Uniqueness in Single-Variable Proofs

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In this paper I show that uniqueness equals existence in single-variable first-order proofs.

In first-order logic, a uniqueness proof is a stricter version of existence written `∃!` instead of `∃`:

$$\exists!\ a\ \{\ p(a)\ \} \qquad <=> \qquad \exists\ a\ \{\ p(a) \land \neg\exists\ b\ \{\ p(b) \land a \neg= b\ \}\ \}$$

When the proof only uses a single variable, the following holds:

$$\exists!\ a\ \{\ p(a)\ \} \qquad <=> \qquad \exists\ a\ \{\ p(a)\ \}$$

This is because:

- If `p(a)` returns `true`, then `p(b)` returns `true`, since `a == b`
- If `p(a)` returns `false`, then `p(b)` returns `false`, since `a == b`

Proof:

$\because \quad \exists\ a\ \{\ p(a) \land \neg\exists\ b\ \{\ p(b) \land a \neg= b\ \}\ \}$
$\therefore \quad \exists\ a\ \{\ p(a) \land \neg\exists\ b\ \{\ p(b) \land \text{false}\ \}\ \}$ Since `a == b`
$\therefore \quad \exists\ a\ \{\ p(a) \land \neg\exists\ b\ \{\ \text{false}\ \}\ \}$
$\therefore \quad \exists\ a\ \{\ p(a) \land \neg\text{false}\ \}$
$\therefore \quad \exists\ a\ \{\ p(a) \land \neg\text{false}\ \}$
$\therefore \quad \exists\ a\ \{\ p(a) \land \text{true}\ \}$
$\therefore \quad \exists\ a\ \{\ p(a)\ \}$

Q.E.D.