Parameter Elimination in Higher Order Existential Paths

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When a higher order existential path of `f` is functionally equivalent to some function `g` quantified over all values of some parameter `a`, then the quantified parameter can be eliminated:

$$\forall\ a\ \{\ \exists f(a) <=> g\ \} \qquad => \qquad \exists f <=> g$$

Notice that `∃f(a)` is interpreted as `∃(f(a))`. This is the precedence rule of the `∃` unary operator.

When analyzing properties of functions, it is desirable to use quantified expressions of this form, because the material implication `=>` means that the quantified expression `∀ a { ∃f(a) <=> g }` contains more information than `∃f <=> g`. However, in theorem proving it might suffice to show that `∃f <=> g` for some step in the proof. Since there is a trade-off between accuracy and simplicity, it is recommended that papers who use quantified expressions also show parameter elimination (if the reader target audience is unfamiliar with parameter elimination in higher order existential paths).

This law is particular useful in non-deterministic path semantics, such as the fields of physics and machine learning, when one wants to guarantee that some random generator will eventually output all values in some range, even if the expected time to do so might be arbitrarily long. For obscure functions, when treating the function as a black box, this property can only be derived analytically and not by looking at the input and output values. Using quantified expressions and then eliminating them, is a way of arriving at a statement about the existential path by splitting the problem into smaller sub-problems.

For example:

$$\forall\ a : A_1,\ b : B_1\ \{\ \exists f(a, b) <=> g\ \}$$

$$f : A_0 \times B_0 \rightarrow C$$

The parameters `a` and `b` can only be eliminated if `$A_0 \subseteq A_1$` and `$B_0 \subseteq B_1$`.

$$\exists f <=> g$$

If only `$A_0 \subseteq A_1$` holds, then one can write:

$$\forall\ b : B_1\ \{\ \exists(f\ b) <=> g\ \}$$

If only `$B_0 \subseteq B_1$` holds, then one can write:

$$\forall\ a : A_1\ \{\ \exists f(a) <=> g\ \}$$