Undecidable Infinitesimals

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In this paper I represent a generalization of propositional logic to boolean functions of real numbers.

A proof `f` in propositional logic returns `true` for all inputs:

    f <=> \true

$$f : \mathrm{bool}^N \to \mathrm{bool}$$

One way of generalizing propositional logic is to keep the definition of the proof and change the type:

    f <=> \true

$$f : \mathrm{real} \to \mathrm{bool}$$

However, in practice this leads to some problems. Recursive sets on the real numbers might converge constructively at infinite depth to a single point, with a consequence that it is impossible to determine whether the point should belong to the set or not. An additional definition is required at the point.

A different generalization of propositional logic is to allow proofs that have undefined infinitesimals.

An undecidable infinitesimal is a number for which the set returns both `true` and `false`.

For example, a well known result in mathematics is that:

$$1 = 0.9999999999...$$

A set of real numbers, determined by an algorithm, might decide that `1` belongs to the set while `0.9999999999...` does not. Although these two descriptions refer to the same number, it is possible for an algorithm to distinguish between them.

Using the semantics of dual numbers, the two numbers are separated by an infinitesimal, such that:

$$f(x) \neg= f(x + \varepsilon) \qquad \vee \qquad f(x) \neg= f(x - \varepsilon)$$

$$f : \mathrm{real} \to \mathrm{bool}$$
$$\varepsilon : \mathrm{real}$$
$$\varepsilon^2 = 0$$

Two sets are equal if they differ, at most, anywhere with intervals smaller than `ε`. The new definition of a proof using this generalization is a change in the definition of equality between sets. The `<=>` in the expression `f <=> \true` operator has a different meaning than functional equality. Using this definition of a proof, it is possible to extract true statements which are impossible to construct in classical propositional logic.