Predicting Primality of Addition

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In this paper I show an example of probabilistic path semantics by deriving the correct algorithm for predicting primality of addition, using some upper bound.

If you add two natural numbers below to equal to some upper bound, and you know the probability of each input being a prime, what is the probability of the output being a prime?

$$\mathrm{add}\{(<= x), (<= x)\}[\mathrm{prime}]_p$$

The probabilistic path is the following:

$$\begin{aligned}
& f[g_{i \to n}]_p := \backslash(b_{i \to n} : [] \wedge [\mathrm{len}]\ |g_{i \to n}|,\ p : [\mathrm{real}] \wedge [\mathrm{len}]\ |g_i|) = \sum j\ 2\hat{}|g_i|\ \{ \\
& \qquad (\exists_p(g_n \cdot f)\{2\hat{}[g_i]\ b_i\}(\beta_j))(b_n) \cdot \prod k\ |g_i|\ \{\ \beta_{jk}((p_k\text{-}(\exists_p g_{ik})(b_k))/(1\text{-}(\exists_p g_{ik})(b_k)))\ \} \\
& \}
\end{aligned}$$

Where one substitutes these variables:

$$\begin{aligned}
& f <=> \mathrm{add}\{(<= x), (<= x)\} \\
& g_{i \to n} <=> \mathrm{prime} \times \mathrm{prime} \to \mathrm{prime} \\
& g_{i0},\ g_{i1} <=> \mathrm{prime}\{(<= x)\}
\end{aligned}$$

Inserting:

$$\begin{aligned}
& \mathrm{add}\{(<= x), (<= x)\}[\mathrm{prime}]_p := \backslash(b_{i \to n} : [] \wedge [\mathrm{len}]\ 3,\ p : [\mathrm{real}] \wedge [\mathrm{len}]\ 2) = \sum j\ 4\ \{ \\
& \qquad (\exists_p(\mathrm{prime} \cdot \mathrm{add}\{(<= x), (<= x)\})\{2\hat{}[\mathrm{prime} \times \mathrm{prime}]\ b_i\}(\beta_j))(b_n) \cdot \\
& \qquad \prod k\ 2\ \{\ \beta_{jk}((p_k\text{-}(\exists_p \mathrm{prime}\{(<= x)\})(b_k))/(1\text{-}(\exists_p \mathrm{prime}\{(<= x)\})(b_k)))\ \} \\
& \}
\end{aligned}$$

Unrolling:

$$\begin{aligned}
& \mathrm{add}\{(<= x), (<= x)\}[\mathrm{prime}]_p := \backslash(b_{i \to n} : [] \wedge [\mathrm{len}]\ 3,\ p : [\mathrm{real}] \wedge [\mathrm{len}]\ 2) = \\
& \qquad (\exists_p(\mathrm{prime} \cdot \mathrm{add}\{(<= x), (<= x)\}))(b_n) \cdot \\
& \qquad (1\text{-}(p_0\text{-}(\exists_p \mathrm{prime}\{(<= x)\})(b_0))/(1\text{-}(\exists_p \mathrm{prime}\{(<= x)\})(b_0))) \cdot \\
& \qquad (1\text{-}(p_1\text{-}(\exists_p \mathrm{prime}\{(<= x)\})(b_1))/(1\text{-}(\exists_p \mathrm{prime}\{(<= x)\})(b_1))) + \\
& \qquad (\exists_p(\mathrm{prime} \cdot \mathrm{add}\{(<= x), (<= x)\}\{[\mathrm{prime}]\ b_0,\ _\}))(b_n) \cdot \\
& \qquad (p_0\text{-}(\exists_p \mathrm{prime}\{(<= x)\})(b_0))/(1\text{-}(\exists_p \mathrm{prime}\{(<= x)\})(b_0)) \cdot \\
& \qquad (1\text{-}(p_1\text{-}(\exists_p \mathrm{prime}\{(<= x)\})(b_1))/(1\text{-}(\exists_p \mathrm{prime}\{(<= x)\})(b_1))) + \\
& \qquad (\exists_p(\mathrm{prime} \cdot \mathrm{add}\{(<= x), (<= x)\}\{_,\ [\mathrm{prime}]\ b_1\}))(b_n) \cdot \\
& \qquad (1\text{-}(p_0\text{-}(\exists_p \mathrm{prime}\{(<= x)\})(b_0))/(1\text{-}(\exists_p \mathrm{prime}\{(<= x)\})(b_0))) \cdot \\
& \qquad (p_1\text{-}(\exists_p \mathrm{prime}\{(<= x)\})(b_1))/(1\text{-}(\exists_p \mathrm{prime}\{(<= x)\})(b_1)) + \\
& \qquad (\exists_p(\mathrm{prime} \cdot \mathrm{add}\{(<= x), (<= x)\}\{[\mathrm{prime}]\ b_0,\ [\mathrm{prime}]\ b_1\}))(b_n) \cdot \\
& \qquad (p_0\text{-}(\exists_p \mathrm{prime}\{(<= x)\})(b_0))/(1\text{-}(\exists_p \mathrm{prime}\{(<= x)\})(b_0)) \cdot \\
& \qquad (p_1\text{-}(\exists_p \mathrm{prime}\{(<= x)\})(b_1))/(1\text{-}(\exists_p \mathrm{prime}\{(<= x)\})(b_1))
\end{aligned}$$

Since one can just invert the probabilities to get non-primality, one can simplify:

add{(<= x), (<= x)}[prime]$_p$([true, true, true]) := \(p : [real] ∧ [len] 2) =
($\exists_p$(prime · add{(<= x), (<= x)}))(true) ·
(1-(p_0-($\exists_p$prime{(<= x)})(true))/(1-($\exists_p$prime{(<= x)})(true))) ·
(1-(p_1-($\exists_p$prime{(<= x)})(true))/(1-($\exists_p$prime{(<= x)})(true))) +
($\exists_p$(prime · add{(<= x), (<= x)}{prime, _}))(true) ·
(p_0-($\exists_p$prime{(<= x)})(true))/(1-($\exists_p$prime{(<= x)})(true)) ·
(1-(p_1-($\exists_p$prime{(<= x)})(true))/(1-($\exists_p$prime{(<= x)})(true))) +
($\exists_p$(prime · add{(<= x), (<= x)}{_, prime}))(true) ·
(1-(p_0-($\exists_p$prime{(<= x)})(true))/(1-($\exists_p$prime{(<= x)})(true))) ·
(p_1-($\exists_p$prime{(<= x)})(true))/(1-($\exists_p$prime{(<= x)})(true)) +
($\exists_p$(prime · add{(<= x), (<= x)}{prime, prime}))(true) ·
(p_0-($\exists_p$prime{(<= x)})(true))/(1-($\exists_p$prime{(<= x)})(true)) ·
(p_1-($\exists_p$prime{(<= x)})(true))/(1-($\exists_p$prime{(<= x)})(true))

Rewriting using a prime counting function `π`:

add{(<= x), (<= x)}[prime]$_p$([true, true, true]) := \(p : [real] ∧ [len] 2) =
($\exists_p$(prime · add{(<= x), (<= x)}))(true) ·
(1-(p_0-π(x)/x)/(1-π(x)/x)) · (1-(p_1-π(x)/x)/(1-π(x)/x)) +
($\exists_p$(prime · add{(<= x), (<= x)}{prime, _}))(true) ·
(p_0-π(x)/x)/(1-π(x)/x) · (1-(p_1-π(x)/x)/(1-π(x)/x)) +
($\exists_p$(prime · add{(<= x), (<= x)}{_, prime}))(true) ·
(1-(p_0-π(x)/x)/(1-π(x)/x)) · (p_1-π(x)/x)/(1-π(x)/x) +
($\exists_p$(prime · add{(<= x), (<= x)}{prime, prime}))(true) ·
(p_0-π(x)/x)/(1-π(x)/x) · (p_1-π(x)/x)/(1-π(x)/x)

Since `add` is commutative, the following holds:

$\exists_p$(prime · add{(<= x), (<= x)}{prime, _}) <=> $\exists_p$(prime · add{(<= x), (<= x)}{_, prime})

So one can rewrite to:

add{(<= x), (<= x)}[prime]$_p$([true, true, true]) := \(p : [real] ∧ [len] 2) =
($\exists_p$(prime · add{(<= x), (<= x)}))(true) ·
(1-(p_0-π(x)/x)/(1-π(x)/x)) · (1-(p_1-π(x)/x)/(1-π(x)/x)) +
($\exists_p$(prime · add{(<= x), (<= x)}{prime, _}))(true) · (
(p_0-π(x)/x)/(1-π(x)/x) · (1-(p_1-π(x)/x)/(1-π(x)/x)) +
(1-(p_0-π(x)/x)/(1-π(x)/x)) · (p_1-π(x)/x)/(1-π(x)/x)
) +
($\exists_p$(prime · add{(<= x), (<= x)}{prime, prime}))(true) ·
(p_0-π(x)/x)/(1-π(x)/x) · (p_1-π(x)/x)/(1-π(x)/x)

This gives us only 3 constant functions of `x` (divided by size of sub-type to get probability):

($\exists_p$(prime · add{(<= x), (<= x)}))(true)	<=>	$k_0(x) / (x+1)^2$
($\exists_p$(prime · add{(<= x), (<= x)}{prime, _}))(true)	<=>	$k_1(x) / ((x+1) \cdot \pi(x))$
($\exists_p$(prime · add{(<= x), (<= x)}{prime, prime}))(true)	<=>	$k_2(x) / \pi(x)^2$

x	$k_0(x)$	$k_1(x)$	$k_2(x)$
10	45	18	4
100	2139	553	16
1000	145507	24740	70
2000	529143	80950	122
3000	1131695	163653	164
4000	1945455	269671	206
5000	2961105	399167	252
6000	4176471	548938	286
7000	5586003	722930	324
8000	7189489	910861	350
9000	8988307	1122678	380
10000	10963863	1355623	410

$$
\begin{aligned}
&\mathrm{add}\{(<= x), (<= x)\}[\mathrm{prime}]_p([\mathrm{true}, \mathrm{true}, \mathrm{true}]) := \backslash(p : [\mathrm{real}] \wedge [\mathrm{len}]\ 2) = \\
&\qquad k_0(x) / (x+1)^2 \cdot (1-(p_0-\pi(x)/x)/(1-\pi(x)/x)) \cdot \\
&\qquad (1-(p_1-\pi(x)/x)/(1-\pi(x)/x)) + k_1(x) / ((x+1) \cdot \pi(x)) \cdot (\\
&\qquad\qquad (p_0-\pi(x)/x)/(1-\pi(x)/x)) \cdot (1-(p_1-\pi(x)/x)/(1-\pi(x)/x)) + \\
&\qquad\qquad (1-(p_0-\pi(x)/x)/(1-\pi(x)/x)) \cdot (p_1-\pi(x)/x)/(1-\pi(x)/x) \\
&\qquad) + k_2(x) / \pi(x)^2 \cdot (p_0-\pi(x)/x)/(1-\pi(x)/x) \cdot (p_1-\pi(x)/x)/(1-\pi(x)/x)
\end{aligned}
$$

For example, when `x = 10000`:

$$\pi(x) = 1229$$

$$
\begin{aligned}
&\mathrm{add}\{(<= 10000), (<= 10000)\}[\mathrm{prime}]_p([\mathrm{true}, \mathrm{true}, \mathrm{true}]) := \backslash(p : [\mathrm{real}] \wedge [\mathrm{len}]\ 2) = \\
&\qquad 10963863 / (10000+1)^2 \cdot (1-(p_0-1229/10000)/(1-1229/10000)) \cdot \\
&\qquad (1-(p_1-1229/10000)/(1-1229/10000)) + 1355623 / ((10000+1) \cdot 1229) \cdot (\\
&\qquad\qquad (p_0-1229/10000)/(1-1229/10000) \cdot (1-(p_1-1229/10000)/(1-1229/10000)) + \\
&\qquad\qquad (1-(p_0-1229/10000)/(1-1229/10000)) \cdot (p_1-1229/10000)/(1-1229/10000) \\
&\qquad) + 410 / 1229^2 \cdot (p_0-1229/10000)/(1-1229/10000) \cdot (p_1-1229/10000)/(1-1229/10000)
\end{aligned}
$$

Creating a table showing probability for various input probabilities:

	$p_0 = 0$	$p_0 = 0.25$	$p_0 = 0.5$	$p_0 = 0.75$	$p_0 = 1$
$p_1 = 0$	0.10790366397479109	0.11119583802820641	0.11448801208162174	0.11778018613503705	0.12107236018845234
$p_1 = 0.25$	0.11119583802820641	0.10818263528525629	0.1051694325423062	0.10215622979935608	0.09914302705640599
$p_1 = 0.5$	0.11448801208162174	0.1051694325423062	0.09585085300299066	0.08653227346367513	0.07721369392435959
$p_1 = 0.75$	0.11778018613503705	0.1021562297993561	0.08653227346367513	0.07090831712799418	0.05528436079231322
$p_1 = 1$	0.12107236018845237	0.09914302705640599	0.0772136939243596	0.055284360792313234	0.0333550276602668