Prime Bound

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In this paper I introduce prime bounds, which behave like primes by the property that every positive power has the same prime bound. This reveals some of the hidden structure behind primes.

Every natural number greater than 1 can be factorized into a non-empty ordered set of primes. A prime is not divisible by any other natural number except 1 and itself. Prime number have been studied for millennia in mathematics. Still the hidden structure behind primes has remained elusive.

The reason the hidden structure behind primes has remained elusive for so long time, might be that the hidden structure has a similar property to primes, while being a distinct feature of natural numbers. The basic problem of primes is that one must separate natural numbers greater than 1 into the set of primes and the set of composite numbers. Perhaps the hidden structure behind primes is something that holds for all natural numbers greater than 1?

I suggest the following definition to satisfy the property above:

$$\text{prime\_bound}(x : \text{nat} \wedge (> 1)) = a \cdot b$$

Where:

n = number of factors of `x` (including repeated primes)

a = the factor of highest lower bound such that `$a^n$ <= x`

b = the factor of lowest upper bound such that `$b^n$ >= x`

This satisfies the following property (conjecture):

$$\text{prime\_bound}(x^n) == \text{prime\_bound}(x) \quad \text{where } n >= 1$$

For primes, `a == b`. Likewise, if `a == b` then `x` is a prime of some positive power:

$$\text{prime\_bound}(x : [\text{prime\_pos\_pow}]\ \text{true}) == \text{prime\_base}(x)^2$$

The even property is preserved for odd numbers:

$$\text{even}(\text{prime\_bound}(x)) == \text{even}(x) \qquad \text{if } !\text{even}(x)$$

Written as a constrained normal path:

$$\text{prime\_bound}\{[\text{even}]\ \text{false}\}[\text{even}] => \backslash\text{false}$$

This holds since `mul[even] <=> or` and the only even prime is `2`, in which case `x` is even.

When `x` is odd, neither `a` nor `b` are even and thus the prime bound is also odd.

While positive powers of `x` preserves the prime bound, it is possible to shift the prime bound up or down using some choice of factors from the set of primes by factorizing `x` and multiplying with `x`. It is currently unknown whether it is possible to do so arbitrarily, meaning that the prime bound generated by the co-domain of such operations are covered by any possible prime bound by continuous sub-sets of the factors (including repeated ones), except sub-sets consisting of a single prime (potentially repeated).

The prime bound factors are continuous in the ordered set of factors of `x`. This means, the set of arbitrary prime bounds by using sub-sets of factors of `x`, are ordered.

For example:

    2 * 3 * 5

The possible continuous non-square prime bounds are: `2 * 3` and `3 * 5`.
The possible contiouous square prime bound are: `2 * 2`, `3 * 3` and `5 * 5`.

The actual prime bound of `2 * 3 * 5` is `3 * 5`. If you take away factors from either end, then you can construct any possible continuous prime bound, including square prime bounds. All square prime bounds are trivially continuous. If you add new factors which already exist in the factorized set of primes, then it is impossible to get any square prime bound unless the set of primes contains only a single prime. It is currently unknown whether one can construct any non-square prime bound arbitrarily, that covers the original possible non-square prime bounds.

You never get `2 * 5` as a possible continuous prime bound because it excludes `3` from `2 * 3 * 5`. This is not a valid prime bound, since a prime bound is always continuous with the set of factors.

Due to this property, when the prime bound is even, the input `x` must be even and there are some constraints on its factorized multi-set of primes. These constraints are that the highest prime and its repetition is bounded by the lowest prime greater than 2. Other constraints are the distribution of primes greater than the lowest prime greater than 2. While the set of `x` with same prime bound can be infinite by repeating the lowest prime greater than 2 indefinitely, the set of primes is finite.

Therefore, when the prime bound is even, one can deduce that the set of primes is finite:

    x : [prime_bound] [even] true   =>   finite_set(set_of_primes(x))

The reason a prime bound is continuous with the ordered set of factors, is because `x` is distinct from any single prime chosen from its factors raised to any positive power, unless there is only a single prime among its factors. With other words, from the perspective of prime bounds, the input is invariant over primes raised to any positive power.

It might be counter-intuitive to reason about numbers greater than 1 and their positive powers as belonging to the same set with respect to prime bounds. An alternative way to think about it is as a topological contraction using prime bound as the definition of equivalence for a topological space. The property of finite sets of even prime bounds is only meaningful in this sense. Otherwise, it would not make sense since any factorization of some natural number is finite in the common finitist perspective. Another perspective gives a non-standard interpretation of Peano’s axioms. Such perspectives should not be confused with the topological interpretation.