How to Fairly Allocate Goods to Players: Mathematical Model and Algorithms

Trung Thanh Nguyen

ORLab, Phenikaa University Viasm

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Outline

Combinatorial Optimization Problem (COP)

Basic definition

$$\max/\min \quad f(x) \tag{1}$$
$$\text{subject to} \quad x \in \mathcal{S}. \tag{2}$$

where S is a set of finite feasible solutions.

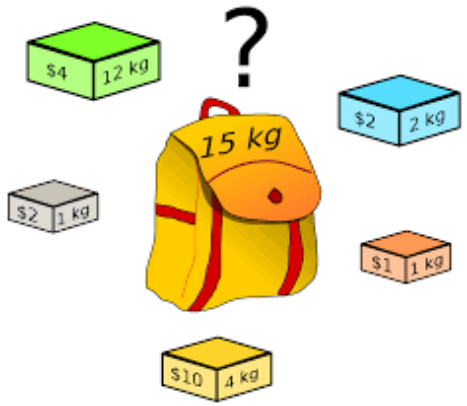
Facts:

- The size of S is often exponential large
- In general, COP is NP-hard to solve exactly

Example of COPs

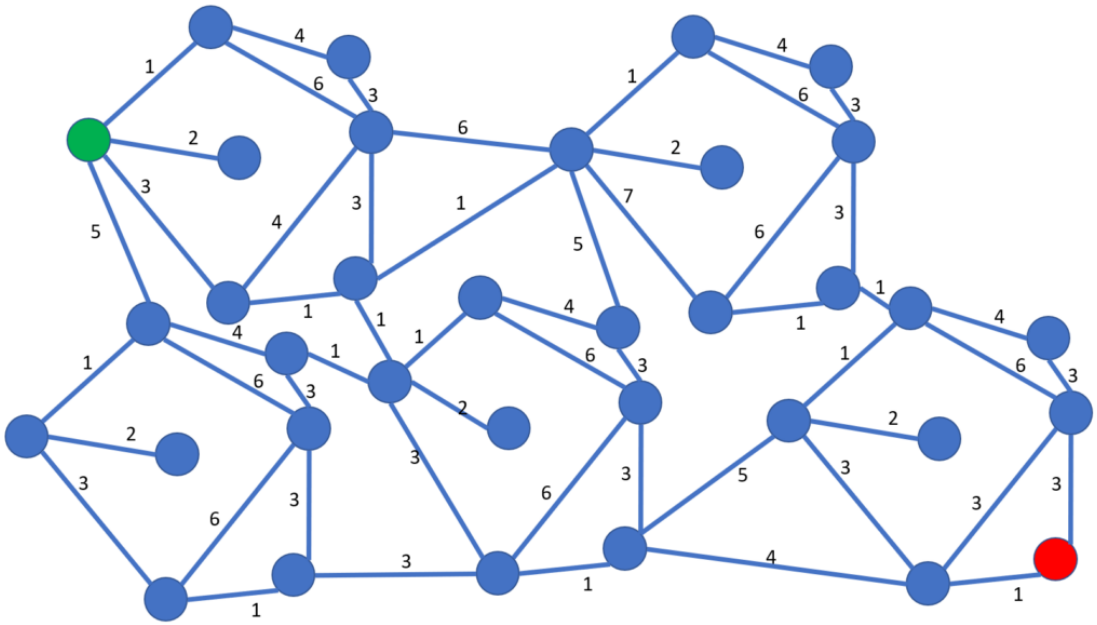
- Partition

- Knapsack



Example of COPs

- Shortest path, TSP



Resource Allocation Problem

Given:
- A set of goods (or objects) $[m] = \{1, \ldots, m\}$
- A set of agents $[n] = \{1, \ldots, n\}$
- Each agent j has a utility function $u_i : 2^{[m]} \to \mathbb{R}_+$
- Goods are indivisible and non-shareable

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Allocation:
- $\pi = (\pi_1, \ldots, \pi_n)$: an ordered partition of $[m]$ into n subsets

Require:
- An allocation that is considered to be "fair" to every agent.

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Applications: assign bandwidth to users in network, courses to students, jobs to machines, etc.
(`https://en.wikipedia.org/wiki/Fair_division_experiments`)

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The concern of economists: - What is a "fair" allocation?

The concern of mathematicians: - How hard is it to find a "fair" allocation?

Concepts of Fairness

Using Collective Utility Function (CUF)

Let $p = (u_1(\pi_1), \ldots, u_n(\pi_n))$ be the vector of agents utilities

Rawls function: $\min_i p_i$ - the utility of the worst-off agents

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Nash function: $\prod_i p_i$ (or $(\prod_i p_i)^{1/n}$) - the product of agents utilities

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Defining fair share of agents

An allocation is fair if the utility of every agent is at least her fair share

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Good news: PROP1 allocations and EF1 allocations does always exist.

Computational Problems

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Objectives based on fair share

Given a problem instance, find an allocation that is maximin fair, proportional fair, or envy-free.

Sub-problems:

- Computing fair share for each agent.
- Deciding whether a fair allocation exists.

Types of Utility Functions

Hardness of fair allocation problems depends heavily on types of utility functions.

additive: $u_i(S) = \sum_{j \in S} u_i(j)$ for every $S \subseteq [m]$, $u_i(j) \equiv u_{ij}$

submodular: $u_i(S \cup V) + u_i(S \cap V) \leq u_i(S) + u_i(V)$, for every $S, V \subseteq [m]$

subadditive: $u_i(S \cup V) \leq u_i(S) + u_i(V)$, for every $S, V \subseteq [m]$

Mathematical models

Max-Min Allocation (MMA) as a Binary Linear Program

$$\text{maximize} \quad \min_{i=1}^{n} \left\{ \sum_{j=1}^{m} u_{ij} x_{ij} \right\} \tag{3}$$

$$\text{subject to} \quad \sum_{i=1}^{n} x_{ij} = 1, \qquad j \in [m], \tag{4}$$

$$x_{ij} \in \{0, 1\} \qquad i \in [n], j \in [m], \tag{5}$$

where $x_{ij} = 1$ if good j is assigned to agent i, and $x_{ij} = 0$ otherwise.

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Partition problem - A special case of MMA

Given a set S of positive number, divide the set into two disjoint subsets so that the difference between the subsets is as small as possible.

$$\text{minimize}_{\emptyset \neq V \subset S} \left| \sum_{j \in V} j - \sum_{j \in S \setminus V} j \right|$$

Known Results

How hard to solve MMA

- MMA is at least as hard as Partition problem, which is NP-hard
- There is no polynomial time algorithm for exactly solving MMA
- The current best algorithm, based on Dynamic Programming, has exponential run-time in n.

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Solve MMA approximately by polynomial time algorithms

- There is no $(\frac{1}{2} + \epsilon)$-approximation for MMA, for any $\epsilon > 0$ (Bezakova and Dani 2005)
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Existing works have mainly focused on special cases where better algorithms can be designed.

Mathematical models

Nash Allocation (NA)

$$\text{maximize} \quad \prod_{i=1}^{n} \left\{ \sum_{j=1}^{m} u_{ij} x_{ij} \right\} \tag{6}$$

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A special case of NA

Given a set S of positive number, divide the set into two disjoint subsets to maximize the following objective

$$\sum_{j \in V} j \cdot \sum_{j \in S \setminus V} j$$

over all V.

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- There is $\frac{1}{2.89}$-approximation for NA (Cole and Gkatzelis 2015)
- There is no 0.99-approximation for NA. (Lee 2017)
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Open question: Narrowing the gap $[0.68; 0.99]$

Maximin Fair Allocation Problem (MFA)

Known results

- Computing MMS for each agent is an NP-hard problem
- An allocation that is maximin fair may not exist for $n \geq 3$. (Kurokawa et al. 2016)
- A $\frac{3}{4}$-maximin fair allocation always exist and can be efficiently computed. (Garg and Taki 2020)

Proportional Fair Allocation Problem (PFA)

Known results

- Computing PS is easy
- PROP allocations do not always exist, even for $n = 2$.
- Deciding if there is a PROP-allocation is an NP-hard problem.
- PROP1-allocations can be computed in strongly polynomial time (Barman, Siddharth; Krishnamurthy 2019)

Envy-Free Allocation Problem (EFA)

Known results

- EF-allocations do not always exist, even for $n = 2$.
- Deciding if there is an EF-allocation is an NP-hard problem.
- EF1-allocations can be computed by using the Round-Robin algorithm

Approximation Schemes for MMA and NA

The case when n is small

Design a polynomial time algorithm that return an allocation whose value is within a factor of $1 - \epsilon$ of the optimum.

Algorithm 1 Pseudo-polynomial-time algorithm

```
1: V_0 := {0};
2: for j := 1 to m do
3:     V_j := ∅;
4:     for each v ∈ V_{j-1} do
5:         V_j := V_j ∪ {v + s_ij · e_i | i = 1, ..., n};
6:     end for
7: end for
8: return Vector v ∈ V_m such that the product of its coordinates is maximal;
```

- $(e_1, \ldots, e_n)$ be the canonical basis of the vector space $\mathbb{R}^n$, where e_i denotes the vector with a 1 in the ith coordinate and 0's elsewhere
- $s_{ij} = u_{ij}$
- V_j the set of all possible assignments of the first j goods to agents, where each assignment is encoded by an n-dimensional vector v.

Approximation Schemes for MMA and NA

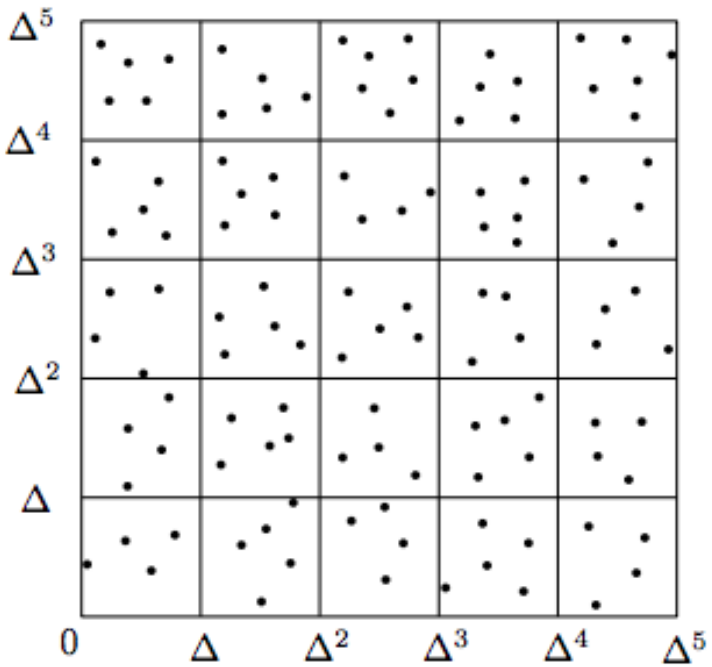
Agent	r_1	r_2	r_3	r_4	r_5
a_1	0	1	2	0	1
a_2	1	1	0	1	0

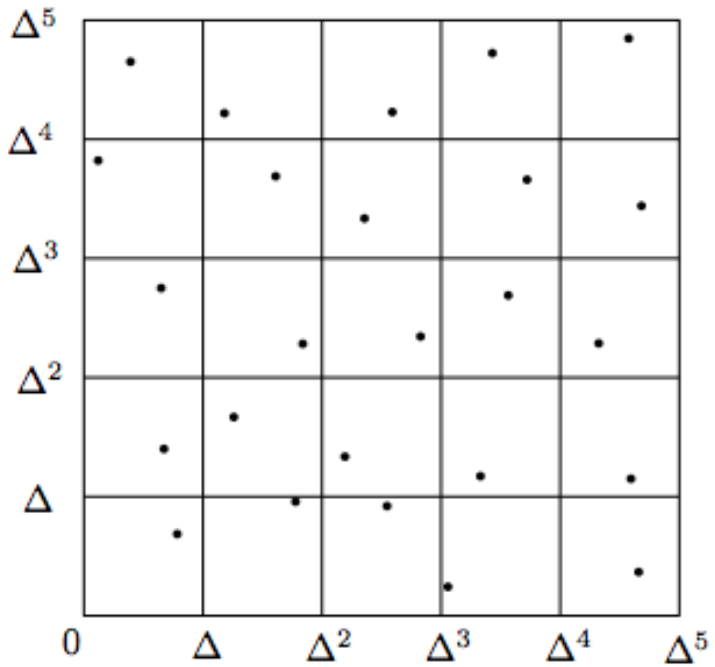
Step	V_j	Size of V_j
$j = 1$	(0, 0), (0, 1)	2
$j = 2$	(1, 0), (0, 1), (1, 1), (0, 2)	4
$j = 3$	(3, 0), (1, 0), (2, 1), (0, 1), (3, 1), (1, 1), (2, 2), (0, 2)	8
$j = 4$	(3, 0), (3, 1), (1, 0), (1, 1), (2, 1), (2, 2), (0, 1), (0, 2) (3, 1), (3, 2), (1, 1), (1, 2), (2, 2), (2, 3), (0, 2), (0, 3)	16
$j = 5$	(4, 0), (3, 0), (4, 1), (3, 1) (2, 0), (1, 0), (2, 1), (1, 1) (3, 1), (2, 1), (3, 2), (2, 2) (1, 1), (0, 1), (1, 2), (0, 2) (4, 1), (3, 1), (4, 2), (3, 2) (2, 1), (1, 1), (2, 2), (1, 2) (3, 2), (2, 2), (3, 3), (2, 3) (1, 2), (0, 2), (1, 3), (0, 3)	32

Approximation Schemes for MMA and NA

Running time of the pseudo-algorithm

$$O(mB^n), \quad \text{where } B = \max_i u_i([m])$$



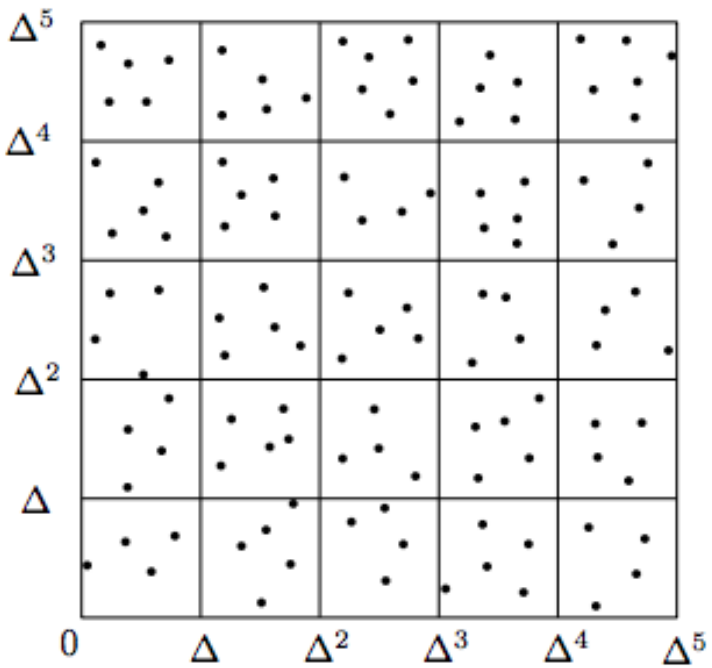


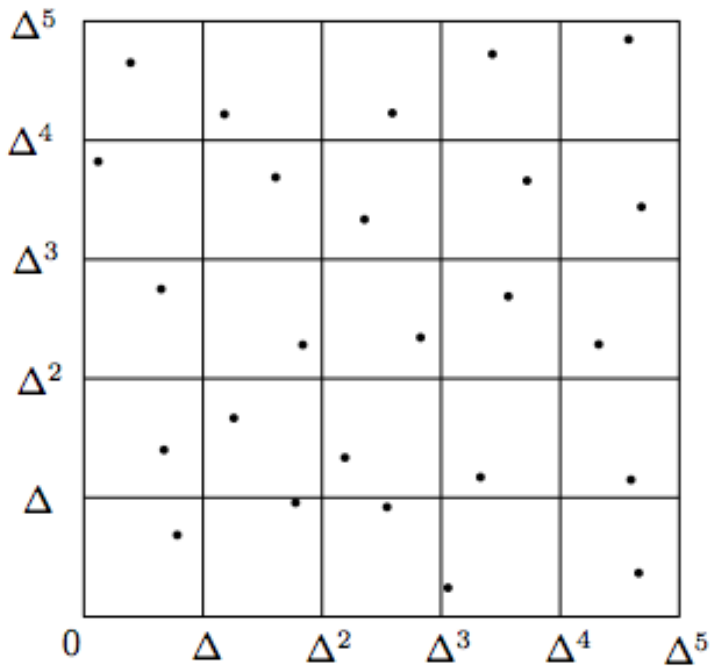
$$\Delta = 1 + \frac{\epsilon}{2m}$$

Approximation Schemes for MMA and NA

Running time of the modified pseudo-algorithm

$$O(mK^{2n}), \quad \text{where } K \leq \left(1 + \frac{2m}{\epsilon}\right) \cdot |I|$$





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Open problems

- Designing a constant approximation algorithm for MMA.
- Narrowing the gap of $[0.68; 0.99]$ for NA.
- Improving $\frac{3}{4}$-maximin fair allocation

THANK YOU FOR YOUR ATTENTION!