UNIVERSITY OF SOUTHERN MINDANAO

MATH121E

Calculus 2

Topic Outline

Applications of Definite Integral

- Plane areas
- Areas Between curves
- Other Applications
    a. Volume
    b. Work
    c. Hydrostatic Pressure

Areas Between Curves

$$A = \int_a^b (Y_{upper} - Y_{lower})\, dx$$ This is for vertical Strip

$$A = \int_a^b (X_{right} - X_{left})\, dy$$ This is for horizontal strip Strip

Areas Between Curves

Steps to Solve the problem

1. Find for the vertex
2. Find the intersection point
3. Draw the curves
4. Solve the area

For vertex= we have the standard form

$F(x) = ax^2+bx+C$, for vertex we have the formula

$$x = \frac{-b}{2a}$$

$F(y) = ay^2+by+C$; vertex is $y = \frac{-b}{2a}$

Areas Between Curves

Examples:

1.) Calculate the area of the region bounded by the curves

$y = x^2$ and $x = y^2$

a. Solve for vertex of the two parabolas

for $y = x^2$; $a=1$; $b=0$

$x = \frac{-b}{2a} = -\frac{0}{2(1)} = 0$

substitute value of x in $y = x^2$

$y = x^2 = 0^2 = 0$

$V_1\ (0,0)$

for $x = y^2$; $a = 1$; $b=0$

$y = \frac{-b}{2a} = \frac{-0}{2(1)} = 0$

substitute in $x = y^2$

$x = 0^2 = 0$

$V_2\ (0,0)$

b. Solve for the point of intersection. Equate the two equations.

① $y = x^2$ and ② $x = y^2$

$y = x^2$; get value of y from the equation ② $x = y^2$.

$x = y^2$

$(x)^{1/2} = (y^2)^{1/2}$

$x^{1/2} = y$

$y = \sqrt{x}$; substitute in equation ①

$y = x^2$

$\sqrt{x} = x^2$

$(\sqrt{x})^2 = (x^2)^2$

$x = x^4$

$0 = x^4 - x$

$0 = x(x^3 - 1)$

$x = 0$ and $x = 1$

substitute in equation ①

$y = x^2$

when $x = 0$;

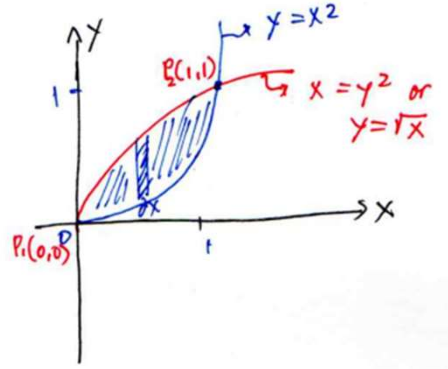
$y = 0^2$; $y = 0$ $\quad P_1(0,0)$

when $x = 1$

$y = 1^2 = 1$ $\quad P_2(1,1)$

Areas Between Curves

c. Draw the graph



d. Solve for Area

$$A = \int_a^b (y_{upper} - y_{lower})\,dx$$

$$A = \int_0^1 \left[(\sqrt{x}) - (x^2)\right] dx$$

$$A = \int_0^1 (x^{1/2} - x^2)\,dx$$

$$A = \frac{x^{3/2}}{3/2} - \frac{x^3}{3}\Big|_0^1$$

$$A = \frac{2}{3}x^{3/2} - \frac{x^3}{3}\Big|_0^1$$

$$A = \frac{2}{3}(1)^{3/2} - \frac{1^3}{3} - 0$$

$$A = \frac{2}{3} - \frac{1}{3}$$

$$\boxed{A = \frac{1}{3} \text{ sq. units}}$$

Areas Between Curves

2. Calculate the area of the region bounded by the ~~line~~ curve $y = x^2 - 4x$ and the x-axis

a. Solve for vertex of the parabola

$y = x^2 - 4x$; $a = 1$ and $b = -4$

$x = \frac{-b}{2a}$

$x = \frac{-(-4)}{2(1)} = \frac{4}{2} = 2$

substitute the value of x in the parabola equation

$y = x^2 - 4x$

$y = 2^2 - 4(2)$

$y = 4 - 8$

$y = -4$

$V(2, -4)$

b. Solve for the point of intersection

$y = x^2 - 4x$; get the x-intercept by letting $y = 0$.

$y = x^2 - 4x$

$0 = x^2 - 4x$

$0 = x(x-4)$

$0 = x(x-4)$

$x = 0$; $x = 4$

Substitute values of x in the equation,

$y = x^2 - 4x$; $x = 0$

$x = 0^2 - 4(0)$

$y = 0$ $\qquad P_1 (0,0)$

$y = x^2 - 4x$; $x = 4$

$y = 4^2 - 4(4)$

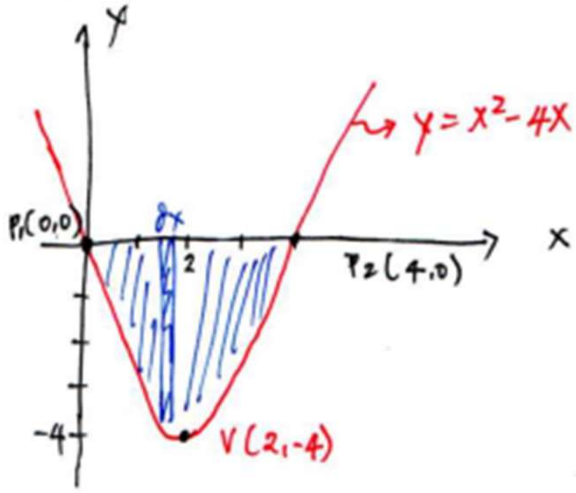
$y = 16 - 16$

$y = 0$

$P_2 (4, 0)$

Areas Between Curves

c. Draw the graph



d. Solve for area

$$A = \int_a^b (y_{upper} - y_{lower})\,\partial x$$

$$A = \int_0^4 \left[0 - (x^2 - 4x)\right]\partial x$$

$$A = \int_0^4 (-x^2 + 4x)\,\partial x$$

$$A = -\frac{x^3}{3} + \frac{4x^2}{2}\Big|_0^4$$

$$A = -\frac{x^3}{3} + 2x^2\Big|_0^4$$

$$A = -\frac{(4)^3}{3} + 2(4)^2 - 0$$

$$A = -\frac{64}{3} + 32$$

$$A = \frac{-64 + 96}{3}$$

$$\boxed{A = \frac{32}{3} \text{ sq. units}}$$

Areas Between Curves

3. Calculate the area of the region bounded by the equations $y = x^2 - 4x$ and $y = 6 - 3x$

a. Solve for the vertex of parabola

$y = x^2 - 4x$

$a = 1$; $b = -4$

$x = -\frac{b}{2a} = -\frac{(-4)}{2(1)} = \frac{4}{2} = 2$

Substitute value of x in

$y = x^2 - 4x$

$y = 2^2 - 4(2) = 4 - 8 = -4$

$V(2, -4)$

b. Intersection at x-axis (roots)

$y = x^2 - 4x$; get the x-intercept by letting $y = 0$

$y = x^2 - 4x$

$0 = x^2 - 4x$

$0 = x(x - 4)$; $x = 0$ and $x = 4$

Substitute this values in below equation:

$y = x^2 - 4x$; $x = 0$

$y = 0^2 - 4(0)$

$y = 0$ $\quad P_1(0,0)$

$y = x^2 - 4x$; $x = 4$

$y = 4^2 - 4(4)$

$y = 16 - 16$

$y = 0$ $\quad P_2(4,0)$

c. Solve for the point of intersection of the curve and the line by equating the two equations

$y = x^2 - 4x$ and $y = 6 - 3x$

$y = x^2 - 4x$ Substitute

$6 - 3x = x^2 - 4x$

$0 = x^2 - 4x + 3x - 6$

$0 = x^2 - x - 6$

$0 = (x - 3)(x + 2)$

$x = 3$ and $x = -2$

Substitute the value of x in the equation of the line

$y = 6 - 3x$

$y = 6 - 3(3)$

$y = 6 - 9$

$y = -3$ $\quad P_3(3,-3)$

$y = 6 - 3x$

$y = 6 - 3(2)$

$y = 6 + 6$

$y = 12$ $\quad P_4(-2,12)$

Areas Between Curves

d. Solve for the x and y intercepts of the line ;

$y = 6 - 3x$ let $y = 0$

$0 = 6 - 3x$

$3x = 6$ $P_5(2,0)$

$\frac{3x}{3} = \frac{6}{3}$

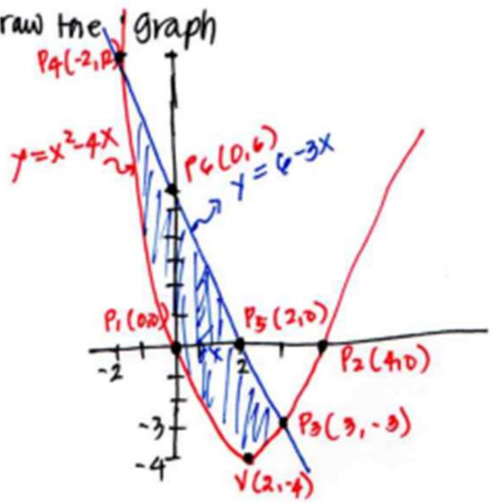
$x = 2$

let $x = 0$; $y = 6 - 3x$

$y = 6 - 3(0)$

$y = 6$ $P_6(0,6)$

e. Draw the graph



f. Solve for area

$$A = \int_a^b (y_{upper} - y_{lower})\,\partial x$$

$$A = \int_{-2}^{3} [6 - 3x - (x^2 - 4x)]\,\partial x$$

$$= \int_{-2}^{3} (6 - 3x - x^2 + 4x)\,\partial x$$

$$= \int_{-2}^{3} (6 + x - x^2)\,\partial x$$

$$= 6x + \frac{x^2}{2} - \frac{x^3}{3} \Big|_{-2}^{3}$$

$$= 6(3) + \frac{(3)^2}{2} - \frac{(3)^3}{3} - \left[6(-2) + \frac{(-2)^2}{2} - \frac{(-2)^3}{3}\right]$$

$$= 18 + \frac{9}{2} - \frac{27}{3} - \left[-12 + \frac{4}{2} + \frac{8}{3}\right]$$

$$= 18 + \frac{9}{2} - 9 - \left[-12 + 2 + \frac{8}{3}\right]$$

$$= 9 + \frac{9}{2} - \left[-10 + \frac{8}{3}\right]$$

$$= \frac{18 + 9}{2} - \left[\frac{-30 + 8}{3}\right]$$

$$= \frac{27}{2} - \left[-\frac{22}{3}\right]$$

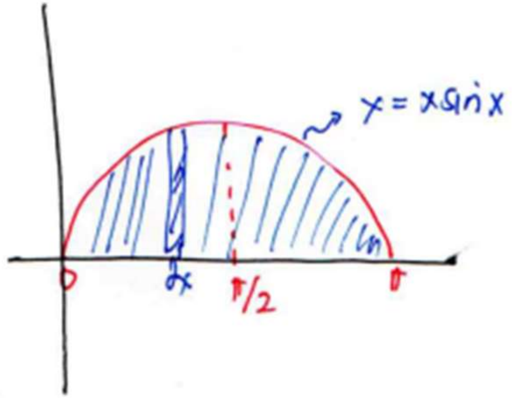
$$= \frac{27}{2} + \frac{22}{3}$$

$$= \frac{81 + 44}{6}$$

$$\boxed{A = \frac{125}{6} \text{ sq. units}}$$

Areas Between Curves

4. Find the area under the first arc of the curve
$y = x \sin x$.



$$A = \int_a^b (y_{upper} - y_{lower})\, dx$$

$$A = \int_0^{\pi} (x \sin x - 0)\, dx$$

$$A = \int_0^{\pi} x \sin x \, dx$$

by integration by parts

let $u = x$ $\quad dv = \sin x\, dx$

$du = dx$ $\quad v = -\cos x$

from $uv - \int v\, du$

$$= x(-\cos x) - \int -\cos x \, dx$$

$$= -x\cos x + \int \cos x \, dx$$

$$= -x\cos x + \sin x \Big|_0^{\pi}$$

$$= -\pi \cos \pi + \sin \pi - [0 \cos 0 + \sin 0]$$

$$= -\pi \cos \pi + \sin \pi - 0 - 0$$

$\pi = 180°$

$$= -180° \cos 180° + \sin 180° - 0$$

$$= -180° (-1) + 0$$

$$= 180° \text{ or } \boxed{\pi \text{ square units}}$$

End of Topic

Thank you

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