Monotonic Properties of Lists

by Sven Nilsen, 2020

A list satisfies the following law when adding a new element:

$$\begin{aligned}
\therefore \quad & \forall\ x : [T],\ y : T,\ p : T \to \text{bool}\ \{ \\
& \quad \exists\ i\ \{\ p(x[i])\ \} \Rightarrow \exists\ i\ \{\ \text{push}(x, y)[i])\ \}\ \wedge \\
& \quad \forall\ i\ \{\ p(\text{push}(x, y)[i])\ \} \Rightarrow \forall\ i\ \{\ p(x[i])\ \} \\
& \}
\end{aligned}$$

$$\because \quad \text{push} : [T] \wedge [\text{len}]\ n \times T \to [T] \wedge [\text{len}]\ n + 1$$

From this comes a deeper consequence: One can think about properties in the context of lists as objects in their own right, independent of the list and which elements it is extended with. These properties form sets for which elements of the list the property holds. These sets can only grow by adding new elements to the list. To express this idea, one can constrain the property to elements which are contained by a list:

$$\forall\ x : [T],\ y : T,\ p : T \to \text{bool}\ \{\ |[p\{\text{contains}(x)\}]\ \text{true}| \le |[p\{\text{contains}(\text{push}(x, y))\}]\ \text{true}|\ \}$$

$$\text{contains} : [T] \to T \to \text{bool}$$

So, the number of such properties that are non-empty sets grows monotonically with list size:

$$\begin{aligned}
& \forall\ x : [T],\ y : T\ \{ \\
& \quad \textstyle\sum\ p : T \to \text{bool}\ \{\ \text{if}\ |[p\{\text{contains}(x)\}]\ \text{true}| > 0\ \{1\}\ \text{else}\ \{0\}\ \}\ \} \le \\
& \quad \textstyle\sum\ p : T \to \text{bool}\ \{\ \text{if}\ |[p\{\text{contains}(\text{push}(x, y))\}]\ \text{true}| > 0\ \{1\}\ \text{else}\ \{0\}\ \} \\
& \}
\end{aligned}$$

One can think about it as the **number of properties increases**. It is possible to count these properties in a different way by lifting the property into an existential property. An existential property of a predicate `p` is existentially quantified over all elements in the list:

$$\text{there\_exists}(x : [T],\ p : T \to \text{bool}) = \exists\ i\ \{\ p[i])\ \}$$

Existential properties are monotonic increasing:

$$\forall\ x : [T],\ y : T\ \{\ |[\text{there\_exists}(x)]\ \text{true}| \ge |[\text{there\_exists}(\text{push}(x, y))]\ \text{true}|\ \}$$

Similarly, one can lift properties in another way such that **number of shared properties decreases**.
A universal property of a predicate `p` is universally quantified over all elements in the list:

$$\text{for\_all}(x : [T],\ p : T \to \text{bool}) = \forall\ i\ \{\ p(x[i])\ \}$$

Universal properties are monotonic decreasing:

$$\forall\ x : [T],\ y : T\ \{\ |[\text{for\_all}(x)]\ \text{true}| \le |[\text{for\_all}(\text{push}(x, y)]\ \text{true}|\ \}$$