Order-Free Non-Determinism

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In this paper I show that some non-deterministic functions using perfect randomness are order-free. This means that the probabilistic existential path encodes the full description of the function. This property only holds if time traveling is impossible, including traveling at the speed of light. At frozen time, limiting restrictions can preserve order-free non-determinism. When a source of randomness is available while the time is frozen, there is a time paradox which is resolved physically with causality.

In a pure function, the output is determined by the input.
When a pure function `f` takes no inputs, it can only have one output:

$$f : () \rightarrow T$$

Assume `f() = a`. The probabilistic existential path `$\exists_p f$` is defined as following:

$$(\exists_p f)(x : T) = \text{if } x == a \ \{ \ 1 \ \} \text{ else } \{ \ 0 \ \}$$

From a such probabilistic existential path, one can construct the function `f`.

If a non-deterministic function `f : () → T` uses a source of perfect randomness,
then its outputs do not correlate with the order of previous outputs.

This means that one can not construct a function `f' : () → T` with the same probabilistic existential path `$\exists_p f \iff \exists_p f'$` which permutes the order of the outputs in a way that is distinguishable from `f`.

Somewhat surprisingly, one can not even use `f` to change the order of the outputs, but this is only true when time traveling is impossible. For example, if one recorded the output of `f` and later swap to non-equal outputs, then one can prove that the outputs could not be generated by `f` when rewinding time and compare the modified outputs with replayed results of `f`. If one can not go back in time, then two copies of `f` can not be used to communicate the output of `f` at specific moment. Calling `f()` twice might produce two different outputs.

When all these conditions apply, it is possible to determine `f` from its probabilistic existential path. One can say that the probabilistic existential path determines `f`, because no other function can be constructed that is distinguishable from `f` with the same probabilistic existential path.

Therefore, if all forms of time traveling is prohibited, then there exists some non-deterministic functions of type `() → T` using perfect randomness that are order-free.

This prohibition includes traveling at speed of light, because:

1. Traveling at the speed of light freezes time for a distant observer
2. Freezing time is a subset of time travel that goes back to a single moment every time
3. Time traveling is prohibited, therefore traveling at speed of light is prohibited

If one could freeze time, which is the same as going back in time to a single moment every time, it is possible to communicate what `f()` will return at that specific moment using two copies of `f`.

However, to construct `f’` such that the probabilistic existential path is the same, one would need some probability of an output `x` that is equal to another probability of a different output `y`.
By swapping `x` with `y`, one can use a “frozen” copy of `f` to construct `f’`:

```
f’() = if f() == x { y } else if f() == y { x } else { f() }
```

Formally, the condition under frozen time that prohibits non-order free functions:

$$\exists\ x, y\ \{\ (\exists_p f)(x) == (\exists_p f)(y)\quad \wedge\quad (\exists_p f)(x)\ \neg{=}\ 0\ \}$$

As one gets closer and closer to the speed of light, a distant observer will see time slows down for the traveler. The traveler will not experience a slow down in time, but a contraction of distances in space. If the output of `f()` is kept constant for some interval of time, two distant observers can communicate the output using two copies of `f`, where `f` “observes” the traveler.

The probabilistic existential path is a way of encoding the probability distribution of `f`.

For example, the probability distribution:

false	true
0.3	0.7

Gives the following `f`:

```
f() = random() <= 0.7
```

If limited time traveling is possible, such as freezing time,
then one can invert the condition to permit order-free non-deterministic functions:

$$\neg\exists\ x, y\ \{\ (\exists_p f)(x) == (\exists_p f)(y)\quad \wedge\quad (\exists_p f)(x)\ \neg{=}\ 0\ \}$$

This is the same as:

$$\forall\ x, y\ \{\ (\exists_p f)(x)\ \neg{=}\ (\exists_p f)(y)\quad \vee\quad (\exists_p f)(x) == 0\ \}$$

For example:

false	true	
0.3	0.7	allowed when traveling at speed of light
0.5	0.5	not allowed when traveling at speed of light

An unbiased coin flip is not order-free when traveling at speed of light. This means that one could flip heads to tails and tails to heads and it would produce a different function. However: If one has access to an unbiased coin flip while time is frozen, then one can construct any rational number relationship between probabilities and swap fractions of outputs with others. E.g. 10% of the `false` outputs can be swapped with 10% of the `true` outputs.

Whether probabilities in the distribution are rational or not, is irrelevant.
A rational fraction of an irrational probability can be easily swapped.

So, access to a random source while time is frozen, means in practice that order-free non-determinism can not be preserved. A probability distribution can not be used to determine `f` uniquely. One could stop here and think there is no more interesting stuff about this case.

Yet, this is actually where it starts to get interesting. For the sake of the argument, let us say that any rational probability swap is prohibited. This has no implications for practical order-free non-determinism, but it sheds some light on the consistency of mathematics vs physics.

When rational probability swaps are prohibited, it means that the only swaps that are permitted are those with irrational probabilities.

If only irrational swaps are permitted at frozen time, then constructing `f’` becomes undecidable, because one would have to flip the coin an infinite number of times to get an irrational number.

However, since time is frozen…
… why is flipping a coin an infinite number times impossible?

This is a time paradox, because the decidability of `f’` depends on the interpretation of time:

- Either flipping the coin an infinite number times is impossible, or
- … flipping the coin an infinite number of times is possible,
  but not within a finite amount of time, or
- … flipping the coin an infinite number of times is possible

A frozen moment of time is a finite amount of time. So, in the second case, it would imply that flipping the coin an infinite number of time is impossible. The second case implies the first for frozen time.

In the physical world, the coin flip must be done by the distant observer watching the traveler moving at a speed close to the speed of light. Observing something frozen in time does not freeze your own actions. So, naturally, a coin flip is possible. On the other hand, the traveler can not flip coins and communicate the results to the distant observer, because this information would also appear frozen in time. This means that the distant observer must be the one flipping the coin. However, due to causality, the distant observer can only flip the coin a finite number of times. During the time it takes to flip the coin, the traveler will have moved outside of sight and there is no signal the distant observer can use to reach the traveler.

A distant observer can observe someone frozen in time, but the observer itself can not go back in time. This is how the time paradox is resolved in the physical world, which corresponds to the second case.

Notice that while the traveler moves of out sight, the distant observer can still flip the coin and make a swap afterwards. Two distant observers can then prove that order-free non-determinism is not valid. However, they can not make irrational swaps and produce the results in time for anyone else. The audience would have to wait, forever, because it takes an infinite amount of time to produce an event of irrational probability.