Permutative Symmetry Paths

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In this paper I introduce permutative symmetry paths, which generalizes groups similarly to sub-types.

A sub-group in Group Theory is just a group that can contains a group. However, one might want to have a theory similar to Group Theory but that is more like the theory of sub-types relative to types. This theory is about discrete symmetries in general, without constraining these symmetries to groups.

A permutative symmetry path of \`f\` by a set of lists \`$x_i$\`, is a set of permutative sorts \`$g_j$\`:

$\therefore \quad f[x_i]_s \iff g_j$

$\because \quad \forall\ i, j\ \{\ f(x_i) == f(g_j(x_i))\ \}$
$\because \quad x_i \iff \{\ x_0, x_1, x_2, \dots, x_{n-1}\ \}$
$\because \quad g_j \iff \{\ g_0, g_1, g_2, \dots, g_{m-1}\ \}$
$\because \quad x : nat \rightarrow [T]$
$\because \quad g : nat \rightarrow [T] \rightarrow [T]$
$\because \quad f : [T] \rightarrow U$

It is assumed here that \`[T]\` has a fixed length.

If \`$x_i$\` contains a single element and elements of \`$g_j$\` satisfies the axioms of Group Theory, then the permutative symmetry path is isomorphic to a group. When \`$x_i$\` contains more than one element and \`$g_j$\` satisfies the axioms of Group Theory, one can treat it as a common group structure over labels by \`f\`.

The arrow \`=>\` can be used to express that a sub-set of the possible permutative sorts \`$g_j$\` are known:

$f[x_i]_s \Rightarrow h_j$ \`$h_j$\` contains a sub-set of \`$g_j$\`

The arrow \`<=>\` is used only when \`$g_j$\` describes the largest possible set of permutative sorts.
There is always some sub-set \`=>\` which is a group, as long it is non-empty.
Every relation, e.g. \`f([a, b, c]) == f([b, c, a])\` forms a sub-group, since \`[b, c, a]\` is its own inverse.
However, there are some cases where \`<=>\` is not a group.
For example, \`f(x) = x != [b, a, c]\`. Here, \`f([a, b, c]) == f([b, c, a]) == f([a, c, b]) == false\`.
\`[b, a, c]\` is composition of permutation sorts \`[a, c, b] . [b, c, a]\` which requires closure in \`$g_j$\`.
By removing \`[b, a, c]\` from the set \`$x_i$\`, one forces \`$f[x_i]_s$\` to not be a group.

For every \`f\`, when a strict sub-set \`$g'_j$\` exists that is a group, there exists a function \`f'\` and such that:

$f'[x_i]_s \iff g'_j$

The function \`f'\` contains less symmetries than the original \`f\`.

If \`f <=> id\`, then \`$g_j$\` contains only the identity sort.

If `f` satisfied the following existential path equation and `$x_i$` is total:

f <=> \u where `u : U`
x_i <=> [T]
$f[x_i]_s$ <=> g_j

Then `$g_j$` satisfies the axioms of Group Theory.

The rest of this paper is about proving this result.

The axioms of Group Theory are:

1. Closure
2. Associativity
3. Identity element
4. Inverse element

Proof of Identity Element (3)

If `$g_j$` contains the largest possible set of permutative sorts, then it contains the identity sort.

Proof of Associativity (2)

Permutation sorts are associative.

Proof of Closure and Inverse Element (1) and (4)

Every permutation in `[T]` results in `$g_j$`, for any `$x_0$, $g_1$, $g_2$`, is covered by some `$g_3$` in `$g_j$`:

$$\begin{tikzcd} x_0 \arrow[r, "g_0"] \arrow[dr, "g_3"'] & x_1 \arrow[d, "g_1"] \\ & x_2 \end{tikzcd}$$

This is because `[T]` contains any possible `$x_i$` which all are assigned the same label `u` by `f`.

Among all those possible `$x_i$`, there exists all possible permutations of the list, since permutation is just a kind of modification. Since all possible permutations exists, the largest set of permutation sorts `$g_j$` includes all permutations. This includes inverse permutations, compositions etc.

Therefore, all axioms of Group theory are satisfied.

Q.E.D.