Associativity as Asymmetric Paths

by Sven Nilsen, 2019

In this paper I show that associativity can be expressed as one of two asymmetric paths.
The two asymmetric paths are provable from each other.

Associativity[1] of a binary operator \`op\` can be expressed in first-order logic[2] as:

$\because \quad \forall\ x, y, z\ \{\ op(op(x, y), z) = op(x, op(y, z))\ \}$

$\because \quad op : T \times T \to T$

In Category Theory[3], it is common to lift this into an equation of functional extensional equality[4]:

$\because \quad op \cdot (op \times id_T) \Leftrightarrow op \cdot (id_T \times op)$

$\because \quad id_T : T \to T$ The identity function[5]

Here, \`·\` means function composition[6] and \`×\` means Cartesian product[7].

In Path Semantics[8], one can express associativity as one of two equivalent asymmetric paths[9]:

$\because \quad (op \times id_T)[id_T \times op \to op] \Leftrightarrow op$
$\because \quad (id_T \times op)[op \times id_T \to op] \Leftrightarrow op$

$\therefore \quad op \times id_T : T \times T \times T \to T \times T$ cheating with a bit syntax sugar \`(T × T) × T ~= T × T × T\`
$\therefore \quad id_T \times op : T \times T \times T \to T \times T$ cheating with a bit syntax sugar \`T × (T × T) ~= T × T × T\`

When not cheating with a bit syntax sugar, the types are:

$\therefore \quad op \times id_T : (T \times T) \times T \to T \times T$
$\therefore \quad id_T \times op : T \times (T \times T) \to T \times T$

Cartesian products are not necessarily associative in all languages, but almost associative:

$(T \times T) \times T \sim= T \times (T \times T)$

This means that these two forms are equivalent, but not necessary equal.
Equivalence means there exists a function that turns one into another:

$\because \quad (fst \cdot fst, (snd \cdot fst, snd)) : (T \times T) \times T \to T \times (T \times T)$
$\because \quad ((fst, fst \cdot snd), snd \cdot snd) : T \times (T \times T) \to (T \times T) \times T$

$\therefore \quad ((fst, fst \cdot snd), snd \cdot snd) \cdot (fst \cdot fst, (snd \cdot fst, snd)) \Leftrightarrow id_{(T \times T) \times T}$
$\therefore \quad (fst \cdot fst, (snd \cdot fst, snd)) \cdot ((fst, fst \cdot snd), snd \cdot snd) \Leftrightarrow id_{T \times (T \times T)}$

The two laws, without cheating with some syntax sugar, would look like this:

$\because$ $(op \times id_T)[(fst \cdot fst, (snd \cdot fst, snd)) \rightarrow id][id_T \times op \rightarrow op] \Leftrightarrow op$
$\because$ $(id_T \times op)[((fst, fst \cdot snd), snd \cdot snd) \rightarrow id][op \times id_T \rightarrow op] \Leftrightarrow op$

This can also be written[10]:

$\because$ $(op \times id_T)[(id_T \times op) \cdot (fst \cdot fst, (snd \cdot fst, snd)) \rightarrow op] \Leftrightarrow op$
$\because$ $(id_T \times op)[(op \times id_T) \cdot ((fst, fst \cdot snd), snd \cdot snd) \rightarrow op] \Leftrightarrow op$

The two laws are provable from each other. Considering the following:

$$(op \times id_T)[(fst \cdot fst, (snd \cdot fst, snd)) \rightarrow id] : T \times (T \times T) \rightarrow T \times T$$
$$(id_T \times op)[((fst, fst \cdot snd), snd \cdot snd) \rightarrow id] : (T \times T) \times T \rightarrow T \times T$$

Notice that this has the same type as by swapping the arguments:

$$id_T \times op : T \times (T \times T) \rightarrow T \times T$$
$$op \times id_T : (T \times T) \times T \rightarrow T \times T$$

If there are two functions `$f_0$` and `$f_1$` of the following types:

$$f_0 : T \times (T \times T) \rightarrow T \times T$$
$$f_1 : (T \times T) \times T \rightarrow T \times T$$

Then there exists two path function product `$g_{01}$` and `$g_{10}$` such that:

$$f_0[g_{01}] \Leftrightarrow f_1$$
$$f_1[g_{10}] \Leftrightarrow f_0$$

This works by using the equivalence of Cartesian products. The solutions are:

$$g_{01} \Leftrightarrow (\ ((fst, fst \cdot snd), snd \cdot snd) \rightarrow id\ )$$
$$g_{10} \Leftrightarrow (\ (fst \cdot fst, (snd \cdot fst, snd)) \rightarrow id\ )$$

Since these are known to be inverses of each other, this proves that swapping the arguments plus the fix for syntax sugar is valid. By swapping the arguments plus the fix for syntax sugar, the two expressions 1) and 2) belong to the same group[11]:

$$(op \times id_T)[(id_T \times op) \cdot (fst \cdot fst, (snd \cdot fst, snd)) \rightarrow op] \quad (1)$$
$$(id_T \times op)[(op \times id_T) \cdot ((fst, fst \cdot snd), snd \cdot snd) \rightarrow op] \quad (2)$$

$$op \times id_T \Leftrightarrow id_T \times op$$
$$(fst \cdot fst, (snd \cdot fst, snd)) \Leftrightarrow ((fst, fst \cdot snd), snd \cdot snd)$$

The combination of these swap operations is structure preserving from the axioms of group theory[11] and correct assuming that the asymmetric path exists. Since the asymmetric path is `op` when `op` is associative, the asymmetric path exists and therefore the two laws are isomorphic.

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