

Formal Definition of Normal Paths

by Sven Nilsen, 2020

In this paper I give the formal definition of normal paths in path semantics.

A homotopy path^[1] H is a map between two continuous functions f and g such that:

$$\forall x \{ H(x, 0) = f(x) \} \quad \wedge \quad \forall x \{ H(x, 1) = g(x) \}$$

$$H : X \times [0, 1] \rightarrow Y$$

$$f : X \rightarrow Y$$

$$g : X \rightarrow Y$$

It follows from this definition that there can be paths between paths, e.g. $X \Leftrightarrow [0, 1]$.

When only the end points are considered, one can use a \mathbb{B} (boolean) instead of $[0, 1]$:

$$(H \text{ false}) \Leftrightarrow f \quad \wedge \quad (H \text{ true}) \Leftrightarrow g$$

$$H : X \times \mathbb{B} \rightarrow Y$$

This means that a homotopy path represents a binary relation^[2] of an ordered pair:

$$(f, g)$$

Where the truth values of these binary relations defines the space of all paths.

When the relation is true , there exists a path, but when the relation is false , there exists no path.

In constructive mathematics^[3] one can not prove that the relation is true from $\neg \text{false}$ and vice versa.

This means that there exists some construction that represents the path: The proof of the path.

In Homotopy Theory^[4], the proof of the path is constructed by the homotopy path H .

However, this is looking at the path from the outside.

From within the path, there is only $X \rightarrow Y$, which can be proved as following.

By using function currying^[5], one can transform this type into:

$$H : X \rightarrow (\mathbb{B} \rightarrow Y) \quad \text{the choice of some function } X \rightarrow \mathbb{B} \text{ determines } X \rightarrow Y$$

The type $X \rightarrow \mathbb{B}$ is a set over X . The complexity^[6] in number of bits of this set is $\log_2(|X|) + 1$.

From within the path, there is only x , which has type X and maps into some value of Y .

Therefore, from within the path, there is only $X \rightarrow Y$, which is an ordinary function type.

The change in linear input complexity from inside to outside is $2^{\log_2(|X|) + 1} - 2^{\log_2(|X|)} = |X|$.

From using this, one can derive that the change in linear output complexity is $|Y|^{|X|} = |X \rightarrow Y|$.

Therefore, there should exist a space similar to “computation” that relates functions to each other.

Where is this space?

The insight of Homotopy Type Theory^[7] is: Dependent types^[8] can model proofs of Homotopy Theory.

This means that the direction of this space, seen from inside these paths, is analogous to dependent types. However, dependent types only describe a single step of “computation” in this direction.

What is beyond dependent types?

Using the analogy of complex numbers^[9]:

1	the base of normal computation
i	the base of path-space computation

For all complex numbers in base ` e `, the following property holds:

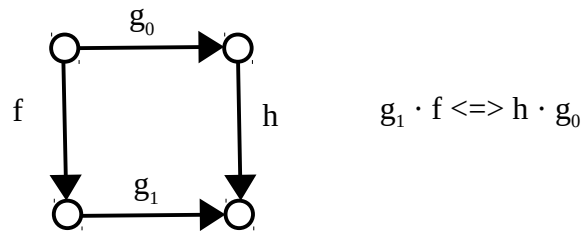
$$Ae + Be = (A + B)e$$

The analogy law for functions in base ` e ` is the following:

$$fe \cdot ge = (f \cdot g)e$$

Once the base of path-space computation is found, one can use normal function composition rules in the new dimension.

Now, if one looks at commutative diagrams^[10]:



For every point, or value, they behave as normal computation, which satisfies ` $fe \cdot ge = (f \cdot g)e$ '. Yet, the point-wise computation is not the space one is looking for, since this is covered by ` $X \rightarrow Y$ '.

From within the paths, the total complexity can be thought of as a product:

$$|X \times \mathbb{B} \rightarrow Y| = |Y|^{|X \times \mathbb{B}|} = |Y|^{2|X|} = |Y|^{|X|} \cdot |Y|^{|X|} = |X \rightarrow Y|^2$$

This means that the “missing parameter” one is looking for is the “function surface” ` $g_0 \rightarrow g_1$ `:

$$H := f[g_0 \rightarrow g_1]$$

The constructive proof is ` h ` to represent ` H ` from within homotopy paths (called “normal paths”):

$$f[g_0 \rightarrow g_1] <=> h \quad \text{Q.E.D.}$$

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