Cubical Types

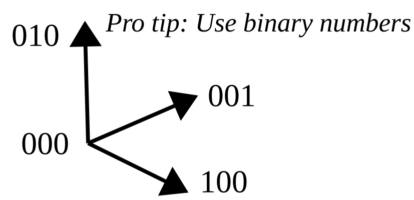
by Sven Nilsen, 2020

In this paper I show that there is a way of modeling cubical types in Path Semantical Logic.

A cubical type[1] means a relation that behaves like the interior of an N-cube[2] compared to its exterior. The exterior of an N-cube consists of a hierarchy of N-1-cubes, N-2-cubes and so on, down to 0-cubes. A 0-cube is just a point. A 1-cube is an edge. A 2-cube is a surface. A 3-cube is a normal cube etc.

When one refers to the interior of an N-cube as a set, the exterior is not included in the set.
The relation representing the interior of an N-cubical type should not imply exterior relations.

To model N-cubes in a logic means points are propositions[3], e.g. \`a\` or \`b\`. An edge is an equality between two propositions \`a=b\`. An equality can be \`true\` without the propositions connected by the equality being \`true\`. Therefore, a 3-cube is modeled as:



$$((x_{000}=x_{100})=(x_{010}=x_{110}))=((x_{001}=x_{101})=(x_{011}=x_{111}))$$

One can prove in normal Propositional Logic that :

$$(a=b)=(c=d) \qquad = \qquad (a=c)=(b=d)$$

This trick works in general for any N-cube. So, it does not matter which order one is modeling the vertices of the cube. Any permutation will suffice.

In Path Semantical Logic[4], all propositions are assigned to a level, where:

- Level 0 – types (usually) – starting with a small letter, e.g. \`T\`
- Level 1 – variables (usually) – starting with a big letter, e.g. \`x\`

To model a cubical type means transporting the N-cube relation from level 1 to 0:

1. One needs a model of the N-cube in level 1
2. One needs an abstract transport[5] model of the N-cube from level 1 to 0

The proof is as following:

$$\frac{((x_{000}=x_{100})=(x_{010}=x_{110}))=((x_{001}=x_{101})=(x_{011}=x_{111})),\quad ((x_{000}(T_{000})=x_{100}(T_{100}))=(x_{010}(T_{010})=x_{110}(T_{100})))=((x_{001}(T_{001})=x_{101}(T_{101}))=(x_{011}(T_{011})=x_{111}(T_{111})))}{((T_{000}=T_{100})=(T_{010}=T_{110}))=((T_{001}=T_{101})=(T_{011}=T_{111}))}$$

The notation \`x(T)\` means \`x=>T\` where \`T\` is at a lower level.

One can also check that no parts of the exterior relations are provable,
hence proving the interior type only and therefore modeling cubical types.

References:

[1] “cubical type theory”
nLab
https://ncatlab.org/nlab/show/cubical+type+theory

[2] “Hypercube”
Wikipedia
https://en.wikipedia.org/wiki/Hypercube

[3] “Propositional calculus”
Wikipedia
https://en.wikipedia.org/wiki/Propositional_calculus

[4] “Path Semantical Logic”
AdvancedResearch, reading sequence on Path Semantics
https://github.com/advancedresearch/path_semantics/blob/master/sequences.md#path-semantical-logic

[5] “Concrete and Abstract Transport”
Sven Nilsen, 2020
https://github.com/advancedresearch/path_semantics/blob/master/papers-wip/concrete-and-abstract-transport.pdf