Monotonic Non-Linear Solutions

by Sven Nilsen, 2020

In this paper I represent a technique of solving some non-linear equations efficiently.

A non-linear equation has a monotonic non-linear solution if there exists a positive monotonic function can be used to compute the unique solutions of the non-linear equation. Since the use of a monotonic function is efficient, the solution of the non-linear equation can also be efficiently computed.

For example:

$$y = x \cdot \ln(x)$$

Solving this for `x` is not possible using algebraic techniques.
Binary search is not possible either, since this function is not monotonic.

However, by substituting `x = x’` and `y = x’ · y’` the following equation can be solved exactly:

$$x' \cdot y' = x' \cdot \ln(x')$$

Finding the roots:

$$x' \cdot \ln(x') - x' \cdot y' = 0$$
$$x' \cdot (\ln(x') - y') = 0$$
$$x' = 0 \quad \vee \quad \ln(x') - y' = 0$$

$$\ln(x') = y'$$
$$x' = \exp(y')$$

This gives the monotonic function (for `y’ >= 0`):

$$y = \exp(y') \cdot y'$$

When `x’ = 0`, `y = exp(0) · 0 = 0`, so this contracts the two roots into one formula.
Since this function is monotonic, one can use binary search to find `y’` efficiently from a desired `y`.

Since `x = x’` one gets:

$$x = \exp(y')$$

Testing: Find `x` for `y = 10`:

$y' \sim= 1.745$	Using binary search on `y = exp(y’) · y’`
$x = \exp(1.745) \sim= 5.7259$	Compute solution
$5.7259 \cdot \ln(5.7259) \sim= 10$	Checking with `y = x · ln(x)`