Commutative Symmetric Paths

by Sven Nilsen, 2017

A commutative symmetric path of `f` by `g` has the following property:

$$f[swap \rightarrow id] \iff f[g]$$

All full commutative symmetries implies a commutative symmetric path:

$$f[swap \rightarrow id] \iff f[g \times g \rightarrow id][id \times id \rightarrow g] \iff f[g]$$

There are two kinds of commutative symmetric paths:

- Fake commutative symmetric path: An another commutative symmetric path to `id` exists
- Real commutative symmetric path: No other commutative symmetric path to `id` exists

For example, multiplication of natural numbers has `id` as commutative operator:

$$mul[swap \rightarrow id] \iff mul[id]$$

Multiplication of non-zero real numbers has another commutative operator `inv`.
This is an example of a “fake” commutative operator, since it also has `id` as commutative operator:

$$b \cdot a = 1/(1/a \cdot 1/b)$$

$$mul[swap \rightarrow id] \iff mul[inv]$$

Multiplication of square matrices has a commutative operator `transpose`:

$$BA = (A^T B^T)^T$$

$$mul[swap \rightarrow id] \iff mul[transpose]$$

Multiplication of invertible square matrices has also a commutative operator `inverse`.
Notice that `id` is not a commutative operator here:

$$BA = (A^{-1} B^{-1})^{-1}$$

$$mul[swap \rightarrow id] \iff mul[inverse]$$

Anti-commutative multiplication has `neg`:

$$b \cdot a = - a \cdot b \qquad \text{because } `(-a) \cdot (-b) = a \cdot b`$$

$$mul[swap \rightarrow id] \iff mul[neg]$$