# Univalent Involutions

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*In this paper I prove that isomorphisms are isomorphic to Univalent Involutions.*

An isomorphism[1] from a Category[2] theoretic view is a morphism `f` with an inverse `g` such that:

$$f \cdot g <=> id_B$$
$$g \cdot f <=> id_A$$

$$f : A \to B$$
$$g : B \to A$$

An involution[3] is a morphism `h` such that:

$$h \cdot h <=> id_T$$

$$h : T \to T$$

Every involution is an isomorphism, but not every isomorphism is an involution.

It turns out that every isomorphism can be turned into a Univalent Involution:

```
h := \(x : T) = if let some(a) = h'A^-1(x) { h'B(f(a)) }
                else if let some(b) = h'B^-1(x) { h'A(g(b)) }
                else { unreachable!() }
```

| | |
|---|---|
| $T := A \mid B$ | `T` is the sum type of `A` and `B` |
| $h'_A : A \to T$ | Lifts `A` into `T` |
| $h'_B : B \to T$ | Lifts `B` into `T` |
| $h'^{-1}_A : T \to opt[A]$ | Takes `A` out of some `T` |
| $h'^{-1}_B : T \to opt[B]$ | Takes `B` out of some `T` |

The univalent involution `h` has the following normal paths (where `opt` is used as a functor):

$$h[h'^{-1}_A \to h'^{-1}_B] <=> opt(f)$$
$$h[h'^{-1}_B \to h'^{-1}_A] <=> opt(g)$$

A Univalent Involution differs from ordinary involutions by the property it can be turned back into a heterogenous isomorphism, kind of like a tuple `(a, b)` can be turned into `a` and `b`:

$$h.0 <=> f$$
$$h.1 <=> g$$

Since equality in Intuitionistic Logic[4] using types is a tuple `(f, g)`,
this particular form of involution is thought to be univalent[5].

## References:


[1] "Isomorphism"
Wikipedia
https://en.wikipedia.org/wiki/Isomorphism

[2] "Category (mathematics)"
Wikipedia
https://en.wikipedia.org/wiki/Category_(mathematics)

[3] "Involution (mathematics)"
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https://en.wikipedia.org/wiki/Involution_(mathematics)

[4] "Intuitionistic logic"
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https://en.wikipedia.org/wiki/Intuitionistic_logic

[5] "Homotopy type theory"
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https://en.wikipedia.org/wiki/Homotopy_type_theory