Natural Loneliness

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In this paper I present a formal definition of "loneliness" for natural numbers.

In mathematical research on number theory, such as prime gap[1], there is an invested interest in proofs of gaps of some kind. Mathematicians continue working on these problems full time, in particular focusing on prime numbers. Given a such large amount of effort into this direction, it might be worth taking a step back and asking whether "gaps" in general are well understood. Perhaps examining the general nature of these problems could reveal new insights usable in proofs.

What is a "gap" in general for natural numbers? The word "gap" often implies some sort of one-dimensional continuous absence of some property. However, this can be misleading because the utility of such gaps in mathematics are in relation to particular objects, e.g. natural numbers. I believe a better word to describe this mathematical nature in general is "loneliness". Loneliness provides an intuition of how gaps work, without the restriction to a particular form of dimensionality.

However, a such definition of loneliness for natural number must be *precise*, due to research purposes, but also *simple*: There is little use of a definition if nobody can wrap their head around it. Furthermore, this definition should also be *efficient*. Number search using computer algorithms should have as low complexity as possible, assuming the generality nature of the research interest.

I suggest the following definition for "natural loneliness" that is *precise*, *simple* and *efficient*:

(f, g, h) Natural loneliness is a triple of two functions \`f, g\` and a list \`h\`

$$\forall\ n\ \{\ h[n] = \min\ x\ \{\ g(x) \land \forall\ i\ [0, n)\ \{\ \neg g(f(i, x))\ \}\ \}\ \}$$

$$f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$$
$$g : \mathbb{N} \to \mathbb{B}$$
$$h : [\mathbb{N}]$$

The first parameter of \`f\` is used as a continuous of absence of some property \`g\`.
This is the "gap" that is studied for primes in number theory.

The function \`g\` defines the property of the set of numbers of interest, e.g. \`prime\`.

The list \`h\` follows from the definition of \`f\` and \`g\`.
For example, in the case of prime gaps, it contains the smallest prime that has a gap of at least \`n\` places to the next prime.

Notice that such lists can be incomplete while satisfying the definition of natural loneliness.
This is intended, since incomplete knowledge is important to be reflected upon using mathematics.

One trivial way of constructing such triples is to use an empty list:

$$\forall\ f, g\ \{\ (f, g, [])\ \}$$

This encodes the intuition that in order to understand natural loneliness, one needs proofs.

Natural loneliness is a *simple* definition due to the ability to reason about the overall property of loneliness in some discrete space using a single triple.

For example, if `h(n) = x`, then it is known that all numbers smaller than `x` that is in `g` will have "neighbours" in the range `[0, n)`. It is known how to construct these neighbours by using `f`.

A lot of the complexity is ignored, by trading it for a linear search given some `x`.

This way of storing information about the discrete space gives back a lot of information compared to the simplicity of the format which stores the knowledge. The definition is *simple*, yet useful.

Natural loneliness is an *efficient* definition to the ability to reuse previous knowledge, when doing number search with computer algorithms, to extend the current state of knowledge.

The next step of searching requires only checking a single case, which is:

g(f(n, x))

If this is `true`, then one must expand the search to larger numbers than `x`.
If this is `false`, then one knows `h[n] = x` and can immediately increment `n` for the next search.

Most complexity is contained within the problem of finding the next e.g. prime.
This sort of complexity is well understood and fits current mathematical research.

This means there is very little complexity in the definition that adds inefficiency.
Therefore, this definition of natural loneliness is *efficient*.

A simple example of natural loneliness, is that `2` is the only even prime, encoded as the triple:

(\(n, x) = x + 1 + 2 · n, prime, [2, 3, 3, 3, 3, …])

Since `2` is the first prime, it is the first number in the list of range `[0, 0)`.

When searching for the next number, one checks `f(0, 2) = 3`, which is a prime.
This means `2` is not lonely for the range `[0, 1)`.
A higher prime number than `2` is needed.
For any `n`, prime(f(n, 3)) = false`.
This means that the list will be infinite, but repeat `3` forever.

Notice how there is no explicit mentioning of "even" or "odd" primes.
However, one can prove from the definition of `f` that no even number greater than `3` is a prime.
Since the next even number after `3` is also the next even number after `2`,
this implies that no even number greater than `2` is a prime.

The triple encodes knowledge about "loneliness" of even primes, even if this knowledge is not directly expressed, it can be proved with relatively few extra steps.

Both the prime gap and even primes can be encoded using such triples.

It also holds for more complex cases, such as flipping a bit `n` in the binary representation.
For example, in this case, `h(60) = 3181`. This means all primes below `3181` have a neighbour prime by flipping one of the first 60 bits. In this sense, `3181` is the most lonely prime up to 60 bits.

References:

[1] “Prime gap”
Wikipedia
https://en.wikipedia.org/wiki/Prime_gap