Matrix Tangent Space

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In this paper I explain how to use matrix tanget space with dual numbers to solve problems, using inversion of a `2 × 2` matrix as an example.

This procedure can be followed to use matrix tangent space to solve problems:

1. Express the problem in the form `A(B) = 0` where `B` is an unknown matrix
2. Set up the equation `det(A(B)) = 0`
3. Insert dual numbers for components of the unknown matrix `B`
4. Separate dual components into its own equation
5. Isolate dual coefficients
6. Separate equations associated with dual coefficients
7. Solve equations and translate into matrix form

Assume that one wants to find the inverse of a matrix `M`:

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

The inverse matrix `$M^{-1}$` has the following property:

$$M^{-1}M = I$$

From this one can create an equation for the inverse:

$$\det(M^{-1}M - I) = 0$$

$$M^{-1}M = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a'a + b'c & a'b + b'd \\ c'a + d'c & c'b + d'd \end{pmatrix}$$

$$M^{-1}M - I = \begin{pmatrix} a'a + b'c - 1 & a'b + b'd \\ c'a + d'c & c'b + d'd - 1 \end{pmatrix}$$

The determinant of this matrix is the following:

$$(a'a + b'c - 1)(c'b + d'd - 1) - (a'b + b'd)(c'a + d'c)$$

Solving:

$$aa'dd' - aa' - d'd + bb'cc' - b'c - bc' - a'bcd' - ab'c'd + 1$$

Insert dual numbers for elements of the inverse matrix, e.g. `a’ => (a’ + Δa’ε)`:

$$
\begin{aligned}
&a(a' + \Delta a'\varepsilon)d(d' + \Delta d'\varepsilon) - a(a' + \Delta a'\varepsilon) - (d' + \Delta d'\varepsilon)d + \\
&b(b' + \Delta b'\varepsilon)c(c' + \Delta c'\varepsilon) - (b' + \Delta b'\varepsilon)c - b(c' + \Delta c'\varepsilon) - \\
&(a' + \Delta a'\varepsilon)bc(d' + \Delta d'\varepsilon) - a(b' + \Delta b'\varepsilon)(c' + \Delta c'\varepsilon)d + 1
\end{aligned}
$$

$$
\begin{aligned}
&(aa'd + ad\Delta a'\varepsilon)(d' + \Delta d'\varepsilon) - (aa' + a\Delta a'\varepsilon) - (dd' + d\Delta d'\varepsilon) + \\
&(bb'c + bc\Delta b'\varepsilon)(c' + \Delta c'\varepsilon) - (cb' + c\Delta b'\varepsilon) - (bc' + b\Delta c'\varepsilon) - \\
&(a'bc + bc\Delta a'\varepsilon)(d' + \Delta d'\varepsilon) - (ab'd + ad\Delta b'\varepsilon)(c' + \Delta c'\varepsilon) + 1
\end{aligned}
$$

$$
\begin{aligned}
&((aa'd + d\Delta a'\varepsilon)d' + (aa'd + d\Delta a'\varepsilon)\Delta d'\varepsilon) - (aa' + a\Delta a'\varepsilon) - (dd' + d\Delta d'\varepsilon) + \\
&((bb'c + bc\Delta b'\varepsilon)c' + (bb'c + bc\Delta b'\varepsilon)\Delta c'\varepsilon) - (cb' + c\Delta b'\varepsilon) - (bc' + b\Delta c'\varepsilon) - \\
&((a'bc + bc\Delta a'\varepsilon)d' + (a'bc + bc\Delta a'\varepsilon)\Delta d'\varepsilon) - ((ab'd + ad\Delta b'\varepsilon)c' + (ab'd + ad\Delta b'\varepsilon)\Delta c'\varepsilon) + 1
\end{aligned}
$$

$$
\begin{aligned}
&(aa'dd' + add'\Delta a'\varepsilon + a'd\Delta d'\varepsilon) - (aa' + a\Delta a'\varepsilon) - (dd' + d\Delta d'\varepsilon) + \\
&(bb'cc' + bcc'\Delta b'\varepsilon + bb'c\Delta c'\varepsilon) - (cb' + c\Delta b'\varepsilon) - (bc' + b\Delta c'\varepsilon) - \\
&(a'bcd' + bcd'\Delta a'\varepsilon + a'bc\Delta d'\varepsilon) - (ab'c'd + adc'\Delta b'\varepsilon + ab'd\Delta c'\varepsilon) + 1
\end{aligned}
$$

Separate dual components into its own equation to get the matrix tangent space:

$$
\begin{aligned}
&(add'\Delta a' + aa'd\Delta d') - a\Delta a' - d\Delta d' + \\
&(bcc'\Delta b' + bb'c\Delta c') - c\Delta b' - b\Delta c' - \\
&(bcd'\Delta a' + a'bc\Delta d') - (adc'\Delta b' + ab'd\Delta c')
\end{aligned}
$$

Isolate dual coefficients:

$$
\begin{aligned}
&add'\Delta a' + aa'd\Delta d' - a\Delta a' - d\Delta d' + \\
&bcc'\Delta b' + bb'c\Delta c' - c\Delta b' - b\Delta c' - \\
&bcd'\Delta a' - a'bc\Delta d' - adc'\Delta b' - ab'd\Delta c'
\end{aligned}
$$

$$
\begin{aligned}
&add'\Delta a' - aa\Delta a' - bcd'\Delta a' + \\
&bcc'\Delta b' - c\Delta b' - adc'\Delta b' + \\
&bb'c\Delta c' - b\Delta c' - ab'd\Delta c' + \\
&aa'd\Delta d' - d\Delta d' - a'bc\Delta d'
\end{aligned}
$$

$$
\begin{aligned}
&\Delta a'(add' - a - bcd') + \\
&\Delta b'(bcc' - c - adc') + \\
&\Delta c'(bb'c - b - ab'd) + \\
&\Delta d'(aa'd - d - a'bc)
\end{aligned}
$$

Separate equations associated with dual coefficients:

$$
\begin{aligned}
&add' - a - bcd' = 0 \\
&bcc' - c - adc' = 0 \\
&bb'c - b - ab'd = 0 \\
&aa'd - d - a'bc = 0
\end{aligned}
$$

Solve equations and translate into matrix form:

add' – a – bcd' = 0
d' = a / (ad – bc) = a / det

bcc' – c – adc' = 0
c' = -c / (ad – bc) = -c / det

bb'c – b – ab'd = 0
b' = -b / (ad / bc) = -b / det

aa'd – d – a'bc = 0
a' = d / (ad – bc) = d / det

In matrix form this becomes:

$$M^{-1} = \frac{1}{(ad - bc)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Qed.