# Idempotency from Commutative Symmetry

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*In this paper I prove that commutative symmetry implies idempotency of the symmetry operator.*

A binary operator `f` is commutative symmetric if there exists a unary operator `g` such that:

$$\forall\ a, b\ \{\ f(a, b) = g(f(b, a))\ \}$$

Here, `g` is called the “symmetry operator”.

When `g <=> id`, the binary operator `f` is commutative.
When `g <=> neg`, the binary operator `f` is anti-commutative.

Commutative symmetry unifies the properties of commutative and anti-commutative operators.

From commutative symmetry, one can prove the following:

$$\forall\ a, b\ \{\ f(a, b) = g(f(b, a))\ \} \quad = \quad \forall\ a, b\ \{\ g(f(a, b)) = f(b, a)\ \}$$

Proof:

| | | |
|---|---|---|
| ∵ | $\forall\ a, b\ \{\ g(f(a, b)) = f(b, a)\ \}$ | |
| ∴ | $\forall\ a, b\ \{\ f(b, a) = g(f(a, b))\ \}$ | using `(x = y) = (y = x)` |
| ∴ | $\forall\ b, a\ \{\ f(a, b) = g(f(b, a))\ \}$ | replacing `a => b` and `b => a` |
| ∴ | $\forall\ a, b\ \{\ f(a, b) = g(f(b, a))\ \}$ | using `∀ x, y { … } = ∀ y, x { … }` |

Now, one can use this to prove that the symmetry operator `g` is idempotent:

$$g^2 \iff id$$

Proof:

| | | |
|---|---|---|
| ∵ | $g(g(f(a, b))$ | |
| ∴ | $g(f(b, a))$ | using `∀ a, b { g(f(a, b)) = f(b, a) }` |
| ∴ | $f(a, b)$ | using `∀ a, b { f(a, b) = g(f(b, a)) }` |

Q.E.D.