Adversarial Discrete Topology

by Sven Nilsen, 2018

Adversarial paths are based on ideas from a discussion with Adam Nemecek. In this paper I introduce the construction of discrete topological spaces using adversarial paths.

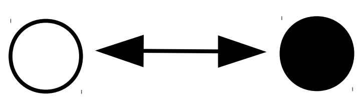
A discrete topological space is a sub-type of discrete spaces where there exists a topological path between every pair of discrete spaces. The topological path can be contracted or expanded in a such way that, seen from inside the space, the identity equals information collected by navigating around.

An adversarial discrete topological space can be constructed using the following language:

\(A : Choice, x : T) = (A ~ 0)(x)	Adversarial existential path
\(A : T → Choice, x : T) = A(x)	Application of a higher order choice
\(p : T → bool) = ∀ x : T { p(x) }	First order logic “for all” of predicate
\(A : Choice, B : Choice) = A <=> B	Logical equivalence of choices
\(p : bool, q : bool) = p ∧ q	Logical AND of predicate
\() = ()	Unit value

Where \`T\` is a generic parameter. There can be multiple types of choices.

Here is an example:

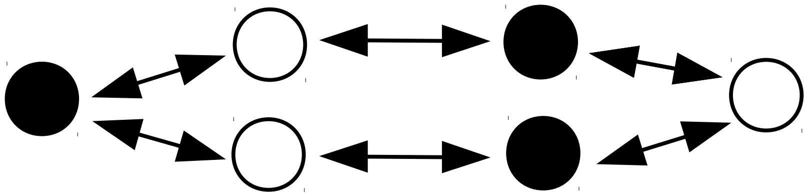


This discrete space consist of two states with two adversarial paths. However, in the language of adversarial discrete topology, it is not possible to express that there are “just two” states. Instead, a more general higher order discrete space is defined, which is contractible to the example above:

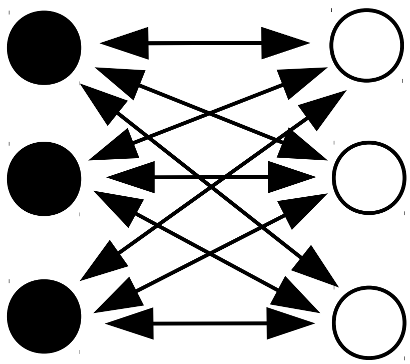
∀ x { (B ~ 0)(x) <=> W }   ∧   ∀ x { (W ~ 0)(x) <=> B }
for every black choice you get white, and, for every white choice you get black

W = “white”, B = “black”

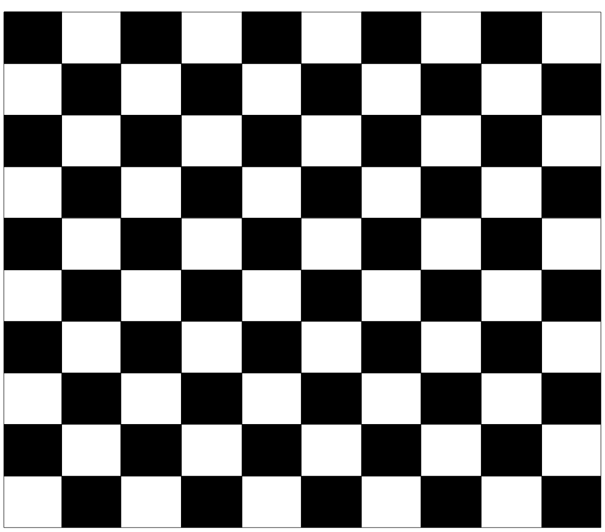
Another discrete space that satisfies the sub-type above:



One can also do this:



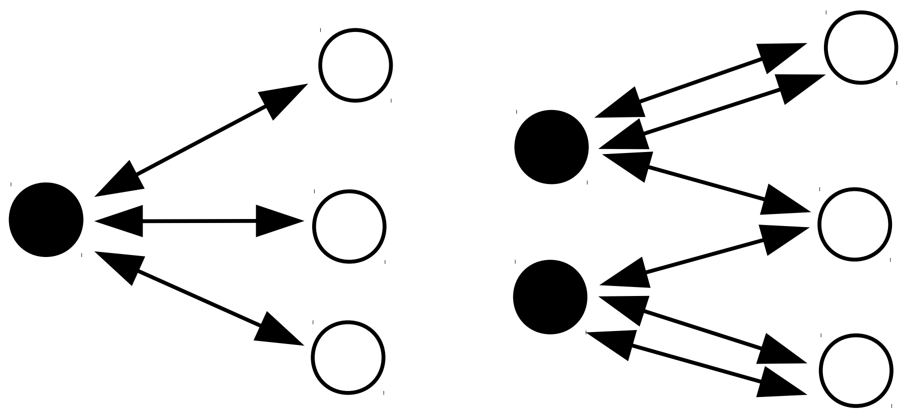
Or, imagine an infinite checker board where you can only move left/right/up/down:



Seen from inside the space, it is not possible to distinguish between any of these spaces.

Notice that the number of edges must be the same for every node of same color.
The number of adverserial paths must be the same for every choice of same type.

However, you can change the number of nodes and edges, as long it is done consistently:



An adversarial path “encodes” what it means to lose track of the state that makes it possible to distinguish these diagrams from each other. Since the existential path of an adversarial path is the only knowledge available, it is only possible to “reconstruct” the knowledge of how a choice influences the state from the topology of the discrete space.

However, the topological property assumes that the number of choices are unknown, which is not true for all types. Instead, types are divided into equivalence classes and then the type information is erased.

It is possible to express that there is just one choice using the unit type `()`, but it is not possible to express that some type is identical to the unit type or not. This means that one sentence is contractible into another from a *homotopical* perspective, which permits proofs that two topological discrete spaces share some discrete space.

In the language of adversarial discrete topology, it is only possible to construct sentences that permits valid reasoning according to the semantics of adversarial paths, with the additional constraint that just enough knowledge about types is erased to get topological reasoning.

With other words, adversarial discrete topology is a language that describes the limits of what agents who navigate adversarial paths can know about the world. They do not know the number of choices they have, except that there is at least one, but they can infer lots of things from the topology alone.

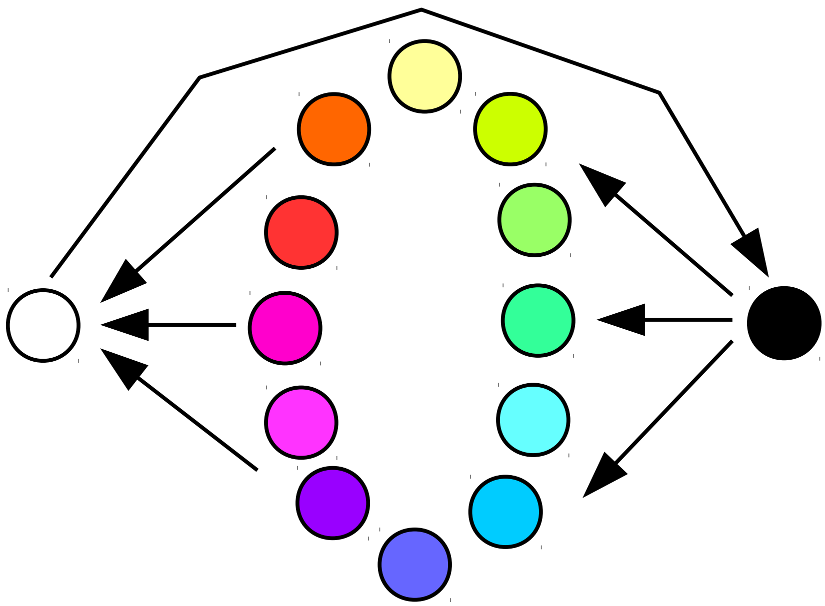
Here is another example:

$$\forall\ x\ \{\ (C(x) \sim 0)(()) \Leftrightarrow W\ \}\quad \wedge\quad \forall\ x\ \{\ (B \sim 0)(x) \Leftrightarrow C(x)\ \}\quad \wedge\quad (W \sim 0)(()) \Leftrightarrow B$$

for every color choice you get white, and, for every black choice you get color, and, white choice gives you black

W = “white”, C = “color”, B = “black”

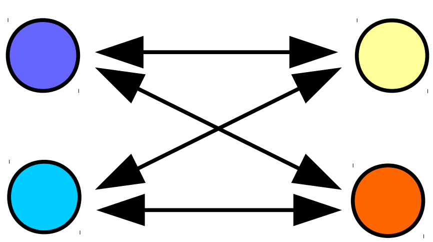
Instead of drawing all arrows, I use just a few ones to give you the general idea:



This discrete space is directed. It can not be turned into a bi-directional one because it quantifies over all choices. These choices are all there is, but you can imagine many variants of this space.

A higher order choice can be thought of as nodes of “many” colors. An agent navigating adversarial paths might be able to tell these from each other, unlike nodes of the same color.

I can use this to construct a topological space which contains all turn-based games with two players:



One side has cool colors and the other side has warm colors.

$$\forall\ x, y\ \{\ (C(x) \sim 0)(y) \Leftrightarrow W(y)\ \} \wedge \forall\ x, y\ \{\ (W(x) \sim 0)(y) \Leftrightarrow C(y)\ \}$$

all cool choices lead to warm colors, and, all warm choices lead to cool colors

C = “cool color”, W = “warm color”

Each side represent a player’s turn to make a move. Depending on what the player does, the next state might bring the game into any state, except that it is the other player’s turn to make a move.

Most games are more complex and contain initial and final states with rules for how to navigate from initial to final states. In addition, the player’s are often assigned an goal which they try to maximize. In some games the players cooperate with each other, in other games they compete.

To construct a turn-based game of `N` players:

$$\backslash(C : [\mathrm{Choice}] \wedge [\mathrm{len}]\ N) = \forall\ i, x, y\ \{\ (C[i](x) \sim 0)(y) \Leftrightarrow C[(i+1)\%N](y)\ \}$$

In this game, each player moves by turn, rotating such that each player knows which other player comes next. While the number of choices might be infinite, a player can optimize their own strategy e.g. by sabotaging the next player in competitive games with many players. Since each player knows that their own actions will not likely influence their own benefit the next time it is their turn, they know that all players might aim for the same strategy, defecting the next player. Even though they could minimize sabotage overall, it depends on all, or most, players to cooperate. If one player stops sabotaging the next one, it might not influence the previous player to stop, so there is little benefit in not continue sabotaging.

A strange thing is that even very little is known about the game in particular, one can already infer a lot of general game strategies from the topology of the discrete space.