quired to point at. If only some workers or firms are matched by μ_0, then the firm's lack of choice can lead to violations of the agreeable core. I resolve this by only applying the TTC to workers and firms who did not both find better partners through the Propose algorithm, with my addition indicated in italics. This modification guarantees that the Exchange stage is a Pareto improvement of μ_1; by selecting a Pareto improvement, I do not create any new acyclic blocking paths in the blocking digraph. The Exchange algorithm is defined in Algorithm 4.

Algorithm 4 Exchange Stage algorithm

set $\mu_2(a) = \mu_1(a)$ for all a.

every w such that $\mu_1(w) = \mu_0(w)$ is activated with $\mu_0(w)$.

while at least one worker is active **do**

every active worker points to his most-preferred of the active firms *who prefer him to her μ_0-worker*.

every active firm points *to her μ_0-worker.*

choose an arbitrary cycle $(w_1, f_2, \dots w_{2k-1} \equiv w_1, f_{2k} \equiv f_2)$ such that every agent points to the next agent in the cycle.

all agents in the cycle sit down.

match every w_k to f_{k+1}.

end while

return μ_2

My first observation is that the Exchange algorithm makes no agents worse off than under μ_1. Workers only point to firms they prefer to μ_0, and by my simplification of workers' preferences, firms can only be pointed at by workers they prefer to μ_0. The result is that at the end of the Exchange algorithm, μ_2 admits no cyclic blocking paths.

Lemma 2. *μ_2 admits no cyclic blocking paths.*

My proof leverages that if w strictly prefers f to $\mu_2(w)$, then f must sit down at least one step *before* w. A cyclic blocking path then implies that the firms in the path sit down on average strictly before the workers in the path sit down. However, because every worker's μ_0-firm is in the path and they sit down in the same step, it must be that the firms in the path sit down on average in the same step as the workers in the path sit down. This contradiction rules out cyclic blocking paths.

4.4 Existence

I am now ready to prove that μ_2 is in the agreeable core.

Proof of Theorem 1: Suppose (toward a contradiction) that μ_2 is not in the agreeable core. Then by Proposition 1 the digraph $(A, \mu_0 \cup \mu_2 \cup I(\mu_2))$ contains a blocking path P. By Lemma 2, P is acyclic. But P is also blocking path in $(A, \mu_0 \cup \mu_1 \cup I(\mu_1))$ because $\mu_2 \cup I(\mu_2) \subseteq \mu_1 \cup I(\mu_1)$ and $I(\mu_2) \subseteq I(\mu_1)$. By Lemma 1, P is not acyclic. This is a contradiction, which proves the claim. □

The importance of the Propose-Exchange algorithm in my proof cannot be understated. However, the algorithm has practical implications because it is also computationally efficient. The Propose stage runs in polynomial time because each worker can make at most $|F| + 1$ proposals. Similarly, one cycle is removed in every iteration of the Exchange stage, and at most $|F|$ cycles can be removed. An efficient algorithm is necessary for implementing the agreeable core in applications.

4.5 Structure

In this subsection, I highlight the difficulty in characterizing the underlying structure of the agreeable core and how it relates to other classes of algorithms commonly used to compute core outcomes. Although the set of stable matches has a well-understood structure which I summarize in the following paragraph, the agreeable core is not as tame. The hurdle in the analysis comes from the Exchange stage. To the best of my knowledge, there are no results from the literature that apply to the agreeable core when every agent is μ_0-matched.

I briefly summarize the main structural results on the set of stable matches. First, a *lattice* is a partially ordered set $(L, \geq)$ such that any two elements of L have a unique *least upper bound*, called the *join* of x and y, and a unique *greatest lower bound*, called the *meet* of x and y. That is, there is a unique $x \vee y$ such that if $z \geq x$ and $z \geq y$ then $z \geq x \vee y$, and there is a unique $x \wedge y$ such that if $x \geq z$ and $y \geq x$ then $x \wedge y \geq z$. A key result in two-sided matching is that the set of stable matches forms a lattice with the partial order $\succsim_W$.[8] The join of two matches μ and ν is the match that gives every worker w his more preferred partner from $\{\mu(w), \nu(w)\}$ and every f her less preferred partner from $\{\mu(f), \nu(f)\}$; the meet is given symmetrically. This implies that there is a conflict of interest between the workers and the firms: if every worker weakly prefers a stable μ to a stable ν, then every firm weakly prefers ν to μ. Moreover, there is a *worker optimal* stable match and a *firm optimal* stable match.

To show that the agreeable core fails to be a lattice, consider the following example. Let $\mu_0(w_1) = f_A$, $\mu_0(w_2) = f_B$, and $\mu_0(w_3) = f_C$, and preferences are given as in Section 4.5. Both the pair w_2 and f_B and the pair w_3 and f_C prefer to participate in a cycle with the pair w_1 and f_A, but w_1 and f_A have opposing preferences over the two possible cycles. Worker w_1 prefers firm f_C and firm f_A prefers worker w_2, and so either cycle may be in the agreeable core. The agreeable core consists uniquely of the $\bar{\mu}$ match and the $\dot{\mu}$ match, a pair which is not ordered by $\succsim_W$. In this example there is no worker optimal match.

Despite the impossibility of recovering a complete lattice over the agreeable core as in the classic model of stability, I show that a narrower result continues to hold. Given that the lattice structure failed in the example because two com-

[8]Donald Knuth attributes this to John H. Conway.

w_1	w_2	w_3
f_C	f_A	f_A
f_B	f_B	f_C
f_A	$\varnothing$	$\varnothing$
$\varnothing$	f_C	f_C

f_A	f_B	f_C
w_2	w_1	w_1
w_3	w_2	w_3
w_1	$\varnothing$	$\varnothing$
$\varnothing$	w_3	w_2

(a) Initial match—μ_0

w_1	w_2	w_3
f_C	f_A	f_A
f_B	f_B	f_C
f_A	$\varnothing$	$\varnothing$
$\varnothing$	f_C	f_C

f_A	f_B	f_C
w_2	w_1	w_1
w_3	w_2	w_3
1	$\varnothing$	$\varnothing$
$\varnothing$	w_3	w_2

(b) w_2, f_A, and f_B's preferred match—$\bar{\mu}$

w_1	w_2	w_3
f_C	f_A	f_A
f_B	f_B	f_C
f_A	$\varnothing$	$\varnothing$
$\varnothing$	f_C	f_C

f_A	f_B	f_C
w_2	w_1	w_1
w_3	w_2	w_3
w_1	$\varnothing$	$\varnothing$
$\varnothing$	w_3	w_2

(c) w_1, w_3, and f_C's preferred match—$\dot{\mu}$

Figure 6: An example showing that the outcomes in the agreeable core cannot be ordered by $\gtrsim_W$.

peting cycles exist in the agreeable core, an astute reader may conjecture that the lattice structure continues to hold for workers and firms who do not lie in such cycles. Suggestively, say that a is a *free agent* in μ if a lies on an acyclic, complete, and alternating path of (A, μ_0, μ). My first proposition justifies my terminology:

Proposition 3. *If μ is in the agreeable core, then there are no blocking pairs among free agents in μ. Moreover, every free agent a in μ weakly prefers $\mu(a)$ to being unmatched.*

The proof of Proposition 3 shows that these agents are "free" to form blocking pairs because each can satisfy a sequence formed by alternating edges from μ_0 and μ. Free agents resemble the agents in the classic model: their μ_0-partner (if any) is not concerned with the partner she finds.

However, an obstacle arises because the free agents depend on μ; that is, a may be a free agent in μ but not in ν. What I can show is that, if μ and ν share the same set of free agents and they agree on the agents who are not free, then $\mu \vee \nu$ is in the agreeable core. Toward that end, I say that μ and ν are *structurally similar* if they have the same set of free agents and $\mu(a) = \nu(a)$ for every agent which is not free. The following lemma shows that structurally similar matches in the agreeable core play nicely with the join and meet operators defined previously:

Lemma 3. *Let μ and ν be structurally similar matches in the agreeable core. Then $\mu \vee \nu$ is a match. The same holds for $\mu \wedge \nu$.*

Notably, $\mu \vee \nu$ may not be structurally similar to μ and ν.[9] The (possible) structural differences between $\mu \vee \nu$ and μ force us to discard any hope of obtaining a lattice-like result. However, the join and meet operators still produce matches in the agreeable core:

Theorem 2. *Let μ and ν be structurally similar matches in the agreeable core. Then $\mu \vee \nu$ and $\mu \wedge \nu$ are both in the agreeable core.*

[9] I have an example demonstrating this (available upon request), but it is too lengthy to include because it involves eight workers and eight firms.

The conflict of interest continues to hold for structurally similar matches. That is, if μ and ν are in the agreeable core *and are structurally similar*, then if every worker weakly prefers μ to ν, then every firm weakly prefers ν to μ. Conversely, in the classic matching framework, $\mu_0(a) = a$ for every agent and thus every agent is free. Every match is then structurally similar and hence my Theorem 2 generalizes standard results.

5 Incentives in the Propose-Exchange algorithm

This section addresses the incentive properties of the Propose-Exchange algorithm. The results provide insight into how robust the PE is to manipulation by participants. This is crucial for implementing the PE in practice because the output of the PE is only guaranteed to be in the agreeable core if the inputs are accurate. I find that while the PE is more susceptible to more kinds of manipulations than either the DA or the TTC, the new manipulations are difficult to execute.

I consider two kinds of manipulations in these subsections. In the first, I allow a worker to arbitrarily misreport his preference.[10] In the second, I allow a worker and a firm to create an artificial initial match, a misreport of μ_0.

For clarity through this section, I write $\succsim'_w$-Propose stage to indicate the operation of the Propose stage on the matching problem when w's preference $\succsim_w$ is replaced by $\succsim'_w$. A similar shorthand is used when μ_0 is replaced by μ'_0.

5.1 Preference Manipulation

In this subsection I discuss preference manipulations by workers. I allow a worker w to misreport his preference $\succsim_w$ by reporting $\succsim'_w$ instead. The intuition is that a worker may benefit from manipulating which agents (including himself) are

[10]It is well-known that a firm can manipulate the DA by misreporting her preference, so I only consider the problem from the worker's perspective.