Dual Number Monotonic Density

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In this paper I represent a technique to construct monotonic functions that supports both discrete and continuous sets by using a dual number for describing linear monotonic density of ranges.

A monotonic function has the property:

$$x >= y \qquad => \qquad f(x) >= f(y)$$

These functions are closely related to efficient representations of sets.

For example if you have a list of natural numbers, you can sort it to get a monotonic function.
This makes it possible to use binary search to determine whether the list contains a natural number.

To store large sets of natural numbers efficiently in memory, one can use a list of ranges:

$$[\ [a_0, b_0),\ [a_1, b_1),\ \dots\ ]$$

However, this assumes that most natural numbers in the set are separated by `1`.
For a set where most natural numbers are separated by `2` or more, this is less efficient.

One could easily extend ranges with a custom step value `$k_i$`:

$$[\ [a_0, k_0, b_0),\ [a_1, k_1, b_1),\ \dots\ ]$$

For continuous sets of real numbers, there is no analogue of extending ranges this way.

However, it turns out that there exists a natural generalization of continuous sets: A monotonic density!

The monotonic density for each range is a dual number:

$$k_i := x_i + y_i\ \varepsilon$$

x_i : real
y_i : real
$\varepsilon^2 = 0$
$(x_i = 0) \veebar (y_i = 0)$

The monotonic density measures the discrete/continuous increase of each range.
There must be a discrete increase or a continuous increase, but not both at the same time.

- When `$x_i = 0$`, the range increases continuously (a "solid" line)
- When `$y_i = 0$`, the range increases discretely (a "dotted" line)

Hence, such monotonic functions supports both discrete and continuous sets.

The problem is: How does one constructs such monotonic functions?

The intuition of such constructions is that if one has two such monotonic functions, one can construct a new one by taking the union of these “sets” and “sorting” them over the monotonic density.

For non-overlapping ranges this is trivial, since the monotonic density is preserved.

For overlapping continuous increase, one computes new continuous increase with this formula:

$y_i' := 1 / (ba_i / y_{iA} + ba_i / y_{iB})$

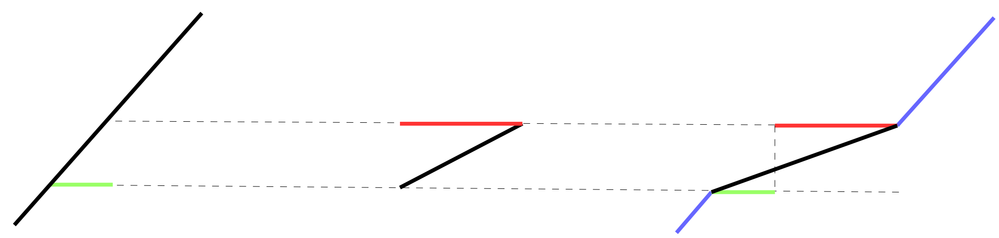
$ba_i := \min(b_{iA}, b_{iB}) – \max(a_{iA}, a_{iB})$
$\max(a_{iA}, a_{iB}) < \min(b_{iA}, b_{iB})$

For example:

[[0, 0 + 1 ε, 10)]	Input A
[[2, 0 + 0.5 ε, 5)]	Input B

[[0, 0 + 1 ε, 2), [2, 0 + 1/9 ε, 5), [5, 0 + 1 ε, 10)]	Sorted Output

Illustrated:



Input A   Input B   Sorted Output

For overlapping discrete increase, one uses a similar algorithm as for natural numbers. If the ranges overlaps perfectly this is trivial. For other cases it can get much more complicated. The difficulty of doing this correctly can be avoided by using only discrete increases of `1`. One can also ignore discrete steps entirely by using empty ranges `[a, 0 + 1 ε, a)` for points.

A discrete increase overlapping with continuous increase results in the same continuous increase. The intuition is that a discrete density can be thought of points. When you add a point to a line, the length the line does not increase, because the point is infinitely small.

Overview	**Discrete**	**Continuous**
Discrete	Same as for natural numbers	Continuous (unchanged)
Continuous	Continuous (unchanged)	Formula for continuous increase