Avatar Logic to Set Theory

by Sven Nilsen, 2021

In this paper I introduce a method of translating Avatar Logic to Zermelo-Fraenkel Set Theory.

The axioms of Avatar Logic[1] might be translated to Zermelo-Fraenkel Set Theory[2]:

$$\mathbf{(a, b) \land b : p \land uniq(b)}$$
$$p(a, b) \land \exists! \, z \; \{ \; p(a, z) \; \}$$

$$\mathbf{(a, q'(b)) \land q'(b) : p}$$
$$p(a, \{\{q\}, \{q, b\}\}) \land \forall \, x \; \{ \; \exists! \, z \; \{ \; p(a, \{\{z\}, \{z, x\}\}) \; \} \; \}$$

Translation must happen for every relation, otherwise it would require extending
Second-Order Logic[3] with tuples, roles and 1-avatars. Per relation requires only First-Order Logic[4].

The translation uses Kuratowski's definiton[5] of an ordered pair `{{x}, {x, y}}` for `x'(y)`.
This representation is chosen because ordered pairs are not used as arguments in Avatar Logic.

Ordered pairs might also be used without `b : p`, but only to mean `(a, b)` as a binary relation.

The `uniq` predicate returns `true` for all atomic symbols, plus those 1-avatars that are optionally
chosen to be behaving uniquely. Both axioms must be applied when the 1-avatar is unique.

In expanded form limited to quantifiers `∀, ∃`, connectives `=>, =, ∈, ∨, ∧`, negation `¬`:

$$\mathbf{(a, b) \land b : p \land uniq(b)}$$
$$p(a, b) \land \exists \, z \; \{ \; p(a, z) \land \neg\exists \, y \; \{ \; p(a, y) \land \neg(y = z) \; \} \; \}$$

$$\mathbf{(a, q'(b)) \land q'(b) : p}$$
$$\begin{array}{l}
p(a, _k) \land \\
\exists \, r \; \{ \\
\quad (r \in _k) \Rightarrow \forall \, c \; \{ \; (c \in _k) \Rightarrow ((c = q) \lor (c = r)) \; \} \land \\
\quad \forall \, d \; \{ \; (d \in r) \Rightarrow ((d = q) \lor (d = b)) \; \} \\
\} \land \\
\forall \, x \; \{ \\
\quad \exists \, z \; \{ \; p(a, _n) \land \neg\exists \, y \; \{ \; p(a, _m) \land \neg(y = z) \; \} \; \} \land \\
\quad \exists \, r \; \{ \\
\qquad (r \in _n) \Rightarrow \forall \, c \; \{ \; (c \in _n) \Rightarrow ((c = z) \lor (c = r) \; \} \land \\
\qquad \forall \, d \; \{ \; (d \in r) \Rightarrow ((d = z) \lor (d = x)) \; \} \\
\quad \} \land \\
\quad \exists \, r \; \{ \\
\qquad (r \in _m) \Rightarrow \forall \, c \; \{ \; (c \in _m) \Rightarrow ((c = y) \lor (c = r)) \; \} \land \\
\qquad \forall \, d \; \{ \; (d \in r) \Rightarrow ((d = y) \lor (d = x)) \; \} \\
\quad \} \\
\}
\end{array}$$

Notice that `p` might be written with uppercase letter in standard First-Order Logic notation.
The variables starting with underscore e.g. `_n`, are introduced to bind the sub-expressions together.

References:

[1] "Avatar Logic"
AdvancedResearch – Summary Page on Avatar Extensions
https://advancedresearch.github.io/avatar-extensions/summary.html#avatar-logic

[2] "Zermelo-Fraenkel set theory"
Wikipedia
https://en.wikipedia.org/wiki/Zermelo%E2%80%93Fraenkel_set_theory

[3] "Second-order logic"
Wikipedia
https://en.wikipedia.org/wiki/Second-order_logic

[4] "First-order logic"
Wikipedia
https://en.wikipedia.org/wiki/First-order_logic

[5] "Kuratowski's definition – Ordered pair"
Wikipedia
https://en.wikipedia.org/wiki/Ordered_pair#Kuratowski's_definition