Semi-Invertible Folded Fields

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In this paper I present a mathematical object that simplifies reasoning about physics.

In physics[1], one might be concerned about the position, velocity etc. of objects. With other words, one wants to reason about physical properties of the world. However, the general problem in physics is that the use of physical properties is very diverse and the amount of data can be extreme. This is why it is important from a language design perspective to develop tools for reasoning.

A Semi-Invertible Folded Field simplifies reasoning about physics because it resembles the notion of a path, yet it also has properties of a group generator[2]. It does not require total invertibility, which simplifies describing real physical systems. It is similar to many mathematical objects used to reason about physics, such as Lie groups[3] and differentiable manifolds[4], but it does not remove the underlying language of physics. In some sense, it is less abstract than other mathematical objects, but at the same time abstract enough to be useful.

First, I will define what I mean by “physics”. It is most common to think of physics as a general language to describe the world. However, when using Semi-Invertible Folded Fields, it is possible to reduce physics to reasoning about positions. With other words, a “physics” is some type with an associated position type:

physics : type
pos : physics $\rightarrow$ type

For example:

classic : physics pos(classic) = real^3	Non-relativistic classical physics
quantum : physics pos(quantum) = real^3 $\rightarrow$ complex	Non-relativistic quantum physics
rel_classic : physics pos(rel_classic) = real^4	Relativistic classical physics
rel_quantum : physics pos(rel_quantum) : real^4 $\rightarrow$ complex	Relativistic quantum physics

A Semi-Invertible Folded Field is a type with two associated properties:

field : type
scale : field × real $\rightarrow$ field
inv : field $\rightarrow$ field

For example, rotation around the z-axis:

rotate_z(angle) : field
scale(rotate_z(angle), s) = rotate_z(angle · s)
inv(rotate_z(angle)) = rotate_z(-angle)

Notice that the rotation around the z-axis is independent of the kind of physics one is reasoning about. Where the kind of physics becomes important, is how one describes motion:

motion : field → (ph : physics) → pos(ph) → real → pos(ph)

With other words, there is one motion that corresponds to non-relativistic classical physics, another motion that corresponds to relativistic quantum physics etc.

In the case of rotation around the z-axis for classical physics, one can define motion as following[5]:

```
motion(rotate_z(angle), classical, x, dt) = row_mat3_transform(
        [
                [cos(angle · dt), -sin(angle · dt), 0],
                [sin(angle · dt), cos(angle · dt), 0],
                [0, 0, 1]
        ],
        x
)
```

For rotation around the z-axis for non-relativistic quantum physics, the motion returns a wave function[6]. A wave function is defined at every position in space, so it does not make sense to think of it as a motion in the classical sense, where the observer follows the position. Instead, one must think of the observer as looking at some fixed point in space:

```
motion(rotate_z(angle), quantum, wave, dt) = \(x) = wave(
                row_mat3_transform(
                [
                        [cos(-angle · dt), -sin(-angle · dt), 0],
                        [sin(-angle · dt), cos(-angle · dt), 0],
                        [0, 0, 1]
                ],
                x
        )
)
```

Notice that the sign of the rotation angle is flipped. This is done to produce the effect of an observer that looks at some fixed point in space, instead of moving along with the rotation.

A folded field is basically a way to take some position and transform it to a new position using some time delta. When a field is scaled, it speeds up or slows down time. When a field is flipped, time is reversed. In many cases, scaling with `-1` produces the inverted field. The reason scaling is separate from inversion is, among other properties, due to allowing fields to be semi-invertible.

Now, what does “semi-invertible” mean?

One property that a field must satisfy, is that when inverting twice, one gets back the original field:

inv(inv(f)) <=> f

This property is called an “involution”[7].

However, although inversion is an involution, does not mean that motion is invertible in that sense.

To explain semi-invertibility, one can think of a field that increases or decreases natural numbers[8]:

```
discrete : physics
pos(discrete) = nat

inc : field
scale(inc, s) = inc
inv(inc) = dec

dec : field
scale(dec, s) = dec
inv(dec) = inc

motion(inc, discrete, x, dt) = x + 1
motion(dec, discrete, x, dt) = if x == 0 { 0 } else { x – 1 }
```

When decreasing `0`, there is no negative natural number, so the position remains at `0`. This means that `0` has two possible past moments using `dec`: `0` and `1`. However, for `inc` there is only a single possible past moment. This difference means that `inc` and `dec` are semi-invertible:

```
f(inv(f)(x)) = x        ∨        inv(f)(f(x)) = x
```

Translated for motion:

```
∀ f : field, ph : physics, x : pos(ph), dt : real {
    motion(f, ph, motion(inv(f), ph, x, dt), dt) = x        ∨
    motion(inv(f), ph, motion(f, ph, x, dt), dt) = x
}
```

Another property that is common to use with semi-invertibility, is the following:

```
inv(f)(f(x)) ¬= x        =>        f(f(x)) = x
```

This means that when the inverted field is not an involution for `x`, one can use the field itself as its own involution. With other words, the past is fully reversible by choosing between the field or its inverse. This property is called “terminal semi-invertibility” and is usually assumed implicitly, unless one explicitly says “non-terminal semi-invertibility”.

Translated for motion:

```
∀ f : field, ph : physics, x : pos(ph), dt : real {
    motion(inv(f), ph, motion(f, ph, x, dt), dt) ¬= x        =>
    motion(f, ph, motion(f, ph, x, dt), dt) = x
}
```

Terminal semi-invertibility makes it possible to reason about motion as “cyclic” in the sense that for any combination of movement forwards or backwards in time, it is always possible to return to the present moment. It does not mean that the past can be determined, but it is kind of like a soft version of determinism[9] or unitary laws of motion[10].

In order to understand quantum physics[11], it is very helpful to think about inversion of fields as a choice. A particle can rotate around the z-axis in one direction, the other, or both[12].

One important higher order field is composition of two fields:

```
∀ f : field, g : field {
	(g · f) : field ∧
	scale(g · f, s) = scale(g, s) · scale(f, s) ∧
	inv(g · f) = inv(f) · inv(g)
}

∀ f : field, g : field, ph : physics, x : pos(ph), dt : real {
	motion(g · f, ph, x, dt) = motion(g, ph, motion(f, ph, x, dt), dt)
}
```

A field is smooth when the following property holds:

```
lim dt → 0 {
	∀ f : field, g : field, ph : physics, x : pos(ph), dt : real {
		motion(g · f, ph, x, dt) = motion(f · g, ph, x, dt)
	}
}
```

Another important higher order field is a “fixed” field:

```
∀ f : field {
	fixed(f) : field ∧
	scale(fixed(f), s) = fixed(scale(f, 1)) ∧
	inv(fixed(f)) = fixed(inv(f))
}

∀ f : field, ph : physics, x : pos(ph), dt : real {
	motion(fixed(f), ph, x, dt) = motion(f, ph, x, 1)
}
```

A “fixed” field is used to perform coordinate transformations without depending on time.

For example, to rotate around the z-axis at a coordinate `v`, one can do the following:

```
fixed(translate(v)) · rotate_z(angle) · inv(fixed(translate(v))
```

This can also be written as a normal path[13]:

```
rotate_z(angle)[fixed(translate(v)]
```

For “fixed” fields, the difference between scaling and inversion is important. Without the ability to distinguish between the two operations, it would be impossible to express the subtleties that makes it work out mathematically. This is an example of intentional mathematical language design.

There are other important higher order fields outside the scope of this paper, such as a vector field, where each field contributes using a coefficient. Inversion of a vector field inverts every sub-field, so one can also define a “power vector field” where every possible choice of inverting individual sub-fields contributes using coefficients. From this, it is easier to reason about Feynman Path Integral[14] of quantum mechanics, as fields existing a-priori where contributions are summed up over arbitrary compositions. A physical system can be described as a ray[15] (position + coefficients).

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