Terminology for Binary Relations

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Antisymmetric relation

A **binary relation** `f : T × T → 𝔹` where `( f(a, b) = f(b, a) ) => (a = b)`.
In matrix form, there can be `true` in upper or lower triangle, but not both.
The diagonal might contain both `true` and `false`.

Associative

A function `f : T × T → U` is commutative when `f(f(a, b), c) = f(a, f(b, c))`.

Binary relation

A function `f : T × T → 𝔹`.
In matrix form, columns are the first argument, rows are second argument and return value is a cell.

Commutative

A function `f : T × T → U` is commutative when `f(a, b) = f(b, a)`.

Connex relation

A **binary relation** `f : T × T → 𝔹` where `f(a, b) ∨ f(b, a)`.
Implies **reflexivity**.

Equivalence relation

A **binary relation** `f : T × T → 𝔹` that is **reflexive**, **symmetric** and **transitive**.

Homogeneous relation

A function `f : T × T → 𝔹`. Also just called a **binary relation**.

Idempotency

A function `f : T × T → U` is idempotent for `x` when `f(x, x) = x`.
If `f` is thought of as a product, then `f(a, a) = a^2 = a`.
Idempotency must not be confused with **involution**.

Involution

A function `f : T → U` is involution when `∀ x { f(f(x)) = x }`.
When `g(f, f) <=> h` where `h` is an identity element of `f`, then `f` is an involution.
Involution must not be confused with **idempotency**.

Join

A join `z : T` of a **binary relation** `f : T × T → 𝔹`, also called “lowest greater bound” or “supremum” of `x : T` and `y : T`, if `f(x, z) ∧ f(y, z) ∧ ∀ w : T { ( f(z, w) ∧ f(y, w) ) => f(z, w) }`.
If there exists a join `z` for `x` and `y`, then it is unique and written `x ∨ y`.
If all pairs of `T` has a join, then the join is a binary operation `g : T × T → T` that is **commutative**, **associative** and **idempotent**. When not all pairs have a join, the join is a **partial binary operator**.

Join-semilattice order

A **binary relation** `f : T × T → 𝔹` that is **reflexive**, **transitive**, **antisymmetric** and has joins.
Implies **partial order**.
Implied by **lattice order**.

Meet

A meet `z : T` of a **binary relation** `f : T × T → 𝔹`, also called “greatest lower bound” or “infimum” of `x : T` and `y : T`, if `f(z, x) ∧ f(z, y) ∧ ∀ w : T { ( f(w, z) ∧ f(w, y) ) => f(w, z) }`.
If there exists a meet `z` for `x` and `y`, then it is unique and written `x ∧ y`.
If all pairs of `T` has a meet, then the meet is a binary operation `g : T × T → T` that is **commutative**, **associative** and **idempotent**. When not all pairs have a meet, the meet is a **partial binary operator**.

Meet-semilattice order

A **binary relation** `f : T × T → 𝔹` that is **reflexive**, **transitive**, **antisymmetric** and has meets.
Implies **partial order**.
Implied by **lattice order**.

Lattice order

A **binary relation** `f : T × T → 𝔹` that is **reflexive**, **transitive**, **antisymmetric** and has joins and meets.
Implies **partial order**, **join-semilattice order** and **meet-semilattice order**.

Partial binary operator

A binary operator `f : T × T → 𝔹` where the trivial path `∀f <¬=> \true`.

Partial order

A **binary relation** `f : T × T → 𝔹` that is **reflexive**, **antisymmetric** and **transitive**.

Preorder

A **binary relation** that is **reflexive** and **transitive**.
An **antisymmetric** preorder is a **partial order**, and a **symmetric** preorder is an **equivalence relation**.

Prewellordering

A **binary relation** `f : T × T → 𝔹` that is **connexive**, **transitive** and **wellfounded**.

Quasiorder

The same as a **preorder**.

Reflexive relation

A **binary relation** `f : T × T → 𝔹` where `f(x, x) : (= true)`.
In matrix form, all cells across the diagonal are `true`.
Implied by **connex** relations.

Semiconnex relation

A **binary relation** `f : T × T → 𝔹` where `(a ¬= b) => ( f(a, b) ∨ f(b, a) )`.

Symmetric relation

A **binary relation** `f : T × T → 𝔹` where `f(a, b) = f(b, a)`.
In matrix form, the transposed matrix is equal to itself `$M^T = M$`.

Total preorder

A **binary relation** `f : T × T → 𝔹` that is **reflexive**, **connexive** and **transitive**.

Total order

A **binary relation** `f : T × T → 𝔹` that is **antisymmetric**, **transitive** and **connexive**.
Implied by **well-order**.

Transitive relation

A **binary relation** `f : T × T → 𝔹` where `f(a, b) ∧ f(b, c) => f(a, c)`.

Well-founded relation

A **binary relation** `f : T × T → 𝔹` where `∀ S ⊆ T { (S ¬= ∅) => (∃ m : S { ∀ s : S { ¬f(s, m) } })}`.

Well-quasi-order

A **binary relation** `f : T × T → 𝔹` that is **reflexive**, **transitive** and such that any infinite sequence `$x_0$, $x_1$, $x_2$, …` from `T` contains `f($x_i$, $x_j$)` where `i < j`.
Implies **preorder**.

Well-order

A **binary relation** `f : T × T → 𝔹` that is **antisymmetric**, **transitive**, **connexive**, and **well-founded**.
Implies **total order**.