

# Permutative Symmetry Paths

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*In this paper I introduce permutative symmetry paths, which generalizes groups similarly to sub-types.*

A sub-group in Group Theory is just a group that contains a group. However, one might want to have a theory similar to Group Theory but that is more like the theory of sub-types relative to types. This theory is about discrete symmetries in general, without constraining these symmetries to groups.

A permutative symmetry path of  $f$  by a set of lists  $x_i$ , is a set of permutative sorts  $g_j$ :

$$\begin{aligned} \therefore & \quad f[x_i]_s \Leftrightarrow g_j \\ \therefore & \quad \forall i, j \{ f(x_i) == f(g_j(x_i)) \} \\ \therefore & \quad x_i \Leftrightarrow \{ x_0, x_1, x_2, \dots, x_{n-1} \} \\ \therefore & \quad g_j \Leftrightarrow \{ g_0, g_1, g_2, \dots, g_{m-1} \} \\ \therefore & \quad x : \text{nat} \rightarrow [T] \\ \therefore & \quad g : \text{nat} \rightarrow [T] \rightarrow [T] \\ \therefore & \quad f : [T] \rightarrow U \end{aligned}$$

It is assumed here that  $[T]$  has a fixed length.

If  $x_i$  contains a single element and elements of  $g_j$  satisfies the axioms of Group Theory, then the permutative symmetry path is isomorphic to a group. When  $x_i$  contains more than one element and  $g_j$  satisfies the axioms of Group Theory, one can treat it as a common group structure over labels by  $f$ .

The arrow  $\Rightarrow$  can be used to express that a sub-set of the possible permutative sorts  $g_j$  are known:

$$f[x_i]_s \Rightarrow h_j \quad \text{`h_j` contains a sub-set of `g_j`}$$

The arrow  $\Leftrightarrow$  is used only when  $g_j$  describes the largest possible set of permutative sorts.

There is always some sub-set  $\Rightarrow$  which is a group, as long it is non-empty.

Every relation, e.g.  $f([a, b, c]) == f([b, c, a])$  forms a sub-group, since  $[b, c, a]$  is its own inverse.

However, there are some cases where  $\Leftrightarrow$  is not a group.

For example,  $f(x) = x \neq [b, a, c]$ . Here,  $f([a, b, c]) == f([b, c, a]) == f([a, c, b]) == \text{false}$ .

$[b, a, c]$  is composition of permutation sorts  $[a, c, b]$  .  $[b, c, a]$  which requires closure in  $g_j$ .

By removing  $[b, a, c]$  from the set  $x_i$ , one forces  $f[x_i]_s$  to not be a group.

For every  $f$ , when a strict sub-set  $g'_j$  exists that is a group, there exists a function  $f'$  and such that:

$$f'[x_i]_s \Leftrightarrow g'_j$$

The function  $f'$  contains less symmetries than the original  $f$ .

If  $f \Leftrightarrow \text{id}$ , then  $g_j$  contains only the identity sort.

If  $f$  satisfied the following existential path equation and  $x_i$  is total:

$$\begin{aligned} f &\Leftrightarrow \lambda u \quad \text{where } u : U \\ x_i &\Leftrightarrow [T] \\ f[x_i]_s &\Leftrightarrow g_j \end{aligned}$$

Then  $g_j$  satisfies the axioms of Group Theory.

The rest of this paper is about proving this result.

The axioms of Group Theory are:

1. Closure
2. Associativity
3. Identity element
4. Inverse element

### Proof of Identity Element (3)

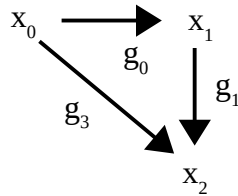
If  $g_j$  contains the largest possible set of permutative sorts, then it contains the identity sort.

### Proof of Associativity (2)

Permutation sorts are associative.

### Proof of Closure and Inverse Element (1) and (4)

Every permutation in  $[T]$  results in  $g_j$ , for any  $x_0, g_1, g_2$ , is covered by some  $g_3$  in  $g_j$ :



This is because  $[T]$  contains any possible  $x_i$  which all are assigned the same label  $u$  by  $f$ .

Among all those possible  $x_i$ , there exists all possible permutations of the list, since permutation is just a kind of modification. Since all possible permutations exists, the largest set of permutation sorts  $g_j$  includes all permutations. This includes inverse permutations, compositions etc.

Therefore, all axioms of Group theory are satisfied.

Q.E.D.