Category Realizable Groupoids

by Sven Nilsen, 2020

In this paper I show that normal paths form a category realizable groupoid.

In the paper “Imaginary Inverse”[1], I showed that introducing an imaginary inverse makes it possible to encode a useful subset of normal paths[2] in any calculus with a composition operator[3].

It is known that groupoids[4] can be viewed as a category[5] augmented with an inverse unary operator. Hence, when the inverse operator is imaginary, any category can be lifted into a groupoid. However, the underlying structure of the category is well preserved, as the realizable part of the groupoid.

Formally, a category `C` is defined as:

- A class `ob(C)` of objects
- A class `hom(C)` of morphisms between objects.
- For every three objects `a`, `b` and `c`,
  a binary operation `$\mathrm{hom}_C(a, b) \times \mathrm{hom}_C(b, c) \to \mathrm{hom}_C(a, c)$` written `$g \cdot f$`

Such that the following axioms hold:

- A morphism is written `$f : a \to b$` for a source object `a` and a target object `b`
- If `$f : a \to b$`, `$g : b \to c$` and `$h : c \to d$` then `$h \cdot (g \cdot f) \iff (h \cdot g) \cdot f$` (Associativity)
- For every object `x` there exists a morphism `$\mathrm{id}_x : x \to x$`
  such that `$\mathrm{id}_x \cdot f \iff f$` and `$g \cdot \mathrm{id}_x \iff g$` (Identity)

A groupoid adds the following axiom:

- For each pair of objects `x` and `y`,
  a function `$\mathrm{inv} : \mathrm{hom}_C(x, y) \to \mathrm{hom}_C(y, x)$` such that
  `$\forall f \in \mathrm{hom}_C(x, y) \{ f \cdot \mathrm{inv}(f) \iff \mathrm{id}_y \ \wedge \ \mathrm{inv}(f) \cdot f \iff \mathrm{id}_x \}$`

Usually, a groupoid is thought of as a small category in which every morphism is bijective[6].
However, when introducing an imaginary inverse, this is no longer true.
The same axiom holds, yet `$\forall x, y \in ob(C), f : x \to y \{ \exists! x \{ \mathrm{inv}(f)(y) = x \} \}$` might be false.

The imaginary inverse `inv` might be thought of as a contravariant functor[7] with a dual[8] image[9]:

$$\mathrm{inv}_C : C \to C^{op} \qquad \mathrm{inv}_G : G \to G \ \text{ where } G = C \mid C^{op} \qquad \mathrm{inv}_C \Rightarrow \mathrm{inv}_G$$

Such that the augmented category of `C` with `$\mathrm{inv}_C$` forms a category realizable groupoid `G` where:

- `$\forall x \in ob(C) \{ \mathrm{inv}_C(x) = x \}$`, hence `$ob(G) \iff ob(C)$` and `$\mathrm{inv}_C \Rightarrow \mathrm{id}_x$` for objects
- `$\forall x, y, z \in ob(C), f : x \to y, g : y \to z \{ \mathrm{inv}_C(g \cdot_C f) \iff \mathrm{inv}_C(f) \cdot_G \mathrm{inv}_C(g) \}$` (Contravariance)
- `$\mathrm{inv}_G \cdot_G \mathrm{inv}_G \iff \mathrm{id}_G$` (Involution)

The reason `inv` is an operation on categories, is because every category can be lifted into a category realizable groupoid. In practice, it is common to not define `inv` for objects, but for morphisms only.

From this definiton of a category realizable groupoid, it is non-trivial that standard techniques for groupoids as small categories can be used, due to not every morphism being bijective.

Therefore, I will prove the necessary to treat category realizable groupoids with standard techniques.

All objects in the groupoid `G` are being covered by objects in `C` (the first Realizable axiom).
Therefore, augmentation of the category `C` consists only of new morphisms:

$f : x \to y \qquad => \qquad inv(f) : y \to x$

Due to involution, these new morphisms do not generate any newer morphisms:

∵ $inv_G(inv_C(f)) = f$
∴ $inv_G(inv_G(f)) = f$ ∵ `$inv_C$ => $inv_G$`
∴ $f = f$
∴ true ∵ by reflection

Therefore, `G` is covered by `ob(C)`, `hom(C)` and `hom($C^{op}$)`.
This can be used to reduce the notation without introducing ambiguity.
For simplicity, I will write `inv` instead of `$inv_G$`, `hom` instead of `$hom_G$` and `·` instead of `$\cdot_G$`.

From the groupoid axiom and realizable axioms, one can derive for any object `x`:

$inv(id_x) <=> id_x$ Abstract identity inverse

Proof:

∵ $inv(id_x)$
∴ $inv(inv(f) \cdot f)$ ∵ `$inv(f) \cdot f <=> id_x$`where `$f \in hom(x, y)$` for some `y` (Groupoid)
∴ $inv(f) \cdot inv(inv(f))$ ∵ `$inv(g \cdot f) <=> inv(f) \cdot inv(g)$` (Realizable contravariance)
∴ $inv(f) \cdot f$ ∵ `$inv(inv(f)) <=> f$` (Realizable involution)
∴ id_x ∵ `$inv(f) \cdot f <=> id_x$` (Groupoid)

∵ $\forall\ f \in hom(x, y)\ \{\ f \cdot inv(f) <=> id_y \ \wedge\ inv(f) \cdot f <=> id_x\ \}$

For any `$f : x \to y$`, one can prove directly using Category identity:

$inv(f \cdot id_x) = inv(f)$
$inv(id_y \cdot f) = inv(f)$

Another proof, using Realizable contravariance, Abstract identity inverse and Category identity:

$inv(f \cdot id_x) = inv(id_x) \cdot inv(f) = id_x \cdot inv(f) = inv(f)$
$inv(id_y \cdot f) = inv(f) \cdot inv(id_y) = inv(f) \cdot id_y = inv(f)$

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