Liar’s Paradox and Complete Functions

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In this paper I show that Liar’s paradox can be interpreted as a member of the powerset of sentences that returns `true` or `false`. This is proved by translating Liar’s paradox to a recursive function, where the interpretation of recursiveness is changed from executing indefinitely to a function solution which constraints satisfy the recursive definition. Interpreted this way, Liar’s paradox is similar to Russel’s paradox. I also show that the combined set of deterministic and indeterministic functions is complete.

A closure can not call itself recursively without a fixed-point combinator, for example `sc` in the Alphabetic List of Functions of Standard Dictionary for Path Semantics. Instead of using a fixed-point combinator, override the interpretation of self-references and consider the following:

f := \(x : bool) = f(x)	`true` for type `bool → bool`	“this sentence is true”

When calling `f(x)` from within the closure, what we mean is that `f` is part of a family of functions, a set, which satisfies the constraint that it returns the value which is returned by `f(x)`. This property is `true` for all functions of type `bool → bool`. This corresponds to the tautology “this sentence is true”. Assuming that “this sentence is true” is true, then the sentence is true, which is correct. Assuming that “this sentence is true” is false, then the sentence is false, which is also correct. Since it works for both sentences that are false and true, it represents all sentences. However, this interpretation only works if one consider the sentence to be a reference to some member of the powerset of sentences.

The Liar’s paradox can be translated similarly:

f := \(x : bool) = ¬f(x)	`false` for type `bool → bool`	“this sentence is false”

There exists no function of type `bool → bool` which returns the inverted value of what it returns. If it returns `false`, it would have to return `true` and vice versa, which is a contradiction.

To make this clearer, one can use a second sentence to speak about the first sentence:

“1+1=2”	“the first sentence is true” is true
	“the first sentence is false” is false

Both sentences “the first sentence is true” and “the first sentence is false” can be used to talk about another sentence. Yet, the second sentense is false.

By making it clear that the sentence to be interpreted is to as if it was about another sentence, then which sentence it means depends on how one defines a set sentences making up the language of what can be said. The identity of this set requires that one can say the same things about each sentence. With other words, if the Liar’s paradox allows interpretation as if it refers to a set of sentences including itself but which set only contain sentences that are either true or false, one gets a contradiction.

This contradiction of the Liar's paradox is similar to Russell's paradox. Assume a set R which contains all sets that do not contain themselves. If R contains itself, then it should not contain itself and vice versa. This is inconsistent. However, if one assumes that a self-referential definition of a set is interpreted as if it was about a member of the powerset of all sets, then it corresponds to the empty set. With other words, Cantor's cardinality hierarchy of higher infinities applies because no matter how we define sets, there is no set in that language which is talked about through Russell's paradox. The identified set lies outside the language. Notice that this interpretation also solves Russell's paradox.

Assume functions of type `T → T`. If the first reference to itself is interpreted as its own identity, and the second reference is interpreted as the second call to itself, then the following property holds for both deterministic and indeterministic functions:

$f := \backslash(x : T) = f(x)$ `true` for both deterministic and indeterministic functions

For all deterministic functions, but for no indeterministic function, the following holds:

$f : \forall\ x \ \{ f(x) == f(x) \}$ `true` for deterministic functions, `false` for indeterministic ones

For all indeterministic functions, but for no deterministic function, the following holds:

$f : \exists\ x \ \{ f(x) \neg= f(x) \}$ `false` for deterministic functions, `true` for indeterministic ones

To talk about any function, one must say something that is either `true` or `false`. This means that the function is complete. One can show that complete functions are either deterministic or indeterministic:

$$\neg\forall\ x \ \{ f(x) == f(x) \} <=> \exists\ x \ \{ f(x) \neg= f(x) \}$$

Since the negation of the definition of deterministic functions is equivalent to the definition of indeterministic functions, the combined set of deterministic and indeterministic functions corresponds to the set of complete functions. With other words, there exists no function which can be talked about using identity as defined in path semantics, that is neither deterministic nor indeterministic.

However, one can talk about a set of functions, like in this particular interpretation of Liar's paradox.