Permutations as Functions

by Sven Nilsen, 2020

In this paper I describe permutations as functions.

A permutation can be thought of as a constrained function `f` that maps to the same type as input:

$$f : T \to T \qquad \forall f <=> \exists f$$

With the constraint that the trivial path is the same as the existential path.
This implies that the trivial path and the existential path have the same size:

$$|\forall f| = |\exists f|$$

Therefore, a permutation is a structure preserving function.
In fact, all total structure preserving functions that maps to the same type are permutations.

The identity function `id` is also a permutation, with a special property for any permutation `f`:

$$id \cdot f <=> f \cdot id <=> f$$

All permutations as functions can be generated from the identity function by either:

- Permuting its inputs while keeping its outputs in same order, called the “left” permutation
- Permuting its outputs while keeping its inputs in same order, called the “right” permutation

There is a natural number associated with permutations `[0, |T|)` in increasing order:

0	0123
1	0132
2	0213
…	
23	3210

If `T` has a total order, all such functions have two natural associated numbers:

- The left number of `f` describes the order it was generated from `id` by permuting inputs
- The right number of `f` describes the order it was generated from `id` by permuting outputs

The identity function has `0` as the same left and right number.

Another function with same left and right number is the last permutation,
since reversing the inputs gives the same function as when reversing the outputs:

01234	43210
43210	01234

In general, all functions with same left and right number have the following property:

f^{-1} <=> f

This is because a function `g` that has same right number as `f`’s left number is the inverse:

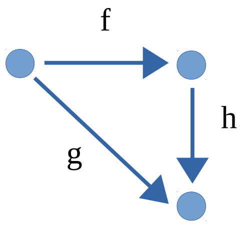
right(g) = left(f)   <=>   g <=> f^{-1}

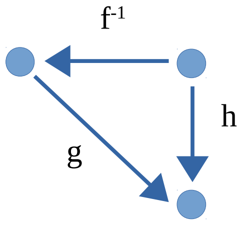
When `f`’s left number is the same as the right number,
the function `g` that has same right number, is `f` itself, therefore `f` is its own inverse.

Since the inverse of a function is unique, `g`’s left number must be `f`’s right number:

left(g) = right(f)   <=> g^{-1} <=> f

From any function `f` to any function `g`, there is an associated function `h`:





g <=> h · f
g · f^{-1} <=> h

For all functions `g` with same right number as `f’`’s left number, `h <=> $f^{-2}$`:

g · f^{-1} <=> h
f^{-1} · f^{-1} <=> h
f^{-2} <=> h

When `f` is its own inverse, `h <=> id`:

h
f^{-2}
f^{-1} · f^{-1}
f^{-1} · f   using `f <=> $f^{-1}$`
id

So, from one right permutation `$f_r$` to another right permutation `$g_r$`,
one can perform a right permutation `$h_r$`:

g_r · f_r^{-1} <=> h_r

Such that this composed permutation generated from `id` gives `g`:

g <=> h · f