Existential Path of Left Recursive Addition

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Assume there is an operator `=>` which takes the left side and puts it into the variable at the right side:

$$x + 1 \Rightarrow x \qquad\qquad x : \mathrm{nat}$$

The `=>` operator repeats the construction infinitely and simplifies:

$$(x + 1) + 1$$
$$((x + 1) + 1) + 1$$
$$(((x + 1) + 1) + 1) + 1$$
$$\ldots$$
$$x + (1 + 1 + 1 + 1 + \ldots) \qquad \text{Applying law of associativity}$$
$$x + \infty \qquad \text{Simplifying}$$

Using the following axiom:

$$\forall\ x : \mathrm{nat},\ y : \mathrm{nat}\ \{\ x + \infty = y + \infty\ \}$$

The expression can be simplified further:

$$x + \infty$$
$$0 + \infty \qquad \text{Using the axiom above, choosing `x = 0`}$$
$$\infty$$

Therefore:

$$x + 1 \Rightarrow x \quad \Leftrightarrow \quad \infty$$

$$x : \mathrm{nat}$$

In general:

$$x + y \Rightarrow x \quad \Leftrightarrow \quad \infty$$

$$x : \mathrm{nat}$$
$$y : \mathrm{nat} \wedge (> 0)$$

When `y = 0`:

$$x + 0 \Rightarrow x \quad \Leftrightarrow \quad x$$

Written as a higher order existential path:

$$\exists(x + y \Rightarrow x) \ \Leftrightarrow \quad \backslash(a : \mathrm{nat} \mid (= \infty)) = \text{if } y == 0\ \{\ a == x\ \}\ \text{else}\ \{\ a == \infty\ \}$$