Formal Definition of Normal Paths

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In this paper I give the formal definition of paths in path semantics.

A homotopy path `H` is a map between two continuous functions `f` and `g` such that:

$$\forall\ x\ \{\ H(x, 0) = f(x)\ \} \quad \wedge \quad \forall\ x\ \{\ H(x, 1) = g(x)\ \}$$

$$H : X \times [0, 1] \to Y$$
$$f : X \to Y$$
$$g : X \to Y$$

It follows from this definition that there can be paths between paths, e.g. `X <=> [0, 1]`.

When only the end points are considered, one can use a `$\mathbb{B}$` (boolean) instead of `[0, 1]`:

$$(H\ \text{false}) \Leftarrow\Rightarrow f \quad \wedge \quad (H\ \text{true}) \Leftarrow\Rightarrow g$$

$$H : X \times \mathbb{B} \to Y$$

This means that a homotopy path represents a binary relation of an ordered pair:

$$(f, g)$$

Where the truth values of of these binary relations defines the space of all paths.
When the relation is `true`, there exists a path, but when the relation is `false`, there exists no path.
In constructive mathematics one can not prove that the relation is `true` from `¬false` and vice versa.
This means that there exists some construction that represents the path: The proof of the path.

In Homotopy Theory, the path is constructed by the homotopy path `H`.
However, this is looking at the path from the outside.
From within the path, there is only `$X \to Y$`, which can be proved as following.

By using function currying, one can transform this type into:

$$H : X \to \mathbb{B} \to Y$$

The type `$X \to \mathbb{B}$` a set over `X`. The complexity in number of bits of this set is `$\log_2(|X|) + 1$`.
From within the path, there is only `x`, which has type `X` and maps into some value of `Y`.
Therefore, from within the path, there is only `$X \to Y$`, which is an ordinary function type.

The change in linear input complexity from inside to outside is `$2^{\log2(|X|) + 1} - 2^{\log2(|X|)} = |X|$`.
From using this one can derive that the change in linear output complexity is `$|Y|^{|X|} = |X \to Y|$`.

Therefore, there should exist a space similar to “computation” that relates functions to each other.

Where is this space?

The insight of Homotopy Type Theory is that dependent types can model proofs of Homotopy Theory.

This means that the direction of this space, seen from inside these paths, is analogous to dependent types. However, dependent types only describe a single step of "computation" in this direction.

What is beyond dependent types?

Using the analogy of complex numbers:

1	the base of normal computation
i	the base of path-space computation

For all complex numbers in base `**e**`, the following property holds:

$$A\mathbf{e} + B\mathbf{e} = (A + B)\mathbf{e}$$

The analogy law for functions in base `**e**` is the following:

$$f\mathbf{e} \cdot g\mathbf{e} = (f \cdot g)\mathbf{e}$$

Once the base of path-space computation is found,
one can use normal function composition rules in the new dimension.

Now, if one looks at commuting squares:

$$\begin{tikzcd} \circ \arrow[r, "g_0"] \arrow[d, "f"'] & \circ \arrow[d, "h"] \\ \circ \arrow[r, "g_1"] & \circ \end{tikzcd} \qquad g_1 \cdot f \Leftrightarrow h \cdot g_0$$

For every point, or value, they behave as normal computation, which satisfies `$f\mathbf{e} \cdot g\mathbf{e} = (f \cdot g)\mathbf{e}$`.
Yet, the point-wise computation is not the space one is looking for, since this is covered by `$X \to Y$`.

From within the paths, the total complexity can be thought of as a product:

$$|X \times \mathbb{B} \to Y| = |Y|^{|X \times \mathbb{B}|} = |Y|^{2|X|} = |Y|^{|X|} \cdot |Y|^{|X|} = |X \to Y|^2$$

This means that the "missing parameter" one is looking for is the "function surface" `$g_0 \to g_1$`:

$$H := f[g_0 \to g_1]$$

The constructive proof is `h` to represent `H` from within homotopy paths (called "normal paths"):

$$f[g_0 \to g_1] \Leftrightarrow h \qquad\qquad \text{Q.E.D.}$$