Non-Cover of Imaginary Inverse

by Sven Nilsen, 2023

In this paper I prove that the imaginary inverse has no cover in logical languages without avatar inequality matching in rules. It means that there are none axioms that can be added in normal logic that describe the imaginary inverse perfectly relatively to the normal inverse. The reason is that the definition of the normal inverse is tautological under the imaginary inverse and hence forces every morphism to be an isomorphism when some cover exists.

A bijective[1] inverse has the following definition:

$$\therefore \quad \mathrm{inv}(f) \cdot f \Longleftrightarrow \mathrm{id}_A \quad \wedge \quad f \cdot \mathrm{inv}(f) \Longleftrightarrow \mathrm{id}_B$$

$$\because \quad f : A \rightarrow B$$

This definition also holds for the imaginary inverse[2]. The difference between the normal inverse and the imaginary inverse is that one must prove these properties for the normal inverse, while for the imaginary inverse these properties hold for every \`f\`.

Theories with the imaginary inverse uses additional axioms to safeguard reversible computation. One common technique in Path Semantics is to use the following (e.g. the Prop library[3]):

inv(f) ~~ g   a proof that \`inv(f)\` has a solution \`g\` by path semantical quality[4]

A “cover” of the imaginary inverse is an axiom, or multiple axioms, such that (using HOOO EP[5]):

(inv(f) == g)^true   =>   inv(f) ~~ g

The cover ensures that whenever \`inv(f)\` has a solution, it can be used as if it has some solution.

The problem is that in normal logical languages without avatar inequality matching in rules (such as in Avalog[7]), it is impossible to avoid the case where \`inv(f) == g\`:

(inv(f) == inv(f))^true   =>   inv(f) ~~ inv(f)

However, one can use some other clever axioms that imply symbolic distinction[6] implicitly to lift equality into quality, hence working around some limitations of the expressiveness of the language.

Here is one example where symbolic distinction is implied implicitly, so it can lift equality:

inv(f) == f   =>   inv(f) ~~ f

The question is: Can adding such clever axioms cover all cases of the imaginary inverse?

The definition of a bijective inverse is a perfect cover. However, since it holds definitionally for the imaginary inverse, it can not be used as a cover. If any other perfect cover exists, then it is tautologically equal to the definition of a bijective inverse. Therefore, there exists no other perfect cover that can be used. Hence, no cover for the imaginary inverse exists without avatar inequality matching in rules. As a result, the cover is undecidable and can only be known partially. However, one can always safely add specific lifting axioms when the normal inverse is provable.

References:

[1] "Bijection"
Wikipedia
https://en.wikipedia.org/wiki/Bijection

[2] "Imaginary Inverse"
Sven Nilsen, 2020
https://github.com/advancedresearch/path_semantics/blob/master/papers-wip/imaginary-inverse.pdf

[3] "Prop"
AdvancedResearch – Propositional logic with types in Rust
https://github.com/advancedresearch/prop

[4] "Path Semantical Quality"
Sven Nilsen, 2021
https://github.com/advancedresearch/path_semantics/blob/master/papers-wip2/path-semantical-quality.pdf

[5] "HOOO Exponential Propositions"
Sven Nilsen, 2022-2023
https://github.com/advancedresearch/path_semantics/blob/master/papers-wip2/hooo-exponential-propositions.pdf

[6] "Symbolic Distinction"
Sven Nilsen, 2021
https://github.com/advancedresearch/path_semantics/blob/master/papers-wip2/symbolic-distinction.pdf

[7] "Avalog"
AdvancedResearch – An implementation of Avatar Logic with Prolog-like syntax
https://github.com/advancedresearch/avalog