Adjoint Paths

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In this paper we formalize adjoint paths using path semantics.

An adjoint path is a relationship between two asymmetric normal paths of `f`:

$$f[g_0 \times id \to id] \Leftrightarrow f[id \times g_1 \to id]$$

$$f : T \to U$$

This relationship is represented by `f`, which is called the "adjoint path".

Since `id` maps to same type `T → T`, it follows that `$g_0$` and `$g_1$` also maps to same type:

$$g_0 : T \to T$$
$$g_1 : T \to T$$

Since these two normal paths are the same, it means that they both use the same function:

$$\because \quad f[g_0 \times id \to id] \Leftrightarrow h \qquad f(x, y) = h(g_0(x), y)$$
$$\because \quad f[id \times g_1 \to id] \Leftrightarrow h \qquad f(x, y) = h(x, g_1(y))$$

$$\therefore \quad h(g_0(x), y) = h(x, g_1(y))$$

The function `$g_0$` is called the "left side" or "left adjoint".
The function `$g_1$` is called the "right side" or "right adjoint".
When the left side `$g_0$` is equal to the right side `$g_1$`, it is called a "self-adjoint operator".

Every function is a self-adjoint path using the `id` function as a self-adjoint operator:

$$f[\mathbf{id} \times id \to id] \Leftrightarrow f[id \times \mathbf{id} \to id] \Leftrightarrow f$$