Undefined Symbols

by Sven Nilsen, 2019

In this paper I show that the core axiom of path semantics treats all undefined symbols as the same.

The core axiom of path semantics[1] is the following:

$$\begin{array}{ll}
\because & F_0(X_0),\ F_1(X_1),\ F_0 > X_0,\ F_0 = F_1 \\
\cline{2-2}
\therefore & X_0 = X_1
\end{array}$$

The association of one collection of symbols, together with a non-circular assumption, is how one asserts a formal association[2], which in turn permits theorem proving from proofs of equality.

At first sight it looks like the core axiom is missing an assumption, namely that `$F_0$` or `$F_1$` are *in use*. Since an associated collection might be simplified to a model in logic as material implication[3] (`=>`), it is easy to think that the axiom should reduce to the following tautology[4] in propositional logic[5]:

$$\begin{array}{ll}
\because & F_0 \Rightarrow X_0,\ F_1 \Rightarrow X_1,\ F_0 = F_1,\ F_0 \vee F_1 \\
\cline{2-2}
\therefore & X_0 = X_1
\end{array}$$

If one of `$F_0$` or `$F_1$` are in use, they are both in use, since one usage value is equal to the other:

$$F_0 = F_1,\ F_0 \vee F_1 \qquad \Leftrightarrow \qquad F_0 = F_1,\ F_0 \wedge F_1$$

From modus ponens[6] on both material implications, one can derive `$X_0 = X_1$`.
It is not wrong to do this, but the core axiom contains another secret by not assuming `$F_0$` or `$F_1$`:

$$\begin{array}{ll}
\because & F_0 \Rightarrow X_0,\ F_1 \Rightarrow X_1,\ F_0 = F_1 \\
\cline{2-2}
\therefore & X_0 = X_1
\end{array}$$

The core axiom in this form must be treated as an assumption:

$$\begin{array}{ll}
\because & (F_0 \Rightarrow X_0,\ F_1 \Rightarrow X_1,\ F_0 = F_1) \Rightarrow (X_0 = X_1) \\
\cline{2-2}
\therefore & \text{<can you prove something weird?>}
\end{array}$$

While this assumption is not a tautology[4], it is *almost* a tautology.
The only two cases which it returns `false` is the following:

$$F_0 = 0,\ F_1 = 0,\ X_0 = 0,\ X_1 = 1 \qquad\qquad F_0 = 0,\ F_1 = 0,\ X_0 = 1,\ X_1 = 0$$

One can see that in these two cases, the symbols that have associated collections are undefined.

It might seem surprising that one can prove the following:

$$\because \quad (F_0 \Rightarrow X_0, F_1 \Rightarrow X_1, F_0 = F_1) \Rightarrow (X_0 = X_1)$$

$$\therefore \quad (F_0 = 0, F_1 = 0) \Rightarrow (X_0 = X_1)$$

If the two symbols that have associated collections are undefined,
then the two associated collections *are equal*.

With other words, when one talks about undefined symbols, everything one can say about one undefined symbol can be said about all the others. Notice that this makes *intuitively sense*. If there is one undefined symbol, e.g. `✻` and another `⋞` and I do not know what these symbols mean, then I can say exactly the same about them: Nothing.

One can also prove that if the associated collections are not equal, then at least one symbol is in use:

$$\because \quad (F_0 \Rightarrow X_0, F_1 \Rightarrow X_1, F_0 = F_1) \Rightarrow (X_0 = X_1)$$

$$\therefore \quad (X_0 \neg= X_1) \Rightarrow F_0 \lor F_1$$

This follows from the previous law of undefined symbols, because one is provable from the other:

$$(F_0 = 0, F_1 = 0) \Rightarrow (X_0 = X_1) \qquad \Leftarrow\Rightarrow \qquad (X_0 \neg= X_1) \Rightarrow F_0 \lor F_1$$

Proof:

$$\because \quad (F_0 = 0, F_1 = 0) \Rightarrow (X_0 = X_1)$$
$$\therefore \quad \neg(X_0 = X_1) \Rightarrow \neg(F_0 = 0, F_1 = 0)$$
$$\therefore \quad (X_0 \neg= X_1) \Rightarrow \neg(F_0 = 0) \lor \neg(F_1 = 0)$$
$$\therefore \quad (X_0 \neg= X_1) \Rightarrow (F_0 \neg= 0) \lor (F_1 \neg= 0)$$
$$\therefore \quad (X_0 \neg= X_1) \Rightarrow F_0 \lor F_1$$
$$\therefore \quad (X_0 \neg= X_1) \Rightarrow F_0 \lor F_1$$

One can also prove that undefined symbols associate with some collections of symbols:

$$\because \quad (F_0 \Rightarrow X_0, F_1 \Rightarrow X_1, F_0 = F_1) \Rightarrow (X_0 = X_1)$$

$$\therefore \quad (F_0 = 0, F_1 = 0) \Rightarrow (F_0 \Rightarrow X_0, F_1 \Rightarrow X_1)$$

However, one can not prove any `X` to be `0` or `1` from undefined symbols of `F`. The associated collections of symbols are unknown. While one can not say anything in particular about undefined symbols, it does not mean that there will be nothing to say, forever. It only means that one can not say anything because it is currently unknown what the symbols mean. This interpretation has another effect: By thinking about unknown values as their type, one can bring back the non-circular assumption, e.g. `true > bool` because `true` must be defined after `bool` is defined. The result is that undefined symbols (modeled as `false` in propositional logic), satisfies the core axiom of path semantics.

References:

[1] "Path Semantics"
Sven Nilsen, 2016-2019
https://github.com/advancedresearch/path_semantics/blob/master/papers-wip/path-semantics.pdf

[2] "Asserting Formal Associations"
Sven Nilsen, 2019
https://github.com/advancedresearch/path_semantics/blob/master/papers-wip/asserting-formal-associations.pdf

[3] "Material implication (rule of inference)"
Wikipedia
https://en.wikipedia.org/wiki/Material_implication_%28rule_of_inference%29

[4] "Tautology (logic)"
Wikipedia
https://en.wikipedia.org/wiki/Tautology_(logic)

[5] "Propositional Logic"
Internet Encyclopedia of Philosophy
https://www.iep.utm.edu/prop-log/

[6] "Modus ponens"
Wikipedia
https://en.wikipedia.org/wiki/Modus_ponens