Questioning the Notion of a Set

by Sven Nilsen, 2020

A set is an object that contains a collection of other objects. It is fundamental to Set Theory, which is used as foundation of classical mathematics. Questioning the notion of a set means also questioning the language of classical mathematics. Since classical mathematics is widely successful, it is reasonable to be critical toward questioning of it. Yet, constructive mathematics was a successful result of questioning the language of classical mathematics. The missing piece seems to be questioning the notion of a set itself, since the language of classical mathematics is derived from it.

The basic problem can be reduced to two types of functions:

nat → bool	Everything seems to work in both classical and constructive mathematics
real → bool	Classical mathematics deviates in a way that seems “too idealistic”

Construction of functions in computer science seems to strictly follow a semantics of constructive mathematics. No matter what programmers do, there seem to be no way around it. Hence, Set Theory has contributed on the side of `nat → bool` while leading to some confusion about `real → bool`. What is the problem with sets of real numbers?

The problem is that no language implemented on a computer can model every set of real numbers.

This has a very simple proof:

$$|real| > |nat|$$

It is impossible to model every set of real numbers on a computer because computer memory consists of bits, which are isomorphic to natural numbers. Since there are more real numbers than natural numbers, even if the computer has infinite memory, no language on a computer can model every set.

To talk about a set is to pretend that every model of it exists. Yet, there exists a proof showing that sets of real numbers can not be modeled using a computer. Therefore, a computer can only talk about sets of real numbers by *pretending* they exist.

If one wanted to develop a logic that accurately describes what a computer can do semantically, it is necessary to deviate from the notion of a set. Hence, questioning the notion of a set is useful.

It is possible to model every set of real numbers on a computer given a procedure that enumerates all sets. One can simply use a single bit which means “the last set from the enumeration procedure” when set to `1` and “the empty set” when set to `0`. However, there are proofs that a such procedure does not exist. Therefore, it is not possible for a computer to model every set of real numbers by “cheating”.

The only way one can consistently talk about sets of real numbers is as a type which is not fully inhabited. This type is meant to approximate the semantics of real numbers. However, an alternative is to talk inconsistently about sets of real numbers in a way that satisfies constructive mathematics.