Integral of Polynomial Product

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In this paper I derive the integral formula for polynomial product.

The integral formual for polynomial product is the following:

$$\int \sum i\ n\ \{ a_i x^i \} \cdot \sum j\ n\ \{ b_j x^j \}\ dx \qquad => \qquad \sum i\ 2n\text{-}1\ \{ (i+1)^{-1} \sum j\ i+1\ \{ a_{i-j} b_j \}\ x^{i+1} \} + C$$

The rest of this paper is a proof of this formula.

When one takes the integral of the form:

$$\int f(x) \cdot g(x)\ dx$$

$$f : T \to T$$
$$g : T \to T$$

One wants to first "normalize" the product such that it becomes a function of a single variable:

$$mul[f^{-1} \times g^{-1} \to id]\{eq\}[id \times unit \to id] : T \to T$$

$$mul[f^{-1} \times g^{-1} \to id] <=> h$$

$$mul(a, b) = h(f^{-1}(a), g^{-1}(b))$$

$$h(f^{-1}(f(x)), g^{-1}(g(x)) <=> h(x, x)$$

$$h\{eq\}[id \times unit \to id] => h'(x)$$

For example:

$$f(x) = k_f x \qquad f^{-1}(x) = k_f^{-1} x$$
$$g(x) = k_g x \qquad g^{-1}(x) = k_g^{-1} x$$

$$mul(a, b) = h(f^{-1}(a), g^{-1}(b))$$
$$mul(a, b) = h(k_f^{-1} a, k_g^{-1} b)$$
$$mul(k_f a, k_g b) = h(a, b)$$ Substituting `a => $k_f$a` and `b => $k_g$b`
$$k_f \cdot k_g \cdot mul(a, b) = h(a, b)$$
$$h <=> (\cdot\ k_f k_g) \cdot mul$$

$$((\cdot\ k_f k_g) \cdot mul)\{eq\}[id \times unit \to id]$$
$$(\cdot\ k_f k_g) \cdot mul\{eq\}[id \times unit \to id]$$
$$(\cdot\ k_f k_g) \cdot (pow\ 2)$$

$$h'(x) = k_f k_g x^2$$

Generalizing:

$f(x) = k_f x^n$ $\quad f^{-1}(x) = k_f^{-1/n} x^{1/n}$
$g(x) = k_g x^m$ $\quad g^{-1}(x) = k_g^{-1/m} x^{1/m}$

$mul(a, b) = h(f^{-1}(a), g^{-1}(b))$
$mul(a, b) = h(k_f^{-1/n} a^{1/n}, k_g^{-1/m} b^{1/m})$
$mul(k_f a^n, k_g b^m) = h(a, b)$
$k_f \cdot k_g \cdot mul(a^n, b^m) = h(a, b)$
$h \Leftrightarrow (\cdot\ k_f k_g) \cdot mul \cdot ((pow\ n), (pow\ m))$

$(\cdot\ k_f k_g) \cdot (mul \cdot ((pow\ n), (pow\ m)))\{eq\}[id \times unit \rightarrow id]$
$(\cdot\ k_f k_g) \cdot (pow\ n + m)$

$h'(x) = k_f k_g x^{n+m}$

Now, one can derive the integral:

$\int \sum i\ n \{ a_i x^i \} \cdot \sum j\ m \{ b_j x^j \}\ dx$
$\int \sum i\ n \{ a_i x^i \cdot \sum j\ m \{ b_j x^j \} \}\ dx$
$\int \sum i\ n \{ \sum j\ m \{ a_i x^i \cdot b_j x^j \} \}\ dx$
$\int \sum i\ n, j\ m \{ a_i x^i \cdot b_j x^j \}\ dx$
$\int \sum i\ n, j\ m \{ a_i b_j x^{i+j} \}\ dx$ "Normalizing" to a single variable
$\sum i\ n, j\ m \{ \int a_i b_j x^{i+j}\ dx \}$
$\sum i\ n, j\ m \{ a_i b_j (i+j+1)^{-1} x^{(i+j+1)} + C \}$ Solving the integral using standard formula
$\sum i\ n, j\ m \{ (i+j+1)^{-1} a_i b_j x^{(i+j+1)} \} + C$

By writing down a table for these terms, one can see that terms of same power lies on a diagonal:

	$\mathbf{a_0x^0}$	$\mathbf{a_1x^1}$	$\mathbf{a_2x^2}$	$\mathbf{a_3x^3}$
$\mathbf{b_0x^0}$	$a_0b_0x^1$	$½\ a_1b_0x^2$	$1/3\ a_2b_0x^3$	$¼\ a_3b_0x^4$
$\mathbf{b_1x^1}$	$½\ a_0b_1x^2$	$1/3\ a_1b_1x^3$	$¼\ a_2b_1x^4$	$1/5\ a_3b_1x^5$
$\mathbf{b_2x^2}$	$1/3\ a_0b_2x^3$	$¼\ a_1b_2x^4$	$1/5\ a_2b_2x^5$	$1/6\ a_3b_2x^6$
$\mathbf{b_3x^3}$	$¼\ a_0b_3x^4$	$1/5\ a_1b_3x^5$	$1/6\ a_2b_3x^6$	$1/7\ a_3b_3x^7$

This pattern can be used to compute the coefficients of the integral polynomial directly:

$\sum i\ n+m-1 \{ (i+1)^{-1} \sum j\ i+1 \{ a_{i-j} b_j \}\ x^{i+1} \} + C$

If the polynomials are represented as vectors with coefficients,
then this formula must use zero when the index is out of bounds.