Soluciones Numéricas para Dinámica de Fluidos con Free*FEM*++

XXXIII CONGRESO MATEMÁTICA CAPRICORNIO
UNIVERSIDAD DE ANTOFAGASTA

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XXXIII COMCA

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(Tercero dia)

Sumário

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Esquemas para Navier-Stokes

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- **Evolutivo**

Sumário

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Navier-Stokes (Espacio × Tiempo) [6], [2], [4] y [5]

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Navier–Stokes (Fluido Laminar):

$$
\begin{cases}
\rho \dfrac{\partial u}{\partial t} - \mu \Delta u + \rho (u \cdot \nabla) u + \nabla p = f & \text{em} \ \ \Omega \times [0, T], \\
\nabla \cdot u = 0 & \text{em} \ \ \Omega \times [0, T], \\
u(0) = u_0 & \text{em} \ \ \Omega \times \{t = 0\} \ , \\
u = \phi_D & \text{em} \ \ \Gamma_D \ \ (\textbf{Dirichlet}).
\end{cases} \tag{1}
$$

El término nolineal...

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$$\mathbf{u}\cdot\nabla\mathbf{u} = \begin{cases} \mathbf{u}^k\cdot\nabla\mathbf{u}^k & (\text{tratamiento totalmente explícito}); \\ \\ \\ \\ \end{cases}$$

El término nolineal...

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$$\mathbf{u}\cdot\nabla\mathbf{u} = \begin{cases} \mathbf{u}^k\cdot\nabla\mathbf{u}^k & (\text{tratamiento totalmente explícito}); \\ \\ \\ \end{cases}$$

El término nolineal...

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$$\mathbf{u}\cdot\nabla\mathbf{u} = \begin{cases} \mathbf{u}^k \cdot \nabla \mathbf{u}^k & (\text{tratamiento totalmente explícito}); \\ \mathbf{u}^{k+1} \cdot \nabla \mathbf{u}^{k+1} & (\text{tratamiento totalmente implícito})\ (\text{sistema no lineal}); \\ \mathbf{u}^k \cdot \nabla \mathbf{u}^{k+1} & (\text{tratamiento semi-implícito estándar}). \end{cases}$$

El término nolineal...

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$$\mathbf{u}\cdot\nabla\mathbf{u} = \begin{cases} \mathbf{u}^k \cdot \nabla \mathbf{u}^k & \text{(tratamiento totalmente explícito);} \\ \\ \mathbf{u}^{k+1} \cdot \nabla \mathbf{u}^{k+1} & \text{(tratamiento totalmente implícito) (sistema no lineal);} \\ \\ \mathbf{u}^k \cdot \nabla \mathbf{u}^{k+1} & \text{(tratamiento semi-implícito estándar).} \end{cases}$$

El problema (**totalmente explícito**):

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$$\left\{ \frac{\rho}{\Delta t}\int_{\Omega_h} u_h^{n+1}\, v_h \; d\Omega_h = -\rho\int_{\Omega_h} (u_h^n\cdot\nabla u_h^n)\, v_h \; d\Omega_h - \mu\int_{\Omega_h} \nabla u_h^n\, \nabla v_h\, d\Omega_h \right.$$

El problema (**totalmente explícito**):

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$$\left\{\begin{array}{l}\dfrac{\rho}{\Delta t}\displaystyle\int_{\Omega_h} u_h^{n+1}\, v_h\ d\Omega_h = -\rho\int_{\Omega_h} (u_h^n\cdot\nabla u_h^n)\, v_h\ d\Omega_h - \mu\int_{\Omega_h} \nabla u_h^n\,\nabla v_h\, d\Omega_h \\[2ex] +\displaystyle\int_{\Omega_h} p_h^n(div\, v_h)\, d\Omega_h + \dfrac{\rho}{\Delta t}\displaystyle\int_{\Omega_h} u_h^n\, v_h\ d\Omega_h + \int_{\Omega_h} f_h^n\, v_h\, d\Omega_h\ , \\ \\ \\ \end{array}\right.$$

El problema (**totalmente explícito**):

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$$
\left\{
\begin{array}{l}
\dfrac{\rho}{\Delta t}\displaystyle\int_{\Omega_h} u_h^{n+1}\, v_h \; d\Omega_h = -\rho \int_{\Omega_h} (u_h^n \cdot \nabla u_h^n)\, v_h \; d\Omega_h - \mu \int_{\Omega_h} \nabla u_h^n \, \nabla v_h \, d\Omega_h \\[2ex]
+ \displaystyle\int_{\Omega_h} p_h^n (div\, v_h)\, d\Omega_h + \dfrac{\rho}{\Delta t}\int_{\Omega_h} u_h^n \, v_h \; d\Omega_h + \int_{\Omega_h} f_h^n \, v_h \, d\Omega_h \, , \\[4ex]
\displaystyle\int_{\Omega_h} (\nabla \cdot u_h^{n+1})\, q_h \, d\Omega_h = 0, \qquad \forall v_h \in \mathbb{V}_h^N \;\; \text{and} \;\; q_h \in W_h^N, \\[4ex]
u_h^{n+1} = 0 \quad \text{on} \quad \partial\Omega_h .
\end{array}
\right. \tag{2}
$$

El problema (totalmente explícito):

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$$\begin{cases} \dfrac{\rho}{\Delta t}\displaystyle\int_{\Omega_h} u_h^{n+1}\, v_h\ d\Omega_h = -\rho\int_{\Omega_h} (u_h^n \cdot \nabla u_h^n)\, v_h\ d\Omega_h - \mu\int_{\Omega_h} \nabla u_h^n\, \nabla v_h\, d\Omega_h \\ + \displaystyle\int_{\Omega_h} p_h^n (div\, v_h)\, d\Omega_h + \dfrac{\rho}{\Delta t}\int_{\Omega_h} u_h^n\, v_h\ d\Omega_h + \int_{\Omega_h} f_h^n\, v_h\, d\Omega_h\ , \\ \\ \displaystyle\int_{\Omega_h} (\nabla \cdot u_h^{n+1})\, q_h\, d\Omega_h = 0, \qquad \forall v_h \in \mathbb{V}_h^N \ \text{ and } \ q_h \in W_h^N, \\ \\ u_h^{n+1} = 0 \quad \text{on} \quad \partial\Omega_h. \end{cases} \tag{2}$$

Ventaja: Velocidad de computacion y simplicidad.

El problema (**totalmente explícito**):

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$$\begin{cases} \dfrac{\rho}{\Delta t}\displaystyle\int_{\Omega_h} u_h^{n+1}\, v_h \; d\Omega_h = -\rho \displaystyle\int_{\Omega_h} (u_h^n \cdot \nabla u_h^n)\, v_h \; d\Omega_h - \mu \displaystyle\int_{\Omega_h} \nabla u_h^n\, \nabla v_h \, d\Omega_h \\ + \displaystyle\int_{\Omega_h} p_h^n (div\, v_h)\, d\Omega_h + \dfrac{\rho}{\Delta t}\displaystyle\int_{\Omega_h} u_h^n\, v_h \; d\Omega_h + \displaystyle\int_{\Omega_h} f_h^n\, v_h \, d\Omega_h\,, \\ \\ \displaystyle\int_{\Omega_h} (\nabla \cdot u_h^{n+1})\, q_h \, d\Omega_h = 0, \qquad \forall v_h \in \mathbb{V}_h^N \ \text{ and } \ q_h \in W_h^N, \\ \\ u_h^{n+1} = 0 \quad \text{on} \quad \partial\Omega_h. \end{cases} \tag{2}$$

Ventaja: Velocidad de computacion y simplicidad.

Limitación: Problemas con la velocidad, conservacion de massa y pression, restricción de estabilidad en el paso de tiempo corto.

El problema (Totalmente implícito):

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$$\left\{\begin{array}{l}\dfrac{\rho}{\Delta t}\displaystyle\int_{\Omega_h} u_h^{n+1}\, v_h\; d\Omega_h \;+\; \rho\int_{\Omega_h}(u_h^{n+1}\cdot\nabla u_h^{n+1})\, v_h\; d\Omega_h + \mu\int_{\Omega_h}\nabla u_h^{n+1}\,\nabla v_h\, d\Omega_h \\ -\displaystyle\int_{\Omega_h} p_h^{n+1}(\nabla\, v_h)\, d\Omega_h = \dfrac{\rho}{\Delta t}\int_{\Omega_h} u_h^{n}\, v_h\; d\Omega_h + \int_{\Omega_h} f_h^{n+1}\, v_h\, d\Omega_h\,, \\ \\ \\ \\ \end{array}\right.$$

El problema (Totalmente implícito):

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$$\left\{\begin{array}{l} \dfrac{\rho}{\Delta t}\displaystyle\int_{\Omega_h} u_h^{n+1}\, v_h \; d\Omega_h \;+\; \rho\int_{\Omega_h} (u_h^{n+1}\cdot\nabla u_h^{n+1})\, v_h \; d\Omega_h + \mu\int_{\Omega_h} \nabla u_h^{n+1}\, \nabla v_h \, d\Omega_h \\ -\displaystyle\int_{\Omega_h} p_h^{n+1}(\nabla\, v_h)\, d\Omega_h = \dfrac{\rho}{\Delta t}\displaystyle\int_{\Omega_h} u_h^{n}\, v_h \; d\Omega_h + \int_{\Omega_h} f_h^{n+1}\, v_h\, d\Omega_h \, , \\ \\ \displaystyle\int_{\Omega_h} (\nabla\cdot u_h^{n+1})\, q_h\, d\Omega_h = 0, \qquad \forall v_h \in \mathbb{V}_h^N \;\; \text{and} \;\; q_h \in W_h^N, \\ \\ u_h^{n+1} = 0 \quad \text{on} \quad \partial\Omega_h \end{array}\right. \tag{3}$$

El problema (Totalmente implícito):

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$$
\begin{cases}
\dfrac{\rho}{\Delta t}\displaystyle\int_{\Omega_h} u_h^{n+1}\, v_h \; d\Omega_h \;+\; \rho \int_{\Omega_h} (u_h^{n+1}\cdot \nabla u_h^{n+1})\, v_h \; d\Omega_h + \mu \int_{\Omega_h} \nabla u_h^{n+1}\, \nabla v_h \, d\Omega_h \\
- \displaystyle\int_{\Omega_h} p_h^{n+1} (\nabla\, v_h)\, d\Omega_h = \dfrac{\rho}{\Delta t}\displaystyle\int_{\Omega_h} u_h^{n}\, v_h \; d\Omega_h + \int_{\Omega_h} f_h^{n+1}\, v_h \, d\Omega_h \,, \\
\\
\displaystyle\int_{\Omega_h} (\nabla \cdot u_h^{n+1})\, q_h \, d\Omega_h = 0, \qquad \forall v_h \in \mathbb{V}_h^N \;\; \text{and} \;\; q_h \in W_h^N, \\
\\
u_h^{n+1} = 0 \quad \text{on} \quad \partial\Omega_h
\end{cases}
\tag{3}
$$

Ventaja: Incondicionalmente estable en el paso de tiempo.

El problema (Totalmente implícito):

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$$
\begin{cases}
\frac{\rho}{\Delta t}\int_{\Omega_h} u_h^{n+1}\, v_h\; d\Omega_h \;+\; \rho\int_{\Omega_h} (u_h^{n+1}\cdot\nabla u_h^{n+1})\, v_h\; d\Omega_h + \mu\int_{\Omega_h} \nabla u_h^{n+1}\,\nabla v_h\, d\Omega_h \\
-\int_{\Omega_h} p_h^{n+1}(\nabla\, v_h)\, d\Omega_h = \frac{\rho}{\Delta t}\int_{\Omega_h} u_h^{n}\, v_h\; d\Omega_h + \int_{\Omega_h} f_h^{n+1}\, v_h\, d\Omega_h\,, \\
\\
\int_{\Omega_h} (\nabla\cdot u_h^{n+1})\, q_h\, d\Omega_h = 0, \qquad \forall v_h \in \mathbb{V}_h^N \;\text{ and }\; q_h \in W_h^N, \\
\\
u_h^{n+1} = 0 \quad \text{on} \quad \partial\Omega_h
\end{cases}
\tag{3}
$$

Ventaja: Incondicionalmente estable en el paso de tiempo.

Limitación: La solucion u_h^{n+1} eres desconocida.

El problema (Semi-implícito):

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$$
\left\{
\begin{array}{l}
\frac{\rho}{\Delta t}\int_{\Omega_h} u_h^{n+1}\, v_h \; d\Omega_h \; + \; \rho\int_{\Omega_h} (u_h^n \cdot \nabla u_h^{n+1})\, v_h \; d\Omega_h + \mu\int_{\Omega_h} \nabla u_h^{n+1}\, \nabla v_h \, d\Omega_h \\
- \int_{\Omega_h} p_h^{n+1} (\nabla\, v_h)\, d\Omega_h = \frac{\rho}{\Delta t}\int_{\Omega_h} u_h^n\, v_h \; d\Omega_h + \int_{\Omega_h} f_h^{n+1}\, v_h \, d\Omega_h \, , \\
\\
\\
\end{array}
\right.
$$

El problema (Semi-implícito):

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$$\begin{cases} \dfrac{\rho}{\Delta t}\displaystyle\int_{\Omega_h} u_h^{n+1}\, v_h\ d\Omega_h\ +\ \rho\int_{\Omega_h} (u_h^n \cdot \nabla u_h^{n+1})\, v_h\ d\Omega_h + \mu\int_{\Omega_h} \nabla u_h^{n+1}\, \nabla v_h\, d\Omega_h \\ -\displaystyle\int_{\Omega_h} p_h^{n+1} (\nabla\, v_h)\, d\Omega_h = \dfrac{\rho}{\Delta t}\displaystyle\int_{\Omega_h} u_h^n\, v_h\ d\Omega_h + \int_{\Omega_h} f_h^{n+1}\, v_h\, d\Omega_h\ , \\ \displaystyle\int_{\Omega_h} (\nabla \cdot u_h^{n+1})\, q_h\, d\Omega_h = 0, \qquad \forall v_h \in \mathbb{V}_h^N \ \text{ and } \ q_h \in W_h^N, \\ u_h^{n+1} = 0 \quad \text{on} \quad \partial\Omega_h \end{cases} \tag{4}$$

El problema (Semi-implícito):

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$$\begin{cases} \dfrac{\rho}{\Delta t}\displaystyle\int_{\Omega_h} u_h^{n+1}\, v_h \; d\Omega_h \; + \; \rho \int_{\Omega_h} (u_h^n \cdot \nabla u_h^{n+1})\, v_h \; d\Omega_h + \mu \int_{\Omega_h} \nabla u_h^{n+1}\, \nabla v_h \, d\Omega_h \\ -\displaystyle\int_{\Omega_h} p_h^{n+1} (\nabla\, v_h)\, d\Omega_h = \dfrac{\rho}{\Delta t}\int_{\Omega_h} u_h^n\, v_h \; d\Omega_h + \int_{\Omega_h} f_h^{n+1}\, v_h \, d\Omega_h \,, \\ \displaystyle\int_{\Omega_h} (\nabla \cdot u_h^{n+1})\, q_h \, d\Omega_h = 0, \qquad \forall v_h \in \mathbb{V}_h^N \;\; \text{and} \;\; q_h \in W_h^N, \\ u_h^{n+1} = 0 \quad \text{on} \quad \partial\Omega_h \end{cases} \tag{4}$$

El problema (Semi-implícito):

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$$
\begin{cases}
\dfrac{\rho}{\Delta t}\displaystyle\int_{\Omega_h} u_h^{n+1}\, v_h\ d\Omega_h\ +\ \rho\int_{\Omega_h} (u_h^n \cdot \nabla u_h^{n+1})\, v_h\ d\Omega_h + \mu\int_{\Omega_h} \nabla u_h^{n+1}\, \nabla v_h\, d\Omega_h \\
-\displaystyle\int_{\Omega_h} p_h^{n+1} (\nabla\, v_h)\, d\Omega_h = \dfrac{\rho}{\Delta t}\displaystyle\int_{\Omega_h} u_h^n\, v_h\ d\Omega_h + \int_{\Omega_h} f_h^{n+1}\, v_h\, d\Omega_h\ , \\
\displaystyle\int_{\Omega_h} (\nabla \cdot u_h^{n+1})\, q_h\, d\Omega_h = 0, \qquad \forall v_h \in \mathbb{V}_h^N \ \text{ and } \ q_h \in W_h^N, \\
u_h^{n+1} = 0 \quad \text{on} \quad \partial\Omega_h
\end{cases}
\tag{4}
$$

Ventaja: Velocidad y conocer u_h^n.

El problema (Semi-implícito):

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$$
\begin{cases}
\dfrac{\rho}{\Delta t}\displaystyle\int_{\Omega_h} u_h^{n+1}\, v_h\, d\Omega_h + \rho \int_{\Omega_h} (u_h^n \cdot \nabla u_h^{n+1})\, v_h\, d\Omega_h + \mu \int_{\Omega_h} \nabla u_h^{n+1}\, \nabla v_h\, d\Omega_h \\
- \displaystyle\int_{\Omega_h} p_h^{n+1} (\nabla\, v_h)\, d\Omega_h = \dfrac{\rho}{\Delta t}\int_{\Omega_h} u_h^n\, v_h\, d\Omega_h + \int_{\Omega_h} f_h^{n+1}\, v_h\, d\Omega_h\,, \\
\displaystyle\int_{\Omega_h} (\nabla \cdot u_h^{n+1})\, q_h\, d\Omega_h = 0, \qquad \forall v_h \in \mathbb{V}_h^N \quad \text{and} \quad q_h \in W_h^N, \\
u_h^{n+1} = 0 \quad \text{on} \quad \partial\Omega_h
\end{cases} \tag{4}
$$

Ventaja: Velocidad y conocer u_h^n.

Limitación: Restricción de estabilidad en el paso de tiempo.

El problema (Semi-implícito):

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$$
\begin{cases}
\dfrac{\rho}{\Delta t}\displaystyle\int_{\Omega_h} u_h^{n+1}\, v_h \; d\Omega_h \;+\; \rho\int_{\Omega_h} (u_h^n\cdot\nabla u_h^{n+1})\, v_h \; d\Omega_h + \mu\int_{\Omega_h}\nabla u_h^{n+1}\,\nabla v_h \, d\Omega_h \\
-\displaystyle\int_{\Omega_h} p_h^{n+1}(\nabla\, v_h)\, d\Omega_h = \dfrac{\rho}{\Delta t}\displaystyle\int_{\Omega_h} u_h^n\, v_h \; d\Omega_h + \int_{\Omega_h} f_h^{n+1}\, v_h\, d\Omega_h \;, \\
\displaystyle\int_{\Omega_h}(\nabla\cdot u_h^{n+1})\, q_h\, d\Omega_h = 0, \qquad \forall v_h\in\mathbb{V}_h^N \;\; \text{and} \;\; q_h\in W_h^N, \\
u_h^{n+1} = 0 \quad \text{on} \quad \partial\Omega_h
\end{cases}
\tag{4}
$$

Ventaja: Velocidad y conocer u_h^n.

Limitación: Restricción de estabilidad en el paso de tiempo.

$$
\boxed{\Delta t \le C\,\frac{h}{\max_{x\in\Omega_h}|u_h^n|}}
$$

Derivada material y fórmula backward Euler

La llamada derivada material (o derivada lagrangiana) del campo de velocidad $\mathbf{u}$ se define como

$$\frac{D\mathbf{u}}{Dt} = \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u}.$$

Derivada material y fórmula backward Euler

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La llamada derivada material (o derivada lagrangiana) del campo de velocidad $\mathbf{u}$ se define como

$$\frac{D\mathbf{u}}{Dt} = \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u}.$$

Aplicando la fórmula backward Euler para aproximar esta derivada en $\mathbf{x}$, tenemos

$$\frac{D\mathbf{u}}{Dt}(\mathbf{x}) \approx \frac{\mathbf{u}^{k+1}(\mathbf{x}) - \mathbf{u}^{k}(\mathbf{x}^{p})}{\Delta t},$$

Derivada material y fórmula backward Euler

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La llamada derivada material (o derivada lagrangiana) del campo de velocidad $\mathbf{u}$ se define como

$$\frac{D\mathbf{u}}{Dt} = \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u}.$$

Aplicando la fórmula backward Euler para aproximar esta derivada en $\mathbf{x}$, tenemos

$$\frac{D\mathbf{u}}{Dt}(\mathbf{x}) \approx \frac{\mathbf{u}^{k+1}(\mathbf{x}) - \mathbf{u}^{k}(\mathbf{x}^{p})}{\Delta t},$$

donde $\mathbf{x}^p = \mathbf{x} - \mathbf{u}^k(\mathbf{x})\Delta t + O(\Delta t^2)$ a lo largo de la característica que llega a $\mathbf{x}$ en el tiempo t^{k+1}.

Método de las características

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La ecuación de momento se escribe como

$$\frac{\mathbf{u}^{k+1}(\mathbf{x}) - \mathbf{u}^{k}(\mathbf{x}^{p})}{\Delta t} - \Delta \mathbf{u}^{k+1}(\mathbf{x}) + \nabla p^{k+1}(\mathbf{x}) = \mathbf{f}^{k+1}(\mathbf{x}).$$

Método de las características

La ecuación de momento se escribe como

$$\frac{\mathbf{u}^{k+1}(\mathbf{x}) - \mathbf{u}^{k}(\mathbf{x}^{p})}{\Delta t} - \Delta \mathbf{u}^{k+1}(\mathbf{x}) + \nabla p^{k+1}(\mathbf{x}) = \mathbf{f}^{k+1}(\mathbf{x}).$$

Para determinar $\mathbf{x}^p$, se sigue hacia atrás la trayectoria $\mathbf{X}(t; s, \mathbf{x})$ de una partícula de fluido en $\mathbf{x}$ en el tiempo $s = t^{k+1}$, resolviendo

$$\begin{cases} \dfrac{d\mathbf{X}}{dt}(t; s, \mathbf{x}) = \mathbf{u}(t, \mathbf{X}(t; s, \mathbf{x})), \quad t \in [t^k, t^{k+1}], \\ \mathbf{X}(s; s, \mathbf{x}) = \mathbf{x}. \end{cases}$$

Método de las características

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La ecuación de momento se escribe como

$$\frac{\mathbf{u}^{k+1}(\mathbf{x}) - \mathbf{u}^k(\mathbf{x}^p)}{\Delta t} - \Delta \mathbf{u}^{k+1}(\mathbf{x}) + \nabla p^{k+1}(\mathbf{x}) = \mathbf{f}^{k+1}(\mathbf{x}).$$

Para determinar $\mathbf{x}^p$, se sigue hacia atrás la trayectoria $\mathbf{X}(t; s, \mathbf{x})$ de una partícula de fluido en $\mathbf{x}$ en el tiempo $s = t^{k+1}$, resolviendo

$$\begin{cases} \dfrac{d\mathbf{X}}{dt}(t; s, \mathbf{x}) = \mathbf{u}(t, \mathbf{X}(t; s, \mathbf{x})), \quad t \in [t^k, t^{k+1}], \\ \mathbf{X}(s; s, \mathbf{x}) = \mathbf{x}. \end{cases}$$

El método de las características consiste en evaluar $\mathbf{u}^k$ en el punto $\mathbf{x}^p \approx \mathbf{x} - \mathbf{u}^k(\mathbf{x})\Delta t$, siguiendo hacia atrás la trayectoria, lo que mejora la estabilidad y precisión numérica.

El problema aproximado :

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$$
\begin{cases}
\dfrac{\rho}{\Delta t}\displaystyle\int_{\Omega_h} u_h^{n+1}\, v_h \; d\Omega_h - \dfrac{\rho}{\Delta t}\displaystyle\int_{\Omega_h} (u_h^n \circ X_h^n)\, v_h \; d\Omega_h + \mu \displaystyle\int_{\Omega_h} \nabla u_h^{n+1}\, \nabla v_h \, d\Omega_h \\
- \displaystyle\int_{\Omega_h} p_h^{n+1} (\nabla\, v_h)\, d\Omega_h = \displaystyle\int_{\Omega_h} f_h^{n+1}\, v_h \, d\Omega_h \; , \\
\displaystyle\int_{\Omega_h} (\nabla \cdot u_h^{n+1})\, q_h \, d\Omega_h = 0, \qquad \forall v_h \in \mathbb{V}_h^N \;\; \text{and} \;\; q_h \in W_h^N, \\
u_h^{n+1} = 0 \quad \text{on} \quad \Gamma_{wall}^h,
\end{cases}
\tag{5}
$$

Aspectos Numericos

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Discretización

- Elementos Taylor-Hood P2-P1 (o Mini P1b-P1)

Aspectos Numericos

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Discretización

- Elementos Taylor-Hood P2-P1 (o Mini P1b-P1)
- Paso de tiempo Euler retrasado;

Aspectos Numericos

Discretización

- Elementos Taylor-Hood P2-P1 (o Mini P1b-P1)
- Paso de tiempo Euler retrasado;
- Sistema lineal con operador CONVECT

Navier-Stokes Flow (I)

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```
mesh Th=square(2^6, 2^6);
fespace Vh(Th, P2), Qh(Th, P1);
Vh u, v, uu, vv, uold, vold;
Qh p, pp;
real nu=0.01, T=1., dt = 0.01;
int m, M= T/dt;

problem NavierStokes([u,v,p], [uu,vv,pp], init=m, solver=LU)
         = int2d(Th)( (u*uu + v*vv)/dt
         + nu*( dx(u)*dx(uu) + dy(u)*dy(uu) + dx(v)*dx(vv) + dy(v)*dy(vv) )
         - p*pp*1.e-6 - p*( dx(uu)  + dy(vv) ) - pp*( dx(u) + dy(v) )
         + ( uold*dx(u) + vold*dy(u) )*uu + ( uold*dx(v) + vold*dy(v) )*vv )
         - int2d(Th)( (uold*uu+vold*vv)/dt )
         + on(1, 2, 4, u=0, v=0)  + on(3, u=1, v=0);

for(m=0; m<M; m++){
                   NavierStokes; uold=u; vold=v;
                   plot(p,value=true, fill=true, wait=false);
                   }
```

Navier-Stokes Flow (II)

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```
mesh Th=square(2^6,2^6);
fespace Vh(Th,P2), Qh(Th,P1);
Vh u,v,uu,vv, uold,vold;
Qh p,pp;
real nu=0.01, T=1., dt = 0.01;
int m, M= T/dt;

problem NS([u,v,p],[uu,vv,pp], init=m, solver=Crout) =
 int2d(Th)( (u*uu+v*vv)/dt + nu*(dx(u)*dx(uu) + dy(u)*dy(uu)
 + dx(v)*dx(vv) + dy(v)*dy(vv))
 - p*pp*1.e-6 - p*(dx(uu) +dy(vv))- pp*(dx(u)+dy(v)))
 - int2d(Th) (( uold(x-uold(x,y)*dt,y-uold(x,y)*dt)*uu
 + vold(x-uold(x,y)*dt,y-uold(x,y)*dt)*vv )/dt)
 + on(1,2,4,u=0,v=0) + on(3,u=1,v=0);

for(m=0;m<M;m++){
  NS; uold=u; vold=v;
  plot(p,value=true, fill=true, wait=false);
}
```

Navier-Stokes Flow (III)

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```
mesh Th=square(2^6,2^6);
fespace Vh(Th,P2), Qh(Th,P1);
Vh u,v,uu,vv, uold,vold;
Qh p,pp;
real nu=0.01, T=1., dt = 0.01;
int m, M= T/dt;

problem NS([u,v,p],[uu,vv,pp], init=m)
      = int2d(Th)( (u*uu+v*vv)/dt
      + nu*(dx(u)*dx(uu) + dy(u)*dy(uu) + dx(v)*dx(vv) + dy(v)*dy(vv))
      - p*pp*1.e-6 - p*(dx(uu) +dy(vv))- pp*(dx(u)+dy(v)))
      - int2d(Th)((convect([uold,vold],-dt,uold)*uu  + convect([uold,vold],-dt,vold)*vv)/dt)
      + on(1,2,4,u=0,v=0) + on(3,u=1,v=0);

for(m=0;m<M;m++){
  NS; uold=u; vold=v;
  plot([u,v],p,value=true, fill=true, wait=false);
}
```

Final Plot

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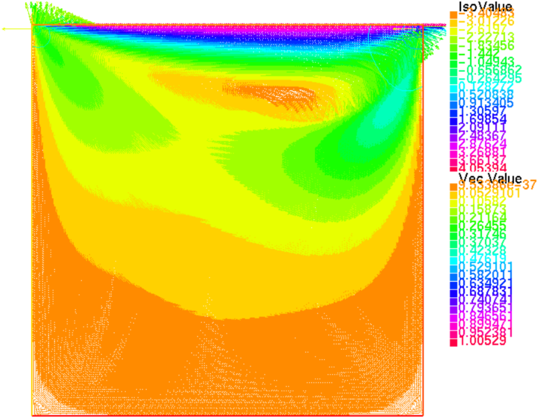

Figura: Velocidad y pression

Geometrias 3D

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CASOS 3D ...

Faces 3D

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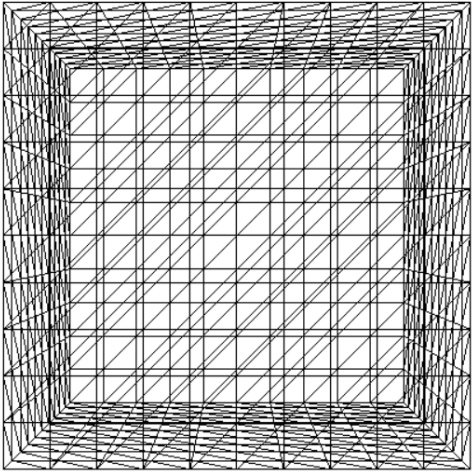
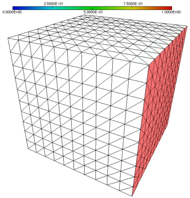
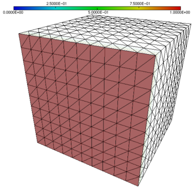
```
load "msh3"
load "medit"
int nn = 10;
mesh3 Th = cube(nn, nn, nn);
medit("Cubo", Th);
fespace Vh(Th, P1);
Vh f1 = (y < 1e-10);       // Face 1
Vh f2 = (x > 1-1e-10);  // Face 2
Vh f3 = (y > 1-1e-10);  // Face 3
Vh f4 = (x < 1e-10);     // Face 4
Vh f5 = (z < 1e-10);     // Face 5
Vh f6 = (z > 1-1e-10);  // Face 6
medit("Face y=0", Th, f1);
medit("Face x=1", Th, f2);
medit("Face y=1", Th, f3);
medit("Face x=0", Th, f4);
medit("Face z=0", Th, f5);
medit("Face z=1", Th, f6);
```



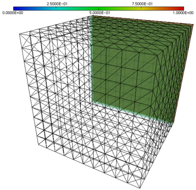
Lembrete: Faces do Domínio 3D



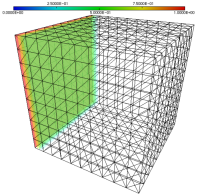
Face y=0



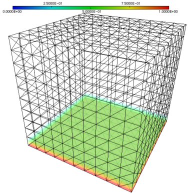
Face x=1



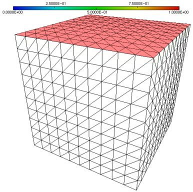
Face y=1



Face x=0



Face z=0



Face z=1

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Importanto Imagens GMSH

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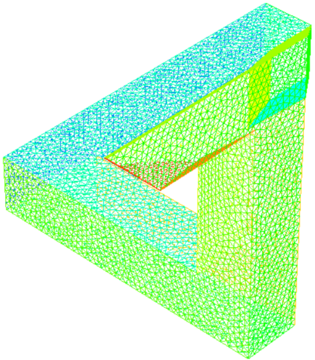
```
load "medit"
load "msh3"
mesh3 Th = readmesh3("Paradox_Design.mesh
    ");
plot(Th, wait=true);
medit("Penrose_Triangle",Th,order=1);
```



Import Mesh Domain - Rotation 3D

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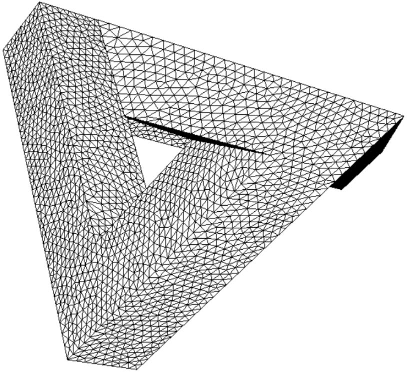

Import Mesh Domain - Rotation 3D

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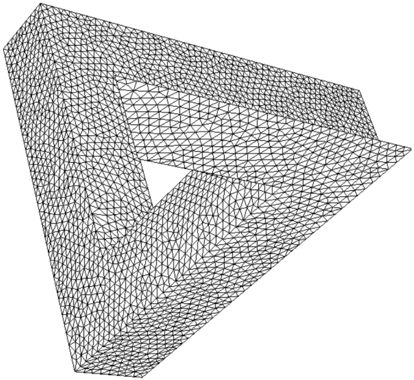
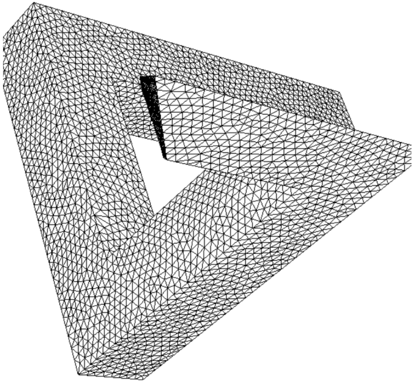


Import Mesh Domain - Rotation 3D



Import Mesh Domain - Rotation 3D

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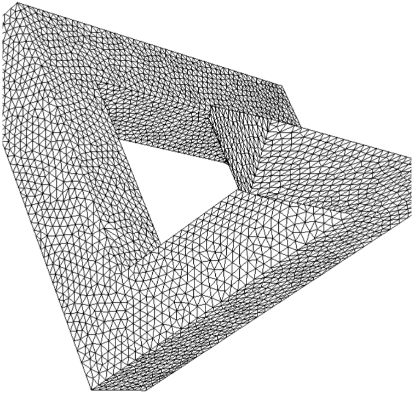

Import Mesh Domain - Rotation 3D

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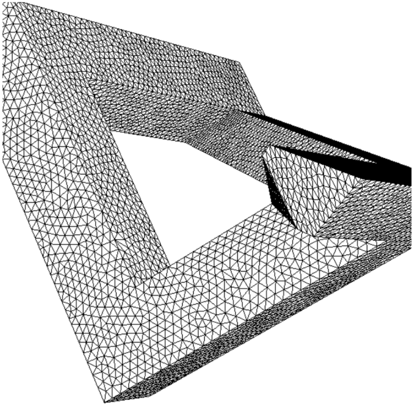

Import Mesh Domain - Rotation 3D

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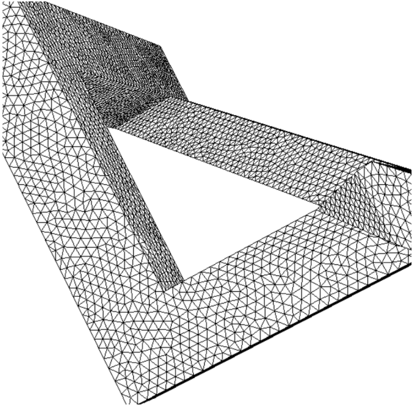
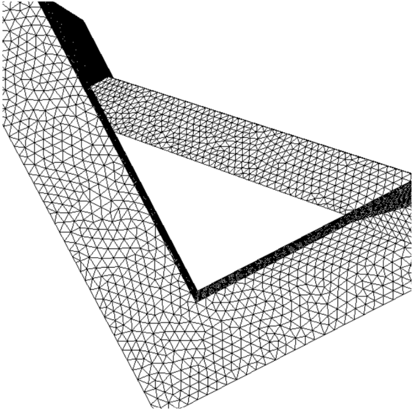


Import Mesh Domain - Rotation 3D



Import Mesh Domain - Rotation 3D

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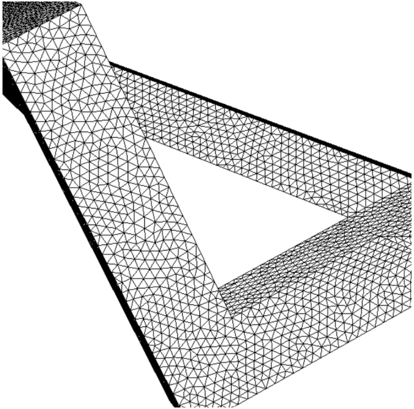

Import Mesh Domain - Rotation 3D

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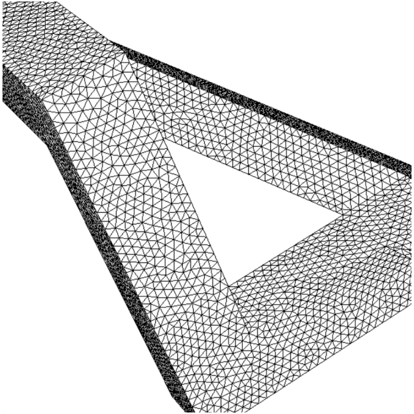

Geometrias Complexas

Imagens Médicas ...

Geometrias Complexas

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Imagens Médicas ...

Geometrias Complexas

Imagens Médicas ...

Imagens Médicas :

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a) MRA image

Imagens Médicas :

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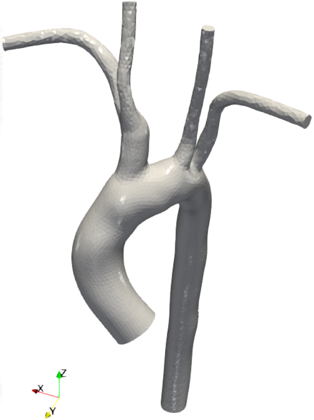
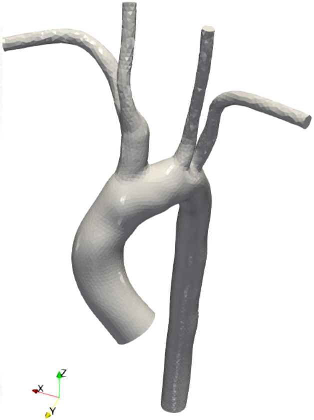
a) MRA image



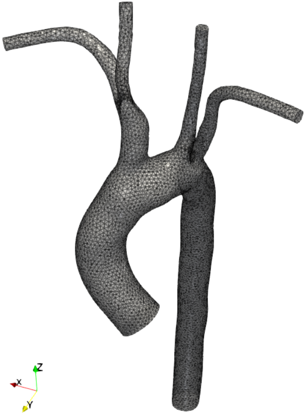
b) Superficie Segmentada

Imagens Médicas :

a) MRA image



b) Superficie Segmentada



c) Malla matemática

Malla del dominio importado (graficada con MEDIT):

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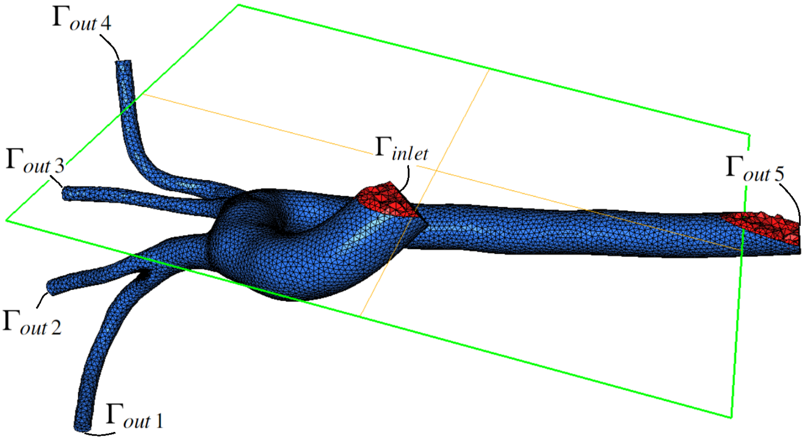

Geometria : $\Omega_h = \bigcup_{T_K \in \mathcal{T}_h} T_K;$ $Dim\,(V_h) = 295488.$

O problema aproximado :

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$$
\left\{
\begin{array}{l}
\dfrac{\rho}{\Delta t}\displaystyle\int_{\Omega_h} u_h^{n+1}\, v_h \; d\Omega_h - \dfrac{\rho}{\Delta t}\displaystyle\int_{\Omega_h} (u_h^n \circ X_h^n)\, v_h \; d\Omega_h + \mu \displaystyle\int_{\Omega_h} \nabla u_h^{n+1}\, \nabla v_h \, d\Omega_h \\
- \displaystyle\int_{\Omega_h} p_h^{n+1} (\nabla\, v_h)\, d\Omega_h - \displaystyle\int_{\Gamma^h_{outlet}} \left(\phi_N^{n+1} n_h\right) v_h \, dS = \displaystyle\int_{\Omega_h} f_h^{n+1}\, v_h \, d\Omega_h \,, \\
\displaystyle\int_{\Omega_h} (\nabla \cdot u_h^{n+1})\, q_h \, d\Omega_h = 0, \\
\forall v_h \in \mathbb{V}_h^N \quad \text{and} \quad q_h \in W_h^N, \\
u_h^{n+1} = \phi_D^{n+1} \quad \text{on} \quad \Gamma^h_{inlet}, \\
u_h^{n+1} = 0 \quad \text{on} \quad \Gamma^h_{wall}, \\
\mu\left(\nabla u_h^{n+1} + \nabla (u_h^{n+1})^T\right) n - p_h^{n+1} n = \phi_N^{n+1} \quad \text{on} \quad \Gamma^h_{outlet},
\end{array}
\right. \tag{6}
$$

Código FreeFEM++ - Parte 1

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```
load "medit"
verbosity=0;

border circle(t=0,2*pi){x=cos(t); y=sin(t);label=1;}
mesh Thx = buildmesh (circle(20));
int[int] rup=[0,3], rdown=[0,2], rmid=[1,1];

mesh3 th=buildlayers(Thx,30,zbound=[0,5],labelmid=rmid,
labelup=rup,labeldown=rdown,
transfo=[x,(4+y)*cos(z*pi/10)-4,(4+y)*sin(z*pi/10)]);

fespace Vh(th,P1b);
fespace Qh(th,P1);
fespace Zh(Thx,P1b); Zh w2;
Vh u,v,w,uh,vh,wh,uold=0,vold=0,wold=0;
Qh p,ph;
real p1=1, dt=0.1, nu=0.005; int n=-1;
```

Código FreeFEM++ - Parte 2

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```
problem a([u,v,w,p],[uh,vh,wh,ph],init=n)=
 int3d(th)((u*uh+v*vh+w*wh)/dt
 + nu*(dx(u)*dx(uh)+dy(u)*dy(uh)+dz(u)*dz(uh)
 + dx(v)*dx(vh)+dy(v)*dy(vh)+dz(v)*dz(vh)
 + dx(w)*dx(wh)+dy(w)*dy(wh)+dz(w)*dz(wh))
 - (dx(uh)+dy(vh)+dz(wh))*p
 - (dx(u)+dy(v)+dz(w))*ph + p*ph*1e-5)
 - int3d(th)((convect([uold,vold,wold],-dt,uold)*uh
  +convect([uold,vold,wold],-dt,vold)*vh
  +convect([uold,vold,wold],-dt,wold)*wh)/dt)
-int2d(th,2)(p1*wh) - int2d(th,3)(p1*vh)
+ on(1,u=0,v=0,w=0)+ on(2,v=0,u=0)+ on(3,u=0,w=0);

for(real t=0;t<2.6;t+=dt){
 p1=cos(t); n+=1; a; uold=u; vold=v; wold=w;
 plot([u,v,w],p, value=true, fill=true, wait=1);
}

medit("Pressure",th,p,order=1);
medit("Velocity",th,[u,v,w],order=1);
```

Código FreeFEM++

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```
load "medit"
verbosity=0;

mesh3 th = readmesh3("AORTA-COMCA2025.mesh"); // MALHA ESTREITA
plot(th, wait=true);
medit("Geometria Arterial", th, order=1);
fespace Vh(th,P1b);
fespace Qh(th,P1);

Vh u,v,w,uh,vh,wh,uold=0,vold=0,wold=0;
Qh p,ph;
real p1=10, p0=0,  dt=0.1, nu=0.005; int n=0;
problem a([u,v,w,p],[uh,vh,wh,ph],init=n, solver=sparsesolver)
        = int3d(th)((u*uh+v*vh+w*wh)/dt
        + nu*( dx(u)*dx(uh)+dy(u)*dy(uh)+dz(u)*dz(uh)
        + dx(v)*dx(vh)+dy(v)*dy(vh)+dz(v)*dz(vh)
        + dx(w)*dx(wh)+dy(w)*dy(wh)+dz(w)*dz(wh) )
        - (dx(uh)+dy(vh)+dz(wh))*p - (dx(u)+dy(v)+dz(w))*ph - p*ph*1e-8 )
```

```
 - int3d(th)(( convect([uold,vold,wold],-dt,uold)*uh
  +convect([uold,vold,wold],-dt,vold)*vh
  +convect([uold,vold,wold],-dt,wold)*wh )/dt)
-int2d(th,5)(p1*wh) - int2d(th,3,4,6,7,8)(p0*vh)
+ on(2,u=0,v=0,w=0)
//+ on(2,v=0,u=0)+ on(3,u=0,w=0)
;
for(real t=0;t<0.6;t+=dt){
p1=cos(pi*t);
 n+=1; a; uold=u; vold=v; wold=w;
plot([u,v,w],p, value=true, fill=true, wait=0);
}

medit("Pressure",th,p,order=1);
medit("Velocity",th,[u,v,w],order=1);
```

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Referencias Bibliográficas

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¡Gracias por su presencia en la parte 3!

¿Preguntas?