# Permutation Group of Functions

by Sven Nilsen, 2020

*In this paper I show that the permutation group of functions satisfies the axioms of Group Theory.*

A permutation `f` on a finite identity map `id := [0, n)` has the following property:

$$\exists\ n : nat\ \{\ f^n \iff id\ \}$$

The same interpretation can be used on functions, therefore `id` corresponds to the identity function.

From this axiom alone, one can derive the axioms of Group Theory over function composition:

| | |
|---|---|
| $f^a \cdot f^b \iff f^{(a+b)\%n}$ | Closure |
| $(f^a \cdot f^b) \cdot f^c \iff f^a \cdot (f^b \cdot f^c)$ | Associativity |
| $f^n$ | Identity element |
| $f^0 \cdot f^0 \iff f^0$ | Inverse element for `n = 0` |
| $f^1 \cdot f^1 \iff f^1$ | Inverse element for `n = 1` |
| $f \cdot f^{n-1} \iff f^{n-1} \cdot f \iff f^n$ | Inverse element for `n > 1` |

Some consequences:

| | |
|---|---|
| $(f^0 \iff id) \iff (f \iff ())$ | The unit element `()` |
| $(f^1 \iff id) \iff (f \iff id)$ | The `id` function |
| $(f^2 \iff id) \iff (f^{-1} \iff f)$ | Self inverse |
| $(f^n \iff id) \iff (f^{-1} \iff f^{n-1})$ | General inverse when `n > 1` |

Any constant can be thought of as constructed by a function of type `() → T`.
Since `$id_{()}$ : () → ()`, it means that `$f^0$ <=> id` is logically equivalent to `f <=> ()`.

Applying the general inverse when `n = 1` would lead to an error:

$$id^{-1} \iff id^0$$

Luckily, since `$id^2$ <=> id`, it also satisfies its own self inverse:

$$(id^2 \iff id) \Rightarrow (id^{-1} \iff id)$$