Glubical Permutations

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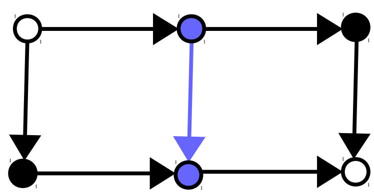
In this paper I introduce a generalization of permutations modeled by gluing n-cubes together.

In the paper "Permutations as Cubical Paths"[1], I showed that permutations can be modeled using paths on n-cubes. I would like to generalize permutations using this model. The name "glubical" comes from the words "glue" and "cubical".

Consider a glue operation where one side of an n-cube is glued together with the side of another n-cube such that the overlapping arrows match and all paths along the arrows lead to a unique vertex, plus the same condition holds when reversing all arrows (there is always one source and one target vertex):



Glubical permutations is about the paths from the source to target vertex in these glued objects.
It turns out that many of these objects can be compactly represented using generalized permutations.

For example, in the graph above, the glue operation corresponds to a glue of permutations:

∴	x, x, y	x, y, x	y, x x	glued 2-cube $(x_0, y_0), (x_1, y_1) \Rightarrow (x, x, y)$
∵	x, y y, x			2-cube

Any glubical permutation is modeled by the *union* of one or more data structures of following kind:

- One or more multiset permutation cycles containing at least two different expressions each
- A lowest path
- A highest path separated from the lowest path everywhere except singularities (corner vertices)

Glue operations that uses a corner vertex is the same as creating two permutation cycles[2]. A multiset permutation[3] means that one expression can occur more than once. Each cycle must contain at least two different expressions and lowest and highest path must be separated such that every square commutes[4]. With other words, no two adjecent vertices are singularities (every line has a quad loop).

The lowest or highest path is said to have some offset when the offset can be non-zero.
With other words, some offset can include zero, but non-offset is only zero. For example:

x, x, x, y, y	has zero offset as the lowest path.
y, y, x, x, x	has zero offset as the highest path.
x, x, y, x, y	has non-zero offset as the lowest path

Any glue operation can be thought of as taking the union of glubical permutations through each vertex in the glued face. When this union is reducible to a single multiset permutation cycle, with some offsets in lowest and highest path, it is called a *base*.

All base glubical permutations are *prime*. Prime means it can not be transformed into a single permutation product with two or more permutation cycles. Prime glubical permutations serve a similar role as prime numbers in number theory. However, there are some glubical permutations that are prime, but not a base.

For example:

x, y, x, y   x, y, y, x   y, x, x, y   y, x, y, x

This can be written:

(x, y), (x, y)

Which are two permutation cycles, hence it is not a prime. By adding the following sort:

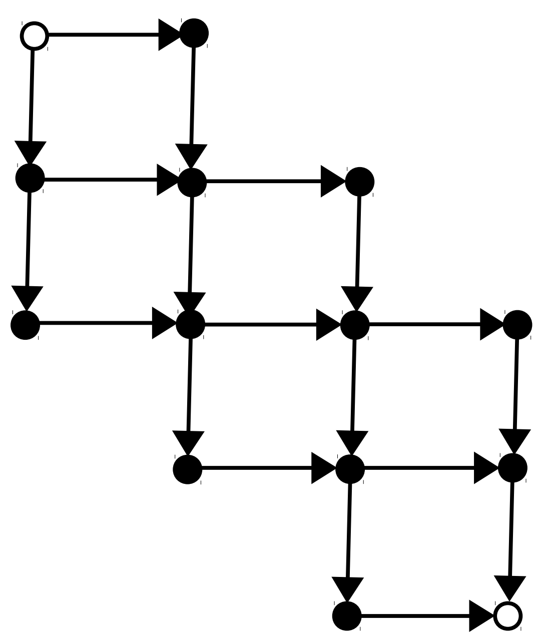
x, x, y, y

This becomes a base, which therefore is also a prime.

When a prime glubical permutation is not a base and not gluable in a single operation using at least one prime that is not a base, it is *special*. All special glubical permutations must be expressed using a union.

For example, the following is a special glubical permutation:

x, y, x, y, x, y, y
x, y, x, y, y, x, y
x, y, x, y, y, y, x
x, y, y, x, x, y, y
x, y, y, x, y, x, y
x, y, y, x, y, y, x
x, y, y, y, x, x, y
x, y, y, y, x, y, x
x, y, y, y, y, x, x
-y, x, x, x, y, y, y
y, x, x, y, x, y, y
y, x, x, y, y, x, y
y, x, x, y, y, y, x
y, x, y, x, x, y, y
y, x, y, x, y, x, y
y, x, y, x, y, y, x
y, x, y, y, x, x, y
y, x, y, y, x, y, x
-y, x, y, y, y, x, x
y, y, x, x, x, y, y
y, y, x, x, y, x, y
y, y, x, x, y, y, x
y, y, x, y, x, x, y
y, y, x, y, x, y, x



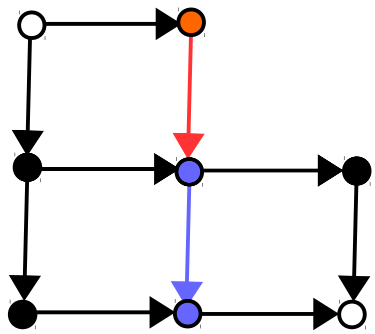
The permutations shown in red are those sorts preventing it from being a base.

The glue operation is *covered* when at least one, of the two sides of the faces being glued together, contains all vertices of the side being covered. The glue operation is *symmetric* when the glued vertices are covered in both sides. The glue operation is *asymmetric* when at least one side is uncovered.

symmetric => covered	Symmetric implies covered
¬covered => ¬symmetric	Uncovered implies asymmetric

It is sufficient to say that the glue operation is uncovered to derive that it is asymmetric.

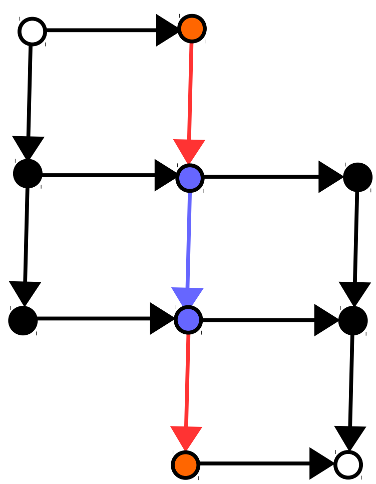
Here is an example of a covered asymmetric glue operation:



x, y, x, y
x, y, y, x
y, x, x, y
y, x, y, x
y, y, x, x

A covered glue operation on n-cubes always includes the target vertex of one of the n-cubes. This property does not hold in general for glubical permutations.

Here is an example of an uncovered asymmetric glue operation, which produces a base:



x, y, x, y, y
x, y, y, x, y
x, y, y, y, x
y, x, x, y, y
y, x, y, x, y
y, x, y, y, x
y, y, x, x, y
y, y, x, y, x

A glubical permutation is said to be covered/uncovered/symmetric/asymmetric when it is a member of the set closed by the corresponding glue operations on 2-cubes, relative to some initial set.

When gluing all vertices together, one is getting the same object out as the arguments. This might be thought of as a form of abstract idempotency[5] `glue_vertices(n : nat){eq}(x : [vertex_len] n, _) = \x`. This form of abstract idempotency depends on the total number of vertices.

References:

[1] "Permutations as Cubical Paths"
Sven Nilsen, 2020
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[2] "Cycle notation"
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[3] "Permutations of multisets"
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[4] "Commutative diagram"
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[5] "Abstract Idempotent Functions"
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