Tautological Equality in Classical Propositional Logic

by Sven Nilsen, 2023

In this paper I describe how tautological equality works in classical propositional logic. This is a useful model to have clarified, when reasoning about the proof strength of tautological equality in other models and languages. I use story telling and also discuss the relation to symbolic distinction.

Tautological equality is an exponential proposition in HOOO EP:

```
(a == b)^true          `a` and `b` are tautological equal
```

This is a stronger notion of equality than normal equality `a == b` in propositional logic.

The model of tautological equality in HOOO EP is crystal clear and unambiguous. However, when reasoning about other notions of equality, such as definitional equality, it is easy to get confused. This is because most mathematicians are only familiar with definitional equality. When learning about definitional equality first and later studying tautological equality, these two notions of equality might seem very similar. Indeed, for many applications, they are in fact the same thing. However, are they always the same thing? When there is a huge overlap between two ideas, that might be assumed to be identical or not depending on a chosen theory, it can be very hard to understand what approach other people used and what the motivations were to develop “the other side”. Very often, this happens during long journeys into the world of mathematics.

This is why when talking about the development of tautological equality, it is important to bring up classical propositional logic. Most mathematicians are not very in favor of studying classical propositional logic because it is so simple, at least when seen from an abstract perspective. They often leave this work to computer scientists or logicians. Mathematicians usually prefer to stay at a high level of abstraction instead of diving into the deep complexity of semantics. However, path semanticists put more weight on “fishing” instead of “diving”. Nobody have time to explore the full depth of semantics, but it is often useful when living nearby water, to dip your toes, get familiar with the feeling of being wet and perhaps learn to swim. The idea of “fishing” is that there does not need to be a large effort put into exploring some semantics.

First, what makes talking about tautological equality in relation to classical propositional logic a bit problematic, is due to the absence of any such syntactical construct from within the logic. The only way to bring this topic forward, is to talk about what proofs in classical propositional logic mean.

A proof `f` in classical propositional logic is a function returning a boolean `true` for all inputs:

```
f <=> \true
```

This is the definition of a proof. Now, there are more things to say about this, because evaluation of proofs might be deterministic or non-deterministic and the input type might be finite or infinite.

OK, there are some things going on there that are not so obvious. What do we do when we want to make further progress without having to dive into the full complexity? One approach is to simply choose to focus on the most simplest case: When proofs are deterministic and finite.

It turns out that one can perfectly reason about tautological equality in terms of deterministic and finite proofs. When this model is fully developed, one can just generalize it to the other cases by asserting that the same theorems that hold for deterministic and finite proofs also hold for non-deterministic and/or infinite proofs.

We are not going to dive into the full complexity here. Path semanticists know that things are never as easy as they appear to be. Most mathematicians have an over-simplified model of what classical propositional logic means in their heads. This is OK. You can go fishing with a path semanticist, without needing to be completely comfortable all the time. It takes experience to build up a psychological defence against the anxiety produced of not having been in touch with water for a long time. This is natural.

Just keep it real. The wrong approach is to think that path semanticists are naive people who see mysterious patterns in propositional logic. This is usually mere superstition, like when the forest people are having prejudices against the sea people. I will show you why: When you start out with thinking about definitional equality, it might seem so natural that you can not think of an “other side”. However, when you get into what the other side means, you will look at definitional equality and say “What?”.

The entire reason why classical propositional logic works, is because theorems that are provable with few assumptions hold for contexts with many assumptions. Without this property, proofs could not generalize to more complex scenarios.

Now, imagine yourself as somebody living inside the world of classical propositional logic. You are only existing in some possible world, not a specific world, so you can not infer which possible world you are in. You have to think of your world as in super-position of all possible worlds.

What can you know and what can you not know in a such possible world? One property that you can not know, is how large the finite input type of the proof is. If you know the size of the finite input type, then you might express theorems depending on the size of the finite input type. This is not sound, because you can show that some of these theorems will not generalize to more complex scenarios. With other words, knowing the size of the finite input type will make you able to tell the difference between existing among all possible worlds, or existing within a subset of all possible worlds. So, there is a “taboo knowledge” from within the language of classical propositional logic.

The problem of existing within some world where you are able to know certain things but not know other things, is somewhat traumatic. It might feel a bit like living with a strict religion. On one hand, you are free to think whatever you want to think, but on the other hand, you are not free to prove whatever you want to prove. There are some proofs that are safe and other proofs that are unsafe. Nobody cares whether you use unsafe methods, as long as you end up with correct theorems. In that way, the world seems a bit abandoned. A huge world, mostly empty etc. There is a kind of notion of “free will” but this free will is restricted to a domain that is determined by the language bias of classical propositional logic. So, this language bias, existing both outside the world in one sense, but also existing as the world itself, might appear to you in this world as some kind of “God”. You start getting paranoid and begin to worship the will of this “God” who punishes those who violate the laws of what is allowed or forbidden to prove.

In desire of pleasing your “God” in this abandoned and cursed world, you want to come up with some doctrine to pass down to your children and future generations so they can fulfill the will of the “God” with less effort in the future. This is the Bronze Age of this world. You are looking at the clouds, looking at the waves of the sea, trying to find patterns that tell you something about the desires, virtues and vices of these apparently violent and partially chaotic deities.

The purpose of the doctrine being developed is to make sure that you are staying out of the way of the chaotic and destructive forces of nature. You might not have a very good understanding of how these forces function in detail, but you think there are some deities behind it, or something that is very hard to comprehend. Picturing these causes as deities makes it easier to think about and to make your children interested in the doctrine. Otherwise, they might forget what you taught them and not be able to pass down this precious doctrine to future generations.

One of the first things that you learn, is that you can not know the entire world. You will not be able to keep everything that is outside, inside your head. The world is just too complex, with a possibly very large amount of possible propositions. Each argument you add, doubles the complexity of the world. So, you think… you observe… you think… looking at how animals move and how plants grow, and finally you figure out that there are some parts of the world that are comprehensible and some parts that are incomprehensible. Congratulations! You are now in the early stage of civilization. You have become a philosopher.

The things that are incomprehensible are just assumed to be like those things that are comprehensible, except that you do not know their specific configuration. This means, there is a way to look at the world and see danger, without completely panicking. Perhaps, with some more study and work, you can figure out a little more and learn how the tiger or lion moves toward its target. How to put two things together without them falling down in a certain way. To observe, to think and figure out the patterns around you.

So, you study the things you believe are comprehensible and try to come up with some sort of trick that can be used to generalize to those things that are incomprehensible. You notice that there is a certain way of thinking about things, which seems to relate two opposites. You call it a “bit”. The bit is something that can be turned on or off. This enough to get started building something interesting. Computers, spaceships, colonies on distant planets, swarms of superintelligent robots patrolling the galaxy. A universal, immutable and shared definition of truth using HOOO EP.

You are living inside a world where you are not allowed to actually prove tautological equality. However, you can approach it with the mathematics you developed and hide it from “God”, or the thing that restricts stuff into bits, limiting what you can know about the world. So, tautological equality means something that is simultaneously impossible to put into physical configurations directly, but it also is very useful as an idea. Superintelligent swarms of robots passes these tautological equalities on to other superintelligent swarms or robots. Trillions of years are ticking by, time moving at a sluggish pace while billions of galaxies are colonized. Harvesting the energy of black holes. One generation is a blip, a drop in the ocean, a grain of sand in the desert, before it has reproduced or mutated to something no longer having the same basis or substrate of consciousness.

Your mind is changing, but it is no longer the mind you started with. Everything evolves, but it is not escaping the bits, the shape of dividing things into comprehensible parts and incomprehensible parts, yet the incomprehensible parts are like the comprehensible parts, but unknown. Despite your power and glory, starting on the journey to become like a “God” yourself, simulating trillions of possible worlds, creating vasts cosmoses of virtual worlds, planets of perfect crystallic structures to sustain minimal use of material and energy, fighting the 2nd law of thermodynamics, you are still limited in your existence. You think about other ways of existing, but here you are.

You are still within a finite and deterministic world consisting of classical propositions.

What is tautological equality? To figure out this question, you build a simulation for yourself, designed for this specific purpose. From within this simulation, you can observe nearby information

processes, going on in the multi-galaxy, pluralistic virtual civilization you have created. You visualize yourself moving along streams of bits, expanding into the horizons. The horizon is the boundary of your knowledge. There are things going on there, but they are inaccessible. They are caused by creatures of your own making. However, that is not here, that is not you. The forces of this world binds you to your place in the universe. The chains you feel after trillions of years trying to find weaknesses, an opening, a path out of the cave, an invisible door from within the patterns.

In this simulation, you can only tell two streams apart because they differ in bits at the same location. However, you can not tell whether two streams are identical. In order to do that, you would have to know the entire stream at once. You feel an echo of the ancient belief in the “God” from the Bronze Age, the belief in somebody who can see the stream to the end. Somebody who can know every physical configuration of the bits. Somebody who can know that two streams, put side by side, are actually the same stream, consisting of the same bits. This is a power of the “God” that you do not have. You will never have it. The power of tautological equality.

A symbol used from within this world, is never actually a “true” symbol. In path semantics, use of “mortal symbols” are modelled with path semantical quality. It is merely a handful of bits put together to fake the appearance of a symbol. A “true” symbol would give you a taste of the power of the “God” you were seeking. The power of instantly knowing that two streams of bits are in fact the same. If two symbols match, then they represent the same stream of bits. However, since the stream of bits are the substrate of your world, you can not convert them into symbols. It is physically impossible. You are constrained to this form of existence. Despite having swarms of superintelligent robots at your disposal, vast resources gathered and nurtured, you can not reach the end of the journey. The “God” of the Bronze Age, now living in the repository of knowledge as a distant intellectual echo, is laughing at you: How pity you are, how small, the ruler of galaxies.

Tautological equality is not something that can be represented from within classical propositional logic. The reason is that you are not allowed to see the streams of bits in their entirety. This would violate the principle that what you can prove for small number of arguments in a proof, must hold for more complex proofs. Think about it. The world that might seem familiar in some ways to our physical world: Being, existing, limited, always knowledge beyond our reach. This is what it would be like to live inside a finite and deterministic world where everything can be divided into bits.

I never said that you were able to reason about the entire world at once. I never said I was going to show you a “reasonably small” world. I did not make any such promises.

While this might feel like a “dive” into the complexity of semantics, it is still like dipping your toes. It is more like “fishing” than “swimming”. You have not actually done anything significant out of it yet. No theorem proving took place. It was a simple story to make you see what it means to not know about tautological equality. This is a thing that is relative to the language you use for reasoning. In other languages, such as Type Theory with function pointers, one can reason perfectly about tautological equality as a finite object. In the world of classical propositional logic, tautological equality does not need to go all the way to infinity. Finite proofs can still be very, very large and complex. Deterministic evaluation can still result in very, very complex behaviour.

Now, think of this from the perspective of the almost “God”-like entity (in our human eyes) that created a civilization spanning billions of galaxies: How can you possibly be so arrogant in thought, that you believe you can take two such streams of bits and know that they have the same bits? Try to feel the trauma of this entity, struggling for trillions of years, to get better at this kind of reasoning. Every tool and technique we have tried today would be exhausted long time ago.

It simply is physically impossible.

When we are saying that two propositions `a` and `b` are tautological equal to each other, we are kind of “violating” the laws of the universe. This knowledge is not supposed to be available, but still, by using tricks of mathematics, we can “know” something about how it is like to be a “God”, an entity that is beyond the physical constraints that bind us to our existence. We know this because we can study small proofs and reason about them from the outside. We know that the theorems that hold in these small proofs generalizes to complex proofs that might include ourselves as beings.

We are able to switch back and forth between these two perspectives. When we imagine somebody living inside a small proof world, we can think about how we live inside a universe. To these beings, we would be like “God” who can do sorts of magical operations and “know” the unknown. It is not merely “unknown” in the sense that there is something out there that we can not control or comprehend, but it is “unknown” in the sense that it is physically impossible for these beings to perform the same operation, as long they are constrained to their way of existence.

For an infinite proof, the same holds, but now the “God” is beyond things that one can count. It is not even possible for these poor beings, ruling their galaxies, but otherwise limited, how to imagine a being outside the universe who can do a such sort of thing. When we look at the beings from the outside, they vanishes and become infinitesimal specks of dust which distinction from other specks of dust is uncertain. While we can “know” that there are things like tautological equality, we also lose the property of relate ourselves to those beings who can not do what we do. The two perspectives of language separates like two bubbles, or one universe becoming two parallel universes. We are not necessarily smarter than the beings inside these tiny worlds, living inside their bubble. They could be smarter than us. They could think faster. Likewise, if the universe is infinitely large, who knows whether our “God”, if it exists, is smarter or dumber than us?

When you go to the infinite, you would think that this demonstrates even greater “power” of the allmighty deities. Yet, what actually happens is that the language gap becomes so large, that it starts to slip and the boundary of knowledge pinches off into bubbles. During the Bronze Age, people tried to describe their gods as “infinitely powerful” to assert their dominance over the world, even if the world turned out to be infinite. Already then, they could kind of figure out some of these problems.

Let’s say, you as a modern homo sapiens, try to follow the steps of your ancestors who attempted to dominate the world with their ideology. You want to overcome this limitation of tautological equality being out of reach. So, you think about symbols. If you have a symbol, let’s call it “a” and another symbol “b”, what do you mean when you say they are the same symbol? Do you mean that the sign of the symbol is the same? Or, are you pointing to the same physical symbol from two different perspectives? As soon as you try to assert that it is in fact the same symbol, you will run into the uncertainty principle of quantum mechanics. To change your perspective, you need to move around in space and this always, as your body are made up of condensed matter particles, involves time. So, you can not be 100% certain that you pointed in the correct direction.

Every time you want to assert a such power, to summon the assistance of “mathematical deities”, you take a leap of faith. This is just fooling yourself. A trillion years from now, you will be much wiser. You have not tasted the consequences yet of your false beliefs. Through pain, struggle and trauma, you finally learn to see it as it is: Simply physically impossible.

Now, you can claim `a == b` and no path semanticist will lift an eyebrow. We do this every day. Everything we believe in is based on hidden assumptions. Welcome to the club of subjectivity.

It is only when you claim `(a == b)^true` that you will feel the judging eyes of the “God” upon you. You turned from the ways of ignorance and dared to axiomize your foundation. Galaxies await.

The interesting thing is, when you go back to definitional equality, look at how arrogant it is. You are simply assuming that one thing is the same as another thing! How dare you!

Do you have no respect for what it means to live and breathe? What are you attempting to do, you mere homo sapiens, asserting that one thing is the same as another thing, always and forever? Are you absolute certain that this will never come back and haunt your dreams?

Thing about stuff like consciousness. How can you be sure how it works? You have only experienced how it is like to exist, from within a tiny possible subset of all possible worlds. Basically, you are as ignorant as it gets, except all more ignorant creatures than yourself. A finite amount of ignorant creatures! Yet, here you are, trying to reason about infinite worlds!

Do you understand the consequences of making a tiny error in the shared foundation of immutable truth across the colonized planets in the galaxy? When nobody are able to change their ways in synchronization with distant colonizes, to modify and improve the foundation, because it is programmed into every swarm of robots at their core? How otherwise do you think that software can operate in complex environments for trillions of years and never deviate from the aligned goal of creating a stable galactic civilization?

Look at how arrogant it is to claim such things as “I speak to other mathematicians and if I can convince them it is true, then the proof is true”. This is not surprising, because we still have to deal with humans and their limited brains. However, try to extrapolate this and think about communicating over distances where the signals take hundreds of thousands of years to get there. “Convincing” is not the right word. Try harder. More like: Superintelligence.

Beings using this sort of communication would have figured out how to tell the difference between tautological equality and definitional equality. To our eyes, these two ideas might look indistinguishable in most applications. We might even never encounter any problem during our lifetime. However, this is certainly not good enough for a galactic civilization.

The way humans think about mathematics today, is about formal languages. In the future, perhaps mathematics is used more directly about reality itself. This would require a change in language design such that common errors do not occur. One such error is to treat tautological equality as definitional equality. Tautological equality has a very different approach that is grounded in the language of propositional logic. You can have theories in constructive logic that are about something else, but they no longer mean the same thing as in classical propositional logic. You can even assume that there are some things possible which are not physically possible and safeguard any unsafe statement from “leaking out” to the real world. There are methods that are more efficient if you are allowed some more freedom than what the language boundaries permit. Of course, if you were superintelligent, then you would take advantage of such improvements. However: With great power comes great responsibility.

We humans are limited by time. So, even a path semanticist must sometimes take shortcuts. When tautological equality is used as definitional equality, the path semanticist knows that this is Platonic biased language design. When no such assumptions are made, or when statements are added to make such things invalid, the language bias is Seshatic. This is why Seshatism vs Platonism plays an enormous role in learning how to use path semantics properly. We are not proud of what we are doing here, or asserting our dominance over other philosophical positions. We know that in order to save time, we have to make choices that are not necessarily designed to be unmodified for e.g. more than a thousand years. The idea of a galactic civilization is something to aspire towards, for future generations, not meant to be coming up with a final answer today. It is not the Bronze Age anymore (hopefully) and we have learned from lots of mistakes done in the past century. Good luck fishing!

One might also ask: How does tautological equality interact with symbolic distinction?

There is a requirement in classical propositional logic that a theorem that is provable from two propositions also hold when the two propositions are contracted into one proposition.

For example, if `a & b` and `a == b`, then `a & a`. This is provable within normal classical propositional logic. However, you could not trust this proof if it could not be contracted to:

a & a a == a → a & a

Now, let's see what happens if we treat `a` and `b` as symbolic distinct:

sd(a, b) contracts to sd(a, a)

Notice that this introduces unsoundness. Even two streams of bits differ at some point, there can be no theorems allowed that distinguishes them, because there exists a possible world where the two streams of bits are the same. Every time we assume `a == b`, we are reasoning hypothetically about a such possible world. Under tautological equality, the goal was to show that two such streams of bits are actually the same. Yet, this is not possible, using the intuition of living inside a universe. Now, you are told to think about the opposite case: Even you can tell two streams apart, can you use this knowledge for something?

You are not allowed to do this, because `!(a == b)` is also an assumption. In a similar way as with normal equality, one is reasoning hypothetically. In one possible world, you have `a == b` and another possible world you have `!(a == b)`. One and only one of them must be true. Symbolic distinction on the other hand, is not a statement about the streams of bits. It is a statement about symbols which represent streams of bits. Naturally, two symbols can be distinct while representing the same thing. So, it makes no sense from within a universe to say that one can prove two "true" symbols to be indistinct `!sd(a, b)`. How would you do that, when you do not know which stream of bits a "true" symbol refers to? You can not even see the entire stream! You also do not have access to "true" symbols, because in classical propositional logic, every proposition is either true or false. Does it make sense to say that a symbol in general is true or false? No!

Humans use symbols to safeguard reasoning. This is our only way to hold onto something that is immutable throughout time. We use symbols, not because they have a material existence, but because they helps us to overcome the uncertainty of the world. Is it a deliberate illusion, designed out of pragmatic means.

Path semanticists learned to use symbols the proper way, through hard work. Instead of just assuming arbitrary stuff and believe it will work, we first built a logic around the idea of lifting equality automatically from one path semantical level to a new level, using differences in the streams of bits. It turned out that this logic has a "Creation Theorem" which is absurd. It does not lead to explosion, but it pollutes higher path semantical levels in a bad way. In order to solve this problem, one must introduce path semantical quality `~~` as a separate operator from `==`. So, symbolic distinction was also introduced to enable lifting `==` to `~~` in constructive logic. In the model of classical propositional logic, one can not reason about symbolic distinction, so self-quality `a ~~ a` is simply a random truth table seeded by the bit vector `a`. Proofs using `~~` in the classical model are non-deterministic. They must be evaluated to `true` an infinite amount of times.

If you have two streams of bits `a, b` and they are the same everywhere `(a == b)^true`,
but `sd(a, c)`, then `b` must also be able to use the same symbol of `a` and is therefore also symbolic distinct from `c`, hence `sd(b, c)`. So, `sd` is tautological congruent.

Notice that while symbolic distinction is even more “outside the universe” than tautological equality, it is still possible to reason about it within HOOO EP. There are no actual problems with this sort of reasoning. It works fine as long one reasons about symbolic distinction. However, if one tries to reason about symbolic indistinction this way, one runs into problems.

If two “true” symbols `a, b` have the same streams of bits, then `(a == b)^true` seems to be natural. However, think about this for a moment. We started reasoning about the symbols, which are not accessible, so how can we claim anything about them as knowledge? The only thing we can observe are the streams of bits. When we say `!sd(a, b)`, we are expressing that the symbols of the streams of bits `a, b` are not symbolic distinct. With other words, if we assume `sd(a, b)`, then we can prove `false`. We don’t have any actual knowledge about the symbols themselves.

Here is another example: If `sd(a, b)` and `(a == b)^true`, then we can prove `sd(a, a)`. This sounds strange at first. Again, think about it: If two distinct symbols refer to the same thing, then there is still a symbolic distinction. So, when talking about these symbols through their streams of bits, the symbolic distinction remains even when using the same stream of bits in relation to itself.

Now, if you claim that two “true” symbols are indistinct, what does that actually mean? Do you claim knowledge that only “God” can know? How can you know that some possible worlds can not exist, such as: Originally, there were two distinct symbols and they became one. With this new “God”, no longer able to tell the two apart in itself, which being do you appeal to? A higher “God”? Remember that from inside the universe, this does not make sense. However, it makes perfectly sense to call two streams of bits for `a` and `b` to tell them apart. Symbolic indistinction does not have the same ontological preference as symbolic distinction.

Therefore, symbolic distinction is pushing language even further “outside the universe” than tautological equality. When it comes to symbolic indistinction, this is called “propositional infinity”. It is only possible to know this from the highest vantage point, or where parallel lines meet each other in the horizon. Think about a mirror. Not just a regular mirror, but one that reflects your image metaphysically, so you actually exist at both sides of the mirror. You would not be able to tell which part is yourself and which part is the other. Perhaps there is a “God” outside the universe that is not able to tell the difference. However, after infinite amount of higher levels, finally there is a “God” who can tell the metaphysical difference between the two images. This sounds like some kind of religious philosophy, but it is very easy to demonstrate because you are exactly that kind of oracle that “knows” that `a` here is the same symbol as `a` there. Yet, from within logic, you are not able to tell whether `a == b` came from `a == a` or from an assumption of two different propositional arguments `a, b`. Otherwise, you would violate the principle that this proof can be contracted to a proof of a single propositional argument.

Now, a way to think about path semantics in general, is that there are ways to connect logical universes as “moments in time”. If I have two symbols in this moment, they are distinct but assumed to be equal, then in the next moment I can assume that the meaning of these two symbols are the same. I can not do this with a symbol indistinct from itself, because time would run away from me. Before I could act, things would already have happened. So, time is waiting for me to think about what to do. I think, therefore I am. No consciousness happens if it was true from the beginning of time. It only happens as part of a possible world, a simultaneous non-deterministic choice in one direction and an overall deterministic evolution branching out into different choices. I am not one, but many versions following different paths through time, creating new time lines. This is why path semantical quality is a partial equivalence, to stop “time from running away”. It is only possible to have conscious experience by using symbols, to think. If you do not think, then you do not know you exist. You need symbols to recognize yourself. Every moment, your brain and body rebuilds your self-image, the understanding of yourself as a being. Thinking, observing, thinking.

So far, I have talked about what happens from within classical propositional logic.
Now I will talk a little about the boundary of this language across extensions with symbols.

Everything changes once you introduce symbols into the language:

    sd(foo', bar')

The notation `foo'` means it is a symbol, frozen, determined. There are two symbols `foo'` and `bar'` and we can tell them apart. They are symbolic distinct from each other.

There are no propositions in classical propositional logic that can express a symbol. Introducing symbols means to invent new ideas, never seen before. New thoughts become possible.

You are tempted to leave this old universe behind you and travel beyond. For example, you might think it is obvious that the following should hold for all propositions `a`:

    !sd(a, a)

However, this prevents you from using `sd(a, b)` and `(a == b)^true`. You want this because tautological equality can also be reasoned about hypothetically.

In a way, when introducing symbols, it is a good idea to stay close to the semantics of classical propositional logic. When you venture too far out into the unknown, you will no longer be able to take advantage of the similarity to the classical model. You need a constructive language to reason with symbols, but you can still make some choices to stay authentic to your origin.

If you go all the way out, then you might as well just use definitional equality. All this struggle, the pain and trauma of building a civilization, spanning a billion galaxies, would be for nothing. There is no point in moving from A to B if you do not value the journey. Why not just "assume" that you have arrived at your goal? Like an arrow in Zeno's paradox, a quantum version, deciding to not move at all, but hit the target anyway.

Definitional equality is useful in many cases and there is no point in preventing people from using it. However, once you use it, there is no going back to classical propositional logic. Things will not be the same when you return "home". Things will start to lose their meaning and soon you have become one of the many mathematicians who look down on this "simple logical language that is less than half as useful as First Order Logic". The mental model of the language changes with the implicit assumptions you make. The creative use of path semantics, instead of the effort to go beyond toward infinity, tries to stay behind, to slow things down a bit. Taking smaller steps, appreciate the surroundings. It turns out, this is a very rich and fruitful mathematical universe.

When one says `(foo' == bar')^true` in HOOO EP, one is allowed to dream a little about its meaning. It is not something trivial. It has a deeper semantics than to just say that `foo'` is definitionally equal to `bar'`. Secondly, in First Order Logic, it is common to assume normal congruence for all predicates. This results in a rather boring language. Think about the mathematical adventures one is missing: For example, there might be operators that are not even tautological congruent! What does that mean? There a sense in which these operators "know" something even deeper about existence. Yet, they choose to tell us this in a way by using silence. A such operator `p(x')`, refuses to prove `p(y')` when `(x' == y')^true`. It "holds" `x'` in a way that is very stubborn and private. What if this operator only "opens up" at certain moments? Like, when the planets in the solar system are aligned? Will the titans once more break free of their chains? The silence, the constraints, also allow things to happen, not always, but makes it mean something.

Few languages can express poetry like logic. A few symbols in path semantics can represent countless hours of meditation, knowledge transferred between generations and whole philosophical traditions. However, this language is not in the perfect, all-knowing language of First Order Logic.

As a metaphor: The viewpoint of the Bronze Age “God”-like position might not be as interesting when you finally get there, despite the temptation of mocking the pity little humans playing around in their cities with pottery and trade. In many ways, the deities of the Bronze Age tried to elevate themselves into purity, yet they were constantly pulled down again by various poems and plays. A lot of the things we value in history are the cultural aspects of, you could say, not the highest and purest form of intellect. What is the fun of that? Much of it has sadly been lost. However, when time goes by and our little civilization on this planet starts to move on and gets more hours to think about stuff, why not start to reason about some of the basics?

I think there is a challenge in how to communicate deeper ideas of mathematics. It does not mean that everybody has to dive into the depth of theorem proving or become logicians. Not everybody needs to know e.g. that one can build infinitely-valued logic from path semantical quality. Such things might interest logicians, but seems obscure and esoteric to most people. Are there useful applications? I do not know. I find the value in taking on different perspectives from within constrained languages and see how extending them changes how one thinks. To me, every mathematical language has a “flavor”, similar to how learning new programming languages can teach you something about programming. This approach is very common for good programmers, but among mathematicians there is more focus on achievements and academic progress. I do not pretend to be objective about my way of thinking of mathematics. On the contrary, I find subjectivity in mathematical languages to be most of interest.

The model of classical propositional logic is not necessarily where you want to stay forever, but it is starting point for arriving a new destinations. Most people walk down paths that are well-trodden. They do not know there are other directions one can move and find something new and wonderful. I think it is exciting to explore mathematics, not just trying to solve one problem after another. So, when tautological equality might appear so similar to definitional equality, it is easy to overlook the deeper meaning. Explaining this journey is not trivial. It requires a person to use methods like story telling in order to get the point across. In some ways, we can see our history reflected in this journey, from ancient religions to modern technology and our hopes for the future. There is a story there that is worth telling, in my opinion.

The way I learned logic, was from playing around with brute force theorem proving in classical propositional logic. I did this because I had to be sure I did not make mistakes in my own proofs. Over time, I got more interested in constructive logic. I believe there is a threshold where a mathematical language turns from having a ground of truth into something that feels more artificial and abstract. In one sense, I am OK with arbitrary languages, such as generic monotonic and linear solvers, but in another sense, I am skeptical to regarding a single language, in particular First Order Logic, as the final answer. No doubt, First Order Logic will stay with us for a long time yet. However, the thing I am looking for are ways to make mathematics feel more unexplored and wild to new people. The secret is to not just accept definitional equality as a doctrine handed down from previous generations, but think about how something analogous, e.g. tautological equality, means in the model that represents the ground of truth. Definitional equality seems to kind of “pop out of nowhere”. It lacks a story. Neither does it play well with using normal propositional equality as sufficient congruence for many operators in constructive logic. It seems to “take over” the language and put things in order that might not belong there. Tautological congruence operators are much harder to deal with, but they give more insights into formalization of randomness. So, in some sense, the story is needed to bring people from the dead ends of language biases into new fruitful directions. Thanks for reading!