Cocyclic N-gon Necklaces

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In this paper I show how to represent cocyclic N-gons using necklaces.

Working with cocyclic N-gons can get quite complex visually as N increases.
It turns out that there is a much simpler way of representing them, ignoring directional labels on edges:

[3]	cocyclic triangles
[2, 2]	cocyclic squares
[1, 4]	cocyclic pentagon
[6], [3, 3], [1, 1, 2, 2]	cocyclic hexagons

The sum of digits is equal to N.

This is a necklace representation (from combinatorics), such that all rotations are equivalent:

[1, 1, 2, 2]
[1, 2, 2, 1]
[2, 2, 1, 1]
[2, 1, 1, 2]

The standard representation is the one that appears first when sorted.

Each digit alternates the direction from the previous digit.
Therefore, the only valid number of digits is:

x : [len] ((= 1) ∨ even)

When constructing the next level of N-gons from a necklace, one can use the following technique:

- Collapse any `1` with its two neighbors, adding `1` to their sum
- Merge any `a > 1` with one of its neighbors, adding `1` to their sum, leave `a – 1`
- Split any `a > 2`, inserting `2`
- Join ends if the number of digits is odd, resort

For example, to find the necklaces of cocyclic heptagons:

[6] => [(**1** + 5)] => [2, 5]
[6] => [(2 + **1** + 3)] => [2, 2, 3] => [2, 5]
[6] => [(4 + **1** + 1)] => [4, 2, 1] => [2, 5]
[3, 3] => [(**1** + 2), 3] => [5, 2] => [2, 5]
[3, 3] => [(2 + **1**), 3] => [2, 5]
[3, 3] => [3, (1 + **1** + 1)] => [3, 1, 2, 1] => [1, 2, 1, 3]
[**1**, 1, 2, 2] => [5, 2] => [2, 5]
[1, 1, (**1** + 1), 2] => [1, 3, 1, 2] => [1, 2, 1, 3]
[1, 1, 2, (**1** + 1)] => [1, 1, 4, 1] => [1, 1, 1, 4]

[6] => [(1 + **1** + 4)] => [1, 2, 4] => [2, 5]
[6] => [(3 + **1** + 2)] => [3, 2, 2] => [2, 5]
[6] => [(5 + **1**)] => [5, 2] => [2, 5]
[3, 3] => [(1 + **1** + 1), 3] => [1, 2, 1, 3]
[3, 3] => [3, (**1** + 2)] => [5, 2] => [2, 5]
[3, 3] => [3, (2 + **1**)] => [5, 2] => [2, 5]
[1, **1**, 2, 2] => [5, 2] => [2, 5]
[1, 1, (1 + **1**), 2] => [1, 1, 1, 4]
[1, 1, 2, (1 + **1**)] => [3, 1, 2, 1] => [1, 2, 1, 3]

[2, 3], [1, 2, 1, 3], [1, 1, 1, 4]