Satisfied Natural Numbers

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In this paper I present a perspective of natural numbers which allows talking about countability for finite numbers in relation to a "measure rod". This generalizes the notion of counting by introducing satisfied and unsatisfied numbers. Furthermore, using this definition, a Hartogs' number is constructed with lower cardinality than real numbers, but is larger than any real.

A natural number `x` is said to be "satisfied" relative to `n` when the following property holds:

$$x == \sum i \{ y_i \} \qquad \text{where} \qquad \exists i \{ y_i < n \}$$

One important property of satisfied numbers, is that they distinguishes theories like general Set Theory, or Peano arithmetic, where there are only satisfied numbers, from theories where products play a central role in representing large numbers.

The precise breaking point of when numbers become hard to count, or comprehend, even though they are finite, is when it becomes hard to produce a satisfactory equal representation from an unsatisfied number.

Furthermore, the measure rod used can be of arbitrary size, meaning that there is an associated smallest possible unsatisfied number for various definitions of measure rods, by imposing additional constraints.

For example, there can be a constraint on the number of terms in the sum.

3 terms and `n = 10`:

$10 + 10 + 10 = 3 \cdot 10 = 30$ is smallest unsatisfied number

`m` terms and `n`:

$m \cdot n$ is smallest unsatisfied number

Notice that this definition, as odd it seems at first, precisely captures the intuition of when a number becomes "larger" than others.

One can also think about this definition as open ended. Surely, in Set Theory and Peano arithmetic, there are no bounds to limit satisfactory numbers, but once somebody tries to reason about what is known about numbers under bounded rationality, the question becomes more interesting, since there will always be some unsatisfied lowest number under constrained representations.

This notion of an unsatisfied lowest number in Set Theory is called a Hartogs' number. Now, the question is: What happens when constructing the lowest unsatisfied number in relation to the cardinality of natural numbers (`$\text{aleph}_0$`)?

$$\text{aleph}_0 + \text{aleph}_0 + \text{aleph}_0 + \text{aleph}_0 + \ldots == \text{aleph}_0^2 \qquad < 2^{\text{aleph}0}$$

This is lower than the cardinality of `$2^{\text{aleph}0}$`, but it is not clear whether this ordinal is in the set of real number cardinality, or less. This might be depending on the Continuum Hypothesis.