Discrete Monotonic Limits

by Sven Nilsen, 2020

In this paper I introduce discrete monotonic limits, which are larger than continuous limits.

A monotonic function has the property:

$$x >= y \quad => \quad f(x) >= f(y)$$

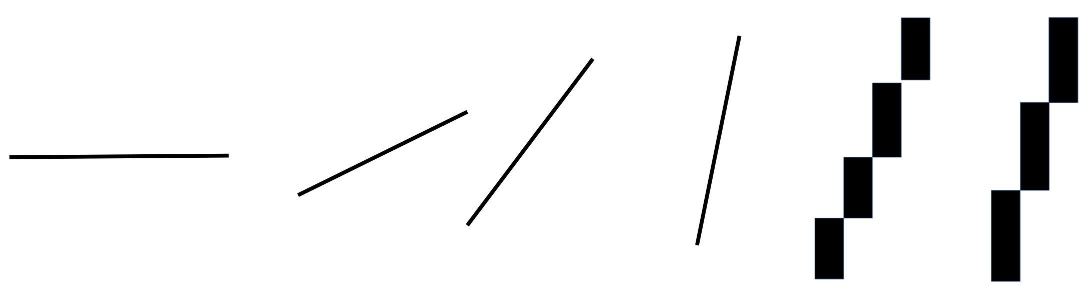
One can construct discrete monotonic functions out of ranges `[$a_i$, $b_i$)`, which can describe discrete sets.
To generalize constructions to support continuous sets, one can use Dual Number Monotonic Density.
A Dual Number Monotonic Density measures the discrete/continuous increase of a range `[$a_i$, $k_i$, $b_i$)`:

$k_i := x_i + y_i\ \varepsilon$

x_i : real
y_i : real
$\varepsilon^2 = 0$
$(x_i = 0) \vee (y_i = 0)$

Dual Number Monotonic Density can be used to reason about semantics of discrete vs continuous sets.
The intuition behind this representation implies a limit where a continuous set becomes discrete.

This idea might be familiar from experienced users of painting software:



If you draw an almost-vertical line in painting software without anti-aliasing,
then when zooming in, the pixels becomes easily noticeable because of regular "jumps".

A monotonic function of type `real → real` can not describe a vertical line, because it would mean that the function returns multiple values for some input value. However, one can get arbitrary close: For any point on a continuous function, there exists a continuous function with a larger increase at the same point. This is done by simply increasing the `$y_i$` component.

There is a way of increasing the steepness of a curve <u>greater</u> than any continuous function:

When the monotonic density is discrete, the `$x_i$` component is non-zero and the `$y_i$` component is zero.
A such dual number is greater than any dual number with a zero `$x_i$` component. However, even if discrete increases are larger than all continuous ones, <u>one can always create a larger discrete increase</u>!