# Geometric Paths

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*In this paper I introduce the concept of geometric paths, used to analyze path-connected spaces. I also introduce the concept of anageometric paths, a generalization of geometric paths.*

A geometric path `f` is a higher order function constructed by an axiom function `f_1`:

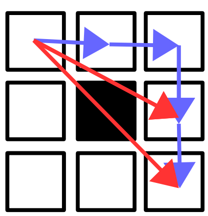
| | |
|---|---|
| $f_1(a, b) \Leftrightarrow \text{continuum}(a, b)$ | Example of an axiomatic that defines |
| $f_1(b, c) \Leftrightarrow \text{continuum}(b, c)$ | conditional continuum between some points. |
| $f_1(\_, \_) \Leftrightarrow \text{false}_1$ | All other points are disconnected. |

$$f := \backslash(a : T, c : T) = \backslash(g : T \to \text{bool}) = f_1(a, c)(g) \vee \exists\, b : g \{ f(a, b)(g) \wedge f(b, c)(g) \}$$

$$f : T \times T \to (T \to \text{bool}) \to \text{bool}$$
$$\text{continuum} := \backslash(a : T, b : T) = \backslash(g : T \to \text{bool}) = g(a) \wedge g(b)$$

It is only permitted to use `continuum` or `false_1` for any pair of inputs in `f_1`.

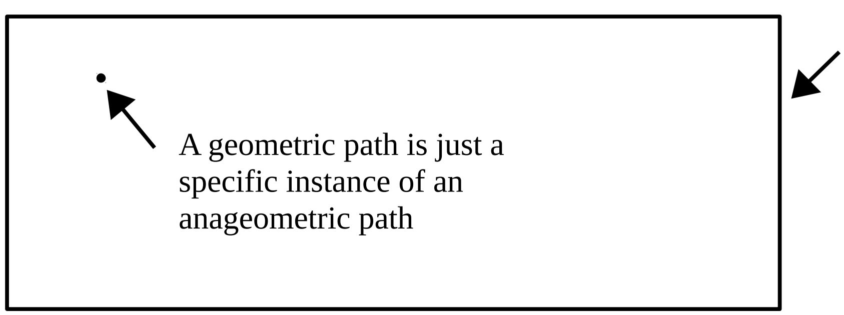
The geometric path can be thought of as connecting a space by composing small arrows:



When there exist a path of blue arrows, then the space is connected by "filling in" with red arrows. The space is undirected, such that all arrows can be flipped around.

A geometric path is a *higher order unlabeled undirected graph* for any function of type `T → bool`.

Each geometric path is part of an infinite family of similar functions, called "anageometric path":




By anageometric path we mean the infinite family of functions similar to a geometric path, constructed by same axiom

The rest of this paper explains how an anageometric path is constructed.

Assume one has a property `g` that defines some sub-type of 3 variables `a`, `b` and `c`:

a : [g] a'  b : [g] b'  c : [g] c'

$g : T \to \text{bool}$

All possible interpretations of the order "abc" can be given descriptive names:

| **abc => a'b'c'** | |
|---|---|
| 000 | external |
| 001 | tip |
| 010 | bridge |
| 011 | tail |
| 100 | head |
| 101 | wall |
| 110 | neg tip |
| 111 | internal |

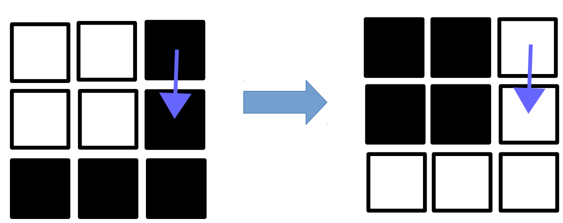
A geometric path is the interpretation "internal", since the property `g` must hold for all points.

Therefore, it is common to write the geometric path the following way:

$f_{111}$  Using the interpretation code `111` in the anageometric path

By swapping `g` with `¬g`, one gets the interpretation "external" (`¬g <=> not · g`):

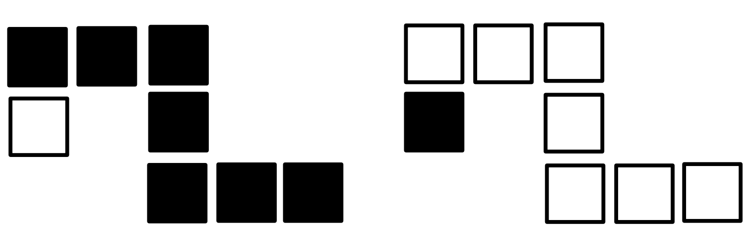
$$f_{000} := \backslash(a : T, b : T) = \backslash(g : T \to \text{bool}) = f_{111}(a, b)(\neg g)$$



The paths in these two spaces are very different, even the relation between the spaces is simple.

A "head" can be thought of as a path starting inside but then immediately goes outside:

$$f_{100} := \backslash(a : T, c : T) = \backslash(g : T \to \text{bool}) = \neg g(a) \wedge f_1(a, b)(\text{true}_1) \wedge \exists\, b : g\ \{\ f_{000}(b, c)(g)\ \}$$
$$f_{011} := \backslash(a : T, b : T) = \backslash(g : T \to \text{bool}) = f_{100}(a, b)(\neg g)$$



The “neg tip” and “tip” are constructed in a similar way:

$f_{110} := \backslash(a : T, c : T) = \backslash(g : T \to \text{bool}) = \neg g(c) \land \exists\, b : g \{ f_{000}(a, b)(g) \land \neg f_{000}(b, c)(\text{true}_1) \}$
$f_{001} := \backslash(a : T, b : T) = \backslash(g : T \to \text{bool}) = f_{110}(a, b)(\neg g)$

The “wall” and “brigde” are constructed the following way:

$f_{101} := \backslash(a : T, c : T) = \backslash(g : T \to \text{bool}) = g(a) \land g(c) \land \exists\, b : \neg g \{ f_1(a, b)(\text{true}_1) \land f_1(b, c)(\text{true}_1) \}$
$f_{010} := \backslash(a : T, b : T) = \backslash(g : T \to \text{bool}) = f_{101}(a, b)(\neg g)$

There is another way of constructing a “wall”, without using `$f_1$`:

$f_{101} := \backslash(a : T, c : T) = \backslash(g : T \to \text{bool}) =$
$\quad \exists\, b : g \{ f_{100}(a, b)(g) \land f_{110}(a, b)(g) \land f_{100}(b, c)(g) \land f_{001}(b, c)(g) \}$

One then generalizes this construction to an interpretation code of arbitrary length:

- Topological equivalent binary numbers up to two bits
- A single bit is interpreted as “wall” or “bridge” respectively

For example:

00000100000 => 00100
00010100111 => 001010011

This topological transform can be described by the following L-system using two rules:

000 => 00 if code is longer than 3
111 => 11 if code is longer than 3

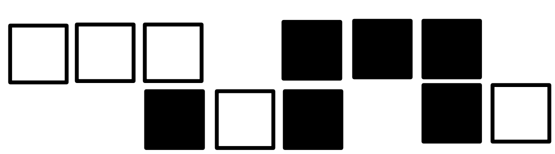
The anageometric path is the **infinite family of functions** from this construction.

$f_x$ where `x` is some interpretation code of the anageometric path

For example, to ask whether some space `f` filled with `g` connects `a` to `b` by the following:

$(a, b) : [f_{1101001}]\ [g]$ true

Here, `a` starts inside the space, then after an internal path goes through a wall, a bridge, an external path and finally inside again:



With no access to `$f_1$`, it is not possible to construct the anageometric path, because one can not find two points that are only locally connected but not connected through any longer path.