Hilbert's Over-Filled Hotel

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In this paper we introduce a version of Hilbert's hotel which is over-filled and closed, all guests need to move, yet a lot of staying guests are rejected renting a room for the night.

You have just settled into a hotel for the night, feeling lucky since you got the last available room, when a servant knocks on your door, apologises for the inconvenience and requests you to move to the next room. Thinking only of a comfy bed and a good night's sleep, you grab your stuff and move.

Next night when you just fell asleep, you are awakened by another servant knocking on your door, apologises for the inconvenience and requests you to move to the next room. This time you demand an explanation. The servant says that in this hotel there are infinite rooms, so by telling every guest to move to the next room, they can make room for another guest.

You accept the idea of moving from room to room each day, after all your luggage is not that big. Besides, you are a little curious about how it feels to explore an infinite hotel. Exciting! The servant ensures you that the service is just as good for all the rooms. OK, no downsides then!

Yet, the impossible happens: One day, there are no more rooms!

Instead, what you find in front of you is a time machine. You go back in time to book the first room *before* you even entered the hotel. You explain to the manager your problem. Despite having infinite rooms, there were no more rooms! The manager apologises and promises to not let anyone, who enters the hotel, to rent a room.

After an infinite moves from room to room, you feel tired and look forward to a good night sleep with no more moving next day. Guess what happens: A servant knocks on your door and ask you to move to the next room. Confused and angry, you grab your stuff and move, *again*. The same thing happens every night. The servants explain that somebody living inside the hotel needs the first room.

How? How can this happen? The only person staying at the hotel is you! There are infinite rooms!

Again, the impossible happens: After an infinite amount of moves, there are no more rooms!

You go back in time, booking the first room earlier than previous time. The hotel manager apologises and promises to not let any more people rent a room, whether they come from inside or outside. A second version of your past self finds the first room occupied and is kicked out of the hotel, looking with confusion on his older self trying to get in a moment later, but being rejected.

The same thing happens again. You go back to an earlier moment when you booked the room a second time. This time, you are either kicked out of the hotel by an *even older* self, or you get in the first room, but is asked to move, *again*. Outside the hotel there are three people staring in confusion at each other.

Despite trying to go back earlier and earlier in time, there is a chance you get kicked out, joining a seemingly infinitely growing crowd, consisting of only yourself, coming from the over-filled hotel.