Cartesian Outer Product of Adversarial Path of List

by Sven Nilsen, 2018

In this paper I show that a Cartesian product of two lists of choices, assuming that a choice leads to a new list of choices, is equivalent to a list of the Cartesian outer product of the two lists of choices. This is based on ideas from a discussion with Adam Nemecek.

Under the assumption that a choice from a list leads to a list of choices as the new resource:

$$[T] \sim 1 : [T]$$

A Cartesian product of two lists of choices is equivalent to a list of Cartesian product of choices:

$$([A], [B]) \Leftrightarrow [(A_0, B_0), (A_0, B_1), \ldots, (A_1, B_0), (A_1, B_1), \ldots]$$

Proof:

$$\begin{aligned}
&(A, B) \sim 1 : [(A, B)] \\
&(A \sim 1, B \sim 1) : [(A, B)] \\
&(A \sim 1 : [A], B \sim 1 : [B]) : [(A, B)] \\
&(A \sim 1, B \sim 1) : ([A], [B]) \wedge [(A, B)] \\
&(A, B) \sim 1 : ([A], [B]) \wedge [(A, B)] \\
&([A], [B]) \Leftrightarrow [(A, B)]
\end{aligned}$$

The most natural way to perform this map is take the Cartesian outer product, since the property:

$$\begin{aligned}
&|([A], [B])| = |[(A, B)]| \\
&|[A], [B]| = |[(A, B)]| \\
&|[A]| \cdot |[B]| = |[(A, B)]|
\end{aligned}$$

This property holds under the Cartesian outer product.

Since making a choice of a Cartesian product takes `nat × nat` while a list takes `nat`:

$$\frac{([A], [B]) \Leftrightarrow [(A, B)] \qquad \wedge \qquad [(A, B)] \sim 0 : nat \to (A, B) \sim 1}{nat \times nat \to (A \sim 1, B \sim 1) \quad \Leftrightarrow \quad nat \to (A, B) \sim 1}$$

It means there exists a map from `nat × nat` to `nat` such that the semantics of choice is preserved.

With other words, a new resource of a Cartesian product of lists of choices is decided from some decision theory producing `nat`, which is sufficient.