# Abstract Idempotent Functions

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*In this paper I present abstract idempotent functions and apply it to unique universal binary relations.*

An abstract idempotent function satisfies the existential path equation:

$f^2$ <=> f · f <=> \f

For all `n > 2`, `$f^n$` is logically equivalent to `\f`:

$f^n$ <=> $f^2$ · $f^{n-2}$ <=> \f · $f^{n-2}$ <=> \f

Abstract idempotent functions has an input and output type inhabited by `f`:

| | |
|---|---|
| \f | proves that `f` returns `f` for some input |
| $f^2$ | proves that `f` takes `f` as input, because `f` returns `f` for some input |

To construct `f` with only access to `f` through the language describing computations, one must have access to at least one other input that `f` takes. If the only input possible is `f`, then it is not possible to construct `f`. However, it possible to construct the abstraction `\f` of `f`. This is why `f` is abstract idempotent and not merely idempotent.

For example, in the theory of unique universal binary relations,
the special `role_of` function is abstract idempotent:

| | |
|---|---|
| role_of(x) = y | Assume that the role of `x` is `y` |
| role_of(x, y) | Written as a pair |
| role_of(y) = role_of | If this pair is a unique universal binary relation,<br>then the role of `y` must be `role_of` |

| | |
|---|---|
| role_of(role_of(x)) = role_of | |
| role_of$^2$(x) = role_of | |
| role_of$^2$ <=> \role_of | `role_of` is abstract idempotent |

Applying the rule of `n > 2`, yields the following unique universal binary relation:

| | |
|---|---|
| role_of(role_of(role_of(x))) = role_of | This holds for all abstract idempotent functions |
| role_of(role_of) = role_of | Substituting `role_of(role_of(x))` with `role_of` |
| role_of(role_of, role_of) | Written as a pair |

The second line proves both that this binary relation is universal and unique.