u\,dv = uv - \int v\,du$

Example (a).

Evaluate $\int x \sin 2x\,dx$

let $u = x$, $du = dx$

$dv = \sin 2x\,dx$, $v = -\frac{1}{2}\cos 2x$

$\int x \sin 2x\,dx$

From the formula,

$\int u\,dv = uv - \int v\,du$

$= x\left(-\frac{1}{2}\cos 2x\right) + \frac{1}{2}\int \cos 2x\,dx$

$= -\frac{1}{2} x \cos 2x + \frac{1}{4}\sin 2x + C$

Example (b).

Evaluate $\int \sec^3\theta\,d\theta$

$\int \sec\theta \sec^2\theta\,d\theta$

let $u = \sec\theta$, $du = \sec\theta\tan\theta$

$dv = \sec^2\theta\,d\theta$, $v = \tan\theta$

From the formula,

$\int u\,dv = uv - \int v\,du$

$= \sec\theta\tan\theta - \int \tan\theta\sec\theta\tan\theta\,d\theta$

$= \sec\theta\tan\theta - \int \sec\theta\tan^2\theta\,d\theta$

$= \sec\theta\tan\theta - \int \sec\theta(\sec^2\theta - 1)\,d\theta$

$= \sec\theta\tan\theta - \int \sec^3\theta\,d\theta - \int \sec\theta\,d\theta$

$\int \sec^3\theta\,d\theta = \sec\theta\tan\theta - \int \sec^3\theta\,d\theta - \int \sec\theta\,d\theta$

$2\int \sec^3\theta\,d\theta = \sec\theta\tan\theta + \int \sec\theta\,d\theta$

$2\int \sec^3\theta\,d\theta = \sec\theta\tan\theta + \ln(\sec\theta + \tan\theta) + C$

$\int \sec^3\theta\,d\theta = \frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\ln(\sec\theta + \tan\theta) + C$

Integration by Parts

Evaluate below:

Example C

$\int e^x \cos x \, dx$

Let $u = e^x$, $dv = \cos x \, dx$

$du = e^x dx$, $v = \sin x$

$\dfrac{du}{dx} = e^x$

$\int e^x \cos x \, dx = e^x \sin x - \int e^x \sin x dx$

$\int e^x \cos x \, dx = e^x \sin x - (-e^x \cos x + \int e^x \cos x \, dx)$

$\int e^x \cos x \, dx = e^x \sin x + e^x \cos x - \int e^x \cos x \, dx$

$\int e^x \cos x \, dx + \int e^x \cos x \, dx = e^x \sin x + e^x \cos x$

$2\int e^x \cos x \, dx = e^x \sin x + e^x \cos x$

$$\int e^x \cos x \, dx = \frac{e^x \sin x + e^x \cos x}{2} + C$$

$\int e^x \sin x dx$

Let $u = e^x$, $dv = \sin x dx$

$du = e^x dx$, $v = -\cos x$

$\int e^x \sin x dx = -e^x \cos x - \int -\cos x \, e^x \, dx$

$\int e^x \sin x dx = -e^x \cos x + \int e^x \cos x \, dx$

Integration by Parts

Evaluate below Integrals

1.) $\int \ln x \, dx$

2. $\int y \cos 4y \, dy$

3. $\int x (2x-1)^7 \, dx$

4. $\int y \cos y \sin^2 y \, dy$

5. $\int \sin x \sin 4x \, dx$

Trigonometric Integrals

Trigonometric Integrals

Trigonometric formulas

$$\sin^2 u = \frac{1-\cos(2u)}{2}$$
$$\cos^2 u = \frac{1+\cos(2u)}{2}$$
$$\tan^2 u = \frac{1-\cos(2u)}{1+\cos(2u)}$$

$$\sin^2 u + \cos^2 u = 1$$
$$1 + \tan^2 u = \sec^2 u$$
$$1 + \cot^2 u = \csc^2 u$$

Example (a).

Evaluate $\int \cos^3 x\,dx$

$= \int \cos^2 x \cos x\, dx$

From the formula:

$\sin^2 x + \cos^2 x = 1$

$= \int (1-\sin^2 x)\cos x\, dx$

Let $u = \sin x$

$du = \cos x\, dx$

$= \int (1-\sin^2 x)\cos x\, dx$

$= \int (1- u^2)\, du$

$= \int du - \int u^2 du$

$= u - \frac{u^3}{3} + C$

$= \sin x - \frac{\sin^3 x}{3} + C$

Trigonometric Integrals

Example (b).

Evaluate $\int \sin^5 x \cos^2 x \, dx$

$= \int \sin^4 x \sin x \cos^2 x \, dx$

$= \int (\sin^2 x)^2 \cos^2 x \sin x \, dx$

$= \int (1-\cos^2 x)^2 \cos^2 x \sin x \, dx$

Let $u = \cos x$

$du = -\sin x dx$

$= \int (1-u^2)^2 u^2 \sin x \, (du/-\sin x)$

$= -\int (1-u^2)^2 u^2 \, du$

$= -\int (1-2u^2 + u^4) u^2 \, du$

$= -\int (u^2 - 2u^4 + u^6) \, du$

$= -\frac{u^3}{3} + \frac{2u^5}{5} - \frac{u^7}{7} + C$

$= -\frac{\cos^3 x}{3} + \frac{2\cos^5 x}{5} - \frac{\cos^7 x}{7} + C$

Example (c).

Evaluate $\int \frac{\sin^3 x \, dx}{\cos^6 x}$

$= \int \frac{\sin^3 x \, dx}{\cos^3 x \cos^3 x}$

$= \int \tan^3 x \sec^3 x \, dx$

$= \int \tan^2 x \sec^2 x \, . \sec x \tan x \, dx$

$= \int (\sec^2 x - 1) \sec^2 x \sec x \tan x \, dx$

let $u = \sec x$

$du = \sec x \tan x \, dx$

$= \int (u^2-1) u^2 \, du$

$= \int (u^4 - u^2) \, du$

$= \frac{u^5}{5} - \frac{u^3}{3} + C$

$= \frac{\sec^5 x}{5} - \frac{\sec^3 x}{3} + C$

Trigonometric Integrals

Evaluate below Integrals

1. $\int \frac{\sin^5 t \, dt}{\cos^2 t}$
2. $\int \sin^5 x \cos^5 x \, dx$
3. $\int \sin^2 x \cos^2 x \, dx$
4. $\int \cos^2 x \cos^3 2x \, dx$
5. $\int \sin^4 x \cos^4 x \, dx$

Trigonometric Substitution

Trigonometric Substitution

When the integrand involves

$a^2 - x^2$, try $x = a \sin\theta$
$a^2 + x^2$, try $x = a \tan\theta$
$x^2 - a^2$, try $x = a \sec\theta$

Example (a).

Evaluate $\int \frac{dx}{(a^2+x^2)^{3/2}}$

let $x = a \tan\theta$
$dx = a \sec^2\theta \, d\theta$

$$= \int \frac{a \sec^2\theta \, d\theta}{(a^2 + a^2 \tan^2\theta)^{3/2}}$$

$$= \int \frac{a \sec^2\theta \, d\theta}{[a^2 (1 + \tan^2\theta)]^{3/2}}$$

$$= \int \frac{a \sec^2\theta \, d\theta}{[a^2 (1 + \tan^2\theta)]^{3/2}}$$

$$= \int \frac{a \sec^2\theta \, d\theta}{a^3 (1 + \tan^2\theta)^{3/2}}$$

$$= \frac{1}{a^2} \int \frac{\sec^2\theta \, d\theta}{(1 + \tan^2\theta)^{3/2}}$$

$$= \frac{1}{a^2} \int \frac{\sec^2\theta \, d\theta}{(\sec^2\theta)^{3/2}}$$

$$= \frac{1}{a^2} \int \frac{\sec^2\theta \, d\theta}{\sec^3\theta}$$

$$= \frac{1}{a^2} \int \frac{d\theta}{\sec\theta}$$

$$= \frac{1}{a^2} \int \cos\theta \, d\theta$$

$$= \frac{1}{a^2} \sin\theta + C$$

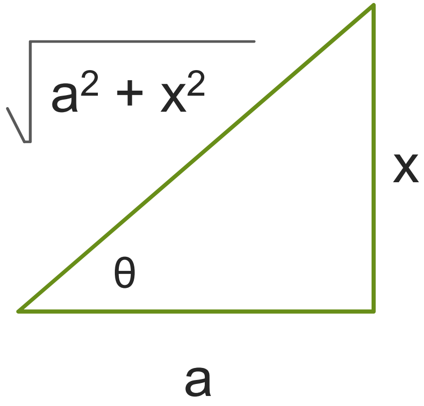
Trigonometric Substitution

Since,

$$\tan\theta = \frac{x}{a}$$

Therefore;

$$\sin\theta = \frac{x}{\sqrt{a^2 + x^2}}$$



Thus:

$$\int \frac{dx}{a^2+x^2)^{3/2}} = \frac{1}{a^2}\frac{x}{\sqrt{a^2 + x^2}} + C$$

Trigonometric Substitution

Example (b).

Evaluate $\int \sqrt{a^2 - x^2}\, dx$

let $x = a \sin\theta$

$dx = a \cos\theta\, d\theta$

$\int \sqrt{a^2 - x^2}\, dx = \int \sqrt{a^2 - a^2\sin^2\theta}\; a\cos\theta\, d\theta$

$= \int \sqrt{a^2 - a^2\sin^2\theta}\; a\cos\theta\, d\theta$

$= \int \sqrt{a^2(1 - \sin^2\theta)}\; a\cos\theta\, d\theta$

$= \int a\cos\theta\; a\cos\theta\, d\theta$

$= a^2 \int \cos^2\theta\, d\theta$

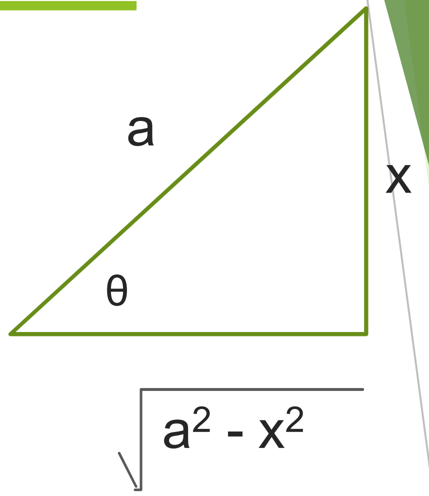
$= a^2 \int \frac{(1 + \cos 2\theta)\, d\theta}{2}$

$= \frac{a^2}{2} [\int d\theta + \int \cos 2\theta\, d\theta$

$= \frac{a^2}{2} + \theta + \frac{1}{2}\sin 2\theta$

$= a^2 \left(\frac{\theta}{2} + \frac{1}{4}\sin 2\theta\right) + C$

Trigonometric Substitution



Since,

$$\sin\theta = \frac{x}{a}$$

Therefore;

$$\cos\theta = \frac{\sqrt{a^2 - x^2}}{a}$$

$$\sin 2\theta = 2\sin\theta\cos\theta = \frac{2x\sqrt{a^2 - x^2}}{a^2}$$

Therefore:

$$\int\sqrt{a^2 - x^2}\,dx = a^2\left(\frac{\theta}{2} + \frac{1}{4}\sin 2\theta\right) + C$$

$$= \frac{a^2}{2}\operatorname{Arcsin}\frac{x}{a} + \frac{a^2}{4}\,\frac{2x\sqrt{a^2 - x^2}}{a^2} + C$$

$$\boxed{= \frac{a^2}{2}\operatorname{Arcsin}\frac{x}{a} + \frac{x}{2}\sqrt{a^2 - x^2} + C}$$

Trigonometric Substitution

Evaluate below Integrals

1. $\int \frac{du}{(u^2 + a^2)^2}$

2. $\int \frac{dz}{z\,(4+z^2)^3}$

3. $\int \frac{dy}{y\,(y^2 + 1)}$

Rational Functions

Rational Functions

A rational function is any function which **can be written as the ratio of two polynomial functions**, where the polynomial in the denominator is not equal to zero. It is a quotient of a polynomial.

Example (a).

Evaluate $\int \frac{(x-1)\,dx}{(x^2+5x+6)}$

$$= \int \frac{(x-1)\,dx}{(x+2)(x+3)}$$

Solution:

let

$$\frac{x-1}{(x+2)(x+3)} = \frac{A}{(x+2)} + \frac{B}{(x+3)}$$

$$x-1 = A(x+3) + B(x+2)$$

when x = -2

$$x-1 = A(x+3) + B(x+2)$$

A= -3

when x = -3

$$x-1 = A(x+3) + B(x+2)$$

B= 4

$$\int \frac{(x-1)\,dx}{(x^2+5x+6)} = \frac{A}{(x+2)}dx + \frac{B\,dx}{(x+3)}$$

$$= -3\int \frac{dx}{(x+2)} + 4\int \frac{dx}{(x+3)}$$

$$= -3\ln(x+2) + 4\ln(x+3 + C$$

Rational Functions

Example (b).

Evaluate $\int \dfrac{5\sin\theta\cos\theta\, d\theta}{\sin^2\theta + 3\sin\theta - 4}$

$$= \int \frac{5\sin\theta\cos\theta\, d\theta}{(\sin\theta+4)(\sin\theta-1)}$$

Let $\int \dfrac{5\sin\theta}{(\sin\theta+4)(\sin\theta-1)} = \dfrac{A}{(\sin\theta+4)} + \dfrac{B}{(\sin\theta-1)}$

$$5\sin\theta = A(\sin\theta-1) + B(\sin\theta+4)$$

If $\sin\theta = 1$, if $\sin\theta = 0$

B=1 A = 4

Thus,

$$\int \frac{5\sin\theta\cos\theta\, d\theta}{\sin^2\theta + 3\sin\theta - 4} = 4\int\frac{\cos\theta\, d\theta}{(\sin\theta+4)} + 1\int\frac{\cos\theta\, d\theta}{(\sin\theta-1)}$$

$$= \mathbf{4\ln(\sin\theta+4) + \ln(\sin\theta-1) + C}$$

Rationalizing Substitution

Rationalizing Substitution

An integration method which is often useful when the integrand is a fraction including more than one kind of root, such as $\frac{\sqrt{x}}{1+\sqrt[3]{x}}$.

A different type of rationalizing substitution can be used to work with integrands such as $\frac{1}{1+e^x}$.

Note: This method transforms the integrand into a rational function, hence the name *rationalizing*.

Rationalizing Substitution

Example: For $\int x\sqrt{5+x^2}\,dx$ let $u = 5 + x^2$.

That means $du = 2x\,dx$

$$\frac{1}{2}du = x\,dx$$

It follows that

$$\int x\sqrt{5+x^2}\,dx = \int \sqrt{u}\cdot\frac{1}{2}du$$

$$= \frac{1}{2}\int u^{\frac{1}{2}}\,du$$

$$= \frac{1}{2}\cdot\frac{2}{3}u^{\frac{3}{2}} + C$$

$$= \frac{1}{3}\left(5+x^2\right)^{\frac{3}{2}} + C$$

Rationalizing Substitution

Example 2: Find $\int \frac{1}{1+e^x}\,dx$.

Let $u = 1+e^x$, so $\ln(u-1) = x$. That means $\frac{1}{u-1}\,du = dx$.

$$\int \frac{1}{1+e^x}\,dx = \int \frac{1}{u}\cdot\frac{1}{u-1}\,du$$

$$= \int\left(\frac{1}{u-1} - \frac{1}{u}\right)du \qquad \text{(partial fractions)}$$

$$= \ln|u-1| - \ln|u| + C$$

$$= \ln\left|\frac{u-1}{u}\right| + C$$

$$= \ln\left|\frac{e^x}{1+e^x}\right| + C \text{ or } \ln\left(\frac{e^x}{1+e^x}\right) + C.$$

Rationalizing Substitution

Evaluate below Integrals

1. $\int \frac{(5x-12)\,dx}{x^3-6x^2+8x}$
2. $\int \frac{(y^3+4)\,dy}{y\,(y+1)}$
3. $\int \frac{3z^2\,dt}{z^4+5z^2+4}$
4. $\int \frac{x^2-5x+3}{x^3-4x^2+3x}$

End of Topic

Thank you

Engr. Febe F. Murillo
College of Engineering and Information Technology