Modal Logic of Equivalences

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In this paper I introduce a modal logic of equivalences.

When there is an equivalence structure of propositions:

$(p_0 \sim= p_1) \sim= (p_2 \sim= p_3)$

p_i : bool

The modal logic of equivalences has the following modalities:

$\Box p = \forall\ i\ \{ p_i \}$ It is “necessary” that `p`
$\Diamond p = \exists\ i\ \{ p_i \}$ It is “possibly” that `p`

This can be combined with sub-types in path semantics, for example:

∵ x := (a ~= b) ~= (c ~= d)

∴ (x : (= a)) = (true ~= false) ~= (false ~= false)

∴ ¬□(x : (= a)) It is not necessary that all `x` is equal to `a`

∴ ◇(x : (= a)) It is possible that some `x` is equal to `a`

Modal logic fits here because since one is constructing equivalences,
it is interesting to reflect over the possible states, or “possible worlds” which are considered equivalent.

This makes it easier to prove things from the axioms of Sized Type Theory.

□(p : (= x)) `p` can be constructed by composing equivalence reflexivity `x ~= x`

¬◇(p : (= x)) `p` can not be constructed by any “known” method by `x`

□p <=> □(p : (= true))

¬◇p <=> □(p : (= false))