Binary Lebesgue Measures

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In this paper I introduce binary Lebesgue measures, which can be used to pose standard problems.

A Lebesgue measure[1] is a standard way to measure subsets of n-dimensional Euclidean space:

- The Lebesgue measure of a line is length
- The Lebesgue measure of a rectangle or flat disc is area
- The Lebesgue measure of a cube or sphere is volume

A binary Lebesgue measure `f` is a function of some type `T`, that computes two real numbers for some inputs such that their sum equals the output of some Lebesgue measure `h` of some type `U` that is generated by a partial normal path, using some function `g`:

$$f[g \rightarrow \text{map}(\text{add})] \Rightarrow \text{some}(h)$$

$$f : T \rightarrow \text{opt}[\mathbb{R}^2] \qquad g : T \rightarrow U \qquad h : U \rightarrow \mathbb{R}$$

$$\text{map} : (\mathbb{R}^2 \rightarrow \mathbb{R}) \rightarrow (\text{opt}[\mathbb{R}^2] \rightarrow \text{opt}[\mathbb{R}])$$
$$\text{some} : (U \rightarrow \mathbb{R}) \rightarrow (U \rightarrow \text{opt}[\mathbb{R}])$$

Here, `=>` is a monadic subset[2] and `map` and `some` are default operators on option types.

For example, when splitting a sphere using a flat plane, the sum of the area of the two halves equals the area of the sphere. Here, `T = (sphere, plane)` and `U = sphere`. However, there are two natural ways of implementing a such algorithm.

A: If the plane does not intersect the sphere, then one can assign area `0` to one half. This will hold for all inputs.

B: If the plane does not intersect the sphere, then one can return `none()`. One only assigns area `0` to one half when the plane intersects the sphere at exactly one point.

Either implementation, both A and B, can be called a binary Lebesgue measure. In either case the Lebesgue measure is the area of a sphere.

When splitting a sphere and calculating the area of each half, there exists a binary Lebesgue measure for every way of assigning the two areas to a tuple e.g. `(area1, area2)`. So, there are possibly infinitely many binary Lebesgue measures for some Lebesgue measure.

The motivation of binary Lebesgue measures is to pose standard problems which solutions require understanding of the underlying semantics of Lebesgue measures in terms of binary set operators:

- Splitting
- Union, Intersection and Subtraction
- Transformations with invariants

The minimum data structures needed to solve these problems correspond to optimal representations.

References

[1] “Lebesgue measure”
Wikipedia
https://en.wikipedia.org/wiki/Lebesgue_measure

[2] “Monadic Subsets”
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https://github.com/advancedresearch/path_semantics/blob/master/papers-wip2/monadic-subsets.pdf