Nash fairness solutions for balanced TSP

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Introduction

The Balanced Traveling Salesman Problem (BTSP)

- Introduced by Larusic and Punnen in 2011 (LP2011).
- Undirected graph $G = (V, E)$ with edge cost $c_{i,j} \in \mathbb{R}_+$.
- Finding a Hamiltonian tour minimizing the max-min distance: difference between the maximum edge cost and the minimum one in a solution tour.

Threshold heuristic for solving BTSP (LP2011)

- Sorting distinct values of edge cost: $z_1 < ... < z_p$.
- Finding a pair (z_i, z_j) satisfying
  1. a Hamiltonian cycle with edge set $\{(i,j) \in E | z_i \leq c_{i,j} \leq z_j\}$.
  2. $z_j - z_i$ as small as possible.

MIP formulation for the BTSP

MIP formulation

$$
\begin{aligned}
\min\ & t && && \text{(1a)}\\
\text{s.t. } & \sum_{j\in[n]} x_{j,i} = \sum_{j\in[n]} x_{i,j} = 1 && \forall i \in [n] && \text{(1b)}\\
& \sum_{i\in H}\ \sum_{j\neq i, j\in H} x_{i,j} \leq |H| - 1 && \forall H \subset V && \text{(1c)}\\
& t \geq u - l && && \text{(1d)}\\
& u \geq c_{i,j} x_{i,j} && \forall i,j \in [n] && \text{(1e)}\\
& l \leq \sum_{j\in[n]} c_{i,j} x_{i,j} && \forall i \in [n] && \text{(1f)}\\
& x_{i,j} \in \{0,1\} && \forall i,j \in [n]. && \text{(1g)}
\end{aligned}
$$

Numerical results for the BTSP (1)

Table 1: Solutions for the BTSP

Instance	Heuristic(LP2011)	MIP formulation
att48	192	**190**
gr48	48	**46**
berlin52	151	**149**
brazil58	1125	**1097**

- Better solutions in several instances with MIP formulation.

Numerical results for the BTSP (2)

Table 2: Optimal solutions for the TSP and the BTSP

Instance	TSP		BTSP	
	Total cost	Max-min	Total cost	Max-min
gr21	2707	328	8630	119
gr24	1272	83	3886	33
fri26	937	118	2458	21
bays29	2020	140	6757	38

- Drawback of optimal solutions for the TSP and the BTSP
  1. TSP: efficient total cost but possibly inefficient max-min distance.
  2. BTSP: efficient max-min distance but possibly inefficient total cost.

Motivation

Motivation

- Solution with better trade-off between total cost and max-min distance.
- Nash Fairness (NF) solutions achieving a Nash equilibrium between two players in the context of the BTSP.

Proportional fairness (Bertsimas et al - 2011)

Nash equilibrium

- Two players problem: the percentage increase in the utility of one player is greater than the percentage decrease in the utility of the other player.
- Multiple players problem: the aggregate proportional change is not positive.

Definition 1

$x^{NF} \in U$ be a NF solution for n players problem if and only if

$$\sum_{j=1}^{n} \frac{u_j(x) - u_j(x^{NF})}{u_j(x^{NF})} \leq 0, \ \forall x \in U,$$

where $u_j(x) > 0, \ \forall j = 1, 2, ..., n, \forall x \in U$.

Characterization of NF solutions for the BTSP

- For multiple players problem, we prefer greater values of utility functions.
- For the BTSP, we prefer smaller values of cost functions (i.e., total cost and max-min distance).
- The aggregate proportional change should be not negative.

Definition 2

Let P, Q be the total cost and max-min distance in a feasible solution for the BTSP, then $(P^*, Q^*) \in \mathcal{S}$ is a NF solution if and only if

$$\frac{P - P^*}{P^*} + \frac{Q - Q^*}{Q^*} \geq 0 \iff \frac{P}{P^*} + \frac{Q}{Q^*} \geq 2,\ \forall (P, Q) \in \mathcal{S},$$

Remark

If $Q = 0$ then (P, Q) is a NF solution.

Existence of NF solutions

Theorem 1

$(P^*, Q^*) = argmin_{(P,Q)\in\mathcal{S}} PQ$ is a NF solution.

Proof

If $P^*Q^* = 0$ then $Q^* = 0$ and (P^*, Q^*) is a NF solution.
If $P^*Q^* > 0$ then $\forall(P, Q) \in \mathcal{S}$ we have $PQ \geq P^*Q^* > 0$. Hence

$$\frac{P}{P^*} + \frac{Q}{Q^*} \geq 2\sqrt{\frac{PQ}{P^*Q^*}} \geq 2,$$

- There always exists a NF solution that minimizes PQ.
- Finding such a solution may be difficult as PQ is a concave function.

Extreme NF solutions

Remark

Let (P, Q) and (P', Q') be two different feasible solutions for the BTSP. The two inequalities

$$\frac{P'}{P} + \frac{Q'}{Q} \geq 2 \;\text{ and }\; \frac{P}{P'} + \frac{Q}{Q'} \geq 2.$$

may be satisfied simultaneously.

- *Efficient Nash Fairness* (ENF) solution with smallest value of P.
- *Balanced Nash Fairness* (BNF) solution with smallest value of Q.
- The ENF solution as well as BNF solution is unique.

Finding extreme NF solutions

- NF solutions can be obtained by minimizing a linear combination of P and Q

$$\mathcal{P}(\alpha) = \min \ \alpha P + Q \text{ s.t } (P, Q) \in \mathcal{S},$$

where $\alpha \in [0, 1]$ is the coefficient to be determined.

Theorem 2

$(P, Q) \in \mathcal{S}$ is a NF solution if and only if (P, Q) is an optimal solution for $\mathcal{P}(\alpha)$ with $\alpha = \frac{Q}{P}$.

Corollary

If $(P, Q) \in \mathcal{S}$ is an optimal solution for $\mathcal{P}(\alpha)$ and $T_\alpha := \alpha P - Q = 0$ then (P, Q) is a NF solution.

Algorithm for finding the ENF solution

Algorithm 1

Input: An instance of the BTSP with a MIP formulation for $\mathcal{P}(\alpha)$.
Output: A Hamiltonian tour corresponding to the ENF solution.

1: $\alpha \leftarrow 1$
2: **repeat**
3: solve $\mathcal{P}(\alpha)$ to obtain (P, Q) and solution tour X
4: $T \leftarrow \alpha P - Q$
5: $\alpha \leftarrow Q/P$
6: **until** $T = 0$
7: return X.

Algorithm for finding the ENF solution (2)

Convergence of Algorithm 1

- $\{\alpha_i\}_{i\geq 0}$ as the sequence constructed by Algorithm 1.
- $\alpha_i \in [0, 1]$ and $\alpha_i \geq \alpha_{i+1}$ $\forall i \geq 0$.
- $\alpha_i = \alpha_{i+1} \iff T_i = 0$.

Number of iterations

- Iterations of Algorithm 1 explore distinct Pareto-optimal solutions.
- Number of distinct Pareto-optimal is bounded by n^4.
- Algorithm 1 terminates after at most n^4 iterations.

Algorithm for finding the BNF solution

Algorithm 2

Input: An instance of the BTSP with a MIP formulation for $\mathcal{P}(\alpha)$.
Output: A Hamiltonian tour corresponding to the BNF solution.

1: $\alpha \leftarrow 0$
2: **repeat**
3:     solve $\mathcal{P}(\alpha)$ to obtain (P, Q) and solution tour X
4:     $T \leftarrow \alpha P - Q$
5:     $\alpha \leftarrow Q/P$
6: **until** $T = 0$
7: return X.

Numerical results

Table 3: Optimal solutions for the TSP, the BTSP and the NFTSP

Instance	TSP		ENFTSP		BTSP		BNFTSP	
	P	Q	P	Q	P	Q	P	Q
burma14	3323	472	4986	134	4986	134	4986	134
ulysses16	6859	1452	7047	1399	14032	868	13670	868
gr17	2085	311	2227	234	4138	119	3346	139
gr21	2707	328	2989	278	8630	115	5945	120
ulysses22	7013	1490	7070	1471	19168	868	7070	1471
gr24	1272	83	1282	81	3886	33	3847	33
fri26	937	118	980	82	2458	21	2447	21
bays29	2020	140	3449	59	6757	38	4558	44
bayg29	1610	86	1817	63	4252	29	3246	35

Numerical results (2)

Table 4: The aggregate proportional change

Instance	Aggregate proportional change	
	ENFTSP vs TSP	BNFTSP vs BTSP
burma14	2.188	0.000
ulysses16	0.011	0.026
gr17	0.265	0.092
gr21	0.085	0.409
ulysses22	0.004	1.130
gr24	0.016	0.010
fri26	0.395	0.004
bays29	0.958	0.346
bayg29	0.251	0.138

Table 5: Running time for solving the TSP, the BTSP and the NFTSP

Instance	TSP	BTSP	ENFTSP		BNFTSP	
	Time	Time	Time	Iter	Time	Iter
burma14	0.10	0.15	1.40	4	0.70	2
ulysses16	0.51	0.70	4.42	3	17.56	3
gr17	0.21	0.35	5.53	3	8.63	4
gr21	0.01	0.68	1.24	3	18.60	3
ulysses22	14.16	16.80	356.97	3	651.38	4
gr24	0.07	1.85	1.17	3	49.17	3
fri26	0.13	2.72	24.39	3	55.06	3
bays29	0.55	7.95	2033.04	5	2471.25	3
bayg29	0.31	4.12	44.92	3	2567.73	4

Conclusion

Our works

- MIP formulation for solving the BTSP.
- Definition of NF solutions for the BTSP.
- Newton-like iterative algorithms for finding extreme NF solutions with polynomial iterations.

Future works

- Improving CPU time for finding extreme NF solutions, e.g find a way for not solving $\mathcal{P}(\alpha)$ from scratch at each iteration.
- Applying for other multi-objective optimization problems with positive objective functions.