Normal Paths in Category Theory

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In this paper I explain how to use normal paths in Category Theory.

A Category is a very useful concept in mathematics that generalizes collections of objects to structures and operations on structures. Categories are often used abstractly in combination with functors, a higher level concept that maps structures from one category to another. The next level up is natural transformations, which corresponds to normal paths in Path Semantics.

One can use the notation from Path Semantics to reason about natural transformations in Category Theory. The motivation for doing so is to leverage the point-free notation for theorem proving.

Assume that one has some category `𝒞`. This category exists abstractly, as some kind of Platonic idealized form of the objects represented in some data structures and some operations. Instead of naming every type and operation directly, we can use the category as an “abstract anchor”. The corresponding “abstract rope” is a functor, which gives us a way to represent the category `𝒞`, or as a shadow projected down from higher dimensions.

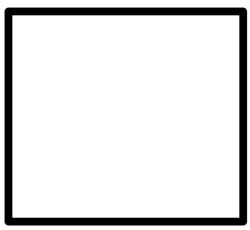
Now, assume there are two functors `F : 𝒞 → Type` and `G : 𝒞 → Type`. These two functors represent two different ways of implementing the category `𝒞` as a computer programs.

For any two objects `A, B` in `𝒞`, one gets 4 different types:

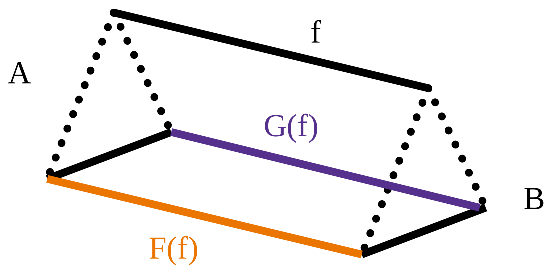
F(A) : Type   G(A) : Type

F(B) : Type   G(B) : Type

Think about these 4 types as 4 corners of a rectangle:



For any morphism `f : A → B` in `𝒞`, one can think about the rectangle above as a “shadow”:



This means that `f` also exists abstractly in `𝒞`, but can be represented as `F(f)` or `G(f)`.

F(f) : F(A) → F(B)   G(f) : G(A) → G(B)

For example, if you are building a software that uses both XML and JSON, then there are two ways to represent the data. The category `$\mathcal{C}$` is an abstract idea of what the data described in these two formats actually mean. This is why I described `$\mathcal{C}$` as an "abstract anchor".

A morphism `$f : A \rightarrow B$` is like an "abstract operator" mapping an object `A` to an object `B`. When `f` is projected down by `F` and `G`, it becomes two concrete operators `F(f)` and `G(f)`.

In Path Semantics, one can write a normal path from `F(f)` to `G(f)` this way:

$$F(f)[g_1 \rightarrow g_2] <=> G(f)$$

Here, `$g_1$` and `$g_2$` define input and output properties of `F(f)` we want to predict with `G(f)`.

Now, the trick is to replace `$g_1$` and `$g_2$` with a natural transformation `$n : F \rightarrow G$` on `A` and `B`:

$$F(f)[n(A) \rightarrow n(B)] <=> G(f)$$

Where:

$$n(A) : F(A) \rightarrow G(A) \qquad\qquad n(B) : F(B) \rightarrow G(B)$$

Using symmetric notation for normal paths, one can write:

$$F(f)[n] <=> G(f)$$

Notice that this does not imply that `n` is a simple map, but it is parameterized by `A` and `B`.

The normal path above only holds for `f` locally. If it holds globally for any `f`, one can write:

$$F[n] <=> G$$

This transforms the functor `F` into `G`.

Now, the notation for normal paths on functors uses global natural transformations and it is just as easy to use as normal paths for functions.

In general, one can take any two functors `$F : \mathcal{C} \rightarrow \mathcal{D}$` and `$G : \mathcal{C} \rightarrow \mathcal{E}$`,
with a global natural transformation `$n : \mathcal{D} \rightarrow \mathcal{E}$`:

$$F[n] <=> G$$

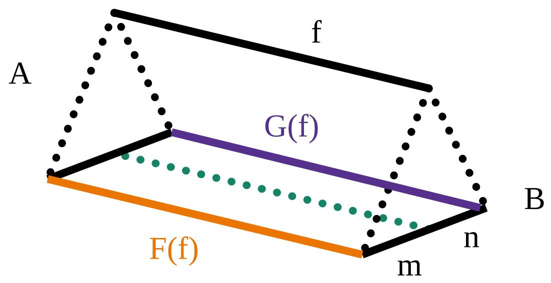
Just like with normal paths for functions, a normal path in Category Theory might be an "open box" without a corresponding "lid" and not have any solution (the solution is imaginary). So, while the normal path maps from some functor, it might not map to some "realized functor". Such normal paths can still be composed. This means one might use a weaker notion of natural transformations that composes to realizable functors.

For example, if `m` and `n` are weak global natural transformations, then `F[m]` can be imaginary:

$$F[m][n] <=> F[n \,.\, m] <=> G$$

It is sufficient that `F[n . m]` is realizable as `G`.

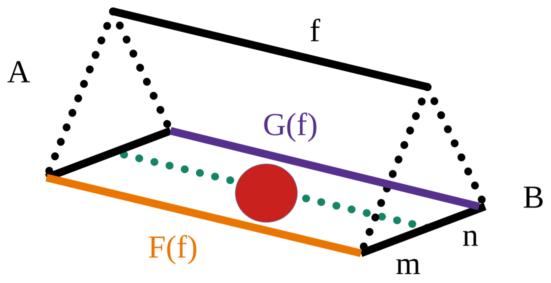
Visualizing this as the shadow of some morphism `f` being projected down from the Platonic realm:



The green line represents `F(f)[m]`, which does not correspond to any functor of `𝒞`.

Therefore, normal paths in Category Theory can take on intermediate forms which do not correspond to any “underlying” abstract form. The Platonic form is imaginary.

One easy way to think about this, is that there is some “hole” blocking the green line:



Naturally, the two projected paths `F(f)` and `G(f)` can go on both sides of the hole, but not in the middle. A path can be transformed/deformed around a hole into another path on the other side.

That’s it! Thanks for reading and I hope you learned something.