Adjoint Commutative Symmetry

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In this paper I show that full commutative symmetry implies a self-adjoint symmetry operator.

Full commutative symmetry is defined as following:

$$f \iff f[\mathrm{swap} \to g] \quad \wedge \quad \exists f \iff \forall g \qquad \text{Non-trivial commutative symmetry}$$
$$f \iff f[g \times g \to \mathrm{id}] \qquad \text{Trivial commutative symmetry}$$

One can write trivial commutative symmetry in the following way:

$$f \iff f[g \times g \to \mathrm{id}] \quad \iff \quad \forall\ a, b \ \{\ f(a, b) = f(g(a), g(b))\ \}$$

Using the form in first-order logic, one can prove the following:

$\because \quad \forall\ a, b \ \{\ f(a, b) = f(g(a), g(b))\ \}$
$\therefore \quad \forall\ a, b \ \{\ f(a, g(b)) = f(g(a), g(g(b)))\ \}$ replacing `b => g(b)`
$\therefore \quad \forall\ a, b \ \{\ f(a, g(b)) = f(g(a), b)\ \}$ using `$g^2$ <=> id`

This proves that full commutative symmetry implies a self-adjoint symmetry operator `g`.

The tactic `b => g(b)` is valid because `∃g <=> ∀g`.
The tactic `$g^2$ <=> id` uses idempotency from non-trivial commutative symmetry.

Alternative proof using path semantical notation:

$\because \quad f \iff f[g \times g \to \mathrm{id}]$
$\therefore \quad f[\mathrm{id} \times g \to \mathrm{id}] \iff f[g \times g \to \mathrm{id}][\mathrm{id} \times g \to \mathrm{id}]$
$\therefore \quad f[\mathrm{id} \times g \to \mathrm{id}] \iff f[(\mathrm{id} \cdot g) \times (g \cdot g) \to (\mathrm{id} \cdot \mathrm{id})]$
$\therefore \quad f[\mathrm{id} \times g \to \mathrm{id}] \iff f[g \times g^2 \to \mathrm{id}]$
$\therefore \quad f[\mathrm{id} \times g \to \mathrm{id}] \iff f[g \times \mathrm{id} \to \mathrm{id}]$ using `$g^2$ <=> id`

It is easy to see that this is an adjoint path.

This can be used to prove that whenever a self-adjoint operator `g` is idempotent,
the trivial commutative symmetry holds.

This also means it is sufficient to get full commutative symmetry when:

1. There is an adjoint path of `f` by a self-adjoint operator `g`
2. The `f` function has non-trivial commutative symmetry `f <=> f[swap → g]`