Natural Continuous Paths

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In this paper I describe an abstract algebraic free L-system that generate maps of unit intervals. This system of applying rules in an arbitrarily manner might be consider "Natural Continuous Paths", because it provides a justification of common mathematical objects, such as geometry or probability.

When constructing programs, mathematical objects, geometric shapes or probabilistic problems, there are some variants that to the mathematically trained mind seem more "natural". For example, the natural path between two points in an Euclidean space might be considered to be a straight line, even if there is no easy way to provide a justification for way we pay particular attention to such choices.

I want to prove mathematically that such choices are not actually arbitrary, but follows from a deep symmetry of systems that have the following properties:

- Generation of new variants with continuous choices: `1 → x_i`
- Folding back choices to the same type: `|_| : real` (measurement)
- Semantics analogue of a set of sentences in a language grammar: `∀ x, y { ||x| y| <= |x| }`

In language grammar, when you make a choice aka Kleene star plus `V^+`, you get a sub-set of a language from the set of all sentences. With other words, you get **less** than the domain you make choices from, even if the initial condition is infinite and included in the new construction.

A justification for naturality is that one makes as *few assumptions as possible*.

I believe that I have found the smallest set of assumptions above. In particular, the idea of making choices from a continuum, observing consequences of choices and organizing them according to some principles that is closely associated with the use of language for reasoning about these objects.

The kind of systems I am talking about are those who permit arbitrary constructions of some kind. One can think of these systems as free L-systems, which permits choosing which rule to apply at each step, instead of applying all rules at every step.

More, these systems must be predictable. We want to construct algebraic systems that predict properties of such systems. Therefore, starting from algebraic initial conditions and then "folding back" the observations of these constructions to algebraic form, is a natural assumption to make.

The major result of this paper is the following:

$$(1 \to x_i) \land \forall\ x, y\ \{\ ||x|\ y| <= |x|\ \} => \forall\ x\ \{\ |x| : (>= 0) \land (<= 1)\ \}$$

$$|\_| : real$$

This proves that "measurements" across these widely applicable systems fall naturally into the range of the unit interval, a possible consequence of reasoning about these systems using language. In path semantics, such choices are called "Natural Continuous Paths". The rest of this paper is the proof.

Assume that there is some abstract system, where two objects `A` and `B` can be produced from `1`:

$1 \rightarrow A$ $\qquad$ $1 \rightarrow B$

$|A|$: real $\qquad$ $|B|$: real

When a rule of the form `1 → X` is used, it is called an “event”.
The `|X|` operation is called “measuring `X`” and is permitted everywhere `X` exists.

A such system has the property that one can not make anything from `0`, which makes it controllable by setting up the initial expression e.g. `1 + 1` which might produce e.g. `$A^4B^3 + A^2B^5$`.

If one assumes e.g. `|A| = 1`, then this is the same as something stated before.

If one assumes e.g. `|A| = 0`, then this is the same as eliminating a sum.
This means that the initial expression `1 + 1 + 1` can generate `1 + 1`, but not vice versa.

If one does not allow `0` to be measured, then `1`, `1 + 1` and `1 + 1 + 1` are separate classes.

So far, I have derived that `0` and `1` are special values in this abstract system.

One can derive the following rules by multiplying a measurement on both sides:

$|A| \rightarrow |A|\,A$ $\qquad$ $|A| \rightarrow |A|\,B$ $\qquad$ $|B| \rightarrow |B|\,A$ $\qquad$ $|B| \rightarrow |B|\,B$

The question is: What is the natural measurement of e.g. `|A| B`?

In language grammar, an element of a set `L` by concatenation with the Kleene star of `L` is a subset of the Kleene star of `L`:

$$LL^* \subset L^*$$

This can be written:

$$L^+ \subset L^*$$

I want a similar construction for this abstract system.

I add the following constraints on measurements:

$||A|\,A| <= |A|$ $\qquad$ $||A|\,B| <= |A|$ $\qquad$ $||B|\,A| <= |B|$ $\qquad$ $||B|\,B| <= |B|$

It means creating some kind of analogue of language grammar for the new constructions.

Notice that these rules compares the measurement over the previously derived rules.

I have not introduced any artificial constructions or justifications except from the desire to follow the semantics of language grammar. Then, I applied this in the most direct way, which seems to have no alternatives.

This can be written more generally:

$$\forall\ x, y\ \{\ ||x|\ y| <= |x|\ \}$$

It turns out that one can prove interesting things about this system that is not mathematically trival.

For example:

$$|4a| <= 4$$
$$4|a| <= 4$$
$$|a| <= 1$$

It is clear that the constraints on measurements produces some interesting consequences.

Another example:

$$|-4a| <= -4$$

Assume that `|a| < 0`, then `|-4a|` must have the sub-type:

$$-4 \cdot (|a| : (< 0)) : (> 0)$$

This is because multiplying a negative number with a negative number is a positive number.
Which is a contradiction with the statement that it’s less than `-4`.

$$|-4a| : (<= -4) \wedge (> 0)$$
$$|-4a| : \text{false}$$

If `a` can not be less than `0`, then it must be greater or equal than `0`.

$$|a| >= 0$$

By combining the conclusion of the two previous examples, one gets:

$$|a| : (>= 0) \wedge (<= 1)$$

Or, expressed in general:

$$\forall\ x\ \{\ |x| : (>= 0) \wedge (<= 1)\ \}$$

I have now proven that:

$$\forall\ x, y\ \{\ ||x|\ y| <= |x|\ \} \qquad => \qquad \forall\ x\ \{\ |x| : (>= 0) \wedge (<= 1)\ \}$$

Holds for this abstract system of “events” and “measurements”.

I call making such modifications to existing systems as “Natural Continuous Paths”.

A Natural Continuous Path is a way of constructing a new variant such that no previous state has produced the same variant with the same method, while “folding” the construction of the new variant back to the type of the variant it was constructed from.

The motivation for Natural Continuous Paths is to justify why certain choices are made when constructing new languages, by analyzing the properties such languages according to some sense of applicability.

Semantics of Natural Continuous Paths is grounded in making choices from a continuum in general, instead of assuming some existing symmetry that permits “narrowing down” the natural choice. This provides a foundation that is computationally cheap and easily provable.

Natural Continuous Paths are not bounded like modifications in the paper “Modification Theory”. They construct new objects, which each might have infinite number of children by similar construction. Therefore, they deserve to be studied as mathematical objects on their own.