Integral of Polynomial Product

by Sven Nilsen, 2020

In this paper I derive the integral formula for polynomial product.

The integral formual for polynomial product is the following:

$$\int \sum i\ n\ \{\ a_i x^i\ \} \cdot \sum j\ n\ \{\ b_j x^j\ \}\ dx \quad => \quad \sum i\ n+m-1\ \{\ (i+1)^{-1} \sum j\ i+1\ \{\ a_{i-j} b_j\ \}\ x^{i+1}\ \} + C$$

The rest of this paper is a proof of this formula.

When one takes the integral of the form:

$$\int f(x) \cdot g(x)\ dx$$

$$f : T \to T$$
$$g : T \to T$$

One wants to first "normalize" the product such that it becomes a function of a single variable:

$$\text{mul}[f^{-1} \times g^{-1} \to \text{id}]\{eq\}[\text{id} \times \text{unit} \to \text{id}] : T \to T$$

$$\text{mul}[f^{-1} \times g^{-1} \to \text{id}] <=> h$$

$$\text{mul}(a, b) = h(f^{-1}(a), g^{-1}(b))$$

$$h(f^{-1}(f(x)), g^{-1}(g(x)) <=> h(x, x)$$

$$h\{eq\}[\text{id} \times \text{unit} \to \text{id}] => h'(x)$$

For example:

$$f(x) = k_f x \qquad\qquad f^{-1}(x) = k_f^{-1} x$$
$$g(x) = k_g x \qquad\qquad g^{-1}(x) = k_g^{-1} x$$

$$\text{mul}(a, b) = h(f^{-1}(a), g^{-1}(b))$$
$$\text{mul}(a, b) = h(k_f^{-1} a, k_g^{-1} b)$$
$$\text{mul}(k_f a, k_g b) = h(a, b) \qquad\qquad \text{Substituting } `a => k_f a` \text{ and } `b => k_g b`$$
$$k_f \cdot k_g \cdot \text{mul}(a, b) = h(a, b)$$
$$h <=> (\cdot\ k_f k_g) \cdot \text{mul}$$

$$((\cdot\ k_f k_g) \cdot \text{mul})\{eq\}[\text{id} \times \text{unit} \to \text{id}]$$
$$(\cdot\ k_f k_g) \cdot \text{mul}\{eq\}[\text{id} \times \text{unit} \to \text{id}]$$
$$(\cdot\ k_f k_g) \cdot (\text{pow } 2)$$

$$h'(x) = k_f k_g x^2$$

Generalizing:

$$f(x) = k_f x^n \qquad\qquad f^{-1}(x) = k_f^{-1/n} x^{1/n}$$
$$g(x) = k_g x^m \qquad\qquad g^{-1}(x) = k_g^{-1/m} x^{1/m}$$

$$mul(a, b) = h(f^{-1}(a), g^{-1}(b))$$
$$mul(a, b) = h(k_f^{-1/n} a^{1/n}, k_g^{-1/m} b^{1/m})$$
$$mul(k_f a^n, k_g b^m) = h(a, b)$$
$$k_f \cdot k_g \cdot mul(a^n, b^m) = h(a, b)$$
$$h <=> (\cdot\ k_f k_g) \cdot mul \cdot ((pow\ n)\ , (pow\ m))$$

$$(\cdot\ k_f k_g) \cdot (mul \cdot ((pow\ n), (pow\ m)))\{eq\}[id \times unit \to id]$$
$$(\cdot\ k_f k_g) \cdot (pow\ n + m)$$

$$h'(x) = k_f k_g x^{n+m}$$

Now, one can derive the integral:

$$\int \sum i\ n\ \{\ a_i x^i\ \} \cdot \sum j\ m\ \{\ b_j x^j\ \}\ dx$$
$$\int \sum i\ n\ \{\ a_i x^i \cdot \sum j\ m\ \{\ b_j x^j\ \}\ \}\ dx$$
$$\int \sum i\ n\ \{\ \sum j\ m\ \{\ a_i x^i \cdot b_j x^j\ \}\ \}\ dx$$
$$\int \sum i\ n, j\ m\ \{\ a_i x^i \cdot b_j x^j\ \}\ dx$$
$$\int \sum i\ n, j\ m\ \{\ a_i b_j x^{i+j}\ \}\ dx \qquad\qquad \text{"Normalizing" to a single variable}$$
$$\sum i\ n, j\ m\ \{\ \int a_i b_j x^{i+j}\ dx\ \}$$
$$\sum i\ n, j\ m\ \{\ a_i b_j (i+j+1)^{-1} x^{(i+j+1)} + C\ \} \qquad \text{Solving the integral using standard formula}$$
$$\sum i\ n, j\ m\ \{\ (i+j+1)^{-1} a_i b_j x^{(i+j+1)}\ \} + C$$

By writing down a table for these terms, one can see that terms of same power lies on a diagonal:

	$a_0 x^0$	$a_1 x^1$	$a_2 x^2$	$a_3 x^3$
$b_0 x^0$	$a_0 b_0 x^1$	$\frac{1}{2}\ a_1 b_0 x^2$	$1/3\ a_2 b_0 x^3$	$\frac{1}{4}\ a_3 b_0 x^4$
$b_1 x^1$	$\frac{1}{2}\ a_0 b_1 x^2$	$1/3\ a_1 b_1 x^3$	$\frac{1}{4}\ a_2 b_1 x^4$	$1/5\ a_3 b_1 x^5$
$b_2 x^2$	$1/3\ a_0 b_2 x^3$	$\frac{1}{4}\ a_1 b_2 x^4$	$1/5\ a_2 b_2 x^5$	$1/6\ a_3 b_2 x^6$
$b_3 x^3$	$\frac{1}{4}\ a_0 b_3 x^4$	$1/5\ a_1 b_3 x^5$	$1/6\ a_2 b_3 x^6$	$1/7\ a_3 b_3 x^7$

This pattern can be used to compute the coefficients of the integral polynomial directly:

$$\sum i\ n+m-1\ \{\ (i+1)^{-1} \sum j\ i+1\ \{\ a_{i-j} b_j\ \}\ x^{i+1}\ \} + C$$

If the polynomials are represented as vectors with coefficients,
then this formula must use zero when the index is out of bounds.