Existential Path in Boolean Path Semantics

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Boolean Path Semantics is path semantics restricted to functions of type `bool^N → bool`. In this paper I describe an interpretation of the existential path in Boolean Path Semantics for theorem proving.

The existential path `∃f{∀f}` in Boolean Path Semantics is a function that tells what `f` returns, constrained by `∀f` (if `f` is a constrained function, then `∀f` is associated with `f`):

$$\exists f\{\forall f\} : \mathrm{bool} \to \mathrm{bool}$$

$$f : \mathrm{bool}^N \to \mathrm{bool}$$
$$\forall f : \mathrm{bool}^N \to \mathrm{bool}$$

One way to interpret `∃f{∀f}` is as a more general proof than proving something to be true:

$(\exists f\{\forall f\})(\mathrm{false}) \Rightarrow \exists\ x : \forall f\ \{\ \neg f(x)\ \}$ “`f` returns `false` for some values of `∀f`”
$(\exists f\{\forall f\})(\mathrm{true}) \Rightarrow \exists\ x : \forall f\ \{\ f(x)\ \}$ “`f` returns `true` for some values of `∀f`”

`x : bool^N` can be e.g. any data that fits in `N` bits of computer memory.
These truth values can be combined to obtain more information than a normal proof does:

	(∃f{∀f})(false)	**¬(∃f{∀f})(false)**
(∃f{∀f})(true)	$\exists\ x : \forall f\ \{\ f(x)\ \} \wedge \exists\ x : \forall f\ \{\ \neg f(x)\ \}$	$\forall\ x : \forall f\ \{\ f(x)\ \}$
¬(∃f{∀f})(true)	$\forall\ x : \forall f\ \{\ \neg f(x)\ \}$	$\neg\exists\ x : \forall f\ \{\ f(x)\ \} \wedge \neg\exists\ x : \forall f\ \{\ \neg f(x)\ \}$

Each of these statements corresponds to a function of type `bool → bool`. There are 4 such functions:

$$(\exists f\{\forall f\} \Leftrightarrow \mathrm{false}_1) \Leftrightarrow \neg\exists\ x : \forall f\ \{\ f(x)\ \} \wedge \neg\exists\ x : \forall f\ \{\ \neg f(x)\ \}$$
$$(\exists f\{\forall f\} \Leftrightarrow \mathrm{not}) \Leftrightarrow \forall\ x : \forall f\ \{\ \neg f(x)\ \}$$
$$(\exists f\{\forall f\} \Leftrightarrow \mathrm{id}) \Leftrightarrow \forall\ x : \forall f\ \{\ f(x)\ \}$$
$$(\exists f\{\forall f\} \Leftrightarrow \mathrm{true}_1) \Leftrightarrow \exists\ x : \forall f\ \{\ f(x)\ \} \wedge \exists\ x : \forall f\ \{\ \neg f(x)\ \}$$

With other words, to prove something, one must show that:

$$\exists f\{\forall f\} \Leftrightarrow \mathrm{id}$$

Since this is equivalent to:

$$\forall\ x : \forall f\ \{\ f(x)\ \}$$

Here, `∀f` can be thought of as an assumption and `f` as statement holding under that assumption.
It can also be thought of as a data structure and `f` as a property that is true for that data structure.