A mini-course on cooperative game theory: the core and the nucleolus, with application to models for trading demands

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Dcember 6, 2023

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Cooperative Transferable Utility Games

- Cooperative TU-games (CTU-games) was introduced by von Neumann and Morgenstern in their book "Theory of Games and Economic Behavior" (1944).
- In CTU-games, there are a set of players N that want to cooperate.
- Each coalition of players $S \subset N$ is defined by one real value, called a payoff, that must be transferred among its players.
- In cooperative games the question is how to transfer in a fair way the payoff of the grand coalition N among each individual player.
- A fair distribution is called a concept solution.
- In this mini-course we will only focus on two concepts: the core and the nucleolus.

Formally a cooperative game is defined by a pair (N, v):

- N is the set of players,
- $\mathrm{v} : 2^N \to \mathbb{R}$, $\mathrm{v}(\emptyset) = 0$.

- v is called the characteristic function, expressing the utility of subset of players.
- A subset of players $S \subseteq N$ is called a coalition.
- The function $\mathrm{v}(S)$ may express the profit or the cost generated by the players in S if they form the coalition S.

Solutions concepts in cooperative games

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- We are interested in two solution concepts: the core and the nucleolus.

We assume that the characteristic function v is :

- **Monotone.** If $S \subseteq T$, then $\mathrm{v}(T) \geq \mathrm{v}(S)$.
- **Superadditive.** If $S \cap T = \emptyset$, then $\mathrm{v}(S \cup T) \geq \mathrm{v}(S) + \mathrm{v}(T)$.

[redacted]

Let x and y be two imputations. We say that y dominates x through the coalition S and we write, $y \succ_S x$, whenever the following hold:

$$y_i > x_i \quad \textit{for each } i \in S,$$
$$\sum_{i \in S} y_i \leq \mathrm{v}(S)$$

The core

[redacted]

The set of undominated imputations of a game (N, v) is called the core and is denoted by $\mathbb{C}(N, \mathrm{v})$.

[redacted]

The core $\mathbb{C}(N, v)$ is the following polytope:

$$
\begin{aligned}
x(N) &= \mathrm{v}(N), \\
x(S) &\geq \mathrm{v}(S), \quad \textit{for } S \subseteq N
\end{aligned}
$$

[redacted]

- By definition each x satisfying the two conditions above cannot be dominated, since if $y \succ_S x$, then $y(S) > \mathrm{v}(S)$. So x is in the core.

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- Now suppose that we have a vector y in the core (y is undominated) that does not satisfies the conditions of the theorem.
- The first condition is satisfied by definition. So assume $y(S) = v(S) - \epsilon$.
- The superadditivity of v implies:

$\square$

[redacted]

- $$\mathrm{v}(N) \geq \mathrm{v}(S) + \mathrm{v}(N \setminus S) \geq \mathrm{v}(S) + \sum_{i \in N \setminus S} \mathrm{v}(i),$$

  so $0 \leq \beta = \mathrm{v}(S) + \sum_{i \in N \setminus S} \mathrm{v}(i)$. Now define $z \in \mathbb{R}^{|N|}$ as follows:

- $z(i) = \begin{cases} y(i) + \frac{\epsilon}{|S|} & \text{if } i \in S, \\ v(i) + \frac{\beta}{|N|-|S|}. & \end{cases}$

So,

$$\begin{aligned} z(N) = z(S) + z(N \setminus S) = & \quad y(S) + \epsilon + \textstyle\sum_{i \in N \setminus S} \mathrm{v}(i) + \beta \\ = & \quad \mathrm{v}(S) + \textstyle\sum_{i \in N \setminus S} \mathrm{v}(i) + \beta = \mathrm{v}(N). \end{aligned}$$

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The core is the set of imputations where each coalition S is more happy with what it gets, $\sum_{i \in S} x_i = x(S)$, than with what it gets by leaving the grand coalition, $v(S)$.

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When the core is empty, we are interested on the set of allocations satisfying each coalition as much as possible, called the least-core. That is the optimal face of the following LP program

$$\begin{aligned}&\max \quad \epsilon \\ &x(N) = \mathrm{v}(N) \\ &x(S) \geq \mathrm{v}(S) + \epsilon, \quad \forall S \neq N\end{aligned}$$

Characterizing the non-emptiness of the core

[redacted]

A cooperative game (N, v) *is balanced if and only if there exists a function* $\lambda : 2^N \setminus \{\emptyset\} \to \mathbb{R}^N_+$, *such that*

$$\sum_{S \subseteq N \setminus \{\emptyset\}} \lambda(S)\mathrm{v}(S) \leq \mathrm{v}(N),$$

$$\sum_{S, i \in S} \lambda(S) = 1, \quad \textit{for all } i \in N.$$

[redacted]

A cooperative game has a non-empty core if and only if is balanced.

Proof of Bondareva-Shapley theorem

We use LP-duality. The core is non-empty if and only if

$$\begin{aligned} \mathrm{v}(N) = & \ \min\{\textstyle\sum x(i) : x(S) \geq \mathrm{v}(S),\ S \subseteq N\}(\textit{duality}) \\ = & \ \ \max\{\lambda(S)v(S) : \textstyle\sum_{S, i \in S} \lambda(S) = 1,\ i \in N\} \end{aligned}$$

For a coalition S and and an imputation $x \in \mathbb{R}^{|N|}$, theirexcess is $e(x, S) = x(S) - v(S)$.

[redacted]

The nucleolus has been introduced by Schmeidler (1969), trying to minimize dissatisfaction of players. It is defined as the allocation that lexicographically maximize the vector $\Theta(x)$ of non-decreasing ordered excess.

Here the excess $e(x, S)$ may be interpreted as the amount of satisfaction of S when x is chosen.

The nucleolus: an example

- Let $N = \{1,2,3\}$ a set of three player with the following characteristic function:

Coalition	v	y_1	$e(y_1, \dot{})$	y_2	$e(y_2, \dot{})$	y_3	$e(y_3, \dot{})$
$\{1\}$	45	45	0	57	12	62	17
$\{2\}$	42	54	12	48	6	48	6
$\{3\}$	40	72	32	66	26	61	21
$\{1,2\}$	99	99	0	105	6	110	11
$\{1,3\}$	117	117	0	123	6	123	6
$\{2,3\}$	98	126	28	114	16	109	11
$\{1,2,3\}$	171	171	-	171	-	171	-

Order the excess vectors y_1, y_2 and y_3 and choose the lexicographically maximum one:

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Order the excess vectors y_1, y_2 and y_3 and choose the lexicographically maximum one:
so $y_1 \prec y_2 \prec y_3$. y_3 **is in fact the nucleolus of our game.**

The nucleolus has nice properties:

- It always exists.
- It is in the core when the core is not empty.
- It is unique

The nucleolus may be computed as a sequence of linear programs :

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\begin{aligned}
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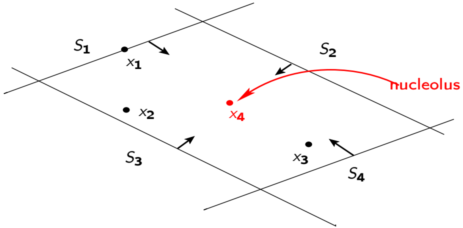
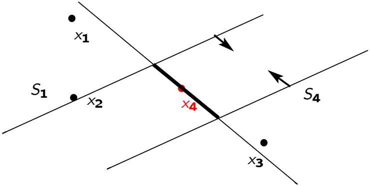
This gives ϵ_2 ... continue ... (at most n times).
In our example $\epsilon_1 = 6$ and $\epsilon_2 = \epsilon_1 + 5$.

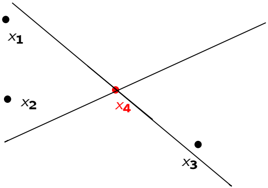
The nucleolus

Approximate Geometric illustration : the distance is viewed as a slack



$x_1 \preceq x_2 \preceq x_3 \preceq x_4$





This is a joint work with Gauthier Stauffer (University of Lausanne) and Gianpaolo Oriolo (University Tor Vergata, Roma). Let

- $N = \{1, \dots, n\}$ a set of n price-taker companies (the players), producing a same commodity product,
- $M = \{1, \dots, m\}$ a set of m markets,
- each company $i \in N$ owns a part d_{ij} of the total demand d_j of market j,
- r_j the product unit price at market j; c_{ij} is the cost of producing/transporting one unit of the product from the company i to the market j.
- q_i the capacity production of company i (possibly $q_i = \infty$, that is $q_i \geq \sum_{j \in M} d_{ij}$).

The objective. *Modelling and proposing efficient sharing principles for a practical collaboration in transportation.*

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This game is defined by the pair (N, v), where if a subset of players $S \subseteq N$ collaborate they get a total profit:

$$\begin{aligned} \mathrm{v}(S) = \max \sum_{i \in S} \sum_{j \in M} (r_j - c_{ij}) y_{ij} & \\ \sum_{i \in S} y_{ij} \leq \sum_{i \in S} d_{ij} & \qquad \forall j \in M \\ \sum_{j \in M} y_{ij} \leq q_i & \qquad \forall i \in S \\ y_{ij} \geq 0 & \quad \forall i \in S, j \in M \end{aligned}$$

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- We are interested in two solution concepts: the core and the nucleolus.

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[redacted]

The core of the production-distribution game is nonempty. We may provide in polynomial time a point in the core.

$$v(N) = \max \sum_{i \in N} \sum_{j \in M} (r_j - c_{ij}) y_{ij}$$

$$\sum_{i \in N} y_{ij} \leq \sum_{i \in N} d_{ij}, \ \forall j \in M \ \beta_j$$

$$\sum_{j \in M} y_{ij} \leq q_i \ \forall i \in N \ \alpha_i$$

$$y_{ij} \geq 0 \ \forall i \in N, j \in M.$$

$$\min \sum_{i \in N} q_i \alpha_i + \sum_{j \in M} (\sum_{i \in N} d_{ij}) \beta_j$$

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$$x^*(S) = \sum_{i\in S} q_i\alpha_i^* + \sum_{j\in M}(\sum_{i\in S} d_{ij})\beta_j^*$$

$$\ge \sum_{i\in S}(\sum_{j\in M} y_{ij}^*)\alpha_i^* + \sum_{j\in M}(\sum_{i\in S} y_{ij}^*)\beta_j^*$$

$$= \sum_{i\in S}\sum_{j\in M} y_{ij}^*(\alpha_i^* + \beta_j^*) \ge \sum_{i\in S}\sum_{j\in M}(r_j - c_{ij})y_{ij}^* = \mathrm{v}(S).$$

$$v(N) = \max \sum_{i\in N}\sum_{j\in M}(r_j - c_{ij})y_{ij}$$

$$\sum_{i\in N} y_{ij} \le \sum_{i\in N} d_{ij},\ \forall j \in M \ \beta_j$$

$$\sum_{j\in M} y_{ij} \le q_i \ \forall i \in N \ \alpha_i$$

$$y_{ij} \ge 0 \ \forall i \in N, j \in M.$$

$$\min \sum_{i\in N} q_i\alpha_i + \sum_{j\in M}(\sum_{i\in N} d_{ij})\beta_j$$

$$\alpha_i + \beta_j \ge r_j - c_{ij},\ \forall i \in N\ , j \in M$$

$$\alpha_i,\ \beta_j \ge 0 \ \forall i \in N, j \in M.$$

Let (α^*, β^*) be an optimal solution of the dual.

We claim that that $x_i^* = q_i\alpha_i^* + \sum_{j\in M}\beta_j^* d_{ij}$, for each $i \in N$, is in the core.

Let y^* the optimal solution of $v(S)$

$$x^*(S) = \sum_{i\in S} q_i\alpha_i^* + \sum_{j\in M}(\sum_{i\in S} d_{ij})\beta_j^*$$

$$\ge \sum_{i\in S}(\sum_{j\in M} y_{ij}^*)\alpha_i^* + \sum_{j\in M}(\sum_{i\in S} y_{ij}^*)\beta_j^*$$

$$= \sum_{i\in S}\sum_{j\in M} y_{ij}^*(\alpha_i^* + \beta_j^*) \ge \sum_{i\in S}\sum_{j\in M}(r_j - c_{ij})y_{ij}^* = v(S).$$

Equality holds for $S = N$ by Complementary Slackness.

In this case the game (N, v) is defined as follows:

If a subset of players $S \subseteq N$ collaborate they get a total profit:

$$
\begin{aligned}
\mathrm{v}(S) = \max \sum_{i \in S} \sum_{j \in M} (r_j - c_{ij}) y_{ij} & \\
\sum_{i \in S} y_{ij} \leq \sum_{i \in S} d_{ij} & \qquad \forall j \in M \\
\cancel{\sum_{j \in M} y_{ij} \leq q_i} & \qquad \cancel{\forall i \in S} \\
y_{ij} \geq 0 & \quad \forall i \in S, j \in M
\end{aligned}
$$

Particular case of the market game of Shapley and Shubik (1969).

$$v(S) = \max \sum_{i \in S} \sum_{j \in M} (r_j - c_{ij}) y_{ij}$$

$$\sum_{i \in S} y_{ij} = \sum_{i \in S} d_{ij} \qquad \forall j \in M$$

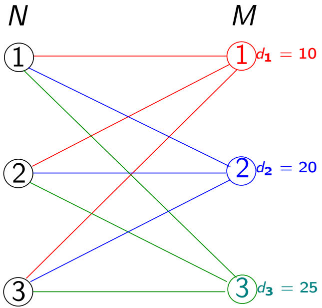
$$y_{ij} \geq 0 \quad \forall i \in S, j \in M$$

This is equivalent to

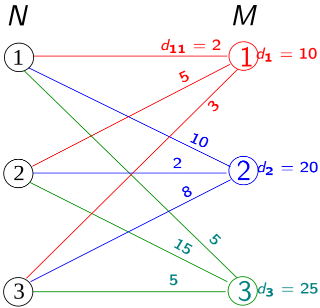
$$v(S) = \sum_{j=1}^{m} d_j(S) \max\{\alpha_{ij} : i \in S\},$$

where $\alpha_{ij} = r_j - c_{ij}$ and $d_j(S) = \sum_{i \in S} d_{ij}$.

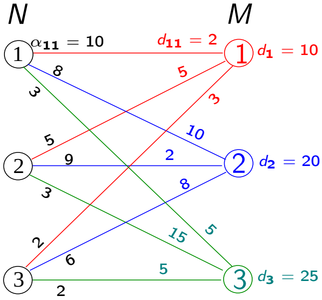
Let N be a set of three companies, the palyers and N a set of three markets. The demands and the profits are as follows:



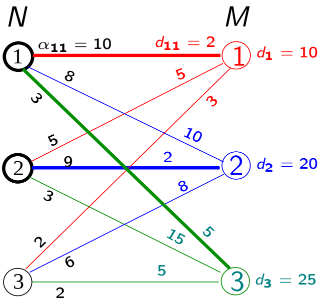
Let N be a set of three companies, the players and N a set of three markets. The demands and the profits are as follows:



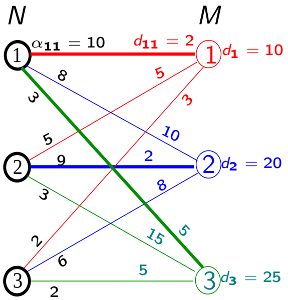
Let N be a set of three companies, the palyers and N a set of three markets. The demands and the profits are as follows:



Let N be a set of three companies, the palyers and N a set of three markets. The demands and the profits are as follows:



$$v(\{1,2\}) = (2+5) \times 10 + (10+2) \times 9 + (5+15) \times 3 = 238.$$



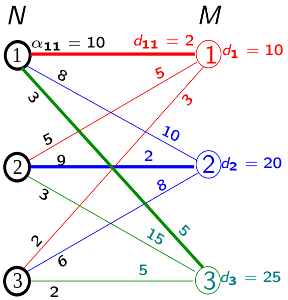
$v(\{1,2,3\}) =$
$10 \times 10 + 20 \times 9 + 25 \times 3 = 355.$

$v(\{1\}) = 10\times2+8\times10+3\times5 = 115.$

$v(\{2\}) = 5 \times 5 + 9 \times 2 + 3 \times 15 = 88.$

$v(\{3\}) = 2 \times 3 + 6 \times 8 + 2 \times 5 = 64.$

$v(\{1\}) + v(\{2\}) + v(\{3\}) = 267.$



$\text{v}(\{1,2,3\}) =$
$10 \times 10 + 20 \times 9 + 25 \times 3 = 355.$

$\text{v}(\{1\}) = 10\times 2+8\times 10+3\times 5 = 115.$

$\text{v}(\{2\}) = 5 \times 5 + 9 \times 2 + 3 \times 15 = 88.$

$\text{v}(\{3\}) = 2 \times 3 + 6 \times 8 + 2 \times 5 = 64.$

$\text{v}(\{1\}) + \text{v}(\{2\}) + \text{v}(\{3\}) = 267.$

For each market $j \in M$, let π^j be a permutation of the elements in N such that $\alpha_{\pi^j(1)j} \geq \alpha_{\pi^j(2)j} \geq, \ldots, \geq \alpha_{\pi^j(n)j}$:

$\pi^1 = (1,2,3)$; $\pi^2 = (2,1,3)$; $\pi^3 = (2,1,3)$.

[redacted]

The core of the game (N, v) *reduces to a single point if and only if* $\alpha_{\pi^j(1)j} = \alpha_{\pi^j(2)j}$, *for each* $j \in M$.

Min-cost spanning tree game [Bird 1976, Megiddo 1987, Galil 1980, Granot and Granot 1992].

- The players are the nodes of a graph $G = (N, E)$, each edge $e \in E$ has an associated positive cost $c(e)$,
- v(S) is the cost of the minimum spanning tree induced by the nodes S.
- Testing core membership is NP complete [Faigle et al. 1997].
- Computing the nucleolus is NP-hard [Faigle et al. 1998].

A flow game [Kalai and Zemel 1982].

- The set of players are the arcs of a network (V, A, c, s, t), where c is the capacity function associated to the arcs, s and t are the source and the sink, respectively.
- v(S) is the value of the maximum st-flow in (V, S, c, s, t).
- When $c(e) = 1$ for each arc, computing the nucleolus is polynomial [Deng et al. 2009].
- For general capacities, computing the nucleolus is NP-hard [Deng et al. 2009].

Matching games [Shapley and Shubik 1972, for bipartite graphs]

- The players are the nodes of an undirected graph $G = (N, E)$, where each edge e is associated with a weight $w(e)$.
- v(S), $S \subseteq N$, is the value of the maximum matching induced by S.
- The nucleolus may be computed in polynomial time :
  - when G is bipartite [Solymosi et al. 1994],
  - when the core is non-empty [Biró et al. 2012],
  - when the core is empty [Könemann et al. 2020].

The shortest path game [Fragnelli et al. 2000]

- The players are the arcs of a directed graph $G = (V, A)$, each arc $a \in A$ is associated with a positive cost. We also have a fixed value r called a revenue, and two special nodes s and t.
- v(S), $S \subseteq A$, is the revenue r minus the cost of the shortest path induced by S.
- The nucleolus may be computed in polynomial time when the core is non-empty and empty [Baïou and Barahona 2019]

The uncapacitated production-distribution game with one market

Definition

Now the size of M is one.

$$\mathrm{v}(S) = \max \sum_{i \in S} (r - c_i) y_i$$

$$\sum_{i \in S} y_i = \sum_{i \in S} d_i$$

$$y_i \geq 0 \quad \forall i \in S,$$

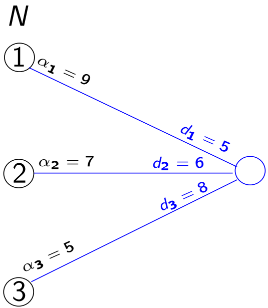
Set $\alpha_i = r - c_i$, and assume $\alpha_1 \geq \alpha_2, \ldots, \geq \alpha_n$. Let $\mathcal{S}_i$ be the subsets containing i but not the elements $j < i$.

$$x(N) = d(N)\alpha_1$$

$$x(S) \geq d(S)\alpha_i \quad \forall S \in \mathcal{S}_i, i = 1 \ldots, n.$$

The uncapacitated production-distribution game with one market

Example



$$
\begin{aligned}
v(\{1,2,3\}) &= 19 \times 9 = 171, \\
v(\{2,3\}) &= 14 \times 7 = 98.
\end{aligned}
$$

The nucleolus may be solved in polynomial time using the framework of Könemann and Toth (2020).

We give a simple cutting-planes algorithm.
For $k = 1, \dots, n$, let P_k be the linear program below, where $\epsilon_0 = 0$ and F_0 consists of the grand coalition equality.

$$\begin{aligned} &\max \quad \epsilon \\ &x(S) \geq \mathrm{v}(S) + \epsilon, \quad \forall S \notin \cup_{j=0}^{k-1} F_j \\ &x(S) = \mathrm{v}(S) + \epsilon_j, \quad \forall S \in F_j, j = 0, \dots, k-1. \end{aligned}$$

Solving P_0.

Solve the following program (Q):

$$\begin{aligned}
&\max \quad \epsilon \\
&x_i \geq d_i\alpha_i + \epsilon, \quad \forall i \in N \\
&x(N) = d(N)\alpha_1.
\end{aligned}$$

[redacted]

Let $(\bar{x}, \bar{\epsilon})$ be the solution of (Q). Find a violated inequality $S \in \mathcal{S}_i$, $\bar{x}(S) < d(S)\alpha_i + \bar{\epsilon}$.

- Define $x'_j = \bar{x}_j - d_j \times \alpha_i$.
- Let π be an increasing ordering of the elements $i+1, \ldots, n$, w.r.t. x', $x'_{\pi(k)} \leq x'_{\pi(k+1)}$.

Initialize A to be the matrix with one row: the grand coalition equality.

1. Set $k := i + 1; S := \{i, \pi(k)\}$
2. If $x'(S) < \bar{\epsilon}$ and $rank(A) > rank(A \cup S)$, then $x(S) \geq d(S)\alpha_i + \epsilon$ is a violated inequality, stop.
3. If $k \leq n - 1$, set $k := k + 1$, $S := S \cup \{\pi(k)\}$ and go to 2.

- If a violated inequality is found add it to (Q) and repeat until no violated inequality exists.
- Update $A := A \cup \{$the row of the inequalities with positive dual variable$\}$. Repeat for P_i, $i = 1, \ldots, n$.

We want to solve the following linear program P_k.

$$\max \quad \epsilon$$
$$x(S) \geq d(S)\alpha_i + \epsilon, \quad \forall S \notin \cup_{j=0}^{k-1},\ S \in \mathcal{S}_i.$$
$$x(S) = \mathrm{v}(S) + \epsilon_j, \quad \forall S \in F_j,\ j = 0, \dots, k-1.$$

$$\min y_N \alpha_1 - \sum_{S \subset N} v(S) y_S$$
$$\sum_{S \notin \cup_{j=0}^{k-1} F_j} y_S = 1,$$
$$-\sum_{S : i \in S} y_S = y_N, \quad \text{for } i = 1, \dots, n,$$
$$y_S \geq 0, \quad S \notin \cup_{j=0}^{k-1} F_j$$

The algorithm for computing the nucleolus.

- Step 1. Set $x_j^0 := d_j\alpha_1$ for $j = 1, \ldots, n$; $i = 2$; $k = 1$;$\epsilon_0 = 0$; $F^0 = \emptyset$;
- Step 2. $\mu^k = \min \{\dfrac{x^{k-1}(S) - v(S) - \epsilon_{k-1}}{|S \setminus F^{k-1}| + 1} : S \subset N \setminus \{1\},\ S \setminus F^{k-1} \neq \emptyset\}$. Let S^k be the argument of μ^k.
- Step 3. Set $x^k(1) \leftarrow x^{k-1}(1) + (n - 1 - |F^{k-1}|)\mu_k$; $x^k(j) \leftarrow x^{k-1}(j) - \mu_k$ for $j \notin F^{k-1} \cup \{1\}$ and otherwise $x^k(j) \leftarrow x^{k-1}(j)$. Set $\epsilon_k \leftarrow \epsilon_{k-1} + \mu_k$ and $F^k \leftarrow F^{k-1} \cup S^k$.
- Step 4. If $F^k \neq N$, then set $k \leftarrow k + 1$; goto step 2, otherwise stop.

[redacted]

At the end of each iteration k, (x^k, ϵ_k) is the optimal solution of P_k. Moreover, the following hold

(i) $x^k(N \setminus \{j\}) = d(N \setminus \{j\})\alpha_1 + \epsilon_k$, *for each* $j \notin F^{k-1} \cup \{1\}$.

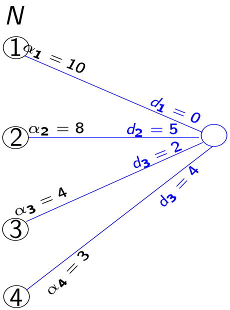
(ii) *All the variables in* F^k *are fixed.*

[redacted]

The proof is done by induction. The feasibility comes by definition. The optimality comes from duality. For $k = 1$, *set* $\alpha = |S^1|$.
By definition $x^1(S_i^1) = d(S_i^1)\alpha_i + \epsilon_1$. *We also have* $x^1(S^j) = d(S^j)\alpha_1 + \epsilon_1$.
Define a dual solution $\bar{y}$ *as follows:* $\bar{y}_N = -\frac{\alpha}{\alpha+1}$; $\bar{y}_{S^1} = \frac{1}{\alpha+1}$ *and* $\bar{y}_{N \setminus j} = \frac{1}{\alpha+1}$ *for each* $j \in S^1$. *For all other sets* $S \subset N$, $\bar{y}_S = 0$. □

[redacted]

The nucleous in the one-market case, may be computed in $O(n^4)$.



The core.

$$
\begin{aligned}
x_1 + x_2 + x_3 + x_4 &= 110\\
x_1 + x_2 + x_3 &\geq 70\\
x_1 + x_2 + x_4 &\geq 90\\
x_1 + x_3 + x_4 &\geq 60\\
x_2 + x_3 + x_4 &\geq 88\\
x_1 + x_2 &\geq 50\\
x_1 + x_3 &\geq 20\\
x_1 + x_4 &\geq 40\\
x_2 + x_3 &\geq 56\\
x_2 + x_4 &\geq 72\\
x_3 + x_4 &\geq 24\\
x_1 &\geq 0\\
x_2 &\geq 40\\
x_3 &\geq 8\\
x_4 &\geq 12
\end{aligned}
$$

$$
\begin{aligned}
&\max \epsilon\\
&x_1 + x_2 + x_3 + x_4 = 110\\
&x_1 + x_2 + x_3 \geq 70 + \epsilon\\
&x_1 + x_2 + x_4 \geq 90 + \epsilon \quad \bullet\\
&x_1 + x_3 + x_4 \geq 60 + \epsilon \quad \bullet\\
&x_2 + x_3 + x_4 \geq 88 + \epsilon\\
&x_1 + x_2 \geq 50 + \epsilon\\
&x_1 + x_3 \geq 20 + \epsilon\\
&x_1 + x_4 \geq 40 + \epsilon\\
&x_2 + x_3 \geq 56 + \epsilon \quad \bullet\\
&x_2 + x_4 \geq 72 + \epsilon\\
&x_3 + x_4 \geq 24 + \epsilon\\
&x_1 \geq 0 + \epsilon\\
&x_2 \geq 40 + \epsilon\\
&x_3 \geq 8 + \epsilon\\
&x_4 \geq 12 + \epsilon
\end{aligned}
$$

$$
\min 110 \times y_N - \sum_{S \subset N} v(S) y_S
$$

$$
\sum_{S \subset N} y_S = 1,
$$

$$
- \sum_{S : i \in S} y_S = y_N, \quad \text{for } i = 1, \dots, n,
$$

$$
y_S \geq 0, \quad S \neq N
$$

$$
\begin{array}{llllll}
\max \epsilon & & & & & \\
x_1 & + x_2 & + x_3 & + x_4 & = 110 & \\
x_1 & + x_2 & + x_3 & & \geq 70 + \epsilon & \\
x_1 & + x_2 & & + x_4 & \geq 90 + \epsilon & \bullet \\
x_1 & & + x_3 & + x_4 & \geq 60 + \epsilon & \bullet \\
 & x_2 & + x_3 & + x_4 & \geq 88 + \epsilon & \\
x_1 & + x_2 & & & \geq 50 + \epsilon & \\
x_1 & & + x_3 & & \geq 20 + \epsilon & \\
x_1 & & & + x_4 & \geq 40 + \epsilon & \\
 & x_2 & + x_3 & & \geq 56 + \epsilon & \bullet \\
 & x_2 & & + x_4 & \geq 72 + \epsilon & \\
 & & x_3 & + x_4 & \geq 24 + \epsilon & \\
x_1 & & & & \geq 0 + \epsilon & \\
 & x_2 & & & \geq 40 + \epsilon & \\
 & & x_3 & & \geq 8 + \epsilon & \\
 & & & x_4 & \geq 12 + \epsilon &
\end{array}
$$

$$
\min 110 \times y_N - \sum_{S \subset N} v(S) y_S
$$

$$
\sum_{S \subset N} y_S = 1,
$$

$$
- \sum_{S : i \in S} y_S = y_N, \quad \text{for } i = 1, \ldots, n,
$$

$$
y_S \geq 0, \quad S \neq N
$$

- $x_1^0 = 0$; $x_2^0 = 50$; $x_3^0 = 20$; $x_4^0 = 40$. $F^0 = \emptyset$.

$$
\begin{aligned}
&\max \epsilon \\
&x_1 + x_2 + x_3 + x_4 = 110 \\
&x_1 + x_2 + x_3 \geq 70 + \epsilon \\
&x_1 + x_2 + x_4 \geq 90 + \epsilon \quad \bullet \\
&x_1 + x_3 + x_4 \geq 60 + \epsilon \quad \bullet \\
&x_2 + x_3 + x_4 \geq 88 + \epsilon \\
&x_1 + x_2 \geq 50 + \epsilon \\
&x_1 + x_3 \geq 20 + \epsilon \\
&x_1 + x_4 \geq 40 + \epsilon \\
&x_2 + x_3 \geq 56 + \epsilon \quad \bullet \\
&x_2 + x_4 \geq 72 + \epsilon \\
&x_3 + x_4 \geq 24 + \epsilon \\
&x_1 \geq 0 + \epsilon \\
&x_2 \geq 40 + \epsilon \\
&x_3 \geq 8 + \epsilon \\
&x_4 \geq 12 + \epsilon
\end{aligned}
$$

$$
\begin{aligned}
&\min 110 \times y_N - \sum_{S \subset N} v(S) y_S \\
&\sum_{S \subset N} y_S = 1, \\
&-\sum_{S: i \in S} y_S = y_N, \quad \text{for } i = 1, \ldots, n, \\
&y_S \geq 0, \quad S \neq N
\end{aligned}
$$

- $x_1^0 = 0$; $x_2^0 = 50$; $x_3^0 = 20$; $x_4^0 = 40$. $F^0 = \emptyset$.
- The set optimizing Step 2 is $S^1 = \{2,3\} : \epsilon_1 = \mu_1 = \frac{x^0(S^1) - v(S^1) - \epsilon_0}{|S^1 \setminus F^0| + 1} = \frac{70-56}{3} = \frac{14}{3}$.

$$
\begin{array}{llllll}
\max \epsilon & & & & & \\
x_1 + & x_2 + & x_3 + & x_4 & = 110 & \\
x_1 + & x_2 + & x_3 & & \geq 70 + \epsilon & \\
x_1 + & x_2 & & + x_4 & \geq 90 + \epsilon & \bullet \\
x_1 & & + x_3 & + x_4 & \geq 60 + \epsilon & \bullet \\
 & x_2 + & x_3 & + x_4 & \geq 88 + \epsilon & \\
x_1 + & x_2 & & & \geq 50 + \epsilon & \\
x_1 & & + x_3 & & \geq 20 + \epsilon & \\
x_1 & & & + x_4 & \geq 40 + \epsilon & \\
 & x_2 + & x_3 & & \geq 56 + \epsilon & \bullet \\
 & x_2 & & + x_4 & \geq 72 + \epsilon & \\
 & & x_3 + & x_4 & \geq 24 + \epsilon & \\
x_1 & & & & \geq 0 + \epsilon & \\
 & x_2 & & & \geq 40 + \epsilon & \\
 & & x_3 & & \geq 8 + \epsilon & \\
 & & & x_4 & \geq 12 + \epsilon &
\end{array}
$$

$$\min 110 \times y_N - \sum_{S \subset N} v(S) y_S$$

$$\sum_{S \subset N} y_S = 1,$$

$$-\sum_{S : i \in S} y_S = y_N, \quad \text{for } i = 1, \ldots, n,$$

$$y_S \geq 0, \quad S \neq N$$

- $x_1^0 = 0$; $x_2^0 = 50$; $x_3^0 = 20$; $x_4^0 = 40$. $F^0 = \emptyset$.
- The set optimizing Step 2 is $S^1 = \{2,3\} : \epsilon_1 = \mu_1 = \frac{x^0(S^1) - v(S^1) - \epsilon_0}{|S^1 \setminus F^0| + 1} = \frac{70-56}{3} = \frac{14}{3}$.
- $x_1^1 = 14$; $x_2^1 = \frac{136}{3}$; $x_3^1 = \frac{46}{3}$; $x_4^1 = \frac{106}{3}$;
- $|S^1 \setminus F^0| = 2$; $\bar{y}_{1234} = -\frac{2}{3}$; $\bar{y}_{23} = \frac{1}{3}$; $\bar{y}_{134} = \frac{1}{3}$; $\bar{y}_{124} = \frac{1}{3}$.

$$
\begin{aligned}
&\max \epsilon\\
&x_1 + x_2 + x_3 + x_4 = 110\\
&x_1 + x_2 + x_3 \geq 70 + \epsilon \; \bullet\\
&x_1 + x_2 + x_4 = 90 + \tfrac{14}{3}\\
&x_1 + x_3 + x_4 = 60 + \tfrac{14}{3}\\
&x_2 + x_3 + x_4 \geq 88 + \epsilon\\
&x_1 + x_2 \geq 50 + \epsilon\\
&x_1 + x_3 \geq 20 + \epsilon\\
&x_1 + x_4 \geq 40 + \epsilon\\
&x_2 + x_3 = 56 + \tfrac{14}{3}\\
&x_2 + x_4 \geq 72 + \epsilon\\
&x_3 + x_4 \geq 24 + \epsilon\\
&x_1 \geq 0 + \epsilon\\
&x_2 \geq 40 + \epsilon\\
&x_3 \geq 8 + \epsilon\\
&\cancel{x_4 \geq 12 + \epsilon}\\
&x_4 \geq \tfrac{82}{3} + \epsilon \; \bullet
\end{aligned}
$$

$$
\begin{aligned}
&\min 110 \times y_N - \sum_{S \subset N} v(S) y_S\\
&\sum_{S \not\subset \textit{fixed coalition}} y_S = 1,\\
&- \sum_{S: i \in S} y_S = y_N, \quad \text{for } i = 1, \ldots, n,\\
&y_S \geq 0, \quad S \neq \text{ fixed coaltion}
\end{aligned}
$$

- $x_1^1 = 14$; $x_2^1 = \frac{136}{3}$; $x_3^1 = \frac{46}{3}$; $x_4^1 = \frac{106}{3}$. $F^1 = \{2, 3\}$.

$$
\begin{array}{l}
\max \epsilon \\
x_1 + x_2 + x_3 + x_4 = 110 \\
x_1 + x_2 + x_3 \geq 70 + \epsilon \; \bullet \\
x_1 + x_2 + x_4 = 90 + \frac{14}{3} \\
x_1 + x_3 + x_4 = 60 + \frac{14}{3} \\
x_2 + x_3 + x_4 \geq 88 + \epsilon \\
x_1 + x_2 \geq 50 + \epsilon \\
x_1 + x_3 \geq 20 + \epsilon \\
x_1 + x_4 \geq 40 + \epsilon \\
x_2 + x_3 = 56 + \frac{14}{3} \\
x_2 + x_4 \geq 72 + \epsilon \\
x_3 + x_4 \geq 24 + \epsilon \\
x_1 \geq 0 + \epsilon \\
x_2 \geq 40 + \epsilon \\
x_3 \geq 8 + \epsilon \\
\cancel{x_4 \geq 12 + \epsilon} \\
x_4 \geq \frac{82}{3} + \epsilon \; \bullet
\end{array}
$$

$$\min 110 \times y_N - \sum_{S \subset N} v(S) y_S$$

$$\sum_{S \not\subset \text{fixed coalition}} y_S = 1,$$

$$-\sum_{S : i \in S} y_S = y_N, \quad \text{for } i = 1, \dots, n,$$

$$y_S \geq 0, \quad S \neq \text{ fixed coaltion}$$

- $x_1^1 = 14;\ x_2^1 = \frac{136}{3};\ x_3^1 = \frac{46}{3};\ x_4^1 = \frac{106}{3}.\ F^1 = \{2,3\}.$
- The set optimizing Step 2 $S = \{2,3,4\};\ \mu_2 = \frac{10/3}{2} = \frac{5}{3};$ $\epsilon_2 := \epsilon_1 + \mu_2 = \frac{14}{3} + \frac{5}{3} = \frac{19}{3}.$

$$
\begin{aligned}
&\max \epsilon\\
&x_1 + x_2 + x_3 + x_4 = 110\\
&x_1 + x_2 + x_3 \geq 70 + \epsilon \quad \bullet\\
&x_1 + x_2 + x_4 = 90 + \tfrac{14}{3}\\
&x_1 + x_3 + x_4 = 60 + \tfrac{14}{3}\\
&x_2 + x_3 + x_4 \geq 88 + \epsilon\\
&x_1 + x_2 \geq 50 + \epsilon\\
&x_1 + x_3 \geq 20 + \epsilon\\
&x_1 + x_4 \geq 40 + \epsilon\\
&x_2 + x_3 = 56 + \tfrac{14}{3}\\
&x_2 + x_4 \geq 72 + \epsilon\\
&x_3 + x_4 \geq 24 + \epsilon\\
&x_1 \geq 0 + \epsilon\\
&x_2 \geq 40 + \epsilon\\
&x_3 \geq 8 + \epsilon\\
&\cancel{x_4 \geq 12 + \epsilon}\\
&x_4 \geq \tfrac{82}{3} + \epsilon \quad \bullet
\end{aligned}
$$

$$
\begin{aligned}
&\min 110 \times y_N - \sum_{S \subset N} v(S) y_S\\
&\sum_{S \not\subset \text{fixed coalition}} y_S = 1,\\
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\end{aligned}
$$

- $x_1^1 = 14$; $x_2^1 = \frac{136}{3}$; $x_3^1 = \frac{46}{3}$; $x_4^1 = \frac{106}{3}$. $F^1 = \{2, 3\}$.
- The set optimizing Step 2 $S = \{2, 3, 4\}$; $\mu_2 = \frac{10/3}{2} = \frac{5}{3}$; $\epsilon_2 := \epsilon_1 + \mu_2 = \frac{14}{3} + \frac{5}{3} = \frac{19}{3}$.
- $x_1^2 = 14 + \frac{5}{3} = \frac{47}{3}$; $x_2^2 = \frac{136}{3}$; $x_3^2 = \frac{46}{3}$; $x_4^2 = \frac{106}{3} - \frac{5}{3} = \frac{101}{3}$.

$$
\begin{array}{l}
\max \epsilon \\
x_1 + x_2 + x_3 + x_4 = 110 \\
x_1 + x_2 + x_3 \geq 70 + \epsilon \;\bullet \\
x_1 + x_2 + x_4 = 90 + \frac{14}{3} \\
x_1 + x_3 + x_4 = 60 + \frac{14}{3} \\
x_2 + x_3 + x_4 \geq 88 + \epsilon \\
x_1 + x_2 \geq 50 + \epsilon \\
x_1 + x_3 \geq 20 + \epsilon \\
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\cancel{x_4 \geq 12 + \epsilon} \\
x_4 \geq \frac{82}{3} + \epsilon \;\bullet
\end{array}
$$

$$
\begin{array}{l}
\min 110 \times y_N - \sum_{S \subset N} v(S) y_S \\
\sum_{S \not\subset \text{fixed coalition}} y_S = 1, \\
-\sum_{S : i \in S} y_S = y_N, \quad \text{for } i = 1, \ldots, n, \\
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\end{array}
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- $x_1^1 = 14$; $x_2^1 = \frac{136}{3}$; $x_3^1 = \frac{46}{3}$; $x_4^1 = \frac{106}{3}$. $F^1 = \{2, 3\}$.
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- $x_1^2 = 14 + \frac{5}{3} = \frac{47}{3}$; $x_2^2 = \frac{136}{3}$; $x_3^2 = \frac{46}{3}$; $x_4^2 = \frac{106}{3} - \frac{5}{3} = \frac{101}{3}$.
- $|S^2 \setminus F^1| = 1$; $\bar{y}_{1234} = -\frac{1}{2}$; $\bar{y}_4 = \frac{1}{2}$; $\bar{y}_{123} = \frac{1}{2}$.

Remarks

- For each market $j \in M$, let (N, v_j) be the uncapacitated one-market game. Then $(N, \mathrm{v} = \sum_{j \in M} \mathrm{v}_j)$ is the uncapacitated multi-market game.
- If $x^j \in \mathbb{C}(N, v_j)$, then $x = \sum_{j \in M} x^j$ belongs to $(N, \mathrm{v} = \sum_{j \in M} \mathrm{v}_j)$.
- Unfortunately, if x^j is the nucleolus of (N, v_j), then $x = \sum_{j \in M} x^j$ may not be the nucleolus of $(N, \mathrm{v} = \sum_{j \in M} \mathrm{v}_j)$.