Asserting Formal Associations

by Sven Nilsen, 2019

I a previous paper[1], I showed that there are *formal* and *informal* associations. A formal association satisfies the assumptions of the core axiom of path semantics[2], while an informal association is missing a non-circular assumption.

In a formal language used for theorem proving, we are implicitly asserting formal associations:

x : bool	Asserting that \`x\` has type \`bool\`
bool : type	Asserting that \`bool\` has type \`type\`
type : type_1	Asserting that \`type\` has type \`$\text{type}_1$\`
type_1 : type_2	Asserting that \`$\text{type}_1$\` has type \`$\text{type}_2$\`
…	

Here, the type of \`type\` belongs to a larger type universe, called \`$\text{type}_1$\`. Each larger universe belongs to an even larger type universe[3], such that is a countably infinite hierarchy of universes.

This assertion of formal associations happens by construction of programming code of the formal language. Type universes is a solution to avoid Girard's paradox[4].

The core axiom of path semantics does not require that a type hierarchy to be countably infinite, only that each term \`a\` in \`a : b\` is non-circular with respect to its type \`b\`, \`a > b\`. However, in a dependent typed language with infinite terms, a finite amount of type universe leads to Girard's paradox. This means that in order to assert formal associations, the type hierarchy must be countably infinite.

Type universes can be thought of as a kind of negative numbers:

$$\text{type}_n > \text{type}_{(n+1)}$$

$$-n > -(n+1)$$

Under a hierarchy of formal assocations, the core axiom of path semantics propagates.

For example:

x = y, x : bool, y : bool (asserting x > bool) =>
bool = bool, bool : type (asserting bool > type) =>
type = type, type : type_1 (asserting type > type_1) =>
type_1 = type_1, type_1 : type_2 (asserting type_1 > type_2) =>
…

In practice, dependent typed theorem provers terminate when they prove two types to be identical.

This is the same as assuming that the path semantical core axiom holds for type hierarcies.
If it did not hold, then dependent typed theorem provers should never terminate.

References:

[1] "Lifted Associations"
Sven Nilsen, 2019
https://github.com/advancedresearch/path_semantics/blob/master/papers-wip/lifted-associations.pdf

[2] "Path Semantics"
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[3] "Universe (mathematics)"
Wikipedia
https://en.wikipedia.org/wiki/Universe_(mathematics)

[4] "Girard's paradox"
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https://en.wikipedia.org/wiki/System_U#Girard's_paradox