# Imaginary Inverse

by Sven Nilsen, 2020

*In this paper I suggest a useful subset of normal paths by introducing an imaginary inverse operator.*

Working on Poi[1], a pragmatic point-free theorem prover assistant, has given me some new insights on how normal paths[2] might be encoded in e.g. Operator Theory[3].

Poi is now able to prove the following, using `inv(f)` for function inverse `$f^{-1}$`:

$$f[g_0 \rightarrow g_1] \iff g_1 \cdot f \cdot g_0^{-1}$$

It is already known that when `$g_0$` is invertible, the normal path has a solution.
Unsurprisingly, laws of the form `$g_1 \cdot f \cdot g_0^{-1}$` appears everywhere in mathematics.

Consider the following case, which is also provable in Poi:

$$\mathrm{concat}[\mathrm{len}] \iff \mathrm{len} \cdot \mathrm{concat} \cdot (\mathrm{len}^{-1} \cdot \mathrm{fst}, \mathrm{len}^{-1} \cdot \mathrm{snd})$$

At first, it might seem wrong that `$\mathrm{len}^{-1}$` is possible to express, which is avoidable when using the standard normal path notation `concat[len]`, since it encodes the logical relationship without implying that every function is invertible.

However, considering that `sqrt(-1) = **i**`[4] is successful in numerical algebra,
why not invent an imaginary inverse?

Formally, `inv` as imaginary inverse is an abstract contravariant functor[5].
"Abstract" here means that `inv(f)` means simply `inv(f)`, but it can have an equivalence `inv(f) ~= g` as in Sized Type Theory[6], which is lifted to equality `inv(f) = g`.

From the perspective of linear logic[7], one could consider an imaginary inversion as a resource that must be spent in order to get a real solution. There are three ways to spend these resources:

1. Find a real solution to `inv(f)`, which might be written `$f^{-1}$` in manual proofs for compatibility
2. Make it disappear by composition e.g. `inv(f) . f <=> id`
3. Make it disappear by substitution e.g. `$g_1$ . f . inv($g_0$) <=> h`

With an imaginary inverse, it should be possible to express all laws using normal paths of total functions in an algebra with composition. From a philosophical point of view, this also challenges the justification of using normal paths, since one can use standard composition algebra with an imaginary inverse. Does this mean that normal paths are merely syntactic sugar[8]?

Normal paths are derived from sub-type paths on functions, which in turn are bootstrapped from the core axiom of path semantics[9]. Since normal paths encode a deeper relationship with path semantics as a theory, I believe there is enough philosophical justification. However, I can imagine e.g. automated theorem provers using the imaginary inverse instead, when reasoning about this subset of normal paths.

## References:


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