Counter-Factual Equality of Primes

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One interesting consequence of equivalences in Sized Type Theory, is that it is possible to reason about e.g. that two primes are equivalent, even if they are not equal. This give rise to counter-factual theorem proving semantics about numbers. Although two numbers are not equal, they can be thought of as equal because equivalences have the same proof consequences as equality.

For example, if `3` and `5` were equal, then `6` and `10` would be equal:

    6 = 2 · 3
    10 = 2 · 5

Proof:

    (· 2)(3 ~= 5)
    (· 2)(3) ~= (· 2)(5)
    6 ~= 10

For the same reason, `18` would be equal to `30`:

    18 = 2 · 3 · 3
    30 = 2 · 3 · 5

    (· 2 · 3)(3 ~= 5)
    (· 2 · 3)(3) ~= (· 2 · 3)(5)
    18 ~= 30

Notice that the shared factors `2 · 3` is treated as a function `(· 2 · 3)` applied to equivalence `3 ~= 5`.

Now, a more complex example where there are only a single shared factor:

    18 = 2 · 3 · 3
    50 = 2 · 5 · 5

    (· 2)(· 3 ~= 5)(3 ~= 5)
    (· 2)((· 3 ~= 5)(3) ~= (· 3 ~= 5)(5))
    (· 2)(((· 3)(3) ~= (· 5)(3)) ~= ((· 3)(5) ~= (· 5)(5)))
    (· 2)((9 ~= 15) ~= (15 ~= 25))
    (· 2)(9 ~= 15) ~= (· 2)(15 ~= 25)
    ((· 2)(9) ~= (· 2)(15)) ~= ((· 2)(15) ~= (· 2)(25))
    (18 ~= 30) ~= (30 ~= 50)

The proof that `18` would be equal to `50` holds over a “surface” of equivalences, which is an equivalence between two equivalences. In this “surface”, the numbers `18` and `50` are connected and therefore they would be equal if `3` and `5` were equal.