All Single Qubits are Constructible

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In this paper I show that all single qubits are constructible.

A quantum non-deterministic function `f` has a complex probability distribution `$\exists_{pc}f$`:

$$\exists_{pc}f : T \rightarrow \mathbb{C}$$

$$f : () \rightarrow T$$

Quantum non-deterministic functions are not constructible in general.

A qubit is a quantum non-deterministic function that returns type `$\mathbb{B}$ = bool`:

$$\text{qubit} : () \rightarrow \mathbb{B}$$

Assume that the complex probability amplitudes of a qubit is represented by `$\alpha$` and `$\beta$`:

$$(\exists_{pc}\text{qubit})(\text{false}) = \alpha$$
$$(\exists_{pc}\text{qubit})(\text{true}) = \beta$$

Normalizing these complex probability amplitudes adds the constraint:

$$|\alpha|^2 + |\beta|^2 = 1$$

All possible partial observations using a function `g` are:

$$g \cdot \text{qubit}$$

$$g : \mathbb{B} \rightarrow \mathbb{B}$$

There are 4 functions of type `$\mathbb{B} \rightarrow \mathbb{B}$`, which yields the probabilistic existential paths:

false_1	$(\exists_p(\text{false}_1 \cdot \text{qubit}))(x : \text{bool}) = \text{if } x \{ 0 \} \text{ else } \{ 1 \}$
not	$(\exists_p(\text{not} \cdot \text{qubit}))(x : \text{bool}) = \text{if } x \{ \lvert\alpha\rvert^2 \} \text{ else } \{ \lvert\beta\rvert^2 \}$
id	$(\exists_p(\text{id} \cdot \text{qubit}))(x : \text{bool}) = \text{if } x \{ \lvert\beta\rvert^2 \} \text{ else } \{ \lvert\alpha\rvert^2 \}$
true_1	$(\exists_p(\text{true}_1 \cdot \text{qubit}))(x : \text{bool}) = \text{if } x \{ 1 \} \text{ else } \{ 0 \}$

These probabilistic existential paths also hold for a fake qubit function:

$$\text{fake\_qubit}() = \text{random}() <= |\beta|^2$$

Since every partial observation of the single qubit agrees with the fake qubit,
by Leibniz' law a single qubit is indiscernible from a fake qubit, hence logically equivalent.
Therefore, all single qubits are constructible (but this does not hold for multiple qubits).