Univalent Involutions

by Sven Nilsen, 2021

In this paper I prove that isomorphisms are isomorphic to Univalent Involutions.

An isomorphism[1] from a Category[2] theoretic view is a morphism \`f\` with an inverse \`g\` such that:

$$f \cdot g \iff id_B$$
$$g \cdot f \iff id_A$$

$$f : A \to B$$
$$g : B \to A$$

An involution[3] is a morphism \`h\` such that:

$$h \cdot h \iff id_T$$

$$h : T \to T$$

Every involution is an isomorphism, but not every isomorphism is an involution.

It turns out that every isomorphism can be turned into a Univalent Involution:

$$\begin{aligned} h := \backslash(x : T) = &\text{ if let some}(a) = h'^{-1}_A(x)\ \{\ h'_B(f(a))\ \} \\ &\text{ else if let some}(b) = h'^{-1}_A(b)\ \{\ h'_A(g(b))\ \} \\ &\text{ else } \{\ \text{unreachable!}()\ \} \end{aligned}$$

$T := A \mid B$	\`T\` is the sum type of \`A\` and \`B\`
$h'_A : A \to T$	Lifts \`A\` into \`T\`
$h'_B : B \to T$	Lifts \`B\` into \`T\`
$h'^{-1}_A : T \to opt[A]$	Takes \`A\` out of some \`T\`
$h'^{-1}_B : T \to opt[B]$	Takes \`B\` out of some \`T\`

The univalent involution \`h\` has the following normal paths:

$$h[h'^{-1}_A \to h'^{-1}_B] \iff f[some]$$
$$h[h'^{-1}_B \to h'^{-1}_A] \iff g[some]$$

A Univalent Involution differs from ordinary involutions by the property it can be turned back into a heterogenous isomorphism, kind of like a tuple \`(a, b)\` can be turned into \`a\` and \`b\`:

$$h.0 \iff f$$
$$h.1 \iff g$$

Since equality in Intuitionistic Logic[4] using types is a tuple \`(f, g)\`,
this particular form of involution is thought to be univalent[5].

References:

[1] “Isomorphism”
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https://en.wikipedia.org/wiki/Isomorphism

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[4] “Intuitionistic logic”
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https://en.wikipedia.org/wiki/Intuitionistic_logic

[5] “Homotopy type theory”
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https://en.wikipedia.org/wiki/Homotopy_type_theory