Idempotency from Commutative Symmetry

by Sven Nilsen, 2020

In this paper I prove that commutative symmetry implies idempotency of the symmetry operator.

A binary operator `f` is commutative symmetric if there exists a unary operator `g` such that:

$$\forall\ a, b\ \{\ f(a, b) = g(f(b, a))\ \} \quad \wedge \quad \exists f \iff \forall g$$

Here, `g` is called the “symmetry operator”.

When `g <=> id`, the binary operator `f` is commutative.
When `g <=> neg`, the binary operator `f` is anti-commutative.

Commutative symmetry unifies the properties of commutative and anti-commutative operators.

From commutative symmetry, one can prove the following:

$$\forall\ a, b\ \{\ f(a, b) = g(f(b, a))\ \} \quad = \quad \forall\ a, b\ \{\ g(f(a, b)) = f(b, a)\ \}$$

In path semantical notation:

$$f \iff f[swap \rightarrow g] \quad \iff \quad f[id \times id \rightarrow g] \iff f[swap \rightarrow id]$$

Proof:

∵ $\forall\ a, b\ \{\ g(f(a, b)) = f(b, a)\ \}$
∴ $\forall\ a, b\ \{\ f(b, a) = g(f(a, b))\ \}$ using `(x = y) = (y = x)`
∴ $\forall\ b, a\ \{\ f(a, b) = g(f(b, a))\ \}$ replacing `a => b` and `b => a`
∴ $\forall\ a, b\ \{\ f(a, b) = g(f(b, a))\ \}$ using `∀ x, y { … } = ∀ y, x { … }`

Now, one can use this to prove that the symmetry operator `g` is idempotent:

$$g^2 \iff id$$

Proof:

∵ $g(g(f(a, b))$
∴ $g(f(b, a))$ using `∀ a, b { g(f(a, b)) = f(b, a) }`
∴ $f(a, b)$ using `∀ a, b { f(a, b) = g(f(b, a)) }`

Strictly said, this only proves `g · g{∃f} <=> id{∃f}`.
However, since `∃f <=> ∀g`, g · g{∃f} <=> g · g{∀g} <=> g · g <=> g^2` and `id{∃f} <=> id{∀g}`.
Under this condition, `$g^2$ <=> id{∀g}` which can be simplified to `$g^2$ <=> id`.

Q.E.D.