

Introduction to classic & constructive

# **Path Semantical Quantum Logic**

## **PSQ**

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# Quick Summary of Logic 1 of 2

- 16 binary operators (AND, OR, IMPLY etc.)
- Propositional Logic is the 0-th order logic
- 1-th order logic adds predicates and quantifiers
- Classical Logic has excluded middle  $a \vee \neg a$
- Constructive Logic has no excluded middle
- Exponential Complexity  $O(2^n)$  for  $n$  arguments

# Quick Summary of Logic 2 of 2

- A classical proof  $f$  is  
 $f : \text{bool}^n \rightarrow \text{bool}$  where  $f \iff \text{true}$   
This means  $f$  returns  $\text{true}$  for all inputs
- A constructive proof  $f$  is a program  
 $f : A$  where  $A$  is a type  
This means  $f$  is a point in the space  $A$

# Extending Logic for PSQ

- Three extensions necessary:
  - 1<sup>st</sup> step: Exponential  $\text{`^`}$  operator
  - 2<sup>nd</sup> step: Qubit  $\text{`~`}$  operator
  - 3<sup>rd</sup> step: Path semantical levels using core axiom
- Related families of logics (excm = excluded middle):
  - **Constructive PSQ: 1, 2, 3**                      PSI: 2, 3
  - **Classical PSQ: 2 + excm**                      PSL: 3 + excm

# Exponential $\wedge$ Operator 1 of 3

- $a \wedge b$  means  $a$  is provable from  $b$
- Similar to  $a \Rightarrow b$ , but can not capture from the environment like lambda/closure expressions
- Can be thought of as like a function pointer
- $\text{uniform}(a) = (a^{\text{true}} \mid \text{false}^a)$
- $\text{theory}(a) = !\text{uniform}(a)$

# Exponential `^` Operator 2 of 3

- $a^b \Rightarrow (a^b)^c$
- $(a \Rightarrow b^a) \Rightarrow b^a$
- $(a^b)^c \Rightarrow a^{(b \ \& \ c)}$
- $(a^b)^a$
- $\text{uniform}(a) \mid \text{theory}(a)$

# Exponential `^` Operator 3 of 3

- `(false^(a == b) | false^(b == c)) => (a == c)^true`
- `(a □ b)^c == (a^c □ b^c)`
- `c^(a □ b) == (c^a □[¬] c^b)`
- `((!a)^b)^(!(a^b))`

# The Qubit $\sim$ Operator 1 of 2

- In classic logic,  $\sim a$  uses  $a$  as a pseudo-random seed such that  $\sim a$  can in principle be equal to any other proposition
- This makes classical proofs probabilistic
- The number of nested applications of the qubit operator defines homotopy levels, e.g.  $\sim \sim a$  has homotopy level 2.



# The Qubit $\sim$ Operator 2 of 2

- In constructive logic,  $\sim a$  is a 1-avatar (new-type) which only allows substitution  $\sim b$  under tautological equality  $(a == b)^{\text{true}}$
- It means, one can not turn  $\sim a$  into  $\sim b$  except under special circumstances where is it provable that  $a == b$  under none assumptions
- Requires Exponential Propositions  $^{\wedge}$

# The Quality $\sim\sim$ Operator

- $\sim(a \sim\sim b) == ((a == b) \& \sim a \& \sim b)$

This proves  $\sim(a \sim\sim a) == \sim a$

- $\sim(a == b) \& \text{theory}(a == b) \Rightarrow (a \sim\sim b)$

This lifts equality into quality,  
which does not hold for reflexivity  $a == a$

- Quality  $\sim\sim$  is used in the core axiom

# The Aquality $\sim!\sim$ Operator

- $\sim(a \sim!\sim b) == ((a == b) \& !\sim a \& !\sim b)$

This proves  $\sim(a \sim!\sim a) == !\sim a$

- Usually accompanied with axiom  $\sim!\sim a == \sim!\sim a$ , but does not hold in all models
- In principle the same as quality, since one can prove same theorems by swapping quality with aquality and vice versa

# Path Semantical Levels 1 of 2

- Each proposition has an associated natural number which is used to prove an order  $a1 < a2$
- Core axiom:

$$\begin{aligned} & ((a1 \Rightarrow a2) \ \& \ (b1 \Rightarrow b2) \ \& \\ & (a1 < a2) \ \& \ (b1 < b2) \ \& \\ & (a1 \sim\sim b1)) \Rightarrow (b1 \sim\sim b2) \end{aligned}$$

# Path Semantical Levels 2 of 2

- Levels can be interpreted as moments in time, or higher universes of types
- Choice of core axioms forces bias toward quality, aquality, or even restoring symmetry of the two, which has deep philosophical implications
- Each level is like a complete language of logic, where quality is used to lift relations from one level to the next – using the core axiom (this structures levels)

# Homotopy Levels 1 of 2

- Homotopy Levels are not Path Semantical Levels, but more like “unstructured relativity of time”
- $| \text{bool}^n \times \text{bool}^n \rightarrow \text{bool}^n |$  counts number of binary functions as measure of complexity
- Rapidly grows  $1, 16, 2^{32}, \dots$  larger than the number of atoms in the universe in level 4
- Humanly incomprehensibly complex and rich semantics

# Homotopy Levels 2 of 2

- $\text{hom\_eq}(2, a, b) == ((a == b) \ \& \ (\sim a == \sim b))$
- $(a \sim \sim b) \Rightarrow \text{hom\_eq}(2, a, b)$
- $(a \sim ! \sim b) \Rightarrow \text{hom\_eq}(2, a, b)$
- $\text{hom\_eq}(2, a, b) == (a \sim \sim b) \mid (a \sim ! \sim b)$  (excm)
- $\text{hom\_eq}(n, a, b)$  aligns qubits in range  $[0, n)$ , such that  $\forall i \in [0, n) \{ \sim^i(a) == \sim^i(b) \}$

# Hypertorus Homotopy 1 of 2

- $\sim a$  might be thought of as creating a “circle” around the point  $a$
- $\sim\sim a$  might be thought of as creating a “torus” around the point  $a$ , or “circle” around “circle” around  $a$
- $\sim^n(a)$  might be thought of as creating a “hypertorus” around the point  $a$



# Hypertorus Homotopy 2 of 2

- When  $\sim a == \sim b$ , the two circles around  $a$  and  $b$  are propositionally equal, which forms a homotopy path
- This notion of homotopy path is restricted to at most one homotopy path between two points
- The points can be unequal, e.g.  $a \neq b$  while being connected by homotopy  $\sim a == \sim b$

# Bonus: Modal Logic 1 of 2

- Uses Exponential  $\text{ }^{\text{ }}$  operator
- Based on theory of Avatar Extensions
- $\text{ }\Diamond p \equiv \text{false}^{\text{ }}(\text{false}^{\text{ }}p)$  is “taboo knowledge”
- Safe to “write”, unsafe to “read”
- Safe invariants using unsafe proofs hide the taboo knowledge from the outside world

# Bonus: Modal Logic 2 of 2

- The `!` operator using `^` inverts every bit
- The `!` operator using `◇` normalises to the inversion in the sense of possible worlds
- Allows avoiding using two different `!` operators
- Might be thought of as a local space of a Möbius strip (or a space with Listing-Möbius shifts)

# Summary

- Extend logic with  $\wedge, \sim, <, \diamond$
- Follows from making core axiom “well behaved”

*One Axiom to rule them all,  
One Axiom to find them,  
One Axiom to bring them all  
and in the End, a new beginning  
there and back again,  
so comes snow after fire,  
and even dragons have their endings*