

Continuous Monotonicity

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In this paper I describe the class of design problems of finding continuous monotonic functions.

The basic building block for continuous functions is the parameterized straight line:

$$\text{line}(a, b) = \lambda(t : \text{real}) = t * (b - a) + a$$

A straight line is continuous monotonic, which means that it progresses toward the end point everywhere without intersecting itself. This can be proved by taking the derivative:

$$\begin{aligned} \lim h \rightarrow 0 \{ & (\text{line}(a, b)(x + h) - \text{line}(a, b)(x)) / h \} \\ \lim h \rightarrow 0 \{ & ((x + h) * (b - a) + a - (x * (b - a) + a)) / h \} \\ \lim h \rightarrow 0 \{ & ((x + h) * (b - a) - x * (b - a)) / h \} \\ \lim h \rightarrow 0 \{ & (x * (b - a) + h * (b - a) - x * (b - a)) / h \} \\ \lim h \rightarrow 0 \{ & (h * (b - a)) / h \} \\ \lim h \rightarrow 0 \{ & b - a \} \\ & b - a \end{aligned}$$

Since `b - a` is a constant, it means that the line has the same “sign” everywhere, which is a proof of that it does not intersect itself. If it did, the “sign” would change at some point. For vectors of `a` and `b`, the “sign” is simply a vector of the signs of individual components. Change of sign in the derivative is used as the practical definition of self intersection, since if `b - a = 0`, then technically there would be a self intersection but without change of sign.

The derivative of a single-argument function in path semantics is a function:

$$d : (\text{real} \rightarrow T) \rightarrow (\text{real} \rightarrow T)$$

When `T <=> real`, under Higher Order Operator Overloading:

$$\text{sign}(d(f)) : \text{real} \rightarrow \text{real}$$

For example:

$$(x^2)' = 2x = \text{mul}(2, x)$$

$$\text{mul}[\text{sign}] \Rightarrow \text{mul}$$

Therefore:

$$\text{sign}(\text{mul}(2, x)) = \text{mul}(\text{sign}(2), \text{sign}(x))$$

When `x > 0`, the sign is positive, so `{(> 0)}(x) = x^2` is continuous monotonic.