

Monotonic Non-Linear Solutions

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In this paper I represent a technique of solving some non-linear equations efficiently.

A non-linear equation has a monotonic non-linear solution if there exists a monotonic function which solution can be used to compute the solution of the non-linear equation. Since the use of a monotonic function is efficient, the solution of the non-linear equation can also be efficiently computed.

For example:

$$y = x \cdot \ln(x)$$

Solving this for x is not possible using algebraic techniques.

Binary search is not possible either, since this function is not monotonic.

However, by substituting $x = x'$ and $y = x' \cdot y'$ the following equation can be solved exactly:

$$x' \cdot y' = x' \cdot \ln(x')$$

Finding the roots:

$$x' \cdot \ln(x') - x' \cdot y' = 0$$

$$x' \cdot (\ln(x') - y') = 0$$

$$x' = 0 \quad \vee \quad \ln(x') - y' = 0$$

$$\ln(x') = y'$$

$$x' = \exp(y')$$

This gives the monotonic function:

$$y = \exp(y') \cdot y'$$

When $x' = 0$, $y = \exp(0) \cdot 0 = 0$, so this contracts the two roots into one formula.

Since this function is monotonic, one can use binary search to find y' efficiently from a desired y .

Since $x = x'$ one gets:

$$x = \exp(y')$$

Testing: Find x for $y = 10$:

$$y' \approx 1.745$$

$$x = \exp(1.745) \approx 5.7259$$

$$5.7259 \cdot \ln(5.7259) \approx 10$$

Using binary search on $y = \exp(y') \cdot y'$

Compute solution

Checking with $y = x \cdot \ln(x)$