Continuous Lattice Functions

by Sven Nilsen, 2020

In this paper I generalize discrete lattice functions to the continuum and discuss its ambiguity set, which models The Continuum Hyptohesis, resulting in its truth value being `true`, given a wildcard set, or `undecidable` if a wildcard set can not be constructed.

A discrete lattice function[1] in boolean representation can be thought of as a function:

$$f : \text{nat} \to \text{bool}$$

Likewise, a continuous lattice function in boolean representation can be thought of as a function:

$$f : \text{real} \to \text{bool}$$

The sum in the monotonic[2] representation is replaced by an integral[3]:

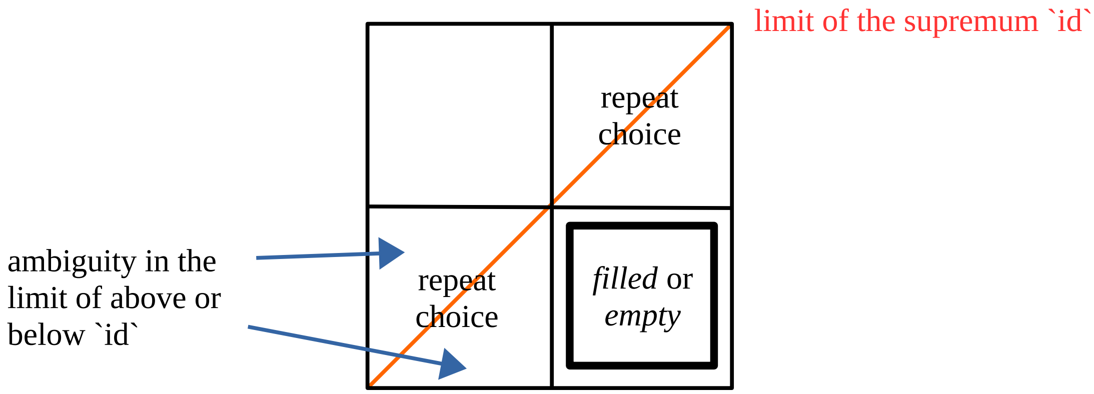
$$g(n) = \int x\ [0, n)\ \{\ f(x) \cdot dx\ \}$$

The monotonic representation has the property that the derivative[4] is in the unit interval `[0, 1]`:

$$\forall\ x\ \{\ 0 <= g'(x) <= 1\ \} \qquad <=> \qquad \forall\ x\ \{\ g'(x) : \text{prob}\ \}$$

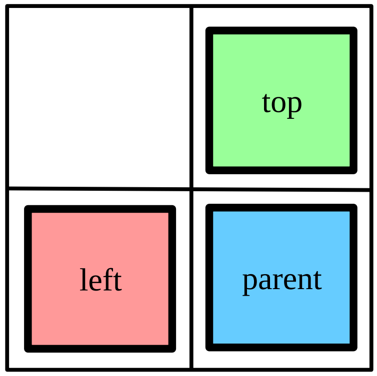
The supremum and infimums[5] are same as for discrete lattice functions (except for the type).

A simplified analytic model of the monotonic representation is to make recursive choices of the following form, under the constraints that no holes or valleys exists to the right or down:



In the limit, there is an ambiguity of this construction of whether the repeated choice should be defined as filled or as empty. In the supremum, one can think about this ambiguity as a set of choices `fill · empty` and `empty · fill`. No more than two *fill* or two *empty* can occur in order. At every point in the real line, the function analogue is identical to `id` because it intersects the points that lies at the `id` line for every possible input. Hence, the ambiguity set is undetectable seen from functional extensionality[6]. With other words, the ambiguity set *does not matter* in applied mathematics.

An interesting thing about this ambiguity set is the problem of determining its size (cardinality). One approach to determine the size of the ambiguity set is to think about the repeated choices as branches of a binary tree[7] relative to the *fill* or *empty* choice:



Every such tree of infinite depth represents a continuous lattice function, because it is impossible to construct a function that is above `id` and also any function that contains holes or valleys to the right or down. This is because every node in the tree has a parent.

One can derive from this that the size of the infinite binary tree at supremum is:

$$2^{\infty} - 1 = \sum x\ [0, \infty)\ \{\ x\ \}$$

Where `∞` the cardinality of the set of natural numbers.

In the ambiguity set itself, there is a supremum (all temporarily above `id`) and infimum (all temporarily below `id`), because it is a lattice too.

For the sequence `$2^n$`:

$$2^{n+1} = \sum x\ [0, n)\ \{\ x\ \} + 1$$

This is the size of the ambiguity set (`n = ∞`) under the interpretation that is temporarily is above `id`. This would mean that its cardinality is greater or equal than the cardinality of the continuum.

However, under the interpretation that the ambiguity set is below `id`, its size (`n = ∞`) would be:

$$2^{n-1} = \sum x\ [0, n-1)\ \{\ x\ \} + 1$$

In this case, the ambiguity set below `id` can be thought of as leafs in the binary tree.

The ambiguity set therefore models The Continuum Hypothesis[8]. In one possible interpretation, there exists no set whose cardinality that is strictly between that of the integers and real numbers. In another possible interpretation, there exist a such set. However, there also exists a continuum of interpretations between the two!

Here is the problem: In the supremum and infimum of the ambiguity set, there exists an algorithm running on a hypercomputer[9], a Turing machine extended with computation of limits, to assign every choice of `fill · empty` or `empty · fill` to two real numbers separated by a hyperreal infintesimal[10].

The two real numbers traces the path through the binary tree to the spot where the member of the ambiguity set is located (all nodes are left or top, but the ambiguity member is at the upper left corner).

This is **not** the case for the continuum of interpretations in general, because if the size could be determined for each possible interpretation, then a **Higher Order Continuum Hypothesis**, where the truth value of The Continuum Hypothesis is a function of interpretation, would be `true` if and only if a such function can be constructed.

For every interpretation of the ambiguity set, the truth value of the Continuum Hypothesis would be uniquely determined by some proof. This holds at the supremum and infimum of the ambiguity set, because one can think about it as adding axioms to ZFC[11] that the Continuum Hypothesis is `false` (above `id`) or `true` (below `id`).

However, if a such axiom could be added to ZFC for every interpretation in the continuum between the supremum and infimum of the ambiguity set, then there would exist an algorithm assinging every real number a natural number.

Direct proof: Adding an axiom to ZFC is a discrete event, but the interpretations of the ambiguity set is a continuum. Using Cantor's diagonal argument[12], one prove that assigning every real number a natural number is impossible.

Proof by contradiction: Assume that it worked anyway. For all interpretations, the truth value of The Continuum Hypothesis would be either `true` or `false`, hence ZFC should be strong enough to derive the meaning of The Continuum Hypothesis itself as a function of the ambiguity set interpretation. Quantifying over all cases, the truth value of The Continuum Hypothesis could be determined. The only valid semantics of this is that The Higher Order Continuum Hypothesis is `true`. However, there would also exist an algorithm to assign every real number a natural number, which is impossible[12].

Therefore, the truth value of The Higher Order Continuum Hypothesis is `false`,
even under the weaker assumption of constructive mathematics[13] (no axiom of excluded middle).

Wait a moment! How can one imagine an interpretation of the ambiguity set? Above `id`, there exists no set with cardinality between natural numbers and real numbers. Below `id`, there exists a such set. The existence of a such set below `id` is a constructive proof of The Continuum Hypothesis being `true` without assuming that it is being `true`! Okay! However, without excluded middle, this still does not prove that The Continuum Hypothesis is not `false`.

This is where The Higher Order Continuum Hypothesis comes to rescue: It proves that there exists no continuous homotopy path from The Continuum Hypothesis being `true` to its truth value being `false`. A homotopy path[14] exists in the sense that the ambiguity set is a lattice, but one can not determine the truth value of The Continuum Hypothesis at every point in the lattice. This property holds for any path, since the homotopy path ensures a continuous transformation into any other path[15]. This provides the equivalent of axiom of excluded middle for this special case.

Therefore, The Continuum Hypothesis is `true` within the simplified analytic model of continuous lattice functions. However, this only holds if one can construct a wildcard set[16]. A wildcard set is the worst case interpretation of the ambiguity set. If no such set exists, then The Continuum Hypothesis is undecidable[16], as when relative consistent to ZFC.

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