Continuous Monotonicity

by Sven Nilsen, 2020

In this paper I describe the class of design problems of finding continuous monotic functions.

The basic building block for continuous functions is the parameterized straight line:

line(a, b) = \(t : real) = t * (b – a) + a

A straight line is continuous monotonic, which means that it progresses toward the end point everywhere without intersecting itself. This can be proved by taking the derivative:

lim h → 0 { (line(a, b)(x + h) – line(a, b)(x)) / h }
lim h → 0 { ((x + h) * (b – a) + a – (x * (b – a) + a)) / h }
lim h → 0 { ((x + h) * (b – a) – x * (b – a)) / h }
lim h → 0 { (x * (b – a) + h * (b – a) – x * (b – a)) / h }
lim h → 0 { (h * (b – a)) / h }
lim h → 0 { b – a }
b – a

Since \`b – a\` is a constant, it means that the line has the same “sign” everywhere, which is a proof of that it does not intersect itself. If it did, the “sign” would change at some point. For vectors of \`a\` and \`b\`, the “sign” is simply a vector of the signs of individual components. Change of sign in the derivative is used as the practical definition of self intersection, since if \`b – a = 0\`, then technically there would be a self intersection but without change of sign.

The derivative of a single-argument function in path semantics is a function:

d : (real → T) → (real → T)

When \`T <=> real\`, under Higher Order Operator Overloading:

sign(d(f)) : real → real

For example:

$(x^2)' = 2x$ = mul(2, x)

mul[sign] <=> mul

Therefore:

sign(mul(2, x)) = mul(sign(2), sign(x))

When \`x > 0\`, the sign is positive, so \`f{(> 0)}(x) = $x^2$\` is continuous monotonic.