Advanced Network Theory

Clos Networks Topology

The Fat-Tree topology is a modern, practical application of a much older and more fundamental concept in network switching theory known as the **Clos Network**. Understanding this theory explains *why* the Fat-Tree is designed the way it is.

The Motivating Problem: Switch Complexity

The simplest conceptual model for a switch that can connect any of its `N` inputs to any of its `N` outputs is a **crossbar switch**.

- A crossbar is an `N x N` grid of connection points (crosspoints).
- Its primary drawback is its high complexity. The number of crosspoints required grows with the square of the number of ports, a **complexity of $O(N^2)$**.
- This becomes prohibitively expensive and physically large for switches with many ports.

The key question that led to the Clos network was: ***can we build a switch with the same non-blocking capability as a crossbar, but with significantly lower complexity?***

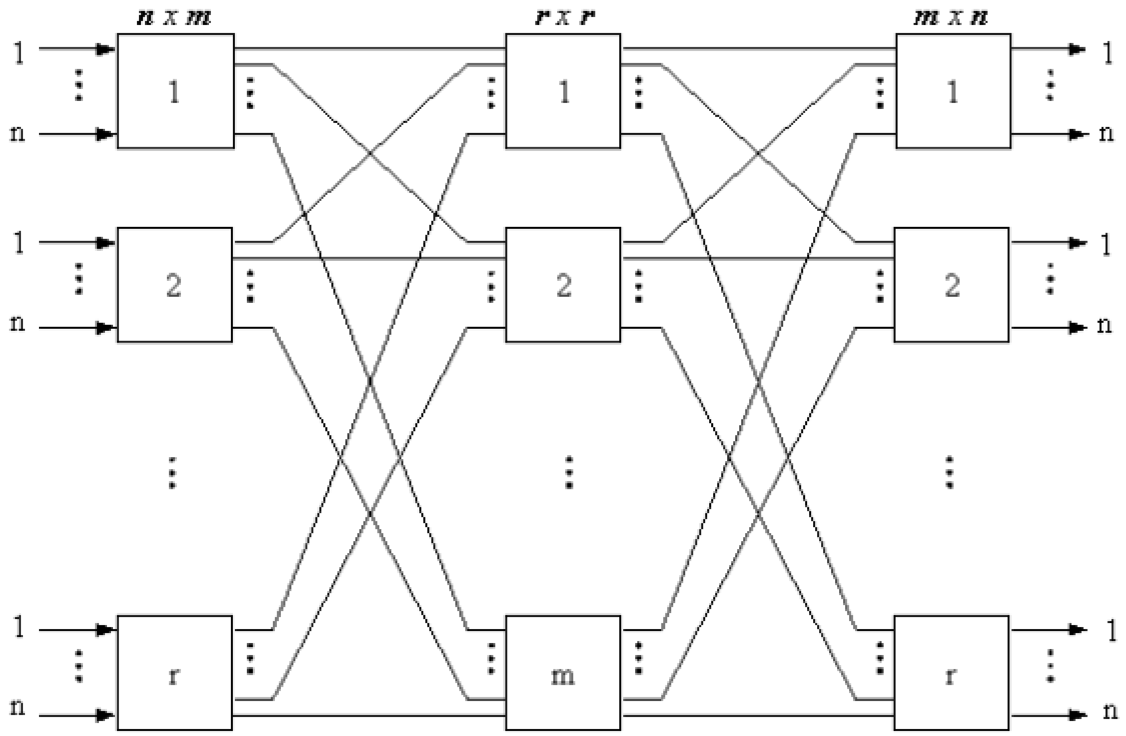
The 3-Stage Clos Network Solution

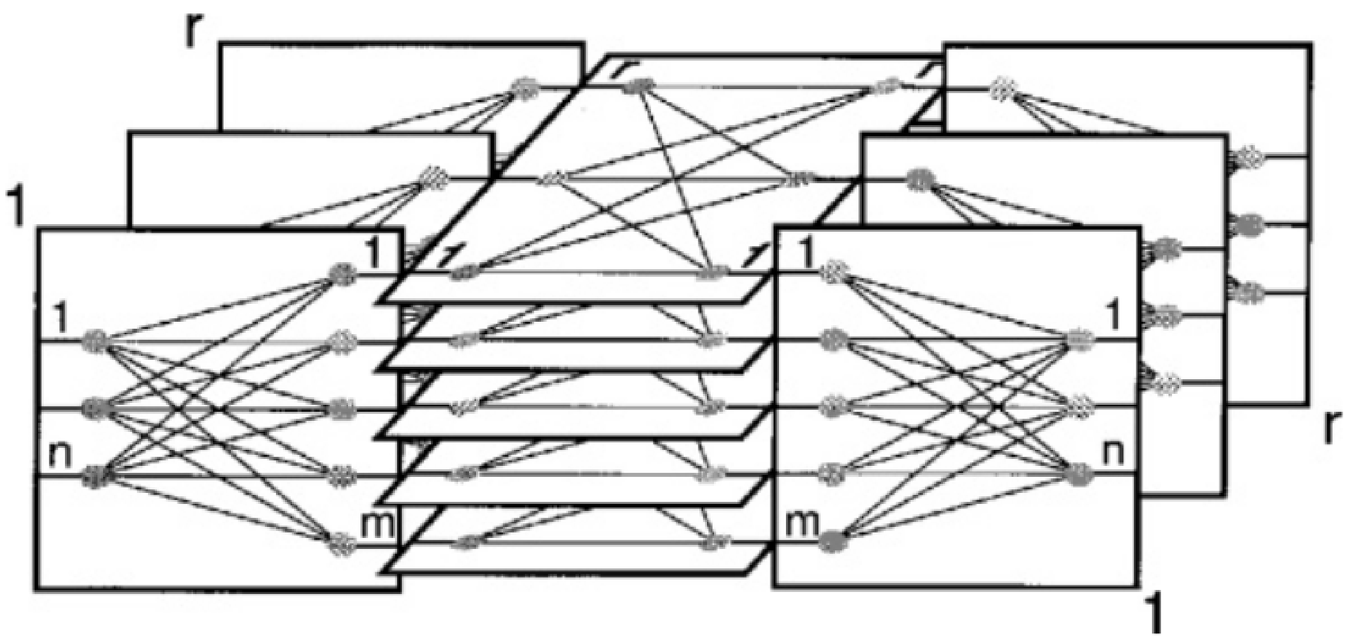
In 1953, Charles Clos proposed a solution using a modular, three-stage design:

1. **Ingress Stage:** Composed of `r` smaller switches, each of size `n x m`.
2. **Middle Stage:** Composed of `m` switches, each of size `r x r`.
3. **Egress Stage:** Composed of `r` switches, each of size `m x n`.

- ***Ingress***: Is the corresponding term for traffic entering a network or device. It is the inbound direction.
- ***Egress***: Refers to the path for traffic exiting a network or a network device. It is the outbound direction.

In this design, the total number of input/output lines for the entire system is `N = n * r`. Each switch in the ingress stage has a connection to every switch in the middle stage, which in turn has a connection to every switch in the egress stage.





Non-Blocking Properties

The performance of a Clos network is defined by its ability to make connections without being "blocked."

A path between input i and output j is an ordinate sequence of switch cross points and inter-stage links that connects i to j; when **connected**, path (i, j) **cannot be used by others**.

There are **two main types**:

- **Strictly Non-Blocking:** A new connection between an idle input and an idle output can *always* be made instantly, without having to alter any existing connections.
  - This is achieved when $m \geq 2n - 1$.

- **Rearrangeably Non-Blocking:** A new connection can always be made, but it might require re-routing (rearranging) existing connections to free up a path. This is acceptable for packet networks.
  - This is achieved with a less strict condition: $m \geq n$.

For data center networks, the **rearrangeably non-blocking property is sufficient**.

Justification for Non-Blocking Conditions

Strictly Non-Blocking: $m \geq 2n - 1$

The goal is to prove that you can always find a free path from an idle input to an idle output without rearranging any existing connections. We do this by considering the "worst-case scenario" for path availability.

Let's say you want to connect an idle input i (on ingress switch S_i) to an idle output j (on egress switch S_j). A path is only available if there is at least one middle-stage switch that is free to connect to *both* S_i and S_j.

1. **Paths used by the ingress switch (S_i):**
   - The ingress switch S_i has n inputs. Since input i is idle, at most $n-1$ other inputs can be busy.
   - In the worst case, these $n-1$ busy inputs are connected to $n-1$ different middle-stage switches.
2. **Paths used by the ingress switch (S_j):**
   - The egress switch S_j has n outputs. Since output j is idle, at most $n-1$ other outputs can be busy.
   - In the worst case, these $n-1$ busy outputs are connected to $n-1$ different middle-stage switches.
3. **The Worst-Case Collision:** The absolute worst case for finding a common free path is if the $n-1$ middle-stage switches used by S_i are **completely different** from the $n-1$ middle-stage switches used by S_j.
4. **Finding a Free Path:** In this worst-case scenario, the total number of middle-stage switches that could be occupied is $(n-1)$ (from S_i) + $(n-1)$ (from S_j), which equals $2n-2$.

To guarantee that there is **at least one** middle-stage switch left over that is completely free, the total number of middle-stage switches, m, must be greater than the maximum possible number of busy ones.

Therefore, the condition is: $m > 2n - 2$, which is equivalent to $m \geq 2n - 1$.

Rearrangeably Non-Blocking: $m \geq n$

For this property, the condition is less strict because the network has the added flexibility of being able to tear down and re-establish existing connections to make room for a new one.

While a formal proof is not provided in the course slides, the intuition is that you no longer need to account for the worst-case scenario where all paths from the input and output switches are disjoint and fixed. Because paths can be rearranged, you only need to ensure there are enough middle-stage switches (m) to accommodate all the potential connections from a single ingress or egress switch (n).

From Clos Network to Fat-Tree

The final conceptual step is to see how this theoretical 3-stage design becomes the Fat-Tree topology we use in data centers:

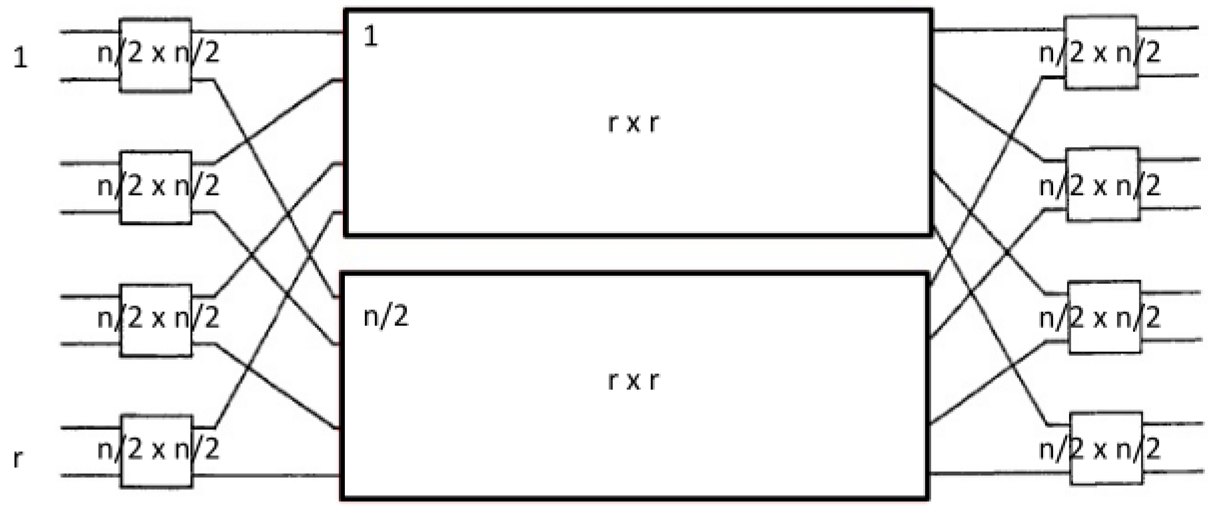
1. We start with a rearrangeably non-blocking Clos network where $m = n$.
2. We can recursively apply the Clos design to the large middle-stage switches to create a deeper, multi-stage network.
3. We then take this feed-forward network and **"fold" it in half** around the central axis.

This "folded-Clos" network is exactly the **Fat-Tree topology**. The ingress and egress stage switches become the **Edge and Aggregation** layer switches within the pods, and the middle stage becomes the **Core** layer. The separate input and output ports become single, bidirectional ports.

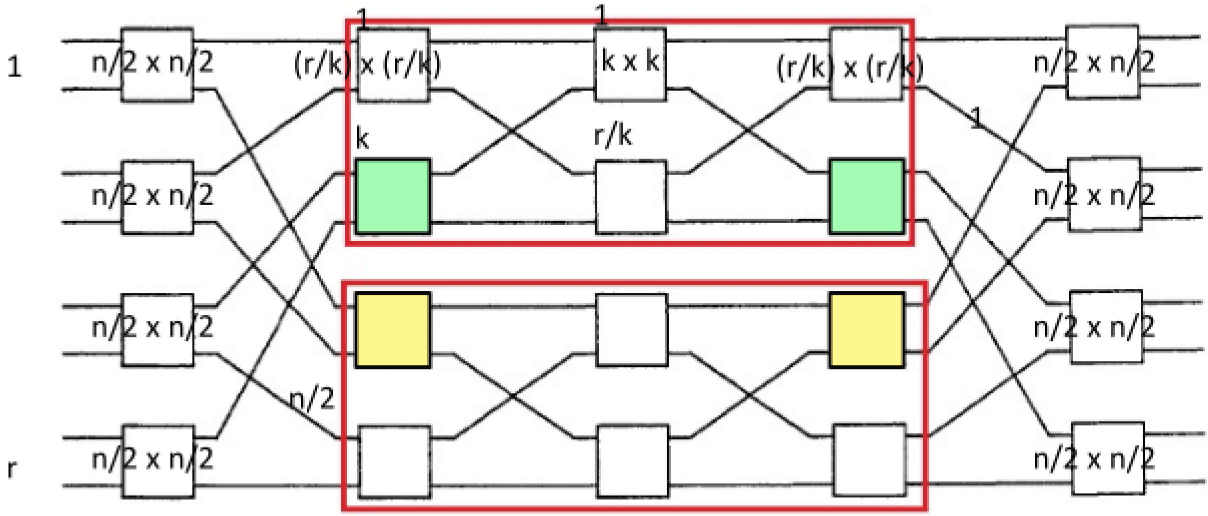
Visual Example

The slides provide a step-by-step visual example of how the theoretical, 3-stage Clos network is transformed into the practical, bidirectional Fat-Tree topology used in data centers.

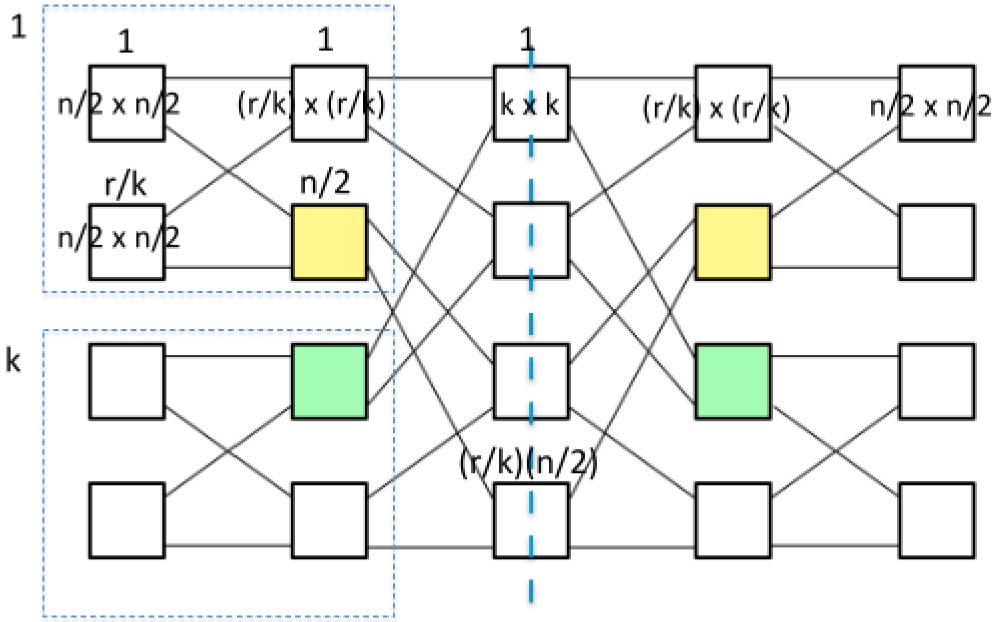
1. **Step 1: Recursive Construction** The process begins with a standard 3-stage Clos network. To build a larger, more scalable fabric, the large switches in the middle stage are themselves replaced with their own, smaller 3-stage Clos networks. This demonstrates the recursive nature of the design.



2. **Step 2: Visual Rearrangement** The resulting multi-stage network is then rearranged visually to align the layers, making the structure clearer before the final transformation.



3. **Step 3: The "Fold"** This is the most critical conceptual step. The entire feed-forward network is **"folded" in half** along its central vertical axis. This fold transforms the architecture:
   - The network becomes **bidirectional**, suitable for server-to-server communication.
   - The formerly separate ingress and egress stages are merged to become the **edge and aggregation layers** within the Fat-Tree pods.
   - The central middle stage becomes the **core layer** of the Fat-Tree.



This visual process shows exactly how the abstract, unidirectional Clos switch fabric becomes the concrete, bidirectional Fat-Tree network topology.

Clos Network: Optimization and the Link to Fat-Tree

This section covers the mathematical details behind the Clos network design, explaining why it's more efficient than a simple crossbar switch and how the theory directly leads to the practical Fat-Tree construction formulas.

Optimizing Switch Complexity

The primary motivation for the Clos network was to reduce the high complexity of a fully-connected crossbar switch.

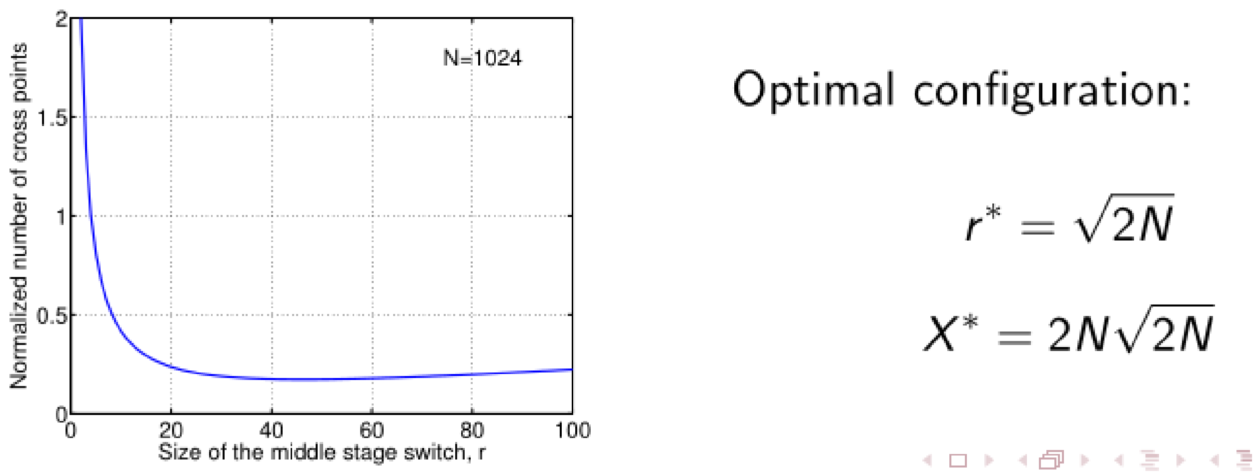
- **Baseline (Crossbar Switch):** A crossbar switch that connects N inputs to N outputs requires N^2 crosspoints.
    - Its complexity grows as $O(N^2)$, which is **not scalable**.
- **Clos Network Complexity:** For a 3-stage, rearrangeably non-blocking Clos network (where $m = n$), the total number of crosspoints, X, is given by the formula:

$$X = 2rn^2 + nr^2$$

- **The Optimization:** For a fixed total number of ports N (where $N = nr$), we can choose the parameters n and r to minimize the complexity X.
    - The optimal complexity for a Clos network is approximately:

$$X^* = 2N\sqrt{2N}$$

- **The Result:** For a **large number of ports** N, the complexity of $2N\sqrt{2N}$ is **significantly lower** than the N^2 complexity of a crossbar switch, making it a much **more scalable** and **cost-effective** design.



The Mathematical Link to the Practical Fat-Tree

The final theoretical step is to show how the general Clos network parameters lead directly to the simple $S = n^3/4$ formula for the number of servers in a Fat-Tree. This is done by adding one practical constraint: **all switches in the network must be identical commodity n-port switches**.

The derivation proceeds as follows:

1. We start with the parameters of a recursively built Clos network topology.
2. To enforce the use of a single switch type, the parameters must satisfy certain relationships. From the example construction in the slides, these are $k = n$ and $r/k = n/2$.
3. Solving these for r (the number of ingress/egress switches in the original Clos model, which corresponds to the number of edge switches per pod multiplied by the number of pods in the Fat-Tree) gives us $r = n^2/2$.

4. We know the total number of servers S is the number of edge switches (r) multiplied by the number of servers per edge switch ($n/2$). So, $S = r \times (n/2)$.
5. Substituting the value of r from step 3 into this equation gives the final result: $S = \left(\frac{n^2}{2}\right) \times \left(\frac{n}{2}\right) = \frac{n^3}{4}$

This proves that the practical Fat-Tree construction is a direct and optimized application of general Clos network theory.

Theoretical Performance Bounds

Beyond just the structure, we can analyze topologies mathematically to understand their theoretical performance limits. The slides present a way to calculate an upper bound on network throughput that is "***application-oblivious***," meaning it **doesn't depend on a specific traffic pattern**.

An Application-Oblivious Throughput Bound

The **normalized throughput** (TH) of a network is limited by the total number of links, the number of flows, and the average length of the paths those flows take. This relationship provides a theoretical upper bound on performance, independent of the specific application traffic.

The main formula for the bound is:

$$TH \leq \frac{l}{\overline{h}\nu_f}$$

Where the variables are defined as:

- TH is the **Normalized Throughput**.
  - It is defined as the rate of the slowest flow (x_i) in the network relative to the capacity of a single link (C):

$$TH \equiv \frac{\min_i x_i}{C}$$

- l is the **total number of links** in the network.
- $\overline{h}$ is the **average path length** (in hops) taken by the flows.
- ν_f is the **total number of active flows** in the network.

The Intuition: This formula shows that to **maximize throughput** (i.e., guarantee the best possible minimum performance for all flows), you need to **minimize the denominator**.

- This means building networks where the **average path length** ($\overline{h}$) is as **short as possible** for the number of flows it needs to support.

Proof of the Bound

The proof relies on comparing the total available network capacity to the total consumed capacity.

1. **Total Available Capacity:** If each of the l **links** has a **capacity** of C, the **total capacity** of the entire network is $l \times C$.

2. **Total Consumed Capacity:** A single flow with **rate** x_i that travels over a **path** of h_i **hops** consumes $x_i \times h_i$ of the network's total capacity.
   - The **total consumed capacity** is the sum over all flows: $\sum x_i h_i$.
3. **The Bound:** Since the **consumed capacity cannot exceed the available capacity**, we have:

   $$\sum x_i h_i \leq l \times C$$

   By the definition of normalized throughput, the rate of every flow x_i is at least $TH \times C$. Substituting this gives the final bound.

Detailed Steps

1. We start with the **Capacity Limit** inequality, which states that the total consumed network capacity cannot exceed the total available capacity:

   $$\sum x_i h_i \leq l \times C$$

2. Next, we use the definition of **Normalized Throughput**. For any flow, its rate x_i is greater than or equal to the minimum throughput rate in the system, so:

   $$x_i \geq TH \times C$$

3. We can now substitute this minimum rate into the "Total Consumed Capacity" sum. This gives us a lower bound on the total consumed capacity:

   $$\sum x_i h_i \geq \sum (TH \times C) h_i$$

4. Since TH and C are constants across all flows, we can factor them out of the sum:

   $$\sum (TH \times C) h_i = TH \times C \times \sum h_i$$

5. Now we can combine the inequalities from steps 1 and 4 into a single chain:

   $$l \times C \geq \sum x_i h_i \geq TH \times C \times \sum h_i$$

6. Focusing on the outer parts of this chain and canceling the link capacity C from both sides, we get:

   $$l \geq TH \times \sum h_i$$

7. Finally, we isolate TH and substitute the definition of average path length ($\sum h_i = \overline{h} \times \nu_f$) to arrive at the final bound:

   $$TH \leq \frac{l}{\overline{h} \nu_f}$$

Here are the updated and expanded notes for that section, incorporating all the details from slides 96-98.

Application to r-Regular Graphs

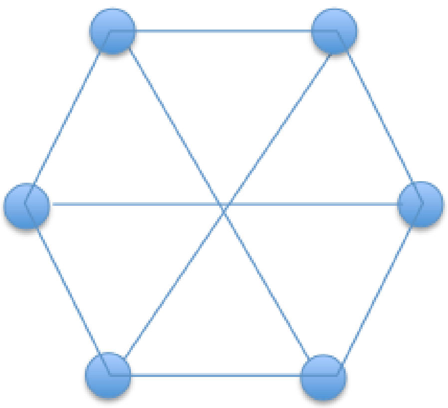
The theoretical throughput bound can be applied to specific, well-defined types of graphs that are used to model data center topologies. One such type is an **r-regular graph**.

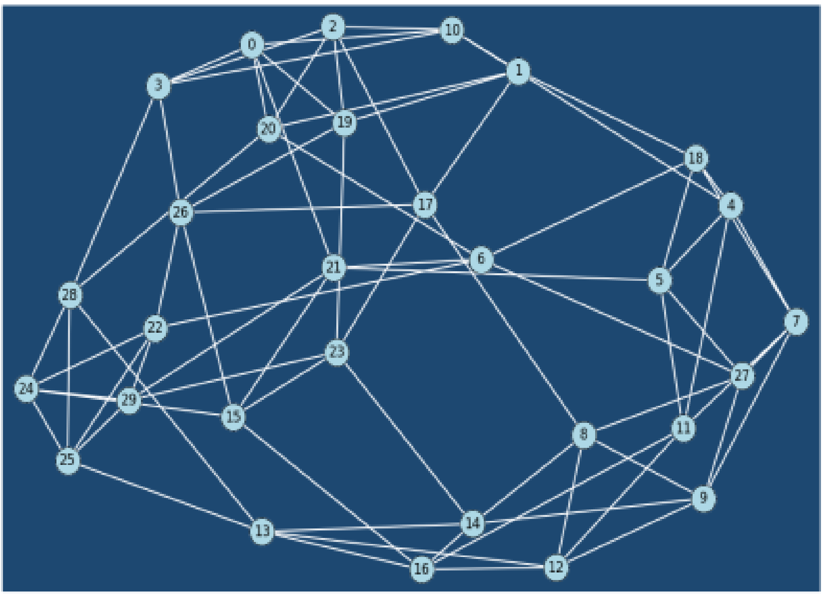
- **Definition:** An r-regular graph is one where every node (switch) has the same number of connections, or degree, which is equal to r.
- **Properties:**
  - The number of switches, S, must be at least $r + 1$.
  - The product $S \times r$ must be an even number.
- **DCN Application:** In this model, each of the S switches uses r of its ports to connect to other switches, and the remaining $n - r$ ports to connect to servers. This gives a total of $N = S(n - r)$ servers in the network.

For this specific structure, the general throughput bound formula can be written as:

$$TH \leq \frac{Sr}{\overline{h}\nu_f}$$

The slides provide visual examples of such graphs, including a 3-regular graph with 6 nodes and a 5-regular graph with 30 nodes.





Theoretical Limit on Path Length: The Moore Bound

The average path length, $\overline{h}$, is a critical parameter for network performance. The **Moore Bound** provides a theoretical *lower limit* on what the average shortest path length can be for a given r-regular graph. It tells us the best possible performance in terms of path length.

The Moore Bound is given by the formula:

$$\overline{h} \geq \frac{\sum_{j=1}^{k-1} jr(r-1)^{j-1} + kR}{N - 1}$$

Where the terms R and k are defined as:

$$R = N - 1 - \sum_{j=1}^{k-1} r(r-1)^{j-1}$$

$$k = 1 + \lfloor \frac{log(N - 2(N-1)/r)}{log(r-1)} \rfloor$$