Semantics of Points

by Sven Nilsen, 2019

In this paper I give a definition of the semantics of points, using a lifted interpretations of symbols into expressions that have sub-types. This definition does not rely on the notion of a space, as is commonly used when trying to model semantics of points. I also explain my motivation for doing so.

Somehow, everybody seem to know how to draw points to illustrate something. However, if you start from a world where there exists only symbols, it is far from trivial how to introduce the idea of points. In particular, if you are used to think about mathematics in terms of functions, the way points are used to illustrate ideas might seem magically efficient and at other times a bit confusing.

One reason path semantics[1] is an efficient language for reasoning about mathematics, is because it makes you express ideas using building blocks built up from the semantics of symbols. It is a technique used to pretend that the world is somewhat easily expressed as computer programs. This is not true, of course, but very often it turns out that something looking intrinsically specific to the kind of world we humans live inside has an analogue, reduced and projected, in the world of symbols. This is also the case for the semantics of points in a way that is not common intuitively understood.

A point is often expressed visually as a circular spot filled with a uniform color:

What is it about points that makes them a such efficient tool for communication? Is it possible to reason about points formally, without having the ability to draw pictures? This inquiry was started by Euclid[2] and an impressive amount of man-hours have gone in this direction since his time. Today, these questions are relevant for developing automated theorem provers and mathematical languages, since the insides of computers are closer to the world of symbols than the world of visualizing ideas in the form of pictures.

We use visualizations in computer graphics all the time, but this is only made possible by defining functions that translates from symbols in computer memory. I do not think of visualization as a "natural" way of computing, but a somewhat natural way of programming or reasoning about theorem proving. I consider visualization as a form of communication that is more higher level and used with more intelligence than is easily expressed in a Turing machine[3].

To solve the problem of finding a useful definition of points, one must make reasonable trade-offs. It is not good to try capturing all the details in how points are used visually. The major focus should be to reason about how points are used. A visual medium is just a specific instance of many ways to communicate using points. The terminology of points spans a wide area and also occurs frequently in natural languages[4], such as the expressions "point of time", "make a point", "data point" or "no point of return". Therefore, I chose a definition such that questions about the implied meaning of points occur naturally as mathematical problems to solve.

The usage of points in a mathematical language is to communicate abstractly that the input to some function is fixed and the function is fixed, without talking about the output that the function returns:

$f(x)$	A unique function takes a unique input	~=	“point”
$\|x\| = 1$	The input `x` is unique		
$\|f\| = 1$	The function `f` is unique		

This definition is interpreted by lifting symbols into expressions. Each expression has a sub-type. The `|<expr>|` operator measures the cardinality of the sub-type. When the cardinality of a sub-type is `1`, the meaning of the symbol is unique.

A natural interpretation of which function `f` to use, when its identity[5] is left unsaid, is the identity function `id`[6]. This makes each unique input map back to themselves. It also translates nicely into what happens when a teacher draws points on a chalkboard in front of students. The points on the chalkboard can refer to points on the chalkboard, which when interpreted with common sense corresponds to the identity function.

Although this definition seems counter-intuitive at first, it has the benefit that it captures how geometry is modeled with homotopy maps[7], as a sub-set of the total usage. Homotopy maps are nice, but feels limiting when comparing to the creative use of points in general communication.

The intention of the definition is to assist the student of path semantics into thinking about points as a very broad and general idea related to functions. One striking insight is that expressing formally that the input to a function is unique, is non-trivial under some interpretations[8], therefore using semantics of points as a tool for generating ideas about path semantics is useful.

It is implicitly understood by common sense[9] that a point can be translated anywhere. A function that takes a unique input returns a unique output, which used again can be input for another function. The mechanism of translating a point corresponds to evaluating a unique function for some unique input. A different kind of translating a point is through a continuous transformation, e.g. using a homotopy map, which I will not go into detail about here.

The fact that the output can be input for another function proves that translation of points is possible. One ends up with the same definition as for points after and before translation. This symmetry[10] corresponds precisely to the symmetry that occurs when illustrating that one point corresponds to another point under a translation. In path semantics, a such symmetry consists of things one can talk about by associating collections of symbols with the two states before and after translation.

However, not only is there a symmetry in the translation when using points or when using functions to compute. There is a symmetry between these two symmetries, as between two languages. According to path semantics, this means that some things that can be said in one language can be said in the other. Although it is not possible to list up all valid expressions associated with the abstract symbols in these two different languages, it is easy to notice that a similar symmetry occurs in use of these languages. The symmetry I refer to is the kind of Wittgenstein[11] philosophized about. This means that one can leave the vague parts of the definition unsaid, while focusing on the fact that the way path semantics is used to talk about unique inputs for unique functions, has an analogue in the communication through visual illustrations using points.

It is funny how obvious it seems to use this line of reasoning in hind sight, yet how points are commonly understood mathematically is often modeled as infinitesimal points in spaces, which is very different.

It is important for the semantics of points that a unique function is part of the picture. If one considers unique inputs by themselves as points, then the idea that a point can be translated is not obvious. The usage of unique functions for unique inputs in a mathematical language translates into the usage of points.

When drawing a point to communicate something, it is not necessary to use the point to refer to a concrete value. However, it is important that the point has a unique value. This is a subtle difference. A point refers to concrete values in general, not to any particular concrete value.

Side note: This is where the idea of contraction as a mechanism in homotopy theory came to mean the very essence of points, as the object of homotopy level `0`. Concrete values can be "contracted" into points, regardless of their inner structure. Although, I am curious about what happens when one goes from homotopy level `0` to homotopy level `1`, which is inhabited by the point and the no-point, seems to me stealing ideas from set theory that does not occur before homotopy level `2`. What I mean by "stealing" is that the language of higher homotopy levels is used to talk about the lower levels. Personally, I prefer to use a semantics grounded in the use of functions. The way I think about homotopy level `1` is as introducing a choice between two unique functions. Homotopy level `2` might be considering all outputs over choices of unique functions. This way of reasoning makes it clear that homotopy levels shifts the parts of expressing mathematical ideas through functions. It feels more natural to me, as one navigates within the same world of symbols.

It is also important that one can talk about unique inputs as a whole, as points, seen from the perspective of language design. For example, with the Cartesian product, if `A` and `B` are expressions, then `(A, B)` is an expression. Still, the Cartesian product is just a "point". It can be considered a unique input for some unique function and therefore satisfies the definition. Notice that the definition of points is used actively to prove properties of the semantics of language design. It is not a "dead" definition. One can also see hints at the connection to type theory here.

It is often more intuitive to use points to illustrate something than talking about concrete values in general. A point is more generic than a concrete value, but it is also easily grasped, as if it was a concrete object. This blurs the distinction between points and concrete values, which makes it easy to swap from one form of expression into the other.

There is a semantic problem with the common intuition of talking about points as infinitesimal objects in a space. One issue is that it connects the definition of the point with the semantics of a space. Just like I prefer to avoid "stealing" from languages of higher homotopy levels, I prefer not to depend on the definition of a space when trying to define points. A space is to me a quantification over points, so it should depend on the definition of points instead of the other way around. The very least, it should be possible to ground the semantics of points without relying on grounding the semantics of space first.

The primary idea of my definition of points, as a way of talking about a certain aspects of functions, is that the semantics of points is connected directly to the general usage of points. It is a Wittgenstein inspired idea (that builds on the notation that natural language means how it is used). However, it is also a serious issue of translating from use of points as a tool for illustration into path semantics. The common intuition of talking about points as infinitesimal objects in a space, gets in the way of translating, because it only defines what points mean in a particular model.

I do not want to translate from a particular model to path semantics, because it put limits to what it says about path semantics, which would lose some of the utility of using intuition about points to study path semantics. Instead, I want to translate from the creative use of points to the creative use of path semantics.

This is why I chose a definition that holds for functions in general. The use of points should give some insights into some use of path semantics, since when talking about how points are used, one can also talk about how functions in general are used.

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