Equations as Algebraic Objects

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It is common to think of algebraic objects as some kind of numbers or generalization of numbers. However, equations are algebraic objects without the corresponding number semantics. Instead, equations preserve truth value under algebraic transformations.

For example, consider the following:

$$x = 3$$

Now, think about this equation as an object `E`:

E = “x = 3”

When adding a number to both sides of an equation, e.g. `+ 2`:

$$x + 2 = 3 + 2$$
$$x + 2 = 5$$

One can think about this as the following algebraic transformation on the equation `E`:

$$E + 2$$

This gives a new equation `E’`:

$$E' = E + 2$$

When we perform operations to equations, we create new equation objects, such that the truth values before and after the transformation are the same:

E : [truth_value] a <=> E’ : [truth_value] a

This property means that equations can be generalized for truth values other than booleans.

Any operator one can do to both sides of the equation that preserves the structure of the truth value, is a valid algebraic transformation. This means that operations that erases information about the truth values are invalid, e.g. when multiplying with `0` at both sides:

$x = 3$	True if and only if `x = 3`
$x \cdot 0 = 3 \cdot 0$	Invalid algebraic transformation of equation
$0 = 0$	Always true

The whole point of transforming equations is to prove that the original equation is `true` if the resulting equation is `true`, or to prove that the original equation is `false` if the resulting equation is `false`. Therefore, it is meaningful only if the truth value is preserved.