Problem Set 4

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1. $a = 3, b = 2, k = 2, b^k = 4, a < b^k, p = 0$
   By applying the Master Theorem, we get:
   $T(n) = \Theta(n^2 \log^0(n)) = \Theta(n^2)$

2. $a = 4, b = 2, k = 2, b^k = 4, a = b^k, p = 0$
   By applying the Master Theorem, we get:
   $T(n) = \Theta\left(n^{\log_2 4} \log^{0+1} n\right) = \Theta(n^2 \log n)$

3. $a = 1, b = 2, k = 2, b^k = 4, a < b^k, p = 0$
   By applying the Master Theorem, we get:
   $T(n) = \Theta(n^2 \log^0(n)) = \Theta(n^2)$

4. $a = 16, b = 4, k = 1, b^k = 4, a > b^k, p = 0$
   By applying the Master Theorem, we get:
   $T(n) = \Theta\left(n^{\log_4 16}\right) = \Theta(n^2)$

5. $a = 2, b = 2, k = 1, b^k = 2, a = b^k, p = 1$
   By applying the Master Theorem, we get:
   $T(n) = \Theta\left(n^{\log_2 2} \log^{1+1} n\right) = \Theta(n \log^2 n)$

6. $a = 2, b = 2, k = 1, b^k = 2, a = b^k, p = -1$
   By applying the Master Theorem, we get:
   $T(n) = \Theta\left(n^{\log_2 2} \log \log n\right) = \Theta(n \log \log n)$

7. $a = 2, b = 4, k = 0.51, b^k \cong 2.028, a < b^k, p = 0$
   By applying the Master Theorem, we get:
   $T(n) = \Theta(n^{0.51} \log^0 n) = \Theta(n^{0.51})$

8. $a = 6, b = 3, k = 2, b^k = 9, a < b^k, p = 1$
   By applying the Master Theorem, we get:
   $T(n) = \Theta(n^2 \log^1 n) = \Theta(n^2 \log n)$

9. $a = 7, b = 3, k = 2, b^k = 9, a < b^k, p = 0$
   By applying the Master Theorem, we get:
   $T(n) = \Theta(n^2 \log^0 n) = \Theta(n^2)$

10. $a = \sqrt{2}, b = 2, k = 0, b^k = 1, a > b^k, p = 1$

By applying the Master Theorem, we get:

$T(n) = \Theta\left(n^{\log_2 \sqrt{2}}\right) = \Theta(n^{0.5}) = \Theta(\sqrt{n})$

11. $a = 3, b = 3, k = 0, b^k = 1, a > b^k, p = 0$

By applying the Master Theorem, we get:

$T(n) = \Theta(n)$

Assume $T(n) = an + b$, then:

(1) $T(1) = a + b = 2$

(2) $3T(n/3) + 1 = 3\left(a \cdot \frac{n}{3} + b\right) + 1 = an + 3b + 1 = an + b$

(3) Given above, we can get $a = \frac{5}{2}, b = -\frac{1}{2}$

Therefore, $T(n) = \frac{5}{2}n - \frac{1}{2}$

12. The equation does not fit for a Master Theorem case. However, we can still find out some rule:

(1) $T(2) = 150 = 2T(1) \to T(1) = 75$

(2) $T(3) = 3 \cdot T(2) = (3 \cdot 2) \cdot T(1) = 3!\,T(1) = 75 \cdot 3!$

Proved by induction, we can generalize that $T(n) = 75 \cdot (n!)$

13. We may consider a recursion tree method.

(1) Assume we need to expand k times of the tree (i.e., tree depth) to expand the leading term $T\left(\frac{9}{10}n\right)$ be $T(1)$. Then:

$$\left(\frac{9}{10}\right)^k \cdot n \le 1 \to \left(\frac{10}{9}\right)^k \ge n \to k \ge \log_{10/9} n$$

Therefore, the tree height k is in the asymptotic notation of $\Theta(\log n)$

(2) From observation:

$$T(n) = T\left(\frac{9}{10}n\right) + T\left(\frac{n}{10}\right) + n = T\left(\frac{81}{100}n\right) + 2T\left(\frac{9}{100}n\right) + T\left(\frac{n}{100}\right) + 2n$$

Each time the recursion tree expands, we will get an extra n term. Since there are $\Theta(\log n)$ times of tree expansion, the asymptotic complexity of this recursion is $\Theta(n \log n)$.

14. The step is similar to Q.13. Let's consider a recursion tree method. Without loss of generality, let's assume that $a \geq 0.5$. If the condition does not hold, we substitute $b = 1 - a$ such that $T(n) = T(bn) + T\big((1-b)n\big) + n$ is still satisfied.
    (1) Assume we need to expand k times of the tree (i.e., tree depth) to expand the leading term $T(an)$ be $T(1)$. Then:

    $$a^k \cdot n \leq 1 \rightarrow \left(\frac{1}{a}\right)^k \geq n \rightarrow k \geq \log_{1/a} n$$

    Since $1/a$ is just a constant, the tree height k is in the notation of $\Theta(\log n)$

    (2) From observation:

    $$T(n) = T(an) + T\big((1-a)n\big) \ + n$$
    $$= T(a^2 n) + 2T(a \cdot (1-a) \cdot n) + T((1-a)^2) + 2n$$

    Each time the recursion tree expands, we will get an extra n term. Since there are $\Theta(\log n)$ times of tree expansion, the asymptotic complexity of this recursion is $\Theta(n \log n)$.