Permutation Group of Functions

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In this paper I show that the permutation group of functions satisfies the axioms of Group Theory.

A permutation `f` on a finite identity map `id := [0, n)` has the following property:

$\exists$ n : nat { f^n <=> id }

The same interpretation can be used on functions, therefore `id` corresponds to the identity function.

From this axiom alone, one can derive the axioms of Group Theory over function composition:

$f^a \cdot f^b$ <=> $f^{(a+b)\%n}$	Closure
$(f^a \cdot f^b) \cdot f^c$ <=> $f^a \cdot (f^b \cdot f^c)$	Associativity
f^n	Identity element
$f^0 \cdot f^0$ <=> f^0	Inverse element for `n = 0`
$f^1 \cdot f^1$ <=> f^1	Inverse element for `n = 1`
$f \cdot f^{n-1}$ <=> $f^{n-1} \cdot f$ <=> f^n	Inverse element for `n > 1`

Some consequences:

(n == 0) == (f <=> ())	The unit element `()`
(f^1 <=> id) <=> (f <=> id)	The `id` function
(f^2 <=> id) <=> (f^{-1} <=> f)	Self inverse
(f^n <=> id) <=> (f^{-1} <=> f^{n-1})	General inverse when `n > 1`

Any constant can be thought of as constructed by a function of type `() → T`.
Since `$id_0$ : () → ()` for `()`, it means that `$f^0$ <=> id` is logically equivalent to `f <=> ()`.

Applying the general inverse when `n = 1` would lead to an undefined case:

id^{-1} <=> id^0

Luckily, since `$id^2$ <=> id`, it also satisfies its own self inverse, which implies `$f^0$ <=> id` for all `f`:

∵ (id^2 <=> id) => (id^{-1} <=> id)
∴ $\forall$ f { f^0 <=> id }