Involution from Commutative Symmetry

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In this paper I prove that commutative symmetry implies involution of the symmetry operator.

A binary operator `f` is commutative symmetric if there exists a unary operator `g` such that:

$$\forall a, b \{ f(a, b) = g(f(b, a)) \} \quad \wedge \quad \exists f \Leftrightarrow \forall g$$

Here, `g` is called the “symmetry operator”.

When `g <=> id`, the binary operator `f` is commutative.
When `g <=> neg`, the binary operator `f` is anti-commutative.

Commutative symmetry unifies the properties of commutative and anti-commutative operators.

From commutative symmetry, one can prove the following:

$$\forall a, b \{ f(a, b) = g(f(b, a)) \} \qquad = \qquad \forall a, b \{ g(f(a, b)) = f(b, a) \}$$

In path semantical notation:

$$f \Leftrightarrow f[swap \rightarrow g] \qquad \Leftrightarrow \qquad f[id \times id \rightarrow g] \Leftrightarrow f[swap \rightarrow id]$$

Proof:

∵	$\forall a, b \{ g(f(a, b)) = f(b, a) \}$	
∴	$\forall a, b \{ f(b, a) = g(f(a, b)) \}$	using `(x = y) = (y = x)`
∴	$\forall b, a \{ f(a, b) = g(f(b, a)) \}$	replacing `a => b` and `b => a`
∴	$\forall a, b \{ f(a, b) = g(f(b, a)) \}$	using `∀ x, y { … } = ∀ y, x { … }`

Now, one can use this to prove that the symmetry operator `g` is an involution:

$$g^2 \Leftrightarrow id$$

Proof:

∵	$g(g(f(a, b))$	
∴	$g(f(b, a))$	using `∀ a, b { g(f(a, b)) = f(b, a) }`
∴	$f(a, b)$	using `∀ a, b { f(a, b) = g(f(b, a)) }`

Strictly said, this only proves `g · g{∃f} <=> id{∃f}`.
However, since `∃f <=> ∀g`, g · g{∃f} <=> g · g{∀g} <=> g · g <=> g^2` and `id{∃f} <=> id{∀g}`.
Under this condition, `$g^2$ <=> id{∀g}` which can be simplified to `$g^2$ <=> id`.

Q.E.D.