Closed Natural Numbers

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The Peano axioms is the most commonly used axiomatic system for natural numbers. However, in modern mathematics, the distinction between countable infinite types and uncountable infinite types is very important, since every countable infinite type has an isomorphism to natural numbers. This makes it desirable to **strengthen natural numbers** in a such way that one can not construct infinity within the system of natural numbers, while still be able to construct higher infinite cardinal types by powerset.

A closed natural number system is a stronger axiomatic system built on Peano axioms, where:

$1 + 1 + 1 + \ldots = 0$	$0 \neg= \infty$
$\|closed\_nat\| = \infty$	$(\infty : closed\_nat) = false$

This means that `0` has a dual meaning, similar to `±∞` in projective reals.
This system **does not imply** that the successor function has a unique model `x : [succ] 0`.

The `closed_nat` type can also be represented with the symbol `$\mathbb{N}_c$`:

$$closed\_nat \iff \mathbb{N}_c$$

The motivation of closed natural numbers comes from the definition of the successor function:

$$succ(n : \mathbb{N}) : \mathbb{N}$$

In path semantics, this implies the trivial path `∀succ` returns `true` for all natural numbers:

$$\forall succ \iff \backslash(x : \mathbb{N}) = true$$

However, from the Peano axioms, the successor function constructs a new element of `ℕ`,
such that for some domain constraint `c` that does not return `true` for all inputs,
the constrained existential path returns `true` for some input which the constraint does not:

$$\forall\ c : \mathbb{N} \to \mathbb{B} \wedge \neg\backslash true\ \{\ \exists\ x\ \{\ (\exists succ\{c\})(x) \wedge \neg c(x)\ \}\ \}$$

This means the largest domain constraint that does not return `true` for all inputs constructs the set including the last element of natural numbers, which can be expressed as following:

$$\exists\ x : \mathbb{N}\ \{\ succ(x) = 1 + 1 + 1 + \ldots\ \}$$

It is necessary from the Peano axioms that `x` exists, but **not necessary** that `x` is **uniquely defined**.

By definition, the Peano axioms implies that the last element is a natural number:

$$1 + 1 + 1 + \ldots : \mathbb{N}$$

However, the construction of this element changes the function identity of `∀succ`, which according to the core axiom of path semantics also changes the function identity of `succ`. Under the interpretation of constrained functions, `succ <=> succ{∀succ}` must hold in order for `succ` to be a function.

The Peano axioms contains an ambiguity under the idea of a last natural number:

If one could construct the last element of the natural numbers, then this element could be used to construct another natural number, hence prove that the element used to construct it was not the "actual" last element of the natural numbers!

This ambiguity might be resolved using one of two options:

1. `succ` is not a function under interpretation of constrained functions:

   $$(\mathrm{succ} \Leftarrow\Rightarrow \mathrm{succ}\{\forall \mathrm{succ}\}) = \mathrm{false}$$

2. There exists no unique natural number which successor is the last natural number:

   $$|x : \mathbb{N}\ \{\ (\mathrm{succ}(x) = 1 + 1 + 1 + \ldots)\ \}| > 1$$

In traditional mathematics, the first option is most common, since the second option seems absurd when misinterpreted as necessity of the following statement:

$$(\exists\ x : \mathbb{N}\ \{\ \mathrm{succ}(x) = 1 + 1 + 1 + \ldots\ \}) = \mathrm{false}$$

By weakening the path semantical definition of function identity for constructors under the interpretation of constrained functions, one can do theorem proving, as long the trivial path is not used.

However, this means that Peano axioms are undecidable consistent/inconsistent under functions.

The closed natural numbers changes the Peano axioms such that `succ` becomes a function.
This happens by resolving the paradox using the second option, which traditional mathematics ignored.
By making the number that is used to construct the last natural number non-unique, one can prove that the ambiguity mentioned above **is in fact an ambiguity**. Things that are true under Peano axioms carries over to the new stronger axiomatic system.

There exists only one choice to do so while preserving the total order for all other numbers than zero:

$$\forall\ x : (\neg = 0)\ \{\ x + 1 > x\ \}$$

This choice sets the last element of natural numbers to be equal to zero:

$$1 + 1 + 1 + \ldots = 0$$

This does not mean that the last element is equal to `0 : ℕ`, but it is equal to `0 :` $\mathbb{N}_c$`.
The zero under closed natural numbers has a dual semantics: It means two things at once.

Zero is both the first and the last closed natural number. However, a traditional mathematician might suspect that this eliminates a total order, under the assumption that the number before the last exists.

It might seem that this choice eliminates a total order for closed natural numbers. However, the only way to express this relation where `x + 1 = 0` is by constructing a negative closed natural number:

$$0 > 0 - 1 \quad \wedge \quad 0 - 1 > 0 \qquad 0 - 1 : \mathbb{N}_c \qquad \forall x : \mathbb{N}_c \{ x + 1 > x \}$$

Here, negative closed natural numbers are constructed by counting back from the last natural number. However, the identity of this number is undecidable. There exists no unique function that maps from this value to the last element of natural numbers. Under closed natural numbers, `succ` is chosen to be among them, without revealing its identity. The function `succ` a function, but its identity is unknown.

To construct a negative closed natural number equals assuming `false`, because violation of core axiom of path semantics. Negative closed natural numbers are inconsistent with path semantics.

Hence, one can not construct closed natural numbers who's successor is known, without knowing their identity. Doing so is inconsistent and can be used to prove anything.

To prevent this inconsistency from happening, quantification over closed natural numbers starts with `0`, then constructs the successors until all elements of the type have been exhausted:

$$\forall x : \mathbb{N}_c \{ p(x) \} \quad \Leftrightarrow \quad \forall x [0, \infty) \{ p(x) \}$$
$$\exists x : \mathbb{N}_c \{ p(x) \} \quad \Leftrightarrow \quad \exists x [0, \infty) \{ p(x) \}$$

$$p : \mathbb{N}_c \to \text{bool}$$

Since zero is no longer double-counted, the total order is restored:

$$\forall x : \mathbb{N}_c \{ x + 1 > x \} \quad \Leftrightarrow \quad \forall x : [0, \infty) \{ x + 1 > x \}$$

This total order is isomorphic to natural numbers under the Peano axioms.

$$\forall x : \mathbb{N}_c \{ p(x) \} \quad = \quad \forall x : \mathbb{N} \{ p'(x) \}$$
$$\exists x : \mathbb{N}_c \{ p(x) \} \quad = \quad \exists x : \mathbb{N} \{ p'(x) \}$$

$\exists g : \mathbb{N}_c \to\to \mathbb{N} \{ p[g \to \text{id}] \Leftrightarrow p' \}$ is a tautology when the isomorphism `g` exists

Therefore, closed natural numbers do not violate the Peano axioms, but **strengthens** them.

This means that theorems that hold in first-order logic for natural numbers also holds for closed natural numbers. It is not needed to change the existing proofs from `ℕ` to `$\mathbb{N}_c$`.

Closed natural numbers are only necessary when introducing the trivial path `∀succ` in proofs.

To preserve the theory of higher infinite cardinals, countable infinity `∞` expresses the size of closed natural numbers, which has no symbol among closed natural numbers.

$$|\mathbb{N}_c| = \infty$$
$$0 \neg= \infty$$
$$(\infty : \mathbb{N}_c) = \text{false}$$