Duality in Path Semantical Logic

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In this paper I show that duality in Path Semantical Logic is obtained by swapping level order.

Duality[1] is a correspondence between the properties of a category[2] `C`
and the dual properties of the opposite category[3] `C^op`.

In Path Semantical Logic[4], propositions[5] are grouped into levels.
An equality at level `N+1` propagates into equality at level `N`.

Therefore, it is natural to investigate whether Path Semantical Logic has an interpretation where *dual* equality at *dual* level `N` propagates into *dual* equality at *dual* level `N+1`.

One approach to obtain a such duality for level 0 and 1, is to reverse the order in the interpretation:

type (level 0)	=>	dual type (level 1)
variable (level 1)	=>	dual variable (level 0)

The *dual* level 0 is level 1 and the *dual* level 1 is 0.
The *dual* standard notation of `a(A)` becomes `A(a)`.

Similarly, when one writes the arguments grouped by levels in a proof, the order is reversed:

(a, b, c) (A, B, C): <proof> becomes (A, B, C) (a, b, c): <proof>

The interpretation of type membership:

a : A becomes A(a) instead of a(A)

In this dual interpretation, when two types `A` and `B` are equal,
there is an equality between every member of these two types.

This means that one can model a singleton type `A` which has only one member.
Therefore, if any variable `x : A`, then `y : A` is sufficient to prove `x = y`.

The semantics of this interpretation corresponds to the notation of a “mere proposition”[6]
in the language of Homotopy Type Theory[7].

References:

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