Modal Logic of Observations

by Sven Nilsen, 2018

In this paper I introduce a modal logic of observations, that despite its simplicity, grounded in pure logic and few assumptions, is surprisingly powerful across ambiguous definitions of observations.

Assume you have made some observations, and you make some conclusions from the observations.

$\square x$ `x` is observed
$\diamond x$ `x` is implied from some observed fact

The main reason to use model logic in this case is to express statements in more concise form:

$\diamond x := \exists\ y : \square\ \{\ y \rightarrow x\ \}$ There exists an observation that implies `x`

$\square x = \square\square x$ Observing `x` equals observing that `x` is observed

Two facts that implies each other are equal and implies observation from each other's observation:

$(x \rightarrow y) \wedge (y \rightarrow x) <=> (x = y) <=> (\square x \rightarrow \square y) \wedge (\square y \rightarrow \square x)$ Derived by substitution

$\square x \wedge (x \rightarrow y) \rightarrow \square x \wedge \diamond y$ `y` is implied from observing `x`

$\square x \wedge (x = y) \rightarrow \square\square x \wedge (\square x \rightarrow \square y) \rightarrow \square x \wedge \diamond\square y$ observing `y` is implied from observing `x`

The statement `◇□y` is stronger than the statement `◇y`. It can be interpreted as "expect to see" `y`.

For example:

"a tall man" = "a high human male"

□"a tall man"
□"a tall man" ∧ ◇□"a high human male"

If you see a tall man, then you expect to see a high human male, and vice versa.

However, if you have the following:

"2" → "square root of 2 is irrational"

□"2"
□"2" ∧ ◇"square root of 2 is irrational"

If you see "2", then mathematics implies its square root is irrational, but you can't **see** that the root is irrational by looking at the number "2". In order to do that, you need to see a proof that the root of 2 is irrational, because that's equivalent to "square root of 2 is irrational".

The point of this is to have a simple *working definition* of the logic of observations, one that can be justified and reasoned about from within logic, and that expects as many observations as possible.

Here is another example:

“2” = “1 + 1”

□”2”	I see `2`
□”2” ∧ ◇□”1+1”	I see `2` and expect to see `1+1`

This might not seem quite right at first glance. What if the math were more complicated?

Or, if you do not know how to add numbers, then in principle you can’t expect to see `1+1` from `2`?

The problem is the following:

1. You can’t infer that `2` implies `1+1` without enough mathematical training
2. You can’t infer that `1+1` implies `2` without enough mathematical training

When you have enough mathematical training learn these two things, then it is easy to see that `1+1` equals `2`, and from this you can expect to see `1+1` when you see `2` and vice versa. It does not mean that you see it, but that you can see it if you look for it.

Which is **exactly** what I expressed earlier, but in logical form:

$$(x \to y) \land (y \to x) \Leftrightarrow (x = y) \Leftrightarrow (\Box x \to \Box y) \land (\Box y \to \Box x)$$

The key to understand this comes from proof semantics: The operation `→` is a very simple proof, which can be derived from memorizing other proofs. So, in order to know that two things are equal, you already need the training, resources, abilities etc. to realize it is true. Which again, gives you the ability to make such observations.

Modal logic of observations implies that there is some underlying structure how agents infer that one fact from another. However, the concrete structure for how this works, can simply be ignored. This structure only matters if there are deviations from the logic, such as customized constraints etc.

With general theorem proving abilities, it is possible to derive a such relationship given enough time, which might be the upper end of the complexity scale here. Theorem proving gives `→` a more *literal* interpretation, in which case the modal logic also hold *literally*.

In informal settings, the operation `→` is given an additional semantics that there exists some “easy association” available. When `→` is given an *informal* semantics, then the modal logic also holds *informally*.

For reasoning about observations in general, it is not necessary to know whether the semantics holds literally or informally, because that depends on individual examples. This makes the model logic surprisingly powerful, despite its simplicity, grounded in pure logic and few assumptions.