CS/COE 1501

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P vs NP

But first, something completely different...

- Some computational problems are *unsolvable*
  - No algorithm can be written that will always produce the correct output
  - One example is the *halting problem*
    - Given a program and an input, will the program halt?
      - Can we write an algorithm to determine whether any program/input pair will halt?

Halting problem example

```
String test = "

public boolean proof_sketch(String program) {
    if (WILL_EVENTUALLY_HALT(program, program)){
        while (true){}
        return false;
    }
    else {
        return true;
    }
}

";
proof_sketch(test);
```

what can possibly happen?

There are a number of other undecidable problems

- But the halting problem is all that we'll cover here

Intractable problems

- Solvable, but require too much time to solve to be practically solved for modest input sizes
  - Listing all of the subsets of a set:
    - $\Theta(2^n)$
  - Listing all of the permutations of a sequence:
    - $\Theta(n!)$

Polynomial time algorithms

- Most of the algorithms we’ve covered so far this term
  - Also the most practically useful of the three classes we’ve just covered…
- Largest term in the runtime is a simple power with a constant exponent
  - E.g., n^2
  - Or a power times a logarithm
    - E.g., $n \lg n$

Consider the following

- The shortest path problem
  - Easily solved in polynomial time
- The longest path problem
  - How long would it take us to find the longest path between two points in a graph?

What if a problem doesn't fall into one of our three categories?

- It can be solved
- There is no proof that a solution requires exponential time
  - … yet
- There is no valid solution that runs in polynomial time
  - … yet

P vs NP

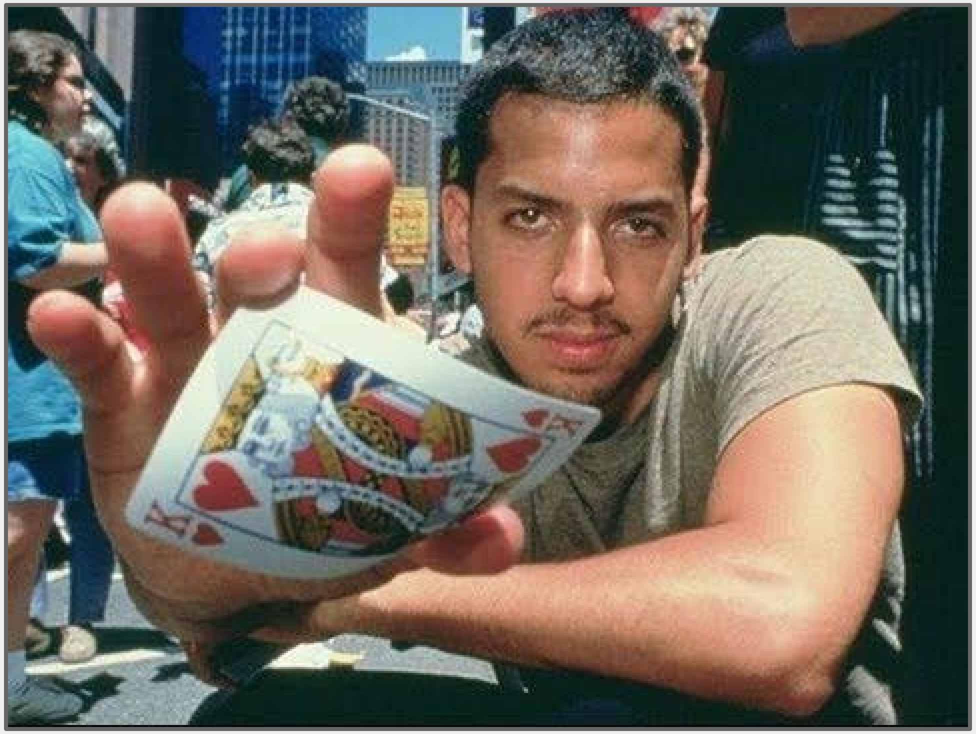
- P
  - The set of problems that can be solved by deterministic algorithms in polynomial time
- NP
  - The set of problems that can be solved by non-deterministic algorithms in polynomial time
    - I.e., solution from a non-deterministic algorithm can be verified in polynomial time

Deterministic vs non-deterministic algorithms

- Deterministic
  - At any point during the run of the program, given the current instruction and input, we can predict the next instruction
  - Running the same program on the same input produces the same sequence of executed instructions
- Non-deterministic
  - A conceptual algorithm with more than one allowed step at certain times and which always takes the right or best step
    - Conceptually, could run on a deterministic computer with unlimited parallel processors
      - Would be as fast as always choosing the right step

Non-deterministic algorithms

- Array search:
  - Linear search:
    - $\Theta(n)$
  - Binary search:
    - $\Theta(\lg n)$
  - Non-deterministic search algorithm:
    - $\Theta(1)$

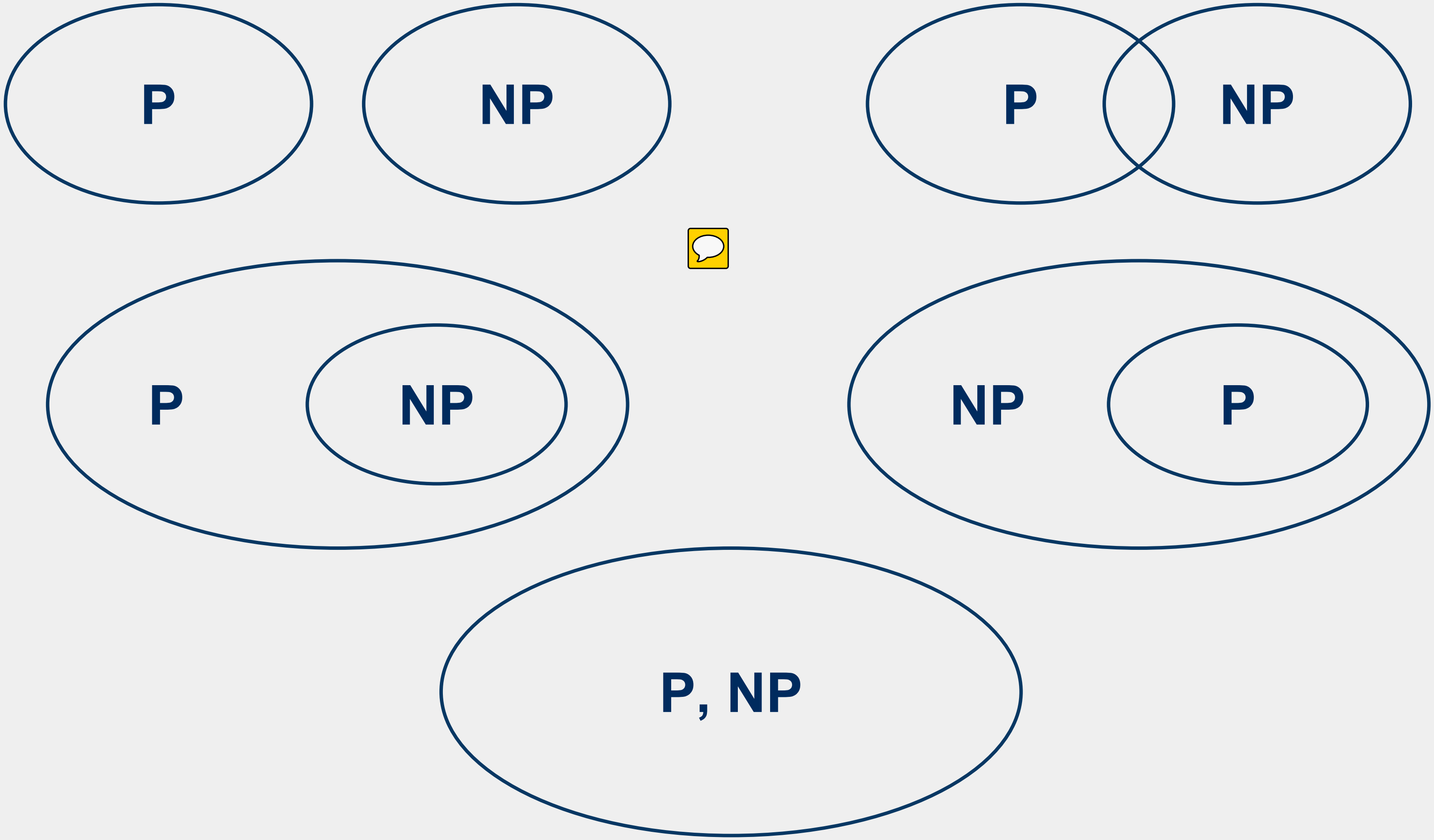


Hamiltonian cycle problem

- A Hamiltonian cycle is a simple cycle through a graph that visits every vertex of the graph
- Can we determine if a given graph has a Hamiltonian cycle?
    - Yes, a brute-force deterministic algorithm would look at every possible cycle of the graph to see if one is Hamiltonian
        - How many possibilities for a complete graph?
            - v!
    - A non-deterministic algorithm would simply return a cycle
- Can we verify a result?
    - Yes! simply look through the returned cycle and verify that it visits every vertex
        - How long will this take?

So we can group problems into P and NP...

- 5 options for how the sets P and NP intersect:



Are any of these clearly impossible?

- Why?

Remember how I kept saying "... yet"

- Either P ⊂ NP or P = NP
  - One of the biggest unsolved problems in computer science
- Can prove that P ⊂ NP by:
  - Proving an NP problem to be intractable
- Can prove P = NP by:
  - Developing a polynomial time algorithm to solve an NP-Complete problem

What if P = NP?

- Most widely-used cryptography would break
  - Efficient solutions would exist for:
    - Attacking public key crypto
    - Attacking AES/DES
    - Reversing cryptographic hash functions
- Operations research and management science would be greatly advanced by efficient solutions to the travelling salesman problem and integer programming problems
- Biology research would be sped up with an efficient solution to protein structure prediction
- Mathematics would be drastically transformed by advances in automated theorem proving

What if P != NP?

- meh
  - Mostly assumed to be the case

OK, but wait...

- What exactly is NP-Complete?
  - That came out of nowhere on the last few slides
    - NP-Complete problems are the "hardest" problems in NP
      - They are all equally "hard"
        - So, if we find a polynomial time solution to one of them, we clearly have a polynomial time solution to all problems in NP

Consider for a moment...

- You've just discovered a new computational problem
    - But you cannot find a polynomial time solution
        - If you can show that the problem is NP-Complete, you know that finding a polynomial time solution boils down to solving P vs NP

Proving NP-Completeness

- Show the problem is in NP
- Show that your problem is at least as hard as every other problem in NP
  - Typically done by reducing an existing NP-Complete problem to your problem
    - In polynomial time

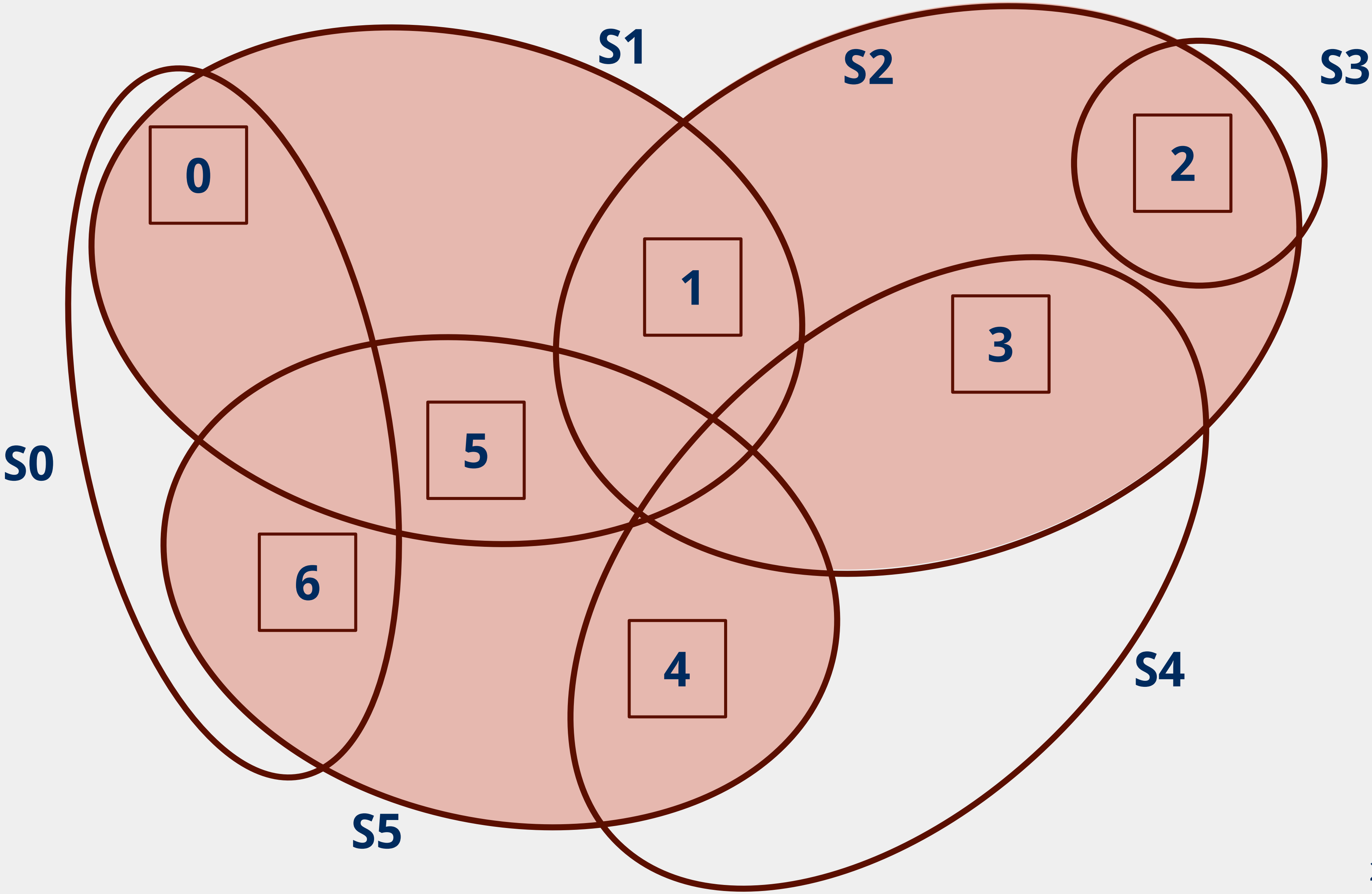
Reduction to show NP-Completeness

- Goal: show that your problem can be used to solve an NP-Complete problem
    - And that the transformation of problem inputs can be performed in polynomial time
- Why does this work?
    - If your algorithm can solve an NP-Complete problem, then a polynomial time solution to your problem with a polynomial time transformation from the NP-Complete problem would mean a polynomial-time solution to an NP-Complete problem

Reduction example

- Assume we've just come up with the set cover problem:
  - Given a set S of elements and a collection $s_1 \dots s_m$ of subsets of S is there a collection of at most k of these sets whose union equals S?

Set cover example



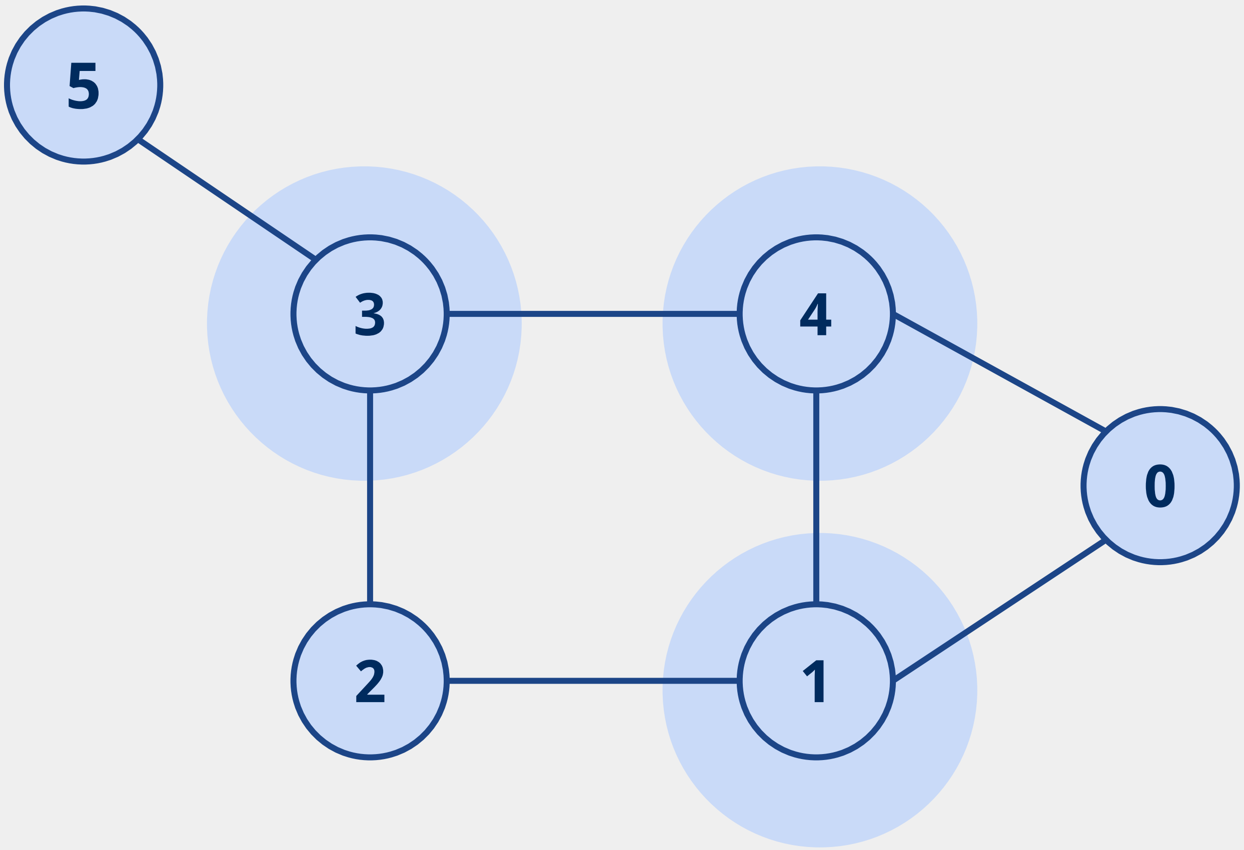
Is set cover NP-Complete?

- First of all is it in NP?
- OK, next step is to find a problem that is known to be NP-Complete and reduce it to an instance of set cover

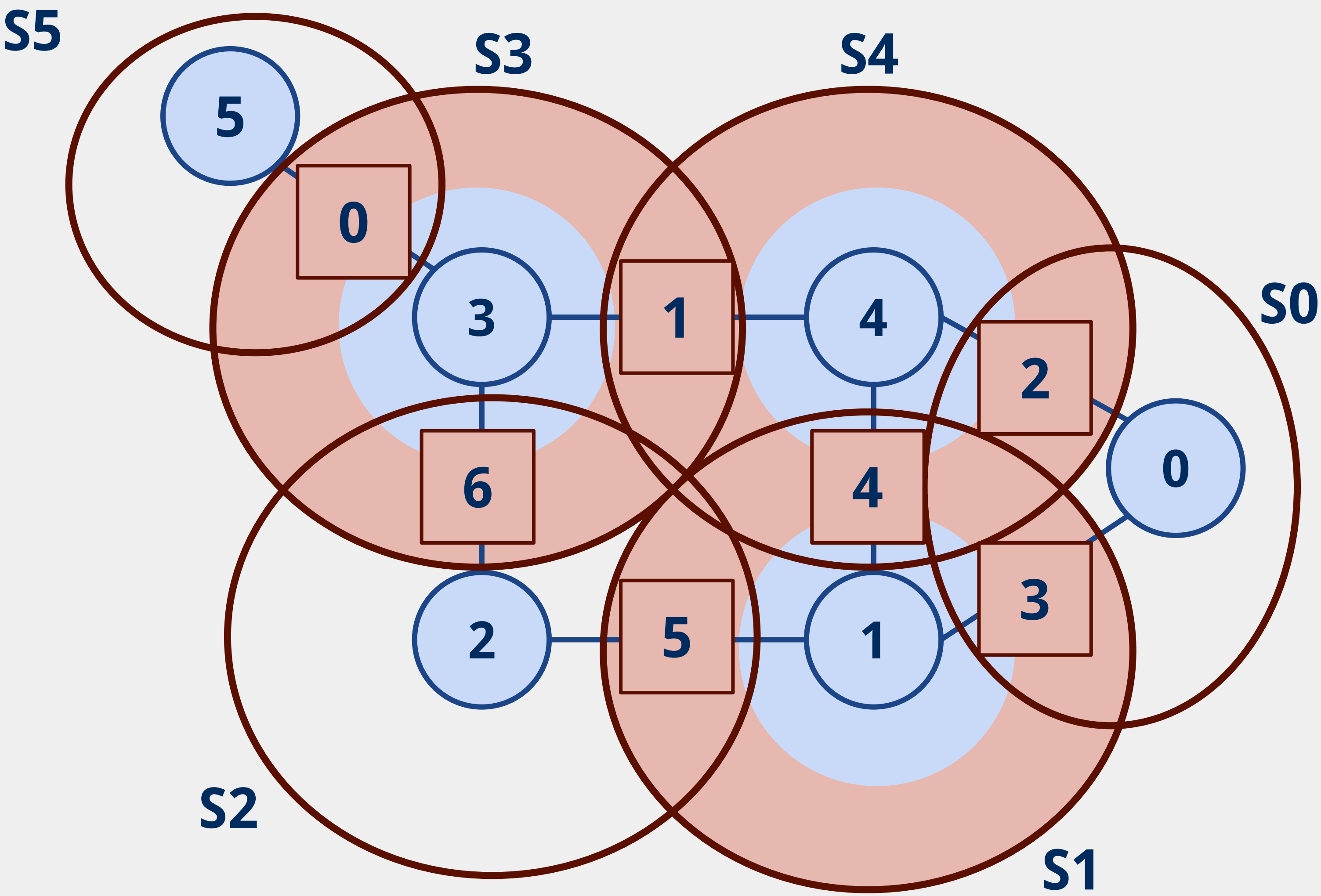
The vertex cover problem

- A vertex-cover of an undirected graph G = (V, E) is a subset V' of V such that if edge (u, w) is an edge of E, then u is in V', w is in V', or both
- Does a vertex-cover exist for a graph G that consists of at most k vertices?

Vertex cover example



Reducing vertex cover to set cover



Reducing vertex cover to set cover

- Given k and a graph to be vertex-covered:
  - Let S = E
  - For each $u_i \in V$: create s_i such that s_i contains all edges in E with u_i as an endpoint
  - Solve the set cover problem for k, S, and $s_1 \ldots s_v$
- Runtime to transform inputs to vertex cover into inputs for set cover?

A final note on reduction

- Be careful about the ordering
  - Always solve the known NP-Complete problem using your new problem
    - You WILL get this mixed up at some point

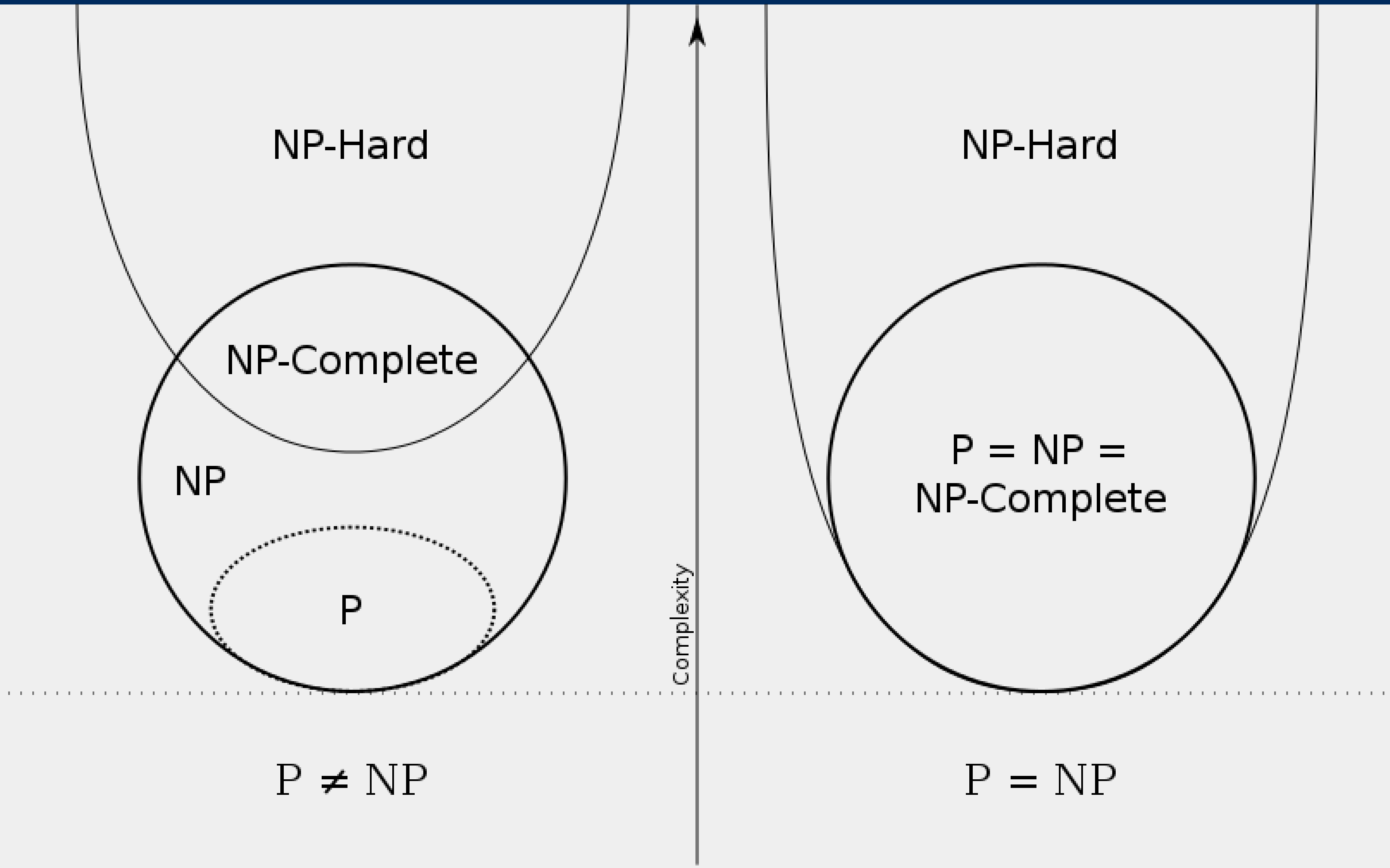
A timeline of P/NP

- 1971 - Cook presents the Cook-Levin theorem, which shows that the boolean satisfiability problem is as hard as every other problem in NP
  - It is NP-Complete, but this term appears nowhere in paper
- 1972 - Karp presents 21 NP-Complete problems via reduction from boolean satisfiability
- Thousands have since been discovered by reducing from those 21

Karp's 21 problems

- Boolean Satisfiability
    - 0–1 integer programming
    - Clique (see also independent set problem)
        - Set packing
        - Vertex cover
            - Set covering
            - Feedback node set
            - Feedback arc set
            - Directed Hamilton circuit
                - Undirected Hamilton circuit
    - Satisfiability with at most 3 literals per clause
        - Chromatic number (aka Graph Coloring Problem)
            - Clique cover
            - Exact cover
                - Hitting set
                - Steiner tree
                - 3-dimensional matching
                - Knapsack
                    - Job sequencing
                    - Partition
                        - Max cut

The landscape



To review

- What are P problems?
- What are NP problems?
- What are NP-Complete problems?
- What about NP-Hard?

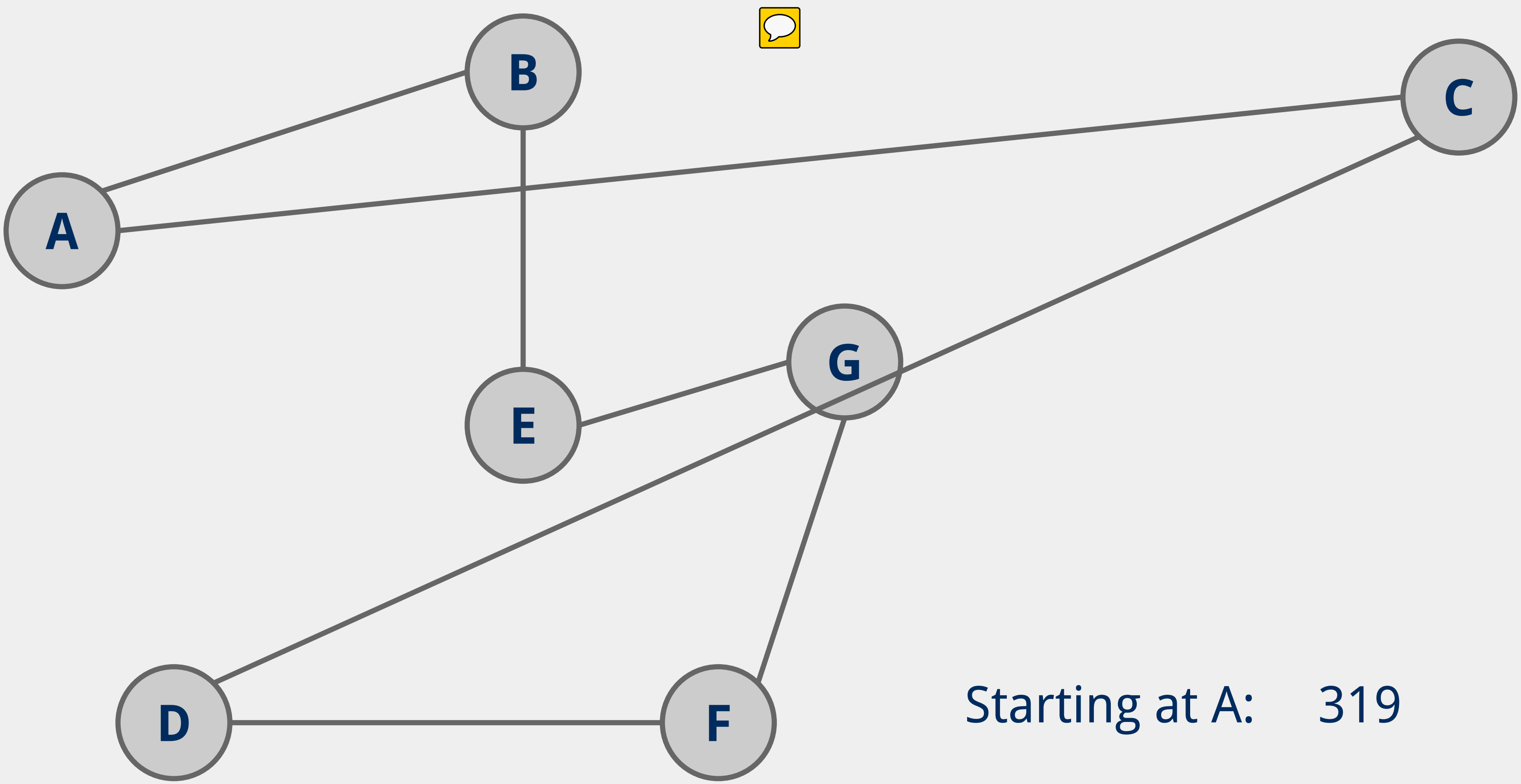
What if you still need to solve an NP-C problem?

- Can't get an exact solution in a reasonable amount of time.
  - What about an approximate solution?
    - Can we devise an algorithm that runs in a reasonable amount of time and gives a close to optimal result?
- Let's look at some heuristics for approximating solutions to the Travelling Salesman Problem:
  - Given a list of cities and the distances between each pair of cities, what is the shortest possible route that visits each city exactly once and returns to the origin city?

Nearest neighbor heuristic

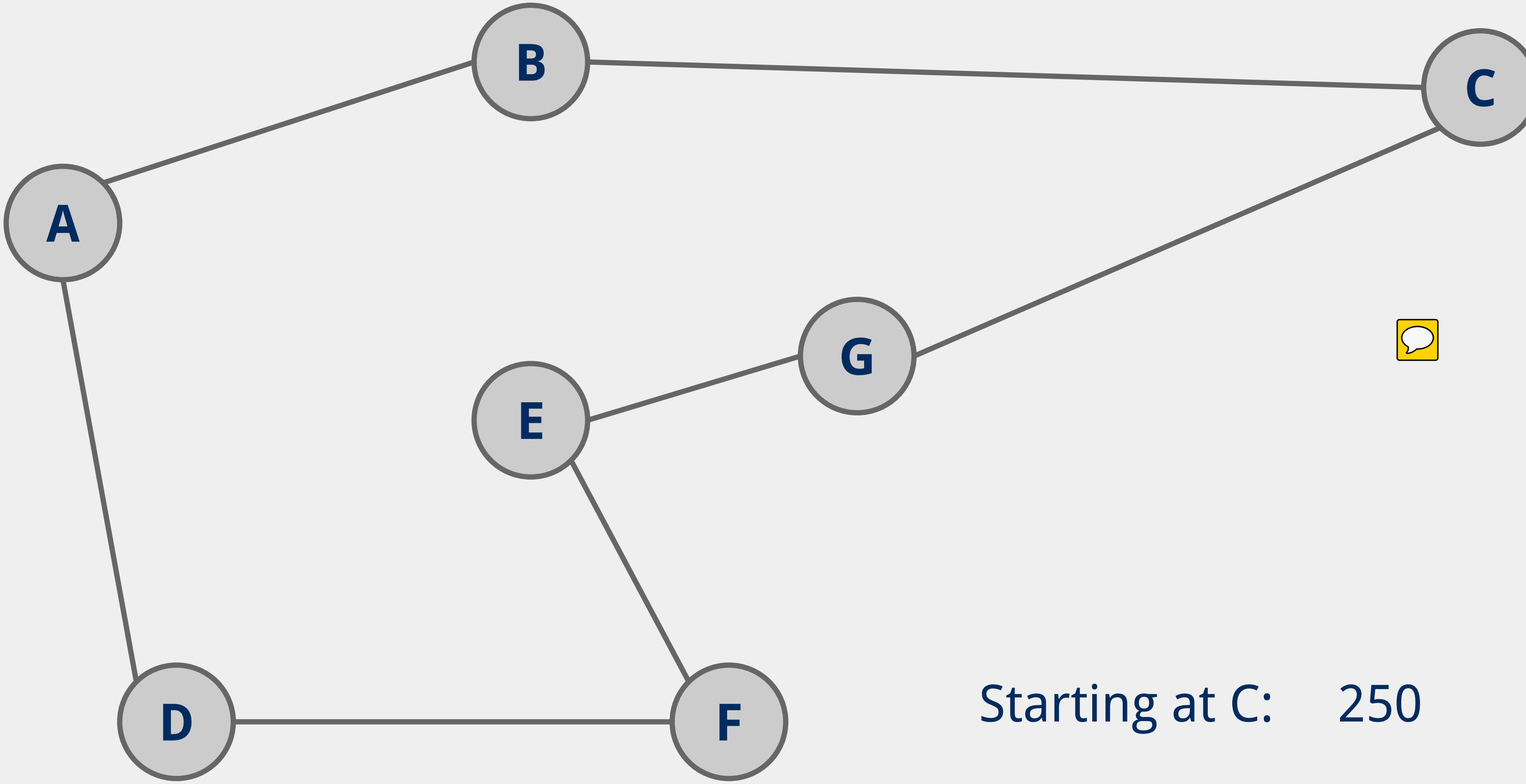
- From each city, visit the nearest city until a circuit of all cities is completed
- Runtime?
  - v^2
- Any other issues?
- How good are solutions generated by this heuristic compared to optimal solutions?

Nearest neighbor example



Starting at A: 319

What about a different starting point?



Starting at C: 250

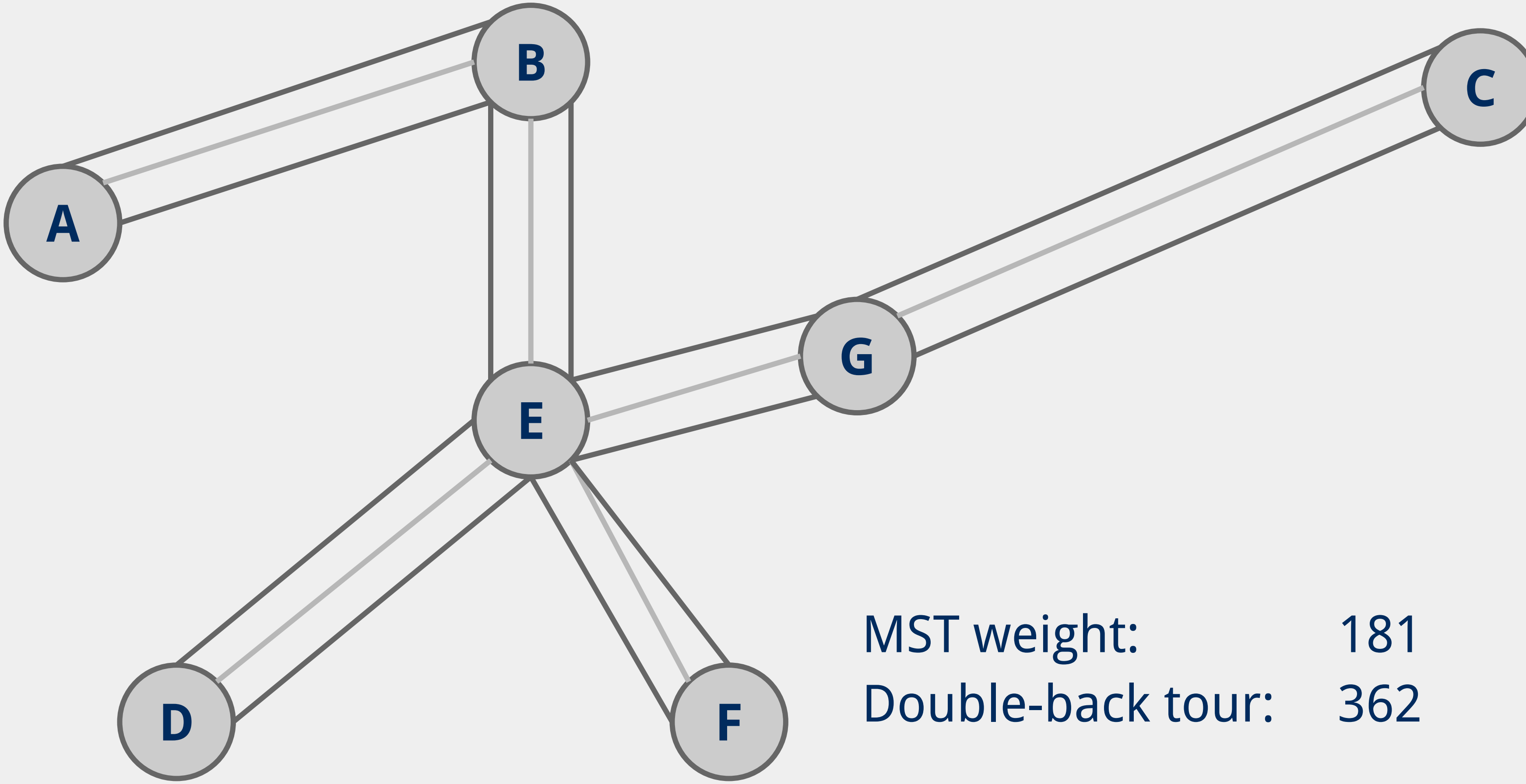
Measuring heuristic algorithm quality

- Let's consider $H_{NN}(C)$ be the length of the tour of the set of cities described by C found by the nearest neighbor heuristic
- Let OPT(C) be the optimal tour for C
- The approximation ratio of the nearest neighbor heuristic is then $H_{NN}(C)/OPT(C)$
  - I.e, how much worst than optimal is nearest neighbor
    - For nearest neighbor, this approximation ratio grows according to log(v)

Let’s aim for a constant approximation ratio

- Consider minimum spanning trees
  - Creates a fully connected graph with minimum edge weight
    - Optimal TSP solution must be more than MST weight
  - Consider the tour produced by a DFS traversal of the MST
    - Travels every edge twice
    - Since MST weight is less than optimal solution, this tour must be less than 2x the optimal solution!

MST Heuristic example

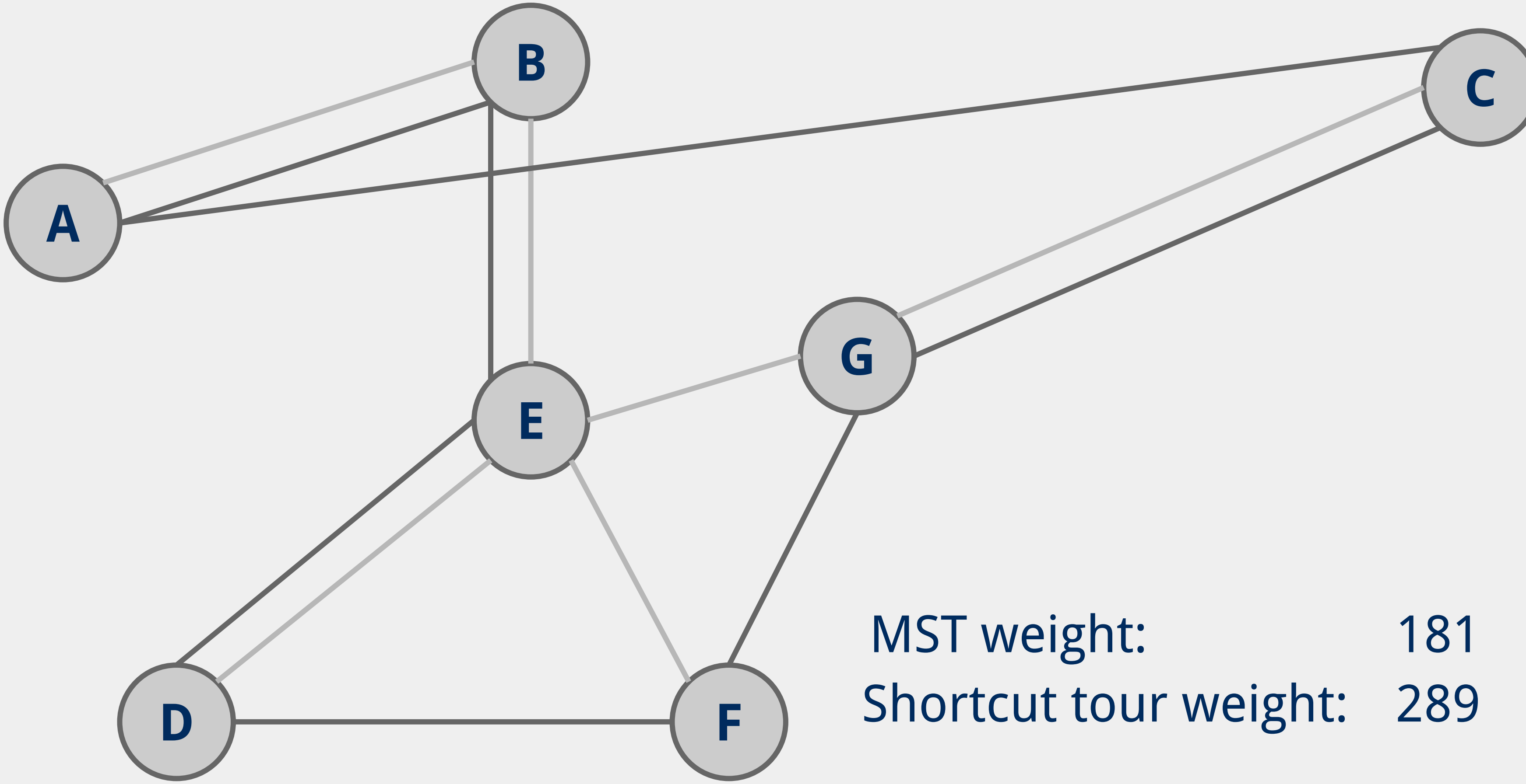


MST weight:	181
Double-back tour:	362

But it visits some cities twice...

- Violates a condition of being a TSP solution
- What about this:
  - Find MST
  - Determine traversal order
  - At each backtrack, simply take the direct route to the next city

MST Heuristic example



MST weight: 181
Shortcut tour weight: 289

Does this maintain our approximation ratio?

- Yes, if we make an additional assumption
  - Distances between "cities" have to abide by Euclidean geometry
    - Specifically, they need to uphold the triangle inequality
      - A direct path between two cities must be shorter than going through a third city