



MATH 121E

CALCULUS II

OBJECTIVES

- To use integration techniques to determine the integral of a function

TOPIC OUTLINE

Integration Techniques

- Integration by Parts
- Trigonometric Integrals
- Trigonometric Substitution
- Rational Functions
- Rationalizing Substitution

The background features abstract green geometric shapes. On the left, a single green triangle points downwards. On the right, a complex arrangement of overlapping green triangles and polygons in various shades of green (from light lime to dark forest green) forms a vertical, jagged shape. A thin, light gray line extends diagonally from the bottom left towards the right, passing through the green shapes.

Integration by Parts

Integration by Parts

Integration by Parts Formula:

$$\int u \, dv = uv - \int v \, du$$

Example (a).

Evaluate $\int x \sin 2x \, dx$

let $u = x$, $du = dx$

$dv = \sin 2x \, dx$, $v = -\frac{1}{2} \cos 2x$

$\int x \sin 2x \, dx$

From the formula,

$$\int u \, dv = uv - \int v \, du$$

$$= x \left(-\frac{1}{2} \cos 2x\right) + \frac{1}{2} \int \cos 2x \, dx$$

$$= -\frac{1}{2} x \cos 2x + \frac{1}{4} \sin 2x + C$$

Example (b).

Evaluate $\int \sec^3 \theta \, d\theta$

$$\int \sec \theta \sec^2 \theta \, d\theta$$

let $u = \sec \theta$, $du = \sec \theta \tan \theta \, d\theta$

$dv = \sec^2 \theta \, d\theta$, $v = \tan \theta$

From the formula,

$$\int u \, dv = uv - \int v \, du$$

$$= \sec \theta \tan \theta - \int \tan \theta \sec \theta \tan \theta \, d\theta$$

$$= \sec \theta \tan \theta - \int \sec \theta \tan^2 \theta \, d\theta$$

$$= \sec \theta \tan \theta - \int \sec \theta (\sec^2 \theta - 1) \, d\theta$$

$$= \sec \theta \tan \theta - \int \sec^3 \theta \, d\theta + \int \sec \theta \, d\theta$$

$$\int \sec^3 \theta \, d\theta = \sec \theta \tan \theta - \int \sec^3 \theta \, d\theta + \int \sec \theta \, d\theta$$

$$2 \int \sec^3 \theta \, d\theta = \sec \theta \tan \theta + \int \sec \theta \, d\theta$$

$$2 \int \sec^3 \theta \, d\theta = \sec \theta \tan \theta + \ln |\sec \theta + \tan \theta| + C$$

$$\int \sec^3 \theta \, d\theta = \frac{1}{2} \sec \theta \tan \theta + \frac{1}{2} \ln |\sec \theta + \tan \theta| + C$$

Integration by Parts

Evaluate below:

Example C

$$\int e^x \cos x \, dx$$

$$\text{Let } u = e^x, \, dv = \cos x \, dx$$

$$du = e^x dx, \, v = \sin x$$

$$\frac{du}{dx} = e^x$$

$$dx$$

$$\int e^x \sin x \, dx$$

$$\text{Let } u = e^x, \, dv = \sin x \, dx$$

$$du = e^x dx, \, v = -\cos x$$

$$\int e^x \sin x \, dx = -e^x \cos x - \int -\cos x \, e^x \, dx$$

$$\int e^x \sin x \, dx = -e^x \cos x + \int e^x \cos x \, dx$$

$$\int e^x \cos x \, dx = e^x \sin x - \int e^x \sin x \, dx$$

$$\int e^x \cos x \, dx = e^x \sin x - (-e^x \cos x + \int e^x \cos x \, dx)$$

$$\int e^x \cos x \, dx = e^x \sin x + e^x \cos x - \int e^x \cos x \, dx$$

$$\int e^x \cos x \, dx + \int e^x \cos x \, dx = e^x \sin x + e^x \cos x$$

$$2 \int e^x \cos x \, dx = e^x \sin x + e^x \cos x$$

$$\int e^x \cos x \, dx = \frac{e^x \sin x + e^x \cos x}{2} + C$$

Integration by Parts

Evaluate below Integrals

1.) $\int \ln x \, dx$

2. $\int y \cos 4y \, dy$

3. $\int x (2x-1)^7 \, dx$

4. $\int y \cos y \sin^2 y \, dy$

5. $\int \sin x \sin 4x \, dx$



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Trigonometric Integrals

Trigonometric Integrals

Trigonometric formulas

$$\sin^2 u = \frac{1 - \cos(2u)}{2}$$

$$\cos^2 u = \frac{1 + \cos(2u)}{2}$$

$$\tan^2 u = \frac{1 - \cos(2u)}{1 + \cos(2u)}$$

$$\sin^2 u + \cos^2 u = 1$$

$$1 + \tan^2 u = \sec^2 u$$

$$1 + \cot^2 u = \csc^2 u$$

Example (a).

Evaluate $\int \cos^3 x dx$

$$= \int \cos^2 x \cos x dx$$

From the formula:

$$\sin^2 x + \cos^2 x = 1$$

$$= \int (1 - \sin^2 x) \cos x dx$$

Let $u = \sin x$

$$du = \cos x dx$$

$$= \int (1 - \sin^2 x) \cos x dx$$

$$= \int (1 - u^2) du$$

$$= \int du - \int u^2 du$$

$$= u - \frac{u^3}{3} + C$$

$$= \sin x - \frac{\sin^3 x}{3} + C$$

Trigonometric Integrals

Example (b).

$$\begin{aligned} & \text{Evaluate } \int \sin^5 x \cos^2 x \, dx \\ &= \int \sin^4 x \sin x \cos^2 x \, dx \\ &= \int (\sin^2 x)^2 \cos^2 x \sin x \, dx \\ &= \int (1 - \cos^2 x)^2 \cos^2 x \sin x \, dx \end{aligned}$$

Let $u = \cos x$

$$du = -\sin x \, dx$$

$$\begin{aligned} &= \int (1 - u^2)^2 u^2 \sin x \, (du / -\sin x) \\ &= -\int (1 - u^2)^2 u^2 \, du \\ &= -\int (1 - 2u^2 + u^4) u^2 \, du \\ &= -\int (u^2 - 2u^4 + u^6) \, du \\ &= -\frac{u^3}{3} + \frac{2u^5}{5} - \frac{u^7}{7} + C \\ &= -\frac{\cos^3 x}{3} + \frac{2\cos^5 x}{5} - \frac{\cos^7 x}{7} + C \end{aligned}$$

Example (c).

$$\begin{aligned} & \text{Evaluate } \int \frac{\sin^3 x \, dx}{\cos^6 x} \\ &= \int \frac{\sin^3 x \, dx}{\cos^3 x \cos^3 x} \end{aligned}$$

$$\begin{aligned} &= \int \tan^3 x \sec^3 x \, dx \\ &= \int \tan^2 x \sec^2 x \cdot \sec x \tan x \, dx \\ &= \int (\sec^2 x - 1) \sec^2 x \sec x \tan x \, dx \\ & \text{let } u = \sec x \\ & \quad du = \sec x \tan x \, dx \\ &= \int (u^2 - 1) u^2 \, du \\ &= \int (u^4 - u^2) \, du \\ &= \frac{u^5}{5} - \frac{u^3}{3} + C \\ &= \frac{\sec^5 x}{5} - \frac{\sec^3 x}{3} + C \end{aligned}$$

Trigonometric Integrals

Evaluate below Integrals

1. $\int \frac{\sin^5 t \, dt}{\cos^2 t}$

2. $\int \sin^5 x \cos^5 x \, dx$

3. $\int \sin^2 x \cos^2 x \, dx$

4. $\int \cos^2 x \cos^3 2x \, dx$

5. $\int \sin^4 x \cos^4 x \, dx$

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Trigonometric Substitution

Trigonometric Substitution

When the integrand involves

$a^2 - x^2$, try $x = a \sin \theta$

$a^2 + x^2$, try $x = a \tan \theta$

$x^2 - a^2$, try $x = a \sec \theta$

Example (a).

Evaluate $\int \frac{dx}{(a^2 + x^2)^{3/2}}$

let $x = a \tan \theta$

$dx = a \sec^2 \theta d\theta$

$$= \int \frac{a \sec^2 \theta d\theta}{(a^2 + a^2 \tan^2 \theta)^{3/2}}$$

$$= \int \frac{a \sec^2 \theta d\theta}{[a^2 (1 + \tan^2 \theta)]^{3/2}}$$

$$= \int \frac{a \sec^2 \theta d\theta}{[a^2 (1 + \tan^2 \theta)]^{3/2}}$$

$$= \int \frac{a \sec^2 \theta d\theta}{a^3 (1 + \tan^2 \theta)^{3/2}}$$

$$= \frac{1}{a^2} \int \frac{\sec^2 \theta d\theta}{(1 + \tan^2 \theta)^{3/2}}$$

$$= \frac{1}{a^2} \int \frac{\sec^2 \theta d\theta}{(\sec^2 \theta)^{3/2}}$$

$$= \frac{1}{a^2} \int \frac{\sec^2 \theta d\theta}{\sec^3 \theta}$$

$$= \frac{1}{a^2} \int \frac{d\theta}{\sec \theta}$$

$$= \frac{1}{a^2} \int \cos \theta d\theta$$

$$= \frac{1}{a^2} \sin \theta + C$$

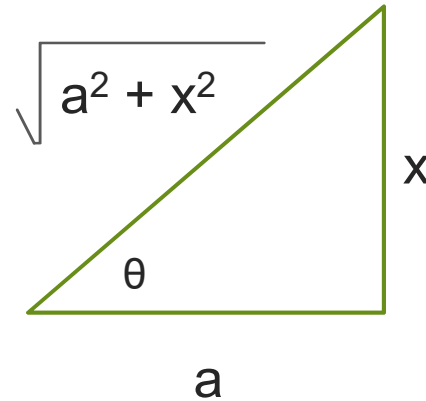
Trigonometric Substitution

Since,

$$\tan \theta = \frac{x}{a}$$

Therefore;

$$\sin \theta = \frac{x}{\sqrt{a^2 + x^2}}$$



Thus:

$$\int \frac{dx}{(a^2 + x^2)^{3/2}} = \frac{1}{a^2} \frac{x}{\sqrt{a^2 + x^2}} + C$$

Trigonometric Substitution

Example (b).

Evaluate $\int \sqrt{a^2 - x^2} \, dx$

let $x = a \sin \theta$

$dx = a \cos \theta \, d\theta$

$$\int \sqrt{a^2 - x^2} \, dx = \int \sqrt{a^2 - a^2 \sin^2 \theta} \, a \cos \theta \, d\theta$$

$$= \int \sqrt{a^2 - a^2 \sin^2 \theta} \, a \cos \theta \, d\theta$$

$$= \int \sqrt{a^2 (1 - \sin^2 \theta)} \, a \cos \theta \, d\theta$$

$$= \int a \cos \theta \, a \cos \theta \, d\theta$$

$$= a^2 \int \cos^2 \theta \, d\theta$$

$$= a^2 \int \frac{(1 + \cos 2\theta)}{2} d\theta$$

$$= \frac{a^2}{2} \left[\int d\theta + \int \cos 2\theta \, d\theta \right]$$

$$= \frac{a^2}{2} + \theta + \frac{1}{2} \frac{\sin 2\theta}{2}$$
$$= a^2 \left(\frac{\theta}{2} + \frac{1}{4} \sin 2\theta \right) + C$$

Trigonometric Substitution

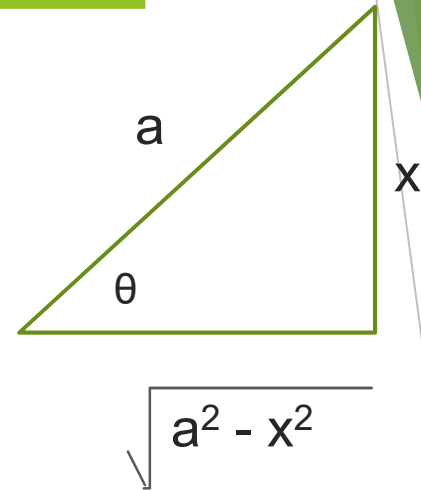
Since,

$$\sin \theta = \frac{x}{a}$$

Therefore;

$$\cos \theta = \frac{\sqrt{a^2 - x^2}}{a}$$

$$\sin 2\theta = 2\sin \theta \cos \theta = \frac{2x\sqrt{a^2 - x^2}}{a^2}$$



Therefore:

$$\int \sqrt{a^2 - x^2} \, dx = a^2 \left(\frac{\theta}{2} + \frac{1}{4} \sin 2\theta \right) + C$$

$$= \frac{a^2}{2} \operatorname{Arcsin} \frac{x}{a} + \frac{a^2}{4} \frac{2x\sqrt{a^2 - x^2}}{a^2} + C$$

$$= \frac{a^2}{2} \operatorname{Arcsin} \frac{x}{a} + \frac{x}{2} \sqrt{a^2 - x^2} + C$$

Trigonometric Substitution

Evaluate below Integrals

1. $\int \frac{du}{(u^2 + a^2)^2}$

2. $\int \frac{dz}{z (4+z^2)^3}$

3. $\int \frac{dy}{y (y^2 + 1)}$



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Rational Functions

Rational Functions

A rational function is any function which **can be written as the ratio of two polynomial functions**, where the polynomial in the denominator is not equal to zero. It is a quotient of a polynomial.

Example (a).

Evaluate $\int \frac{(x-1) dx}{(x^2+5x+6)}$

$$= \int \frac{(x-1) dx}{(x+2)(x+3)}$$

Solution:

let

$$\frac{x-1}{(x+2)(x+3)} = \frac{A}{(x+2)} + \frac{B}{(x+3)}$$

$$x-1 = A(x+3) + B(x+2)$$

when $x = -2$

$$x-1 = A(x+3) + B(x+2)$$
$$A = -3$$

when $x = -3$

$$x-1 = A(x+3) + B(x+2)$$
$$B = 4$$

$$\begin{aligned} \int \frac{(x-1) dx}{(x^2+5x+6)} &= \int \frac{A}{(x+2)} dx + \int \frac{B}{(x+3)} dx \\ &= -3 \int \frac{dx}{(x+2)} + 4 \int \frac{dx}{(x+3)} \\ &= -3 \ln(x+2) + 4 \ln(x+3) + C \end{aligned}$$

Rational Functions

Example (b).

Evaluate $\int \frac{5 \sin \theta \cos \theta d\theta}{\sin^2 \theta + 3 \sin \theta - 4}$

$$= \int \frac{5 \sin \theta \cos \theta d\theta}{(\sin \theta + 4)(\sin \theta - 1)}$$

$$\text{Let } \int \frac{5 \sin \theta}{(\sin \theta + 4)(\sin \theta - 1)} = \frac{A}{(\sin \theta + 4)} + \frac{B}{(\sin \theta - 1)}$$

$$\frac{5 \sin \theta}{(\sin \theta + 4)(\sin \theta - 1)} = \frac{A(\sin \theta - 1) + B(\sin \theta + 4)}{(\sin \theta + 4)(\sin \theta - 1)}$$

If $\sin \theta = 1$, if $\sin \theta = 0$

$$B = 1 \quad A = 4$$

Thus,

$$\int \frac{5 \sin \theta \cos \theta d\theta}{\sin^2 \theta + 3 \sin \theta - 4} = 4 \int \frac{\cos \theta d\theta}{(\sin \theta + 4)} + 1 \int \frac{\cos \theta d\theta}{(\sin \theta - 1)}$$

$$= 4 \ln (\sin \theta + 4) + \ln (\sin \theta - 1) + C$$

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Rationalizing Substitution

Rationalizing Substitution

An integration method which is often useful when the integrand is a fraction including more than one kind of root, such as $\frac{\sqrt{x}}{1+\sqrt[3]{x}}$.

A different type of rationalizing substitution can be used to work with integrands such as $\frac{1}{1+e^x}$.

Note: This method transforms the integrand into a rational function, hence the name *rationalizing*.

Rationalizing Substitution

Example: For $\int x\sqrt{5+x^2} \, dx$ let $u = 5 + x^2$.

That means $du = 2x \, dx$
 $\frac{1}{2} du = x \, dx$

It follows that

$$\int x\sqrt{5+x^2} \, dx = \int \sqrt{u} \cdot \frac{1}{2} du$$

$$= \frac{1}{2} \int u^{\frac{1}{2}} du$$

$$= \frac{1}{2} \cdot \frac{2}{3} u^{\frac{3}{2}} + C$$

$$= \frac{1}{3} (5+x^2)^{\frac{3}{2}} + C$$

Rationalizing Substitution

Example 2: Find $\int \frac{1}{1+e^x} dx$.

Let $u = 1 + e^x$, so $\ln(u - 1) = x$. That means $\frac{1}{u-1} du = dx$.

$$\begin{aligned}\int \frac{1}{1+e^x} dx &= \int \frac{1}{u} \cdot \frac{1}{u-1} du \\ &= \int \left(\frac{1}{u-1} - \frac{1}{u} \right) du \quad (\text{partial fractions}) \\ &= \ln|u-1| - \ln|u| + C \\ &= \ln \left| \frac{u-1}{u} \right| + C \\ &= \ln \left| \frac{e^x}{1+e^x} \right| + C \quad \text{or} \quad \ln \left(\frac{e^x}{1+e^x} \right) + C.\end{aligned}$$

Rationalizing Substitution

Evaluate below Integrals

$$1. \int \frac{(5x-12) dx}{x^3-6x^2+8x}$$

$$2. \int \frac{(y^3+4) dy}{y(y+1)}$$

$$3. \int \frac{3z^2 dt}{z^4+5z^2+4}$$

$$4. \int \frac{x^2-5x+3}{x^3-4x^2+3x}$$

End of Topic

Thank you

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