UNIVERSITY OF SOUTHERN MINDANAO

MATH121E

Calculus 2

Topic Outline

Multiple Integral

- Double Integral
- Triple Integral

DOUBLE INTEGRAL

Multiple Integral: Double Integral

What is the difference between double and triple integrals?

A double integral is used for integrating over a two-dimensional region**, while a triple integral** is used for integrating over a three-dimensional region.

Multiple Integral: Double Integral

Double integral is defined as the integrals of a function in two variables over a region in R2, i.e. the real number plane. The double integral of a function of two variables, say f(x, y) over a rectangular region can be denoted as:

$$\iint R\, f(x, y)\, dA = \iint R\, f(x, y)\, dx\, dy$$

Double integrals are used **to calculate the area of a region, the volume under a surface, and the average value of a function of two variables over a rectangular region**

Multiple Integral: Double Integral

Unit of area in rectangular coordinates

$$\Delta A = \Delta x \Delta y$$

$$\downarrow$$

$$dA = dx\,dy$$

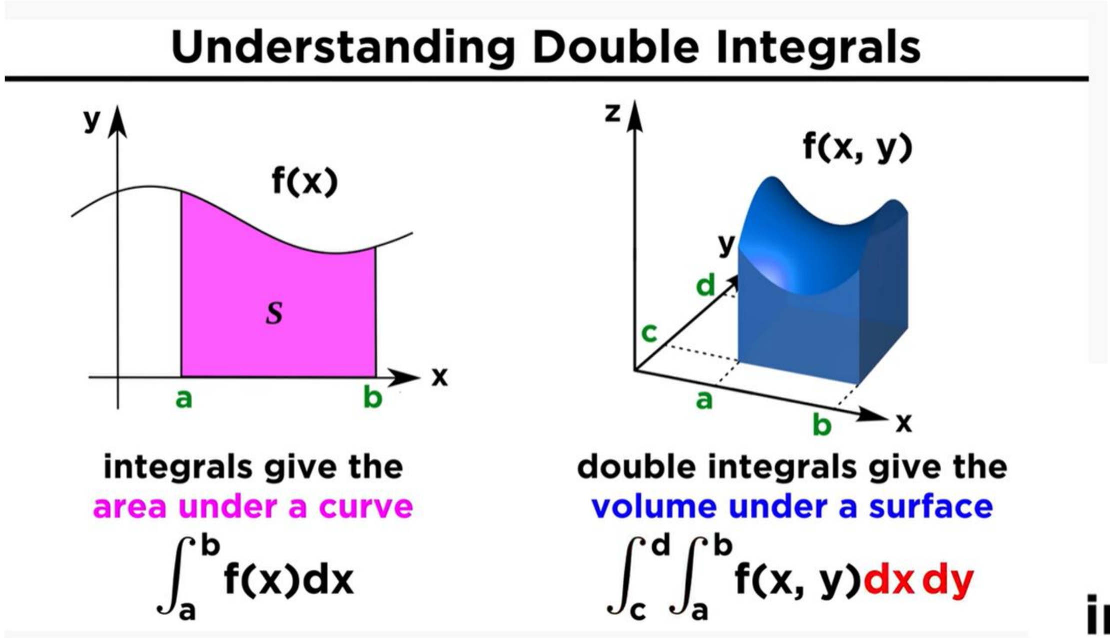
Area of region R: $\iint_R dA$

$dx\,dy$

or

$dy\,dx$

Multiple Integral: Double Integral

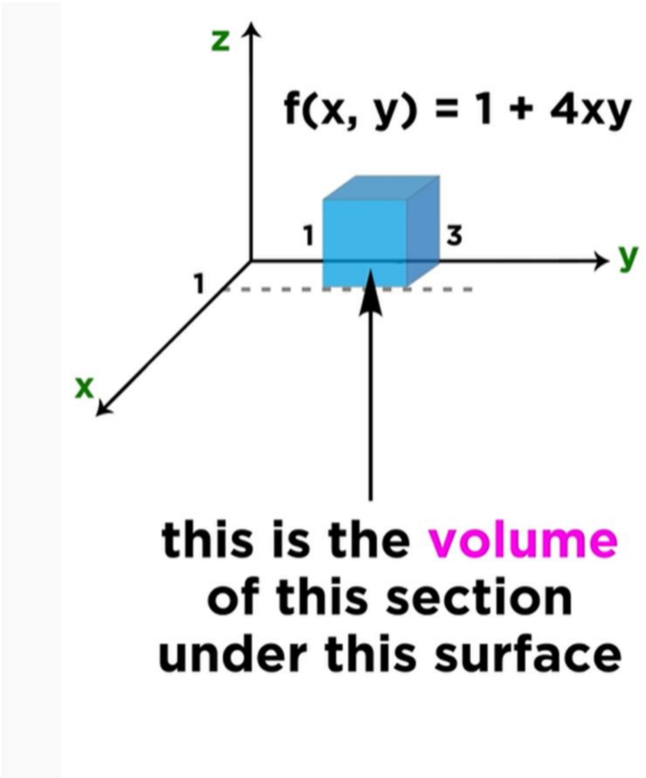


$$\int_c^d \left(\int_a^b f(x, y)dx \right) dy$$

integrate with respect to x and treat y as a constant

then simply integrate the result with respect to y

Multiple Integral: Double Integral



$$\int_1^3 \int_0^1 (1 + 4xy)\,dx\,dy$$

$$\int_1^3 (1 + 2y)\,dy$$

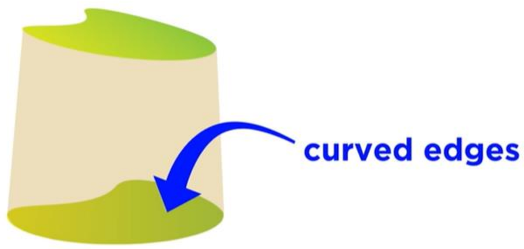
$$y + y^2 \Big|_1^3$$

$$[(3) + (3)^2] - [(1) + (1)^2]$$

$$12 - 2 = 10$$

Multiple Integral: Double Integral

Understanding Double Integrals



sometimes the integration domain depends on the variables we are integrating over

$$\iint f(x, y)\,dx\,dy$$

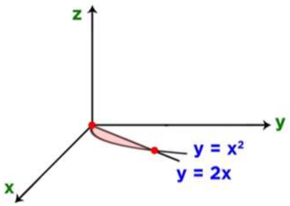
we may have to be careful about bounds and order of integration

bounds of first integration can be in terms of other variable

bounds of second integration must be in terms of numbers

Multiple Integral: Double Integral

Practice Evaluating Double Integrals



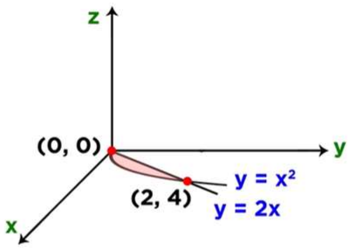
$$2x = x^2$$

$$0 = x^2 - 2x$$

$$0 = x(x - 2)$$

$$x = 0,\ x = 2$$

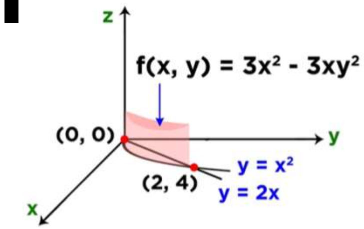
let's establish where the boundary **begins** and **ends**



$$x = 0,\ x = 2$$

$$y = 2(0) = 0$$

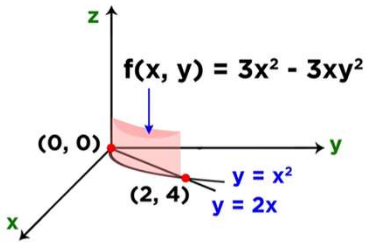
$$y = 2(2) = 4$$



$$\int_0^2 \int_{x^2}^{2x} f(x, y)\,dy\,dx$$

constant bounds must be on outside $\int_0^2 \int_{x^2}^{2x} (3x^2 - 3xy^2)\,dy\,dx$

one option for calculating this volume



outermost bound must involve constants

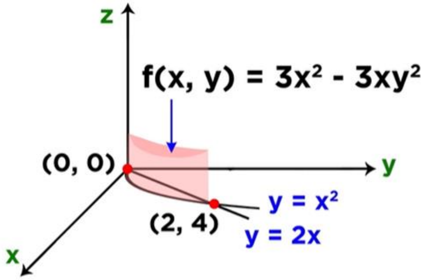
inner bounds may involve variables

1 ⇨ $\int_0^2 \int_{x^2}^{2x} (3x^2 - 3xy^2)\,dy\,dx$

2 ⇨ $\int_0^4 \int_{y/2}^{\sqrt{y}} (3x^2 - 3xy^2)\,dx\,dy$

Choose option that is not complicated which is the option 1

Multiple Integral: Double Integral

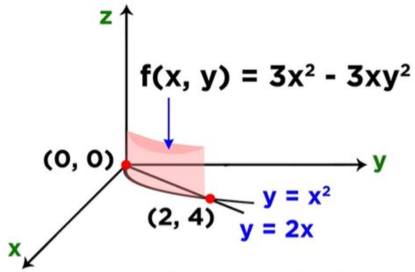


$$\int_0^2 \int_{x^2}^{2x} (3x^2 - 3xy^2)\,dy\,dx$$

$$3x^2y - xy^3 \Big|_{x^2}^{2x}$$

$$[3x^2(2x) - x(2x)^3] - [3x^2(x^2) - x(x^2)^3]$$

$$\int_0^2 (6x^3 - 11x^4 + x^7)\,dx$$



in actuality most of this volume should sit below the xy plane

$$\int_0^2 \int_{x^2}^{2x} (3x^2 - 3xy^2)\,dy\,dx$$

$$\int_0^2 (6x^3 - 11x^4 + x^7)\,dx$$

$$(6x^4/4 - 11x^5/5 + x^8/8) \Big|_0^2$$

$$[6(2)^4/4 - 11(2)^5/5 + (2)^8/8)] - [6(0)^4/4 - 11(0)^5/5 + (0)^8/8)]$$

$$= -14.4 \text{ or } -72/5$$

Multiple Integral: Double Integral

RECTANGULAR COORDINATES

FUBINI'S THEOREM OR ITERATED INTEGRALS

If $f(x, y)$ is continuous on $R = [a, b] \times [c, d]$ then,

$$\iint_R f(x,y)\, dA = \int_a^b \int_c^d f(x,y)\, dy\, dx = \int_c^d \int_a^b f(x,y)\, dx\, dy$$

POLAR COORDINATES

$$x = r\cos\theta \qquad y = r\sin\theta \qquad r^2 = x^2 + y^2$$

$$dA = r\, dr\, d\theta$$

$$\iint_D f(x,y)\, dA = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$$

The concept for this double integral is we need to integrate twice.

If you are to Integrate in terms of y, x are considered constant. Same as if you will integrate in terms of x, y are considered constant.

Multiple Integral: Double Integral

EXAMPLE:

1. Determine the rectangular and polar form of $\iint 2x - 4y^3 \; dA$.

SOLUTION:

RECTANGULAR:

$$\iint (2x - 4y^3)\, dxdy$$

$$= \int (2x - 4y^3)\, dx$$

$$= \int (2xdx - 4y^3 dx)$$

$$= 2\int xdx - 4y^3 \int dx$$

$$= 2\left(\frac{1}{1+1}x^{1+1}\right) - 4y^3(x)$$

$$= 2\left(\frac{1}{2}x^2\right) - 4y^3(x)$$

$$= x^2 - 4xy^3$$

$$= \int (x^2 - 4xy^3)\, dy$$

$$= \int (x^2 dy - 4xy^3 dy)$$

$$= x^2 \int dy - 4x \int y^3 dy$$

$$= x^2(y) - 4x\left(\frac{1}{3+1}y^{3+1}\right)$$

$$= x^2(y) - 4x\left(\frac{1}{4}y^4\right)$$

$$= x^2 y - xy^4 + C \text{ or } xy(x - y^3) + C$$

Multiple Integral: Double Integral

POLAR:

$x = r\cos\theta \qquad y = r\sin\theta \qquad r^2 = x^2 + y^2$

$dA = r\,dr\,d\theta$

$\int u^n dx = \frac{1}{n+1}u^{n+1} + C, n \neq -1$

$$\iint (2x - 4y^3)\, dA$$
$$2x = 2r\cos\theta$$
$$4y^3 = 4(r\sin\theta)^3 = 4r^3\sin^3\theta$$
$$\iint (2r\cos\theta - 4r^3\sin^3\theta)\, r\,dr\,d\theta$$
$$\iint (2r^2\cos\theta - 4r^4\sin^3\theta)\, dr\,d\theta$$
$$= \int (2r^2\cos\theta - 4r^4\sin^3\theta) dr$$
$$= \int 2r^2\cos\theta\, dr - 4r^4\sin^3\theta\, dr$$
$$= 2\cos\theta\int r^2 dr - 4\sin^3\theta\int r^4 dr$$
$$= 2\cos\theta\left(\frac{1}{2+1}r^{2+1}\right) - 4\sin^3\theta\left(\frac{1}{4+1}r^{4+1}\right)$$
$$= 2\cos\theta\left(\frac{1}{3}r^3\right) - 4\sin^3\theta\left(\frac{1}{5}r^5\right)$$
$$= \frac{2}{3}r^3\cos\theta - \frac{4}{5}r^5\sin^3\theta$$

$$= \frac{2}{3}r^3\cos\theta - \frac{4}{5}r^5\sin^3\theta$$
$$= \int\left(\frac{2}{3}r^3\cos\theta - \frac{4}{5}r^5\sin^3\theta\right)d\theta$$
$$= \int\left(\frac{2}{3}r^3\cos\theta\, d\theta - \frac{4}{5}r^5\sin^3\theta\, d\theta\right)$$
$$= \frac{2}{3}r^3\int\cos\theta\, d\theta - \frac{4}{5}r^5\int\sin^3\theta\, d\theta$$
$$= \frac{2}{3}r^3\int\cos\theta\, d\theta - \frac{4}{5}r^5\int(\sin\theta)(\sin^2\theta)\, d\theta$$
$$= \frac{2}{3}r^3\int\cos\theta\, d\theta - \frac{4}{5}r^5\int(\sin\theta)(1-\cos^2\theta)\, d\theta$$
$$= \frac{2}{3}r^3\int\cos\theta\, d\theta - \frac{4}{5}r^5\int(\sin\theta - \sin\theta\cos^2\theta)\, d\theta$$
$$= \frac{2}{3}r^3\int\cos\theta\, d\theta - \frac{4}{5}r^5\left[\int\sin\theta\, d\theta - \int\sin\theta\cos^2\theta\, d\theta\right]$$

$$u = \cos\theta$$
$$du = -\sin\theta\, d\theta$$
$$-du = \sin\theta\, d\theta$$

$$= \frac{2}{3}r^3\int\cos\theta\, d\theta - \frac{4}{5}r^5\left[\int\sin\theta\, d\theta - \int u^2\,(-du)\right]$$
$$= \frac{2}{3}r^3\int\cos\theta\, d\theta - \frac{4}{5}r^5\left[\int\sin\theta\, d\theta + \int u^2 du\right]$$
$$= \frac{2}{3}r^3(\sin\theta) - \frac{4}{5}r^5\left[-\cos\theta + \left(\frac{1}{2+1}u^{2+1}\theta\right)\right]$$
$$= \frac{2}{3}r^3(\sin\theta) - \frac{4}{5}r^5\left[-\cos\theta + \left(\frac{1}{3}u^3\right)\right]$$
$$= \frac{2}{3}r^3(\sin\theta) - \frac{4}{5}r^5\left[-\cos\theta + \left(\frac{1}{3}(\cos\theta)^3\right)\right]$$
$$= \frac{2}{3}r^3(\sin\theta) - \frac{4}{5}r^5\left[-\cos\theta + \left(\frac{1}{3}\cos^3\theta\right)\right]$$
$$= \frac{2}{3}r^3\sin\theta + \frac{4}{5}r^5\cos\theta - \frac{4}{15}r^5\cos^3\theta + C$$

SUBSCRIBE

Multiple Integral: Double Integral

Example 1: $\int_0^1 \int_1^2 xy \, dy \, dx$

$$1.)\ \int_0^1\int_1^2 xy\,dy\,dx = \int_0^1 x\left[\int_1^2 y\,dy\right]dx = \int_0^1 \left(\frac{3}{2}x\right)dx$$

$$\Rightarrow \int_1^2 xy\,dy = x\int_1^2 y\,dy$$

$$= x\left[\frac{y^2}{2}\right]_1^2 = x\left[\frac{2^2}{2} - \frac{1^2}{2}\right]$$

$$= x\left[2 - \frac{1}{2}\right] = x\left[\frac{4-1}{2}\right] = (x)\left(\frac{3}{2}\right) = \frac{3}{2}x$$

$$= \int_0^1 \frac{3}{2}x\,dx = \frac{3}{2}\int_0^1 x\,dx$$

$$= \frac{3}{2}\left[\frac{x^2}{2}\right]_0^1 = \frac{3}{2}\left[\frac{1^2}{2} - \cancel{\frac{0^2}{2}}\right]$$

$$\boxed{= \frac{3}{4}}$$

$$\int_0^1\int_1^2 xy\,dy\,dx \ \boxed{= \frac{3}{4}}$$

Multiple Integral: Double Integral

Example 2: $\int_0^1 \int_y^{y^2} (x+2y)\,dx\,dy$

$$\int_y^{y^2} (x+2y)\,dx = \int_y^{y^2} x\,dx + \int_y^{y^2} 2y\,dx$$

$$= \frac{x^2}{2}\Big|_y^{y^2} + 2y\left[(x)\right]_y^{y^2}$$

$$= \left[\frac{(y^2)^2}{2} - \frac{(y)^3}{2}\right] + 2x\left[y^2 - y\right]$$

$$= \frac{1}{2}\left[(y^2)^2 - y^2\right] + 2y\left[y^2 - y\right]$$

$$= \frac{1}{2}\left[y^4 - y^2\right] + 2y\left[y^2 - y\right]$$

$$= \frac{1}{2}y^4 - \frac{1}{2}y^2 + 2y^3 - 2y^2$$

$$= \frac{1}{2}y^4 + 2y^3 - \frac{5}{2}y^2 = \int_y^{y^2} (x+2y)\,dx$$

$$= \int_0^1 \left(\frac{1}{2}y^4 + 2y^3 - \frac{5}{2}y^2\right) dy$$

$$= \frac{1}{2}\int_0^1 y^4\,dy + 2\int_0^1 y^3\,dy - \frac{5}{2}\int_0^1 y^2\,dy$$

$$= \frac{1}{2}\left[\frac{y^5}{5}\right]_0^1 + 2\left[\frac{y^4}{4}\right]_0^1 - \frac{5}{2}\left[\frac{y^3}{3}\right]_0^1$$

$$= \frac{1}{2}\left(\frac{1}{5}\right)\left[y^5\right]_0^1 + \frac{1}{2}\left[y^4\right]_0^1 - \frac{5}{2}\left(\frac{1}{3}\right)\left[y^3\right]_0^1$$

$$= \frac{1}{10}\left[1^5 - \cancel{0^5}\right] + \frac{1}{2}\left[1^4 - \cancel{0^4}\right] - \frac{5}{6}\left[1^3 - \cancel{0^3}\right]$$

$$= \frac{1}{10} + \frac{1}{2} - \frac{5}{6} = \frac{(1)(2) + 1(10)}{20} - \frac{5}{6} = \frac{2+10}{20} - \frac{5}{6} = \frac{12}{20} - \frac{5}{6} = \frac{3}{5} - \frac{5}{6}$$

$$= \frac{(3)(6) - 5(5)}{30} = \frac{18-25}{30} = \boxed{\frac{-7}{30}}$$

Multiple Integral: Double Integral

Example 3:

3) $\int_1^2 \int_0^{4x} xy\,dy\,dx$

$= \int_0^{4x} xy\,dy = x\int_0^{4x} y\,dy = x\left[\frac{y^2}{2}\right]_0^{4x} = x\left(\frac{1}{2}\right)\left[y^2\right]_0^{4x}$

$= \frac{1}{2}x\left[(4x)^2 - 0^2\right] = \frac{1}{\cancel{2}}x\left(\overset{8}{\cancel{16}}x^2\right) = 8x^3$

$= \int_1^2 8x^3\,dx = 8\int_1^2 x^3\,dx = 8\left[\frac{x^4}{4}\right]_1^2 = \overset{2}{\cancel{8}}\left(\frac{1}{\cancel{4}}\right)\left[x^4\right]_1^2$

$= 2\left[2^4 - 1^4\right] = 2(16-1) = 2(15) \boxed{=30}$

Multiple Integral: Double Integral

Reference:

https://www.youtube.com/watch?v=jDecy4Dvv5o

https://www.youtube.com/watch?v=UubU3U2C8WM

TRIPLE INTEGRAL

Multiple Integral: Triple Integral

TRIPLE INTEGRAL

Triple integrals work in the same manner. We think of all the y's and z's as constants and integrate with respect to x or we think of all x's and z's as constants and integrate with respect to y or we think of all x's and y's as constants and integrate with respect to z.

RECTANGULAR COORDINATES

$$\boxed{\iiint\limits_E f(x,y,z)\,dV}$$

$$\boxed{B = [a,b] \times [c,d] \times [r,s]}$$

$$\boxed{\iiint\limits_B f(x,y,z)\,dV = \int_r^s \int_c^d \int_a^b f(x,y,z)\,dx\,dy\,dz}$$

SUBSCRIBE

Multiple Integral: Triple Integral

Unit of volume in rectangular coordinates

$$\Delta V = \Delta x \Delta y \Delta z$$

$$\downarrow$$

$$dV = dx\,dy\,dz$$

Volume of domain D: $\iiint dV$ — $dx\,dy\,dz$ / $dz\,dy\,dx$

End of Topic

Thank you

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