Non-Trivial Commutative Symmetry

by Sven Nilsen, 2020

In this paper I introduce non-trivial commutative symmetry.

Commutativity and anti-commutativity are important mathematical properties of binary operators.
However, from the perspective of path semantics, these two properties can be treated as one property:

$$\forall\ a, b\ \{\ f(a, b) = g(f(b, a))\ \} \qquad \wedge \qquad \exists f <=> \forall g$$

In path semantical notation:

$$f <=> f[swap \rightarrow g] \qquad \wedge \qquad \exists f <=> \forall g$$

This generalized property of commutativity is called “non-trivial commutative symmetry”,
or just “commutative symmetry” for a short version.

The motivation for this is to prove properties that are more generic.

The condition `∃f <=> ∀g` is weaker than `f` having an identity element, but serves a similar role.

Strictly said, `∃f <=> ∀g` is implied by `∀ a, b { f(a, b) = g(f(b, a)) }`,
because for every output of `f(a, b)`, there must be an output of `g` which gets mapped from
`∀g` which comes from `f(b, a)`. For every output of `f(a, b)` there is an output of `f(b, a)`,
which is a tautology when `a` and `b` are enumerated from the same type. Therefore, `∃f <=> ∀g`.

However, since `∃f <=> ∀g` is not easy to see, it is defined explicitly to be used in theorem proving.

One can use “commutative symmetry” to refer to “non-trivial commutative symmetry”.
The reason for this is that it is closer to the standard usage of commutativity and anti-commutativity.

There is a “trivial commutative symmetry” which can be added, which allows stronger proofs:

$$f[g \times g \rightarrow id] <=> f$$

However, trivial commutative symmetry is not necessary for generalized commutativity.

When trivial commutative symmetry is added, one uses “full commutative symmetry”.
A consequence of full commutative symmetry is a “commutative symmetric path”:

$$f[swap \rightarrow id] <=> f[g]$$

Beware commutative symmetric paths that do not imply full commutative symmetry.