Higher Order Non-Determinism

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In this paper I formalize higher order non-determinism.

When one writes `0`, it can mean a lot of things. For example:

$0 : \mathbb{N}$	`0` is a natural number
$0 : \mathbb{Z}$	`0` is an integer
$0 : \mathbb{R}$	`0` is a real number
$0 : \mathbb{C}$	`0` is a complex number

This means it is possible to define a new type, e.g. for natural numbers:

$\therefore$ $0 : ?\mathbb{N}$ — `0` is a higher order non-deterministic natural number

$\because$ $? : T \rightarrow \text{type}$ — `?` is a higher order non-determinism type constructor

A higher order non-deterministic number has similar semantics to an unknown variable.
It means that constants are "lifted" to represent unknown variables.

It is known that the number is equal to itself:

$$0 = 0 \qquad x = x$$

However, it is not directly known that two non-equal numbers are the same or not:

$$0 = 1 \qquad x = y$$

So, why not just use `x`, `y`, `i`, `j` and so on instead of higher order non-deterministic numbers?

The motivation is that when reasoning about generic non-deterministic functions, there is a duality between algorithms that describe sampling behavior over time and the statistical limit. Higher order non-determinism makes it possible to express both algorithms using the same code.

For example, in quantum non-determinism, it is common to sum over complex distributions:

$$\sum\ i : ?\mathbb{N},\ j : ?\mathbb{N}\ \{\ x_i \cdot x_j^*\ \}$$

$$x : [\mathbb{C}]$$

This sum corresponds to the following equation:

$$\sum\ \_\ \infty\ \{\ x[\text{floor}(\text{random}() \cdot \text{len}(x))] \cdot x[\text{floor}(\text{random}() \cdot \text{len}(x))]^*\ \} = \sum\ i, j\ \{\ x_i \cdot x_j^*\ \}$$