Linear Polarized Light

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In this paper I formalize a simple model for linear polarized light.

Linear polarized light is a building block for understanding the more general theory of photons[1].
In this paper I will focus on the classical physics of light as a wave.

The formalization of linear polarized light consists of two parts:

1. The external model
2. The internal model

The External Model

The external model only deals with how linear polarized light is embedded in 3D space.

The external model is an extended ray[2][3] with an "up" vector:

ray_pos : $\mathbb{R}^3$	the position of the light beam
ray_dir : $\mathbb{R}^3$	the direction of the light beam
ray_up : $\mathbb{R}^3$	the up direction of the light beam

The reason one needs an "up" vector, is because one needs a coordinate system[4] to describe how the linear polarization angle is embedded in 3D space. The up vector is orthogonal to the ray direction:

ray_dir · ray_up = 0 where `·` is the dot product

There is a corresponding left direction which is derived from the ray direction and up direction:

ray_left = ray_up × ray_dir where `×` is the cross product

The electric field is associated with the coordinate system of the up vector.
The magnetic field is associated with the coordinate system of the left vector.
Both `ray_dir` and `ray_up` have length 1.
This means the triple cross product cancels (proof in appendix):

ray_dir × ray_up × ray_dir = ray_up
ray_up × ray_dir × ray_up = ray_dir

The cross product is associative in this case for two reasons (proofs in appendix).

The cancelling of the triple cross product can be used to restore directions:

ray_dir × ray_left = ray_up
ray_left × ray_up = ray_dir

The Internal Model

The internal model only deals with with the information stored intrinsically in linear polarized light.

> amplitude : ℝ
> frequency : ℝ
> phase : ℝ
> linear_polarization : ℝ

The electric and magnetic field magnitude over time is given by:

> field_magnitude(time : ℝ) = amplitude * sin(phase + time * frequency * τ)

Here, `*` means multiplication with real numbers.
This field magnitude is a toy model which is substituted by more realistic models in experiments.
Using a toy model is sound as long it is not referenced externally beyond `field_magnitude`.

Linear polarization is an angle which describes the orientation of the electric wave orthogonal to the ray direction, where the magnetic wave is oriented 90 degrees to the right relative to the electric wave:

> electric_wave_orientation = linear_polarization
>
> magnetic_wave_orientation = linear_polarization + τ / 4

This follows from the magnetic field being associated with the left vector in the external model.

The linear polarization in the internal model might be described using a complex number.
The reason for doing so is to split up the polarization into components in some coordinate system.
This is particularly useful when working with polarization filters.

By translating effects of polarization filters to the internal model, one can ignore the external model.

Integrating External and Internal Models

To understand the intrinsic properties of linear polarized light as embedded in space, one needs to integrate the internal model and the external model.

An angle `φ` in the internal linear polarized coordinates is translated to a vector in the external model:

> external_polarization(φ : ℝ) = cos(φ) * ray_up + sin(φ) * ray_left

This can be used to calculate the external electric and magnetic waves:

> external_electric_wave(time : ℝ) =
>   field_magnitude(time) * external_polarization(electric_wave_orientation)
>
> external_magnetic_wave(time : ℝ) =
>   field_magnitude(time) * external_polarization(magnetic_wave_orientation)

Propagation Through Space

Light propagates through vacuum at the speed of light `c`.
When propagating through a uniform medium, the speed is some fraction `n` of the speed of light.

Since the propagation through a uniform medium has a constant speed `n * c`,
this can be used to calculate the external electric and magnetic waves
at some time and distance along the direction of the ray.

To make calculations simpler, one can use a `propagated` function:

$$\text{propagated}(\text{time} : \mathbb{R}, \text{distance} : \mathbb{R}) = \text{time} - \text{distance} / (n * c)$$

Any function of kind `time → …` can be lifted into `(time, distance) → …`.

$$\text{propagated\_lift}\{T\}(f : \mathbb{R} \to T) \to (\mathbb{R}^2 \to \mathbb{R}) = f \,.\, \text{propagated}$$

The `.` is function composition.

propagated_field_magnitude <=> field_magnitude . propagated
propagated_external_electric_wave <=> external_electric_wave . propagated
propagated_external_magnetic_wave <=> external_magnetic_wave . propagated

Appendix

Proof of Cancelling in Cross Product

The cancelling in cross products happens for orthogonal vectors of length 1:

∵	$a \times (b \times a) = b$	
∴	$(a \cdot a) * b - (a \cdot b) * a = b$	
∴	$(a \cdot a) * b = b$	using `a · b = 0`
∴	$a \cdot a = 1$	

Two Proofs of Conditional Associativity for Cross Product

First, I derive an equation to express the condition of associativity for cross product:

∵	$a \times (b \times c) = (a \times b) \times c$	
∴	$(a \cdot c) * b - (a \cdot b) * c = -(c \cdot b) * a + (c \cdot a) * b$	triple cross product relation to dot product
∴	$(a \cdot c) * b - (c \cdot a) * b - (a \cdot b) * c = -(c \cdot b) * a$	moving to other side
∴	$-(a \cdot b) * c = -(c \cdot b) * a$	using `a · c = c · a`
∴	$(a \cdot b) * c = (c \cdot b) * a$	flip sign on both sides

One can prove that the condition holds for orthogonal vector:

∵	$(a \cdot b) * c = (c \cdot b) * a$	
∴	$0 = 0$	using `a · b = 0` and `c · b = 0`
∴	true	
	Q.E.D.	

Or, one can prove that the condition holds when the first argument is the same as the third:

∵	$(a \cdot b) * c = (c \cdot b) * a$	
∴	$(a \cdot b) * a = (a \cdot b) * a$	using `c = a`
∴	true	
∴	Q.E.D.	

References:

[1] “Photon”
Wikipedia
https://en.wikipedia.org/wiki/Photon

[2] “Ray (optics)”
Wikipedia
https://en.wikipedia.org/wiki/Ray_(optics)

[3] “Line (geometry) – Ray”
Wikipedia
https://en.wikipedia.org/wiki/Line_(geometry)#Ray

[4] “Coordinate system”
Wikipedia
https://en.wikipedia.org/wiki/Coordinate_system