Homotopy Physics and Derivation of Homotopy Maps

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In this paper I show why homotopy maps are sufficient for the Path Integral Formulation.

A homotopy[1] between two continuous functions `f` and `g` from a topological space `X` to a topological space `Y` is defined to be a continuous function:

$$H : X \times I \rightarrow Y$$

$$I \Leftrightarrow \mathbb{R} \wedge (\geq 0) \wedge (\leq 1)$$

such that:

$$H(0) \Leftrightarrow f \qquad\qquad H(1) \Leftrightarrow g$$

In practice, one might work with homotopy maps of some type `T`:

$$H : I^N \rightarrow T$$

$$N : \text{nat}$$

This is because the products of the unit interval `$I^N$` satisfies many relevant properties of homotopies.

In the Path Integral Formulation[2], one is specifically interested in homotopy maps of type:

$$H : I \rightarrow \mathbb{C}^M$$

$$M : \text{nat}$$

Assume `f` is a continuous function of time to space and `g` maps space-time to complex numbers:

$$f : \mathbb{R} \rightarrow \mathbb{R}^3 \qquad\qquad g : \mathbb{R} \times \mathbb{R}^3 \rightarrow \mathbb{C}$$

Using `a` as start time parameter and `b` as end time parameter, one can construct a homotopy map:

$$H := \backslash(x : I) = g(x, f(a + (b - a) \cdot x))$$

Which has the type `$I \rightarrow \mathbb{C}$` and satisfies the properties needed to calculate probabilities.
For cases where multiple complex numbers are needed per point, it suffices to use `$I \rightarrow \mathbb{C}^M$`.

In Path Integral Formulation of Homotopy Physics[3], one generalises this to homotopy maps of type:

$$H : I^N \rightarrow \mathbb{C}^M$$

References:

[1] "Homotopy"
Wikipedia
https://en.wikipedia.org/wiki/Homotopy

[2] "Path integral formulation"
Wikipedia
https://en.wikipedia.org/wiki/Path_integral_formulation

[3] "Homotopy Physics and Path Integral Formulation"
Wikipedia
https://github.com/advancedresearch/path_semantics/blob/master/papers-wip2/homotopy-physics-and-path-integral-formulation.pdf