Many-Value Logics and Involutions

by Sven Nilsen, 2021

In this paper I discuss the properties of involutions in many-value logics.

Many-value logics[1] of 2^N values constructed by orthogonal[2] involutions[3] are of special importance. This is because they can naturally be encoded in binary.

Involutions are important in Avatar Extensions[4], with a rule such that involutions of products do not generate new elements that are not covered by products of elements under involutions.

$$-(a * b) = (-a) * b = a * (-b) \qquad \text{Involutions of products must be covered}^{[5]}$$

A many-value logic can have multiple involutions:

$$\mathrm{not}_0, \mathrm{not}_1, \ldots , \mathrm{not}_{n-1}$$

An involution means that composed with itself, it is the same as the identity map:

$$\mathrm{not} \;.\; \mathrm{not} \iff \mathrm{id}$$

The composition of any two othogonal involutions commute:

$$\forall\, i, j \;\{\; \mathrm{not}_i \;.\; \mathrm{not}_j \iff \mathrm{not}_j \;.\; \mathrm{not}_i \;\}$$

When two different orthogonal involutions are composed, they form a new involution.
These involutions are equivalent to moving along diagonals in a hypercube[6].

Starting with some identity element `T` of `and`:

$$\forall\, x \;\{\; \mathrm{and}(x, T) = \mathrm{and}(T, x) = x \;\}$$

One can produce a canonical representation:

$$(\text{if } b_{n-1} \;\{\mathrm{id}\} \text{ else } \{\mathrm{not}_{n-1}\})(...((\text{if } b_1 \;\{\mathrm{id}\} \text{ else } \{\mathrm{not}_1\})((\text{if } b_0 \;\{\mathrm{id}\} \text{ else } \{\mathrm{not}_0\})(T))))$$

Where `b` is a bit vector. Hence, all elements are representable in binary encoding.

The bits in the bit vector might be chosen with different encodings to reflect the interpretation of the logic. For example, in Answered Modal Logic[7], the identity element `T` is represented as `01` which would correspond to `11` in the canonical form. Answered Modal Logic chooses a different encoding because `11` is used to represent “unknown unanswered”, which is kind of like a set of the possible answered knowledge states.

References:

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