Standard Integration Techniques

Note that at many schools all but the Substitution Rule tend to be taught in a Calculus II class.

***u* Substitution :** The substitution $u = g(x)$ will convert $\int_a^b f(g(x))\,g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$ using $du = g'(x)\,dx$. For indefinite integrals drop the limits of integration.

Ex. $\int_1^2 5x^2\cos(x^3)\,dx$

$u = x^3 \Rightarrow du = 3x^2dx \Rightarrow x^2dx = \frac{1}{3}du$

$x = 1 \Rightarrow u = 1^3 = 1 \;::\; x = 2 \Rightarrow u = 2^3 = 8$

$$\int_1^2 5x^2\cos(x^3)\,dx = \int_1^8 \tfrac{5}{3}\cos(u)\,du$$

$$= \tfrac{5}{3}\sin(u)\Big|_1^8 = \tfrac{5}{3}\left(\sin(8) - \sin(1)\right)$$

Integration by Parts : $\int u\,dv = uv - \int v\,du$ and $\int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du$. Choose u and dv from integral and compute du by differentiating u and compute v using $v = \int dv$.

Ex. $\int x\mathbf{e}^{-x}\,dx$

$u = x \quad dv = \mathbf{e}^{-x} \quad \Rightarrow \quad du = dx \quad v = -\mathbf{e}^{-x}$

$$\int x\mathbf{e}^{-x}\,dx = -x\mathbf{e}^{-x} + \int \mathbf{e}^{-x}\,dx = -x\mathbf{e}^{-x} - \mathbf{e}^{-x} + c$$

Ex. $\int_3^5 \ln x\,dx$

$u = \ln x \quad dv = dx \quad \Rightarrow \quad du = \frac{1}{x}dx \quad v = x$

$$\int_3^5 \ln x\,dx = x\ln x\Big|_3^5 - \int_3^5 dx = \left(x\ln(x) - x\right)\Big|_3^5$$

$$= 5\ln(5) - 3\ln(3) - 2$$

Products and (some) Quotients of Trig Functions

For $\int \sin^n x\cos^m x\,dx$ we have the following :

1. ***n* odd.** Strip 1 sine out and convert rest to cosines using $\sin^2 x = 1 - \cos^2 x$, then use the substitution $u = \cos x$.
2. ***m* odd.** Strip 1 cosine out and convert rest to sines using $\cos^2 x = 1 - \sin^2 x$, then use the substitution $u = \sin x$.
3. ***n* and *m* both odd.** Use either 1. or 2.
4. ***n* and *m* both even.** Use double angle and/or half angle formulas to reduce the integral into a form that can be integrated.

For $\int \tan^n x\sec^m x\,dx$ we have the following :

1. ***n* odd.** Strip 1 tangent and 1 secant out and convert the rest to secants using $\tan^2 x = \sec^2 x - 1$, then use the substitution $u = \sec x$.
2. ***m* even.** Strip 2 secants out and convert rest to tangents using $\sec^2 x = 1 + \tan^2 x$, then use the substitution $u = \tan x$.
3. ***n* odd and *m* even.** Use either 1. or 2.
4. ***n* even and *m* odd.** Each integral will be dealt with differently.

Trig Formulas **:** $\sin(2x) = 2\sin(x)\cos(x)$, $\cos^2(x) = \frac{1}{2}\left(1 + \cos(2x)\right)$, $\sin^2(x) = \frac{1}{2}\left(1 - \cos(2x)\right)$

Ex. $\int \tan^3 x\sec^5 x\,dx$

$$\int \tan^3 x\sec^5 x\,dx = \int \tan^2 x\sec^4 x\tan x\sec x\,dx$$

$$= \int \left(\sec^2 x - 1\right)\sec^4 x\tan x\sec x\,dx$$

$$= \int \left(u^2 - 1\right)u^4\,du \qquad (u = \sec x)$$

$$= \tfrac{1}{7}\sec^7 x - \tfrac{1}{5}\sec^5 x + c$$

Ex. $\int \frac{\sin^5 x}{\cos^3 x}\,dx$

$$\int \frac{\sin^5 x}{\cos^3 x}\,dx = \int \frac{\sin^4 x\sin x}{\cos^3 x}\,dx = \int \frac{(\sin^2 x)^2\sin x}{\cos^3 x}\,dx$$

$$= \int \frac{(1-\cos^2 x)^2\sin x}{\cos^3 x}\,dx \qquad (u = \cos x)$$

$$= -\int \frac{(1-u^2)^2}{u^3}\,du = -\int \frac{1-2u^2+u^4}{u^3}\,du$$

$$= \tfrac{1}{2}\sec^2 x + 2\ln\left|\cos x\right| - \tfrac{1}{2}\cos^2 x + c$$