Univalent Involutions

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In this paper I prove that isomorphisms are isomorphic to Univalent Involutions.

An isomorphism[1] from a Category[2] theoretic view is a morphism \`f\` with an inverse \`g\` such that:

$$f \cdot g \Longleftrightarrow id_B$$
$$g \cdot f \Longleftrightarrow id_A$$

$$f : A \rightarrow B$$
$$g : B \rightarrow A$$

An involution[3] is a morphism \`h\` such that:

$$h \cdot h \Longleftrightarrow id_T$$

$$h : T \rightarrow T$$

Every involution is an isomorphism, but not every isomorphism is an involution.

It turns out that every isomorphism can be turned into a Univalent Involution:

$$\begin{aligned} h := \backslash(x : T) = &\text{ if let some}(a) = h'_A{}^{-1}(x) \; \{ \; h'_B(f(a)) \; \} \\ &\text{ else if let some}(b) = h'_A{}^{-1}(b) \; \{ \; h'_A(g(b)) \; \} \\ &\text{ else } \{ \text{ unreachable!}() \; \} \end{aligned}$$

$T := A \mid B$	\`T\` is the sum type of \`A\` and \`B\`
$h'_A : A \rightarrow T$	Lifts \`A\` into \`T\`
$h'_B : B \rightarrow T$	Lifts \`B\` into \`T\`
$h'_A{}^{-1} : T \rightarrow opt[A]$	Takes \`A\` out of some \`T\`
$h'_B{}^{-1} : T \rightarrow opt[B]$	Takes \`B\` out of some \`T\`

The univalent involution \`h\` has the following normal paths (where \`opt\` is used as a functor):

$$h[h'_A{}^{-1} \rightarrow h'_B{}^{-1}] \Longleftrightarrow opt(f)$$
$$h[h'_B{}^{-1} \rightarrow h'_A{}^{-1}] \Longleftrightarrow opt(g)$$

A Univalent Involution differs from ordinary involutions by the property it can be turned back into a heterogenous isomorphism, kind of like a tuple \`(a, b)\` can be turned into \`a\` and \`b\`:

$$h.0 \Longleftrightarrow f$$
$$h.1 \Longleftrightarrow g$$

Since equality in Intuitionistic Logic[4] using types is a tuple \`(f, g)\`,
this particular form of involution is thought to be univalent[5].

References:

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https://en.wikipedia.org/wiki/Isomorphism

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[4] "Intuitionistic logic"
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[5] "Homotopy type theory"
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https://en.wikipedia.org/wiki/Homotopy_type_theory