Higher Order Operator Overloading and Self Reference

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Higher Order Operator Overloading states that when you have functions of the same input, you can overload any operator on the return values of the functions:

$$g(f_0, f_1) := \backslash(a : A) = g(f_0(a), f_1(a))$$

This makes it possible to treat functions as values you can compute with, constructing new programs.

Assume that you have the following function (the set of points on a unit circle):

$$circle := \backslash(x : real, y : real) = x^2 + y^2 <= 1$$

Here, there are two arguments, `x` and `y`.

To reference `x` and `y` one can do the following:

$$x := \backslash(x : real, y : real) = x$$

$$y := \backslash(x : real, y : real) = y$$

With Higher Order Operator Overloading, the following expression re-constructs the unit circle:

$$x^2 + y^2 <= 1 \quad <=> \quad \backslash(x : real, y : real) = x^2 + y^2 <= 1$$

What happens here is that `x` and `y` are defined as functions that refers to themselves in the context of the function argument tuple `(x : real, y : real)` that is used to construct new functions.

This technique is called “self reference” and states that:

- If we have a function for every input ...
- … and overload all operators with Higher Order Operator Overloading …
- … then the original function can be re-constructed by re-interpreting the original function

By passing `x` and `y` as self referential functions, the following is true:

$$circle(x, y) <=> circle$$

One can use this technique to prove the commutative property of the `circle` function in a new way:

$$circle(y, x) <=> circle \quad <=> \quad \forall\ x, y \{ circle(x, y) = circle(y, x) \}$$