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Non cooperative fuzzy games in normal form: A survey☆

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Abstract

In this paper, we present a review of most of the approaches for solving non-cooperative fuzzy games in normal form. Applications of these games are reviewed as well. We also present a new class of fuzzy games in normal form: the class of non-cooperative games involving fuzzy parameters. Finally, some potential directions of research are provided throughout the paper and in the conclusion.
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1. Introduction

Game theory is a formal way to analyze interaction among a group of rational agents who behave strategically. The early works on game theory go back to the 18th century [40]. Game theory has been formulated as an independent theory in the fundamental book "Theory of Games and Economic Behavior" by Von Neumann and Morgenstern in 1944 [69]. Since that time game theory has developed very fast, and later it has received worldwide recognition. Indeed, game theory has been successfully applied in different areas as competition, voting, auctions, research and development races, cartel behavior, artificial intelligence, e-commerce, biology and more. Games are classified into two major classes: cooperative games and non-cooperative games. In the former the players focus on coalition formation, however, in the latter they focus on the choice of strategies. In this paper, we deal with non-cooperative games in normal (or strategic) form (NCGNF).

As a matter of fact, traditional game theory is anchored on binary (Aristotelian) logic and the fully rational behavior assumption. Fuzzy logic is able to accommodate many of the binary-logic related dilemmas in crisp game theory, e.g. Prisoner's Dilemma game. In general, in real games the players do not behave as fully rational decision makers, rather they follow the principle of "bounded rationality" [63]. Fuzzy logic and "If–Then" rules that are used in fuzzy controllers are an adequate tool for a formal representation of such behavior. Moreover, one of the major limitations of the traditional non-cooperative game theory [52,69] is that it assumes that all the data of the game are known exactly by all players. This assumption seems to be restrictive. In real-world game situations, it often happens that the players are not able to evaluate exactly the outcomes of different strategy profiles and/or their own preferences or the preferences of the other players. In [34,65] games involving uncertainty of probabilistic type have been investigated. However,

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real-game situations may involve other types of uncertainty such as fuzzy, fuzzy-stochastic, etc. By assigning a degree to which an element of a set possesses the defining characteristics of that set, fuzzy sets [74] provide an excellent tool to model imprecisely defined situations. This becomes crucial when problems are vaguely defined in linguistic terms which many real-world problems are apt to be.

Aubin [6] was the first to introduce fuzzy sets in cooperative game theory in 1974. Later, in 1978, Butnariu [19–22] has introduced for the first time fuzzy sets in non-cooperative games. The work of Butnariu has been further developed by Billot [12]. In the Butnariu–Billot formulation each player's beliefs about the action of the other players is modeled as a fuzzy set. Buckley [17] used Zadeh and Bellman's [10] principle of decision making in a fuzzy environment to formulate a multiple goals non-cooperative game under uncertainty. Garazic and Cruz [31] used an approach based on the principle of fuzzy controller to study games with fuzzy preferences. Campos [23] was the first to study non-cooperative games with fuzzy payoffs. In the last decade, this class of games has been intensively studied by many researchers [8,9,17,18,24–26,42,46,48,49,54,55,61,64,66–68,72]. Recently, in a significant contribution, Arfi [4] introduced linguistic fuzzy logic games and showed their great potential for applications in social sciences.

The recent advances in fuzzy game theory show that the formulation of games in fuzzy logic and/or fuzzy-set framework is a significant departure from traditional game theory. We think that the time has come for a survey of the advances of the theory of fuzzy non-cooperative games in normal form (FNCGNF). The fact that such a survey does not exist in literature has motivated us to write this paper.

The main objective of this paper is to present a survey of the most important results on FNCGNF and their application, and provide some possible directions of research. A new class of fuzzy games is presented as our contribution to this field of research: the non-cooperative games with fuzzy parameters. We do not pretend that our survey is exhaustive. By reviewing most of the publications that appeared on FNCGNF theory and its application, we came to the general conclusion that this theory is still at its beginning and it has a promising future in both further developing game theory and its application to social sciences and many other areas.

The remainder of the paper is organized as follows. Section 2 is an introduction to crisp NCGNF. Section 3 discusses general approaches for FNCGNF. Section 4 presents the most important approaches for solving matrix and bimatrix games with fuzzy payoffs and finite two-person games with fuzzy payoffs and fuzzy goals. Section 5 presents the class of non-cooperative games with fuzzy parameters. Section 6 discusses the applications of FNCGNF. Section 7 concludes the paper and provides research directions.

2. Crisp non-cooperative game in normal form

An n-person NCGNF is defined as follows:

$$G_1 = (X_i, f_i(x))_{i\in I}, \tag{1}$$

where $I = \{1, 2, \dots, n\}$ is the set of players; X_i is the set of strategies of the i-th player, $i = 1, \dots, n$, $x = (x_1, x_2, \dots, x_n) \in X = \prod_{i\in I} X_i$, $x_i \in X_i$ is a strategy of the i-th player, $i = 1, \dots, n$; X is the set of strategy profiles; $f_i : X \to R$, is the payoff function of the i-th player, $i = 1, \dots, n$ and R is the set of real numbers. The game takes place as follows. Once each player has chosen his strategy, a strategy profile $x = (x_1, x_2, \dots, x_n)$ is obtained, then the players receive the payoffs $f_i(x)$, $i = 1, \dots, n$, respectively.

Notation. Let $x^0 = (x_1^0, x_2^0, \dots, x_n^0) \in X$, for any player i and $x_i \in X_i$,

$$(x^0//x_i) = (x_1^0, x_2^0, \dots, x_{i-1}^0, x_i, x_{i+1}^0, \dots, x_n^0),$$

i.e. the strategy profile $(x^0//x_i)$ is obtained from x^0 by replacing the component x_i^0 of x^0 by x_i.

Assumption 2.1. In NCGNF it is assumed that

1. The players are rational.
2. There are no enforceable agreements between players.
3. The players know all the data of the game (1).

Under Assumption 2.1, Nash [52] has introduced the basic solution concept to game (1), the Nash equilibrium.

Definition 2.1. A strategy profile $x^0 \in X$ is said to be *Nash equilibrium* of the game (1) if it verifies the following condition:

$$f_i(x^0//x_i) \leqslant f_i(x^0) \quad \text{for all } x_i \in X_i \quad \text{and} \quad i \in I. \tag{2}$$

Remark 2.1. The condition (2) means that it is impossible for any player to improve his payoff by unilaterally deviating from the Nash equilibrium x^0. In other words, Nash equilibrium is a self-enforcing strategy profile.

Definition 2.2. Assume that there exists a real number a such that

$$\sum_{i=1}^{n} f_i(x) = a \quad \text{for all } x \in X.$$

Then the game (1) is said to be a constant-sum game. In this case we have

$$\sum_{i=1}^{n} f_i(x) = 0 \quad \text{for all } x \in X,$$

the game (1) is said to be a zero-sum game.

In a zero-sum game what some players lose is exactly what is gained by the other players.

Definition 2.3. The game (1) is said to be finite if the strategy sets X_i, $i = 1, \ldots, n$ of all players are finite.

Theorem 2.1 (*Nash [52]*). *If the game* (1) *is finite*, *it has a Nash equilibrium in mixed strategies*.

Theorem 2.2 (*Nash [52]*). *If the game* (1) *satisfies the following conditions*:

(1) *The set X_i, is a non empty*, *compact and convex subset of R^{n_i}, $i = 1, \ldots, n$.*
(2) *The functions $f_i(x)$, $i = 1, \ldots, n$ are continuous*; *the function $t_i \to f_i(x//t_i)$ is quasi-concave for all $x \in X$, $i = 1, \ldots, n$.*

Then the game (1) *has a Nash equilibrium.*

2.1. Two-person finite games

In this section, we will present matrix and bimatrix matrix games. This section will be useful for presentation and easy understanding of fuzzy matrix and bimatrix games that we will present in Section 4.

2.1.1. Matrix game

A matrix game is a finite two-person zero-sum game. In such a game there are two players, Players I and II. Player I is the maximizing player and Player II is the minimizing player. Since it is a zero-sum game, it can formally be represented by the following game:

$$G_2 = (X_1, X_2, A), \tag{3}$$

where $X_1 = \{1, 2, \ldots, m\}$ and $X_2 = \{1, 2, \ldots, n\}$ are the sets of *pure* strategies of Players I and II, respectively; A is an $m \times n$ matrix representing the payoffs of Player I

$$A = \begin{pmatrix} a_{11} & \cdots\cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots\cdots & a_{mn} \end{pmatrix} \tag{4}$$

Player I chooses the rows and Player II chooses the columns of (4). If the pair of pure strategies (i, j) is chosen by the players, then Player I gets the payoff a_{ij} and Player II gets $-a_{ij}$. A pair of strategies (i^0, j^0) is a Nash equilibrium of the game (3) if and only if

$$a_{ij^0} \leqslant a_{i^0j^0} \leqslant a_{i^0j} \quad \text{for all } (i, j) \in X_1 \times X_2.$$

The value $a_{i^0 j^0}$ is a saddle point of the payoff matrix (4). It often happens that the game (3) does not have Nash equilibrium in pure strategies. In this case, the players can resort to the mixed strategies which are probability distributions over the sets of pure strategies. The game (3) is extended to the following matrix game in mixed strategies:

$$G_3 = (S^m, S^n, A), \tag{5}$$

where $S^m = \{p = (p_1, p_2, \ldots, p_m),\ p_i \geqslant 0,\ i = 1, \ldots, m,\ \sum_{i=1}^m p_i = 1\}$ and $S^n = \{p = (q_1, q_2, \ldots, q_n),\ q_j \geqslant 0,\ j = 1, \ldots, n,\ \sum_{j=1}^n q_j = 1\}$. The *expected payoff* of Players I and II are, respectively, $p^T Aq$ and $-p^T Aq$, where T is the transposition operation. A Nash equilibrium $(\overline{p}, \overline{q})$ of the game (5) can be characterized by the following inequalities:

$$p^T A\overline{q} \leqslant \overline{p}^T A\overline{q} \leqslant \overline{p}^T Aq \quad \text{for all } (p, q) \in S^m \times S^n. \tag{6}$$

According to Theorem 2.1, the game (5) has a Nash equilibrium. By solving the following pair of dual linear programming problems [56]:

$$\begin{array}{llll} \text{Min} & x^T e_m & \text{Max} & y^T e_n, \\ & x^T A \geqslant e_m & & Ay \geqslant e_n, \\ & x \geqslant 0 & & y \geqslant 0, \end{array} \tag{7}$$

where all components of e_m, e_n are ones. Let $(\overline{x}, \overline{y})$ be a solution to (7), then setting

$$\overline{p} = \frac{\overline{x}}{\sum_{i=1}^m x_i} \quad \text{and} \quad \overline{q} = \frac{\overline{y}}{\sum_{j=1}^n y_j}, \tag{8}$$

$(\overline{p}, \overline{q})$ is a Nash equilibrium of the game (5).

2.1.2. Bimatrix game

Let us consider the following bimatrix game:

$$G_4 = (S^m, S^n, A, B), \tag{9}$$

where S^m and S^n are the sets of mixed strategies of Players I and II, respectively, defined as in the game (5)

$$\boldsymbol{A} = (a_{ij})_{\substack{1 \leqslant i \leqslant m \\ 1 \leqslant j \leqslant n}} = \begin{pmatrix} a_{11} & \cdots\cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots\cdots & a_{mn} \end{pmatrix} \quad \text{and} \quad B = (b_{ij})_{\substack{1 \leqslant i \leqslant m \\ 1 \leqslant j \leqslant n}} = \begin{pmatrix} b_{11} & \cdots\cdots & b_{1n} \\ \vdots & \ddots & \vdots \\ b_{m1} & \cdots\cdots & b_{mn} \end{pmatrix},$$

are the payoff matrices of Players I and II, respectively. When Players I and II choose the pair of pure strategies (i, j), they get, respectively, the payoffs a_{ij} and b_{ij}. The expected payoff of the Players I and II are $p^T Aq$ and $p^T Bq$, respectively. A Nash equilibrium of the game (9) can be characterized by the following inequalities:

$$p^T A\overline{q} \leqslant \overline{p}^T A\overline{q}, \quad \forall p \in S^m \quad \text{and} \quad \overline{p}^T Bq \leqslant \overline{p}^T B\overline{q}, \quad \forall q \in S^n. \tag{10}$$

2.2. Multiobjective non nooperative game in normal form

Introduced by Blackwell in 1956 [13], these games are a generalization of the game (1) when players have multiobjective payoff functions i.e.

$$f_i(x) = (f_{i_1}(x), \ldots, f(x)_{i_s}), \quad i = 1, \ldots, n. \tag{11}$$

The players evaluate the strategy profiles by several criteria not only by one as in game (1).

Definition 2.4. A strategy profile $\overline{x} \in X$ is said to be weighted Nash equilibrium of the game (1) with the multiobjective payoff functions (11), if for each player i there exists positive weights $\lambda_{i_p} > 0$, $p = 1, \ldots, s$ such that $\overline{x}$ is a Nash equilibrium of the following single objective game of type (1):

$$G_5 = \left(X_i, \sum_{p=1}^{s} \lambda_{i_p} f_{i_p}(x) \right)_{i \in I}. \tag{12}$$

3. General approaches to games in fuzzy environment

The data of the traditional game (1) are crisp; such formulation is far from being realistic. Indeed, in real games, it often happens that players are not able to exactly evaluate the data of the game and express precisely their preferences and choices. Fuzzy set theory and fuzzy logic are an adequate tool to handle the vagueness of human judgment and imprecision in evaluation of real-game situations. There are two types of approaches dealing with fuzziness in games: the approaches based on fuzzy preferences, strategies and perceptions or beliefs of players, and the approaches based on fuzzy evaluation of the payoffs. In this section, we deal with the first type of approaches. In Section 4, we treat the second type of approaches.

3.1. Butnariu's approach

This approach was introduced in a series of three papers [19–21]. According to Butnariu, "A game consists of a set of exchanges of information between two or more partners or players. These exchanges have specific rules which are determined by objective possibilities of transmitting and receiving information, i.e. by external medium of exchanges. By information, we understand any object which can be transmitted, identified and received such as money, words, scientific ideas…etc." From this point of view, Butnariu's model is more general than the traditional Von Neumann–Morgenstern model.

Formally, Butnariu defines an n-person non-cooperative fuzzy game as follows:

$$G_6 = (X_i, Y_i, E_i)_{i \in I}, \tag{13}$$

where

(a) $I = \{1, 2, \dots, n\}$, is the set of players, $X_k = \{a_{k_1}, a_{k_2}, \dots, a_{k_{nk}}\}$, is the set of pure strategies of player $k \in I$.
(b) $Y_k \subset R^{n_k}$ is a crisp set called strategic composition of player k in the game (13); if $w^k = (w_1^k, w_2^k, \dots, w_{n_k}^k) \in Y_k$, then w_j^k is called investment of player k in his pure strategy a_{kj}; a n-vector $w = (w^1, w^2, \dots, w^n) \in Z = \prod_{k \in I} Y_k$ is called strategic choice in the game (13).
(c) E_k is a fuzzy set on Z, for any $w = (w^1, w^2, \dots, w^n) \in Z$, $E_k(w)$ is the possibility degree of the strategic choice w from the point of view of player k.
(d) Let $Z_k = \prod_{i \in I-\{k\}} Y_i$, a fuzzy set A_k on Z_k expresses the believe of the player k about the behavior of the remaining players in I–$\{k\}$ based on the information he receives from them. The pair $s_k = (A_k, w^k)$ is called strategic conception of player k in the game (13), w^k is a choice (investment) determined by the informational conjuncture and A_k. A play in the game (13) is an n-vector $s = (s_1, s_2, \dots, s_n)$. The Problem of the "best information exchange" in the game (13) is equivalent to the problem of finding the "best play" in (13).

Definition 3.1. Let $s_k = (A_k, w^k)$ and $s'_k = (A'_k, w'^k)$ be two strategic conceptions of player k in the game (13). We say that s_k is preferable to s'_k denoted by $s_k \succ s'_k$ iff

$$E_k[A_k](w^k) \geqslant E_k[A'_k](w'^k),$$

where $E_k[A_k](w^k) = \sup\{A_k(z_k)E_k(z_k, w^k), z_k \in Z_k\}$. We say that a play s is better than a play s' denoted by $s \succ s'$ iff $s_k \succ s'_k$ for all $k \in I$.

Definition 3.2. A possible solution to the game (13) is a play $s^* = (s_1^*, s_2^*, \dots, s_n^*)$ with, $s_k^* = (A_k^*, w^{*k})$, $k \in I$, such that for any play $s = (s_1, s_2, \dots, s_n)$ with $s_k = (A_k^*, w^k)$, $k \in I$, cannot be preferable to s^* i.e. $E_k[A_k^*](w^{*k}) \geqslant E_k[A_k^*](w^k)$, $\forall w^k \in Y_k$, $k \in I$.

Definition 3.3. An equilibrium point of the game (13) is a possible solution $s^* = (s_1^*, s_2^*, \ldots, s_n^*)$, $s_k^* = (A_k^*, w^{*k})$, $k \in I$ that verifies

$$A_k^*(w^1, w^2, \ldots, w^{k-1}, w^{k+1}, \ldots, w^n) = \begin{cases} 1 & \text{if } w^i = w^{*i}, \ \forall i \in I - \{k\}, \\ 0 & \text{elsewhere,} \end{cases} \tag{14}$$

i.e. each player k is certain that $(w^{*1}, w^{*2}, \ldots, w^{*k-1}, w^{*k+1}, \ldots, w^{*n})$ will be played.

Definition 3.4. The fuzzy individual preference of the player k in the game (13) is a fuzzy relation $\tilde{E}_k$ on the product $W \times W$ of the set of plays of the game

$$\tilde{E}_k(s, \overline{s}) = E_k[A_k](\overline{w}^k) \vee E_k[A_k](w^k) \prod_{i \in I} E_i[\overline{A}_i](\overline{w}^i).$$

For any two plays s and $\overline{s}$ with $s_j = (A_j, w^j)$ and $\overline{s}_j = (\overline{A}_j, \overline{w}^j)$ for any $j \in I$, the fuzzy relation $\tilde{E}_k(s, \overline{s})$ measures the preference of player k of the play s compared with the play $\overline{s}$.

Definition 3.5. A fuzzy preference relation $\tilde{R}$ in $W \times W$ is defined by

$$\tilde{R}(s, \overline{s}) = \prod_{i \in I} \tilde{E}_i(s, \overline{s}), \quad \forall s, \overline{s} \in W.$$

For any plays $s, \overline{s} \in W$ verifying the relation (14), the fuzzy relation

$$R^*(w, \overline{w}) = \tilde{R}(s, \overline{s}),$$

is called restricted preference in the game (13).

Theorem 3.1. *Let s^*, with $s_k^* = (A_k^*, w^{*k})$, $k \in I$ be a play verifying (14), then s^* is an equilibrium point of (13) $\Leftrightarrow$ s^* is a fixed point of the relation R^* i.e.*

$$R^*(w^*, w^*) \geqslant R^*(w^*, w), \quad \forall w \in Z.$$

Remark 3.1. It is interesting to note that in Butnariu's model there are no payoff functions. The preferences and believes of players are modeled as fuzzy sets. Indeed, Butnariu considers that in real-game situations the behavior of players may be driven not only by "physical" outcomes like money but by moral and psychological aspects as well. Butnariu's approach was further developed by Billot [12]. Butnariu's approach seems to be promising for social sciences for it examines fuzzyness in social beings (people, societies, etc) and beliefs about actions of social beings. However, Haller and Sarangi [35] pointed out that Butnariu's model is restrictive, because it requires full information about the beliefs of players which may be difficult to obtain in real games.

Remark 3.2. Ponsard [58] considers a non-cooperative n-persons game with more or less precise preferences and uncertain payoffs. Each player has a set of mixed strategies at his disposal. Then a closed and convex fuzzy point-to-set mapping is defined on the product set of strategies. Then, by using a Butnariu's theorem [22], the existence of a fixed point for this point-to-set fuzzy mapping is proved. Thus, an n-person non-cooperative fuzzy game with mixed strategies has at least one equilibrium point.

3.2. Billot's approach

This approach is summarized in [5] as follows. Unlike the model originally developed by Butnariu [19–21], Billot [11] considers an ordinal game that differs from the standard game-theoretic formulation only by allowing for fuzzy lexicographic preferences. He introduces an axiom called the axiom of local non-discrimination according to which a player is assumed to be indifferent between two very close options. However, such indifference is not unique and its intensity is allowed to take values between 0 and 1. This degree of intensity is captured using a membership function. Next, he shows that under the above axiom, a fuzzy lexicographic preorder can be represented by a continuous utility function defined on a connected referential set $X \subseteq R$.

A game of the form (1) is considered. It is assumed that the strategy spaces are real and convex and the payoff functions are continuous. Next, he introduces a transformation function that orders the strategies lexicographically based on the payoffs they yield. Recall that under the axiom, a fuzzy lexicographic preorder can be represented by a continuous utility function. Since this utility function can now be defined on the set of strategies, Billot calls this a strategic utility function. He then proves an existence result for two-person zero-sum games under fairly simple conditions. It is shown that if the axiom is satisfied, the strategy space is compact and convex, and the payoff function is continuous, then an equilibrium will exist. In terms of technicalities Billiot's work is interesting since it proves an existence result and the equilibria derived using lexicographic preferences contain the usual Nash equilibria. The drawbacks lie in the fact that it is necessary to assume lexicographic preferences and that results are applicable mainly to zero-sum games.

3.3. Buckley's approach

Buckley [17] has investigated a two-party non-cooperative game which involves both uncertainty and multiple goals. Uncertainty arises from the players not knowing the utility functions. Multiple objectives appear as the result of the payoff being a vector of prizes and the players attempt to attain various goals for each prize separately. A fuzzy set/fuzzy programming solution concept to the game is proposed. A comparison of a Bayesian player to one that employs fuzzy set techniques is done. Some of the advantages of the fuzzy set method are pointed out. A computation procedure is provided as well. Let us now briefly present the model. Denote by Players I and II the two parties in conflict. Let $v = (v_1, v_2, \ldots, v_r)$, $w = (w_1, w_2, \ldots, w_r)$ be the vectors of outcomes for Players I and II, respectively. Let $p = (p_1, p_2, \ldots, p_m)$, $q = (q_1, q_2, \ldots, q_n)$ be the mixed strategies of Players I and II, respectively. The model does not involve utility functions that transform the vectors v, w into one number. Player I (II) set multiple, possibly conflicting, goals for each component of $v(w)$. Let G_{ik} denote a function that measures the attainment of the k-th goal for the i-th component v_i in the payoff vector v for Player I, $1 \leqslant k \leqslant L_i$, $1 \leqslant i \leqslant r$. The goal functions for Player II will be H_{ik}, $1 \leqslant k \leqslant M_i$, $1 \leqslant i \leqslant r$. Each player wishes to maximize his goal functions. Let S^m and S^n be the sets of mixed strategies of Players I and II, respectively. Depending on each goal statement, Player I defines a conditional fuzzy set F_{ik} in S^m for each goal. The membership function for F_{ik} is $\mu_{ik}(G_{ik}(p, q)/q)$. The fuzzy set F_{ik} is conditioned on q for each value of $q \in S^n$. The conditional fuzzy set is conditioned on the value of q because: (i) the fuzzy set must be in Player I's decision space S^m; (ii) Player I has no control over q; and (iii) it is natural to be able to vary the shape of the membership function depending on the Players II's mixed strategy q. Next, Player I expresses his uncertainty as to what q Player II will use by defining a fuzzy set F in S^n with membership function $\mu(q)$. It is assumed that Player I uses past information about Player II plus knowledge on the raw score outcomes of the conflict to construct $\mu(q)$. Based on Bellman and Zadeh's principle of decision making in a fuzzy environment [10], Player I uses F and F_{ik} to obtain fuzzy sets in S^m with membership function

$$\mu_{ik}(p) = \mathrm{Max}_q(\mathrm{Min}(\mu_{ik}(G_{ik}(p, q)/q), \mu(q))).$$

Player I wishes to achieve all of his goals, therefore, based on Bellman and Zadeh's principle, the fuzzy decision is

$$A = A_{11} \cap A_{12}, \ldots,$$

whose membership function is

$$\mu(p) = \mathrm{Min}(\mu_{11}(p), \mu_{12}(p), \ldots).$$

A mixed strategy p^* that maximizes $\mu(p)$ is defined as an optimal strategy for Player I. The described model is extended to the case where the conflict repeats over time.

Note that Song and Kandel [64] propose a very similar approach to Buckley's. The difference is that in [64] we have: (i) $\mu_{ik}(p) = \mathrm{Max}_q(G_{ik}(p, q)\mu(q))$, (ii) $\mu(p) = \sum w_{11}\mu_{11}(p) + w_{12}\mu_{12}(p) + \cdots$, where $w_{11}, w_{12}, \ldots$ are the weights that the Player I assigns to his goals, respectively, and (iii) the utilities of players are known.

3.4. Garazic and Cruz approach

Garazic and Cruz [31] propose an approach based on the design of a fuzzy controller as follows: fuzzification → inference → defuzzification.

Strategies are first fuzzyfied. Then a fuzzy preference matrix is generated using "If–Then" rules in the inference stage. Finally, after defuzzification, a crisp game is obtained, the Nash equilibrium of which are considered as solutions to the initial game. A theorem of existence of Nash equilibrium is proved based on usual assumptions in NCGNF.

Remark 3.3. West and Linster [70] have used a similar approach. The main difference between the two approaches is that fuzzy rules are used for the choice of strategies, the strategies are assumed to be crisp. They interestingly find that for fuzzy rules using the most recent histories in two-player repeated games with infinite strategy spaces, the play converges to the analytical Nash equilibria. In [5] it is pointed out that West and Linster did not provide a theoretical foundation of their fuzzy rule based approach.

Remark 3.4. The Garazic and Cruz' approach seems to have a greater potential for application in social sciences than Butnariu's, because it is based on fuzzy controller that is well known and widely used in engineering. However, from computational point of view, for infinite games the fuzzification, inference and defuzzification stages may be difficult and complex. The authors did not discuss this potential problem for infinite games.

3.5. Aristidou and Sarangi's approach

The Aristidou and Sarangi's approach [5] develop a simple static normal form fuzzy game where both payoffs and strategies of players are modeled as fuzzy sets. The model builds on the decision making principle in a fuzzy environment of Bellman and Zadeh [10]. Aristidou and Sarangi consider that their work provides a theoretical foundation to the work of West and Linster [70] that we cited in Remark 3.3. Aristidou and Sarangi start with a crisp normal form game of type (1) with only two players. Then they define a fuzzy version of this game as follows. (i) For each player i, the *constraint* set is defined by

$$\mu_i : X_i \to [0, 1],$$

where μ_i represents the player's perceived or fuzzy constraint, $\mu_i(s_i)$ expresses the perceived degree of feasibility of the strategy s_i. The membership function μ_i is called *perception constraint*. (ii) For each player i, a *goal function* is defined by

$$\gamma_i : X \to [0, 1].$$

This represents the *aspiration level* of the player i.e. the player's fuzzy goal function. Thus, the following two-person FNCGNF is obtained:

$$G^f = (\mu_i, \gamma_i)_{i\in\{1,2\}}.$$

Next, based on the two functions introduced for each player, the *decision set* is determined, which is, in the words of Bellman and Zadeh, the *confluence of goals and constraints* defined by $\delta_i : X \to [0, 1]$ with

$$\delta_i(s) = \text{Min}\{\mu_i(s_i), \gamma_i(s))\},$$

$\delta_i(s)$ is basically the intersection of the set of goals and constraints facing a player. It can be interpreted as follows: for a particular choice $\hat{s}_2 \in X_2$, of Player II, $\delta_1(., \hat{s}_2)$ represents Player I's response to this strategy using the Belmann and Zadeh principle. This means that Player I must follow the above rule when computing his goal function. Essentially, it says that for any strategy profile each player first requires that his constraints and goals are satisfied simultaneously. Thus, the decision set expresses the degree of compatibility between players' perceptions and his goals. A Nash equilibrium for the fuzzy game is defined as follows: A strategy profile $s^* = (s_1^*, s_2^*)$ is a Nash equilibrium in the game G^f if for all $i \in N$, we have

$$\delta_1(s^*) \geqslant \delta_1(s_1', s_2^*) \quad \text{and} \quad \delta_2(s^*) \geqslant \delta_2(s_1^*, s_2') \quad \text{for all } (s_1', s_2') \in X_1 \times X_2.$$

Theorem 3.2. *For the game G^f, if the strategy sets X_i are non empty, convex and compact, and $\delta_i(s)$ is continuous and strictly concave in a player's own strategy, then G^f has at least one Nash equilibrium.*

Remark 3.5. In Section 6, we will see an application of this approach in economics, namely, duopoly. It is worth noting that the computation of a Nash equilibrium of the game G^f may be difficult because the function $\delta_i(s)$ is defined by the minimum operator, which is non smooth in general. Moreover, the authors start from the traditional crisp game in normal form (1), but they did not use the information that the payoff functions f_i provide about the outcomes of the game when they define $\gamma_i(s)$ in the fuzzy game G^f. It could be better and more natural if they defined the goal function γ_i on the set $f(X)$ of payoffs, with $f = (f_1, f_2)$. As we see, the approach of Aristidou and Sarangi is simple and closer to the standard game-theoretic framework. It seems to be more realistic than the Butnariu–Billot approach. It may be used also to model different degrees of bounded rationality in games. However, we note that Aristidou and Sarangi are not the first to extend the Bellman and Zadeh principle of decision making in a fuzzy environment [10] to game situations. In 1984, Buckley [17] has, in a very similar way (see Section 3.3), already used this principle to model a multiple goals non-cooperative conflict under uncertainty, but with finite sets of strategies. Moreover, in 2001 Nishizaki and Sakawa [55] used this principle for solving matrix and bimatrix games with fuzzy goals and fuzzy payoffs (see Section 4.3.1) by a very similar approach. The difference between the two approaches is that Nishizaki and Sakawa treated matrix and bimatrix games with single or multiple objectives only and they did not fuzzify the set of strategies. Aristidou and Sarangi did not make reference to the works of Buckley, Nishizaki and Sakawa.

3.6. Arfi's approach

In a significant contribution Arfi [4] formulates a game-theoretic approach to study non-cooperative strategic situations anchored in linguistic fuzzy logic. This is a considerable contribution to fuzzy non-cooperative game theory. Arfi's new approach does not use Boolean (Aristotelian) logic, but instead uses fuzzy logic, which simply is based on the starting point that logic is inherently (constitutively) vague or fuzzy. Truth values are not only true and false, but can also assume in-between values, e.g. a statement can be more or less true or false–very true, less true, extremely false, approximately false, etc. Arfi's work differs from existing approaches to fuzzy games in the sense that most of the works published on fuzzy games adopt a fuzzy-set framework, which is conventionally developed with the notion of gradual membership to sets. In Arfi's approach a *linguistic fuzzy logic game* is defined with linguistic fuzzy strategies, linguistic fuzzy preferences, and the rules of reasoning and inferences of the game operate according to linguistic fuzzy logic, not Boolean logic. This leads to the introduction of new notions of fuzzy domination and Nash equilibrium which are not based on the usual 'greater than' relation ordering, but rather on a more general form of relation termed linguistic fuzzy relation. Each agent models others as linguistic fuzzy rational agents and tries to find a linguistic fuzzy Nash equilibrium that will achieve the highest linguistic fuzzy payoff. Moreover, Arfi provides a theoretical framework for his model, the De Morgan Lattice. Arfi shows that linguistic fuzzy logic games have a great potential for application in social sciences by examples.

Because of shortage of space, we limit the presentation Arfi's approach to the steps that one has to follow in order to obtain a linguistic fuzzy logic game from a crisp game.

(i) Fuzzify the initial crisp choices of each player into linguistic fuzzy choices: the choices become linguistic variables assuming linguistic values—termed as nuanced choices, each having a feasibility degree.
(ii) Fuzzify the crisp orderings of the alternative strategies into linguistic orderings: the ranking preferences become linguistic variables assuming linguistic values—termed as nuanced preferences, each with a feasibility degree.
(iii) Fuzzify the two-valued logic conjunction "and" into linguistic fuzzy logic conjunction.
(iv) Fuzzify the two-valued logic disjunction "or" into linguistic fuzzy logic disjunction.
(v) Fuzzify the two-valued logic implication into linguistic fuzzy logic implication.
(vi) Fuzzify the rules of the game: changing them from inferences based on two-valued logic to inferences based on linguistic fuzzy logic.

Arfi's approach seems to have a greater potential for application in social sciences than Garazic and Cruz's, because it is completely based on linguistic fuzzy logic.

4. Non cooperative games with fuzzy payoffs

Such games are of the form

$$G_7 = (X_i, \tilde{f}_i(x))_{i\in I}, \tag{15}$$

where $\tilde{f}_i(x)$, $i \in I$ are fuzzy numbers. As far as we know, in literature there are no significant results concerning this general form. The most important results are obtained for two-person finite games. We present the main results on fuzzy matrix and bimatrix games. Note that the approaches of Sections 4.1.1–4.1.4 are discussed in details in [9].

4.1. Fuzzy matrix games

In this section we present the basic idea of the most important approaches for solving matrix games with fuzzy payoffs. A matrix game with fuzzy payoffs is a finite two-person zero-sum game with fuzzy payoffs. It can be represented by a game similar to the game (5), the only difference is that the payoff matrix is fuzzy

$$\begin{aligned} &G_8 = (S^n, S^m, \tilde{A}),\\ &\tilde{A} = (\tilde{a}_{ij})_{\substack{1\leqslant i\leqslant m\\ 1\leqslant j\leqslant n}} = \begin{pmatrix} \tilde{a}_{11} & \cdots\cdots & \tilde{a}_{1n}\\ \vdots & \ddots & \vdots\\ \tilde{a}_{m1} & \cdots\cdots & \tilde{a}_{mn}\end{pmatrix}. \end{aligned} \tag{16}$$

4.1.1. Ranking of fuzzy numbers approach: Campos' model

The first model using ranking of fuzzy numbers was introduced by Campos [23]. It is based on the approach for solving crisp matrix games (7) and (8). Campos transforms the problem of finding a solution to the fuzzy matrix game (16) into the following pair of fuzzy linear programming problems:

$$\begin{array}{llll} \text{Max } v & & \text{and Min } w & \\ \quad p^T\tilde{A}\,\tilde{\geqslant}\, ew & & \quad \tilde{A}q\,\tilde{\leqslant}\, ew & \\ \quad e^Tp = 1, & q\geqslant 0 & \quad e^Tq = 1, & q\geqslant 0. \end{array} \tag{17}$$

Here and onwards e represents a vector of corresponding dimension, whose components are ones. Using Yager's method for expressing the double fuzzy constraints in terms of adequacies $\tilde{t}$ and $\tilde{k}$ [73], the pair of fuzz optimization problems (17) is transformed into the following pair of crisp linear programming problems:

$$\begin{aligned} &\text{Min} \quad \sum_{i=1}^m u_i\\ &\qquad \sum_{i=1}^m (a_{ij}^L + a_{ij} + a_{ij}^U)u_i \geqslant 3 - (t_j^L + t_j + t_j^U)(1-\alpha), \quad j = 1,\ldots,n,\\ &\qquad \alpha\in[0,1], \quad u_i\geqslant 0, \quad i = 1,\ldots,m \end{aligned} \tag{18}$$

and

$$\begin{aligned} &\text{Max} \quad \sum_{j=1}^n s_j\\ &\qquad \sum_{j=1}^n (a_{ij}^L + a_{ij} + a_{ij}^U)s_j \geqslant 3 - (k_i^L + k_i + k_i^U)(1-\beta), \quad i = 1,\ldots,m,\\ &\qquad \beta\in[0,1], \quad s_j\geqslant 0, \quad j = 1,\ldots,n, \end{aligned} \tag{19}$$

where $\tilde{a}_{ij} = (a_{ij}^L, a_{ij}, a_{ij}^U)$, $\tilde{t} = (t_j^L, t_j, t_j^U)$ and $\tilde{k} = (k_i^L, k_i, k_i^U)$ are triangular fuzzy numbers (TFN). Here a Yager's index is used for defuzzification with adequacies $\tilde{t}$ and $\tilde{k}$ [73].

Remark 4.1. Campos' approach has been further developed in [24,25]. Note that in [8] certain difficulties in the Campos' approach on operations on fuzzy inequalities are reported.

4.1.2. Multiobjective approach: Li's model

Li [42] used the following ordering of TFNs.

Definition 4.1. Let $\tilde{a} = (a^L, a, a^U)$ and $\tilde{b} = (b^L, a, b^U)$ be two TFNs. Then

$$\tilde{a} \tilde{\leqslant} \tilde{b} \quad \text{if } a^L \leqslant b^L, \quad a \leqslant b \quad \text{and} \quad a^U \leqslant b^U.$$

Definition 4.2. Let $\tilde{v} = (v^L, v, v^U)$ and $\tilde{w} = (w^L, w, w^U)$ be two TFNs. Then $(\tilde{v}, \tilde{w})$ is called *reasonable solution* of the game (16) if there exist $(\overline{p}, \overline{q}) \in S^m \times S^n$ such that

(i) $\overline{p}^T \tilde{A} q \tilde{\geqslant} \tilde{v}, \quad \forall q \in S^n$,
(ii) $p^T \tilde{A} \overline{q} \tilde{\leqslant} \tilde{w}, \quad \forall p \in S^m$.

Let $V = \{\tilde{v}/\tilde{v}$ satisfying (i)$\}$ and $W = \{\tilde{w}/\tilde{w}$ satisfying (ii)$\}$.

Definition 4.3. An element $(\tilde{v}^*, \tilde{w}^*) \in V \times W$ is called a solution of the game (16) if

(1) $\tilde{v}^* \tilde{\geqslant} \tilde{v}, \quad \forall \tilde{v} \in V$,
(2) $\tilde{w}^* \tilde{\leqslant} \tilde{w}, \quad \forall \tilde{w} \in W$.

The problem of finding a solution to the game (16) is then transformed into a problem of solving the following pair of crisp multiobjective problems:

$$\begin{aligned} \text{Max} \quad & (v^L, v, v^U) \\ \text{s.t.} \quad & \sum_{i=1}^{m} a_{ij}^L p_i \geqslant v^L, \quad j = 1, \ldots, n, \\ & \sum_{i=1}^{m} a_{ij} p_i \geqslant v, \quad j = 1, \ldots, n, \\ & \sum_{i=1}^{m} a_{ij}^U p_i \geqslant v^U, \quad j = 1, \ldots, n, \\ & e^T p = 1, \quad p \geqslant 0 \end{aligned} \tag{20}$$

and

$$\begin{aligned} \text{Min} \quad & (w^L, w, w^U) \\ \text{s.t.} \quad & \sum_{j=1}^{n} a_{ij}^L q_j \leqslant w^L, \quad i = 1, \ldots, m, \\ & \sum_{j=1}^{n} a_{ij} q_j \leqslant w, \quad i = 1, \ldots, m, \\ & \sum_{j=1}^{n} a_{ij}^U q_j \leqslant w^U, \quad i = 1, \ldots, m, \\ & e^T q = 1, \quad q \geqslant 0. \end{aligned} \tag{21}$$

Note that Li's approach is limited to TFNs. In [43] a two-level linear programming method is proposed to solve the pair of problems (20) and (21).

4.1.3. Duality and ranking function approach: Bector et al. model

Bector et al. [8] use duality in fuzzy linear programming and Yager's first index [73] to introduce a solution for a matrix game with fuzzy payoffs. They define two types of solutions for the game (16): the reasonable solution and the solution.

Definition 4.4 (*Reasonable solution*). Let $\tilde{v}, \tilde{w}$ be two fuzzy numbers. Then $(\tilde{v}, \tilde{w})$ is called reasonable solution of the game (16) if there exists $x^* \in S^m$, $y^* \in S^n$ satisfying

(i) $x^{*T}\tilde{A}y \succcurlyeq \tilde{v}, \quad \forall y \in S^n$,
(ii) $x^T\tilde{A}y^* \preccurlyeq \tilde{w}, \quad \forall x \in S^m$.

If $(\tilde{v}, \tilde{w})$ is a reasonable solution of (16) then $\tilde{v}$ (respectively, $\tilde{w}$) is called a reasonable value for Player I (respectively, Player II).

Definition 4.5 (*Solution of the game (16)*). Let T_1 and T_2 be the sets of all reasonable values $\tilde{v}$ and $\tilde{w}$ for Players I and II, respectively, where $\tilde{v}, \tilde{w}$ are fuzzy numbers. Let there exist $\tilde{v}^* \in T_1$ and $\tilde{w}^* \in T_2$ such that

(i) $F(\tilde{v}^*) \geqslant F(\tilde{v}), \quad \forall \tilde{v} \in T_1$,
(ii) $F(\tilde{w}^*) \leqslant F(\tilde{w}), \quad \forall \tilde{w} \in T_2$.

Let x^*, y^* be the strategies corresponding to $\tilde{v}^*, \tilde{w}^*$, respectively. Then $(x^*, y^*, \tilde{v}^*, \tilde{w}^*)$ is called solution of the game (16) where $\tilde{v}^*$ (respectively, $\tilde{w}^*$) is the value of the game (16) for Player I (respectively, Player II) and x^* (respectively, y^*) is called optimal strategy for Player I (respectively, Player II).

$F : N(R) \to R$ is a defuzzification function, where $N(R)$ is the set of real fuzzy numbers. Based on Definitions 4.4–4.5, the following pair of fuzzy linear programming problems for Players I and II is introduced:

$$\begin{array}{ll} \text{Max} & F(\tilde{v}) \\ \text{s.t.} & x^T\tilde{A}y \succcurlyeq_{\tilde{p}} \tilde{v}, \quad \forall y \in S^n, \ x \in S^m \end{array}$$

and

$$\begin{array}{ll} \text{Min} & F(\tilde{w}) \\ \text{s.t.} & x^T\tilde{A}y \preccurlyeq_{\tilde{q}} \tilde{w}, \quad x \in S^m, \ y \in S^n, \end{array}$$

$x^T\tilde{A}y \succcurlyeq_{\tilde{p}} \tilde{v}$, and $x^T\tilde{A}y \preccurlyeq_{\tilde{q}} \tilde{w}$ are double fuzzy inequalities with adequacies $\tilde{p}$ and $\tilde{q}$, respectively. Then using a resolution method proposed in [73], the last pair of problems is transformed into the following crisp pair of problems:

$$\begin{array}{ll} \text{Max} & V \\ \text{s.t.} & \displaystyle\sum_{i=1}^{m} F(\tilde{a}_{ij})x_i \geqslant V + (1-\gamma)F(\tilde{p}), \quad j = 1, \dots, n, \\ & \displaystyle\sum_{i=1}^{m} x_i = 1, x \geqslant 0, \quad 0 \leqslant \gamma \leqslant 1 \end{array} \tag{22}$$

and

$$\begin{array}{ll} \text{Min} & W \\ \text{s.t.} & \displaystyle\sum_{j=1}^{n} F(\tilde{a}_{ij})y_j \leqslant W + (1-\rho)F(\tilde{q}), \quad i = 1, \dots, m, \\ & \displaystyle\sum_{j=1}^{n} y_j = 1, y \geqslant 0, \quad 0 \leqslant \rho \leqslant 1. \end{array} \tag{23}$$

Note here that by solving these two problems, we get only crisp values V^* and W^* for Players I and II, respectively. The actual fuzzy values for Players I and II will be "close to" V^* and W^*, respectively. In particular when the defuzzification function F is Yager's first index [73], the crisp values V^* and W^* will be the "centroid" or "average" values of the fuzzy values for Players I and II, respectively.

The following example has been treated in the approaches of Sections 4.1.1–4.1.3.

Example 4.1. Consider the following matrix game with fuzzy payoffs:

$$\tilde{A} = \begin{pmatrix} \tilde{180} & \tilde{156} \\ \tilde{90} & \tilde{180} \end{pmatrix},$$

where $\tilde{180} = (175, 180, 190)$, $\tilde{156} = (150, 156, 158)$, and $\tilde{90} = (80, 90, 100)$ are TFNs.

(i) By solving the pair of problems (18)–(19) with adequacies $\tilde{p}_1 = \tilde{p}_2 = (0.08, 0.10, 0.11)$, $\tilde{q}_1 = \tilde{q}_2 = (0.15, 0.14, 0.17)$ and the first Yager's index [73], Campos [23] obtains the following solution $x^* = (x_1^*, x_2^*) = (0.77, 0.23)$, $y^* = (y_1^*, y_2^*) = (0.23, 0.77)$, the value of the game is around 160.81.

(ii) Solving the pair of problems (20) and (21) by the two-level linear programming method proposed in [43], we get the following Li's solution $x^* = (x_1^*, x_2^*) = (0.7895, 0.2105)$, $\tilde{v}^* = (v^{*L}, v^*, v^{*U}) = (155, 161.05, 164.736)$, $y^* = (y_1^*, y_2^*) = (0.2105, 0.7895)$ and $\tilde{w}^* = (w^{*L}, w^*, w^{*U}) = (155.264, 161.05, 171.052)$.

(iii) Bector et al. [8] assume that the defuzzification function $F(.)$ is Yager's first index i.e., for any fuzzy number $\tilde{z}$, $F(\tilde{z})$ is the centroid of $\tilde{z}$ and the adequacies are $\tilde{p}_1 = \tilde{p}_2 = (0.08, 0.10, 0.11)$ and $\tilde{q}_1 = \tilde{q}_2 = (0.15, 0.14, 0.17)$. By solving the pair of problems (22) and (23), we get $(x^*, y^*, V^*, W^*) = ((x_1^*, x_2^*), (y_1^*, y_2^*), V^*, W^*) = ((0.7725, 0.2275), (0.2275, 0.7725), 160.91, 160.65)$, $\gamma^* = 0$, and $\rho^* = 0$.

Note that the solution obtained by Campos [23] matches the solution obtained by Bector et al. [8]. Moreover, in (ii) and (iii) the value for Player I is not equal to the value for Player II.

4.1.4. Fuzzy order approach: Maeda's model

Maeda [48] used the following ordering of fuzzy numbers.

Definition 4.6. Let $\tilde{a}$ and $\tilde{b}$ be two fuzzy numbers, and $[\tilde{a}]^\alpha$, $[\tilde{b}]^\alpha$ their α-cuts. Then the relation

$$\tilde{a} \succcurlyeq \tilde{b} \Leftrightarrow (a_L^\alpha, a_U^\alpha) \geqslant (b_L^\alpha, b_U^\alpha), \quad \text{for all } \alpha \in [0, 1].$$

is called *fuzzy max order*. The strict fuzzy max order "$\succeq$" and strong fuzzy max order "$\succ$" are defined similarly. Here a_L^α and a_U^α are the lower and upper bounds of $[\tilde{a}]^\alpha$.

Definition 4.7. A pair of mixed strategies $(\overline{p}, \overline{q})$ is said to be a *minmax equilibrium strategy* of the fuzzy matrix game (16) if

(i) $p^T \tilde{A} \overline{q} \preccurlyeq \overline{p}^T \tilde{A} \overline{q}, \quad \forall p \in S^m$,
(ii) $\overline{p}^T \tilde{A} \overline{q} \preccurlyeq \overline{p}^T \tilde{A} q, \quad \forall q \in S^n$.

Definition 4.8. A pair of mixed strategies $(\overline{p}, \overline{q})$ is said to be a *non-dominated minmax equilibrium strategy* of the fuzzy matrix game (16) if

(i) There does not exist any $p \in S^m$, such that $\overline{p}^T \tilde{A} \overline{q} \preccurlyeq p^T \tilde{A} \overline{q}$,
(ii) There does not exist any $q \in S^n$, such that $\overline{p}^T \tilde{A} q \preccurlyeq \overline{p}^T \tilde{A} \overline{q}$.

Theorem 4.1. *Assume that the entries $\tilde{a}_{ij}$ of the fuzzy matrix $\tilde{A}$ are symmetric TFNs, i.e. $\tilde{a}_{ij} = (a_{ij}, h_{ij})$. Let $A = (a_{ij})$, $H = (h_{ij})$ and $A(\lambda) = A + (1-2\lambda)H$, $A(\mu) = A + (1-2\mu)H$ with $\lambda, \mu \in [0, 1]$. Then a pair $(\overline{p}, \overline{q})$ is a non-dominated minimax equilibrium of the fuzzy matrix game (16) if and only if there exist positive real numbers $\lambda, \mu \in]0, 1[$ such that $(\overline{p}, \overline{q})$ is Nash equilibrium of the following crisp bimatrix game*:

$$G_9 = (S^m, S^n, A(\lambda), -A(\mu)).$$

As we see this approach is limited since Theorem 4.1 is based on the assumption that the entries of the fuzzy matrix $\tilde{A}$ are symmetric TFNs.

4.1.5. Liu and Kao's model

Liu and Kao [46] develop a solution method for matrix games with fuzzy payoffs based on the extension principle [10]. Let us consider the fuzzy matrix game (16). Then the problem for Player I (Player II) is to find the optimal solution of the first (second) problem in (17). Using the extension principle [75], Liu and Kao define the value of the game as a fuzzy number $\tilde{Z}$ with the following membership function:

$$\mu_{\tilde{Z}}(z) = \mathrm{Sup}_{a_{ij}} \mathrm{Min}\{\mu_{\tilde{a}_{ij}}(a_{ij}), \quad \forall i, j/z = Z(a)\}.$$

For a fixed value of the cut level α, the upper bound $\tilde{Z}_\alpha^U$ and lower bound $\tilde{Z}_\alpha^L$ of $\tilde{Z}$ are computed by solving the following pair of two-level problems, respectively,

$$\tilde{Z}_\alpha^U = \mathrm{Max}_{(a_{ij})_\alpha^L \leqslant a_{ij} \leqslant (a_{ij})_\alpha^U, \quad \forall i,j} \mathrm{Max}_p\, v$$
$$\begin{cases} \text{s.t. } \sum_{i=1}^m a_{ij} p_i \geqslant v, \quad j = 1, \ldots, n, \\ \sum_{i=1}^m p_i = 1, \\ p_i \geqslant 0, \quad i = 1, \ldots, m \text{ and } v \text{ unrestricted in sign.} \end{cases} \tag{24}$$

and

$$\tilde{Z}_\alpha^L = \mathrm{Min}_{(a_{ij})_\alpha^L \leqslant a_{ij} \leqslant (a_{ij})_\alpha^U, \ \forall i,j} \mathrm{Max}_p\, v$$
$$\begin{cases} \text{s.t. } \sum_{i=1}^m a_{ij} p_i \geqslant v, \quad j = 1, \ldots, n, \\ \sum_{i=1}^m p_i = 1, \\ p_i \geqslant 0, \quad i = 1, \ldots, m \text{ and } v \text{ unrestricted in sign.} \end{cases} \tag{25}$$

A similar pair of two-level problems can be formulated for Player II. Based on duality theorem, the inner program of (25) is replaced by its dual to make it a minimization problem. After this replacement, both inner and outer problems have the same direction for optimization. Then using the variable substitution of $t_{ij} = a_{ij} p_i$ and $l_{ij} = a_{ij} q_j$ the following pair of linear programming problems is obtained:

$$\tilde{Z}_\alpha^U = \mathrm{Max}\, v$$
$$\begin{cases} \text{s.t. } \sum_{i=1}^m t_{ij} \geqslant v, \quad j = 1, \ldots, n, \\ \sum_{i=1}^m p_i = 1, \\ (a_{ij})_\alpha^L p_i \leqslant t_{ij} \leqslant (a_{ij})_\alpha^U p_i, \quad i = 1, \ldots, m, \quad j = 1, \ldots, n, \\ p_i \geqslant 0, t_{ij}, v \text{ unrestricted in sign}, \quad i = 1, \ldots, m, \quad j = 1, \ldots, n \end{cases} \tag{26}$$

and

$$\tilde{Z}_\alpha^L = \mathrm{Min}\, u,$$
$$\begin{cases} \text{s.t. } \sum_{j=1}^n l_{ij} \leqslant u, \quad i = 1, \ldots, m, \\ \sum_{j=1}^n q_j = 1, \\ (a_{ij})_\alpha^L q_j \leqslant l_{ij} \leqslant (a_{ij})_\alpha^U q_j, \quad i = 1, \ldots, m, \quad j = 1, \ldots, n, \\ q_j \geqslant 0, l_{ij}, \ u \text{ unrestricted in sign}, \quad i = 1, \ldots, m, \quad j = 1, \ldots, n. \end{cases} \tag{27}$$

The objective value $\tilde{Z}_\alpha^L$ together with $\tilde{Z}_\alpha^U$ obtained from the problems (26) and (27), constitute the α-cut $[\tilde{Z}_\alpha^L, \tilde{Z}_\alpha^U]$ where the value $\tilde{Z}$ of the fuzzy game lies at the specified α-level. A similar transformation can be applied to the pair of problems related to Player II, the same pair of problems as (26) and (27) is obtained for this player. This shows that the fuzzy value calculated for the two players is the same. Note that most of the approaches for solving the game (16) do not provide the same fuzzy or crisp value for both players. Let $R(z) = (Z_\alpha^U)^{-1}$ and $L(z) = (Z_\alpha^L)^{-1}$ be the right and left shape function of the fuzzy value $\tilde{Z}$, then $\tilde{Z}$ can be obtained as

$$\mu_{\tilde{z}}(z) = \begin{cases} L(z), & z_{\alpha=0}^L \leqslant z \leqslant z_{\alpha=1}^L, \\ 1, & z_{\alpha=1}^L \leqslant z \leqslant z_{\alpha=1}^U, \\ R(z), & z_{\alpha=1}^U \leqslant z \leqslant z_{\alpha=0}^L. \end{cases}$$

In most cases, the equations $R(z) = (Z_\alpha^U)^{-1}$ and $L(z) = (Z_\alpha^L)^{-1}$ cannot be found analytically. However, for different possibility levels α, $\tilde{Z}_\alpha^L$ and $\tilde{Z}_\alpha^U$ can be calculated to approximate the shapes of $R(z)$ and $L(z)$, respectively.

4.1.6. Buckley and Jowers' approach

Buckley and Jowers [18] consider a matrix game with fuzzy payoffs and fuzzy mixed strategies for both players. Following the notations of the game (16), a fuzzy mixed strategy is defined based on fuzzy probabilities. A fuzzy mixed strategy for Player I is $\tilde{x} = (\tilde{x}_1, \ldots, \tilde{x}_m)$, where $\tilde{x}_i$ is a triangular fuzzy number with support [0,1] or a real number, for all i, and for any cut level $\alpha \in [0, 1]$, there exist $x_i \in [\tilde{x}_i]^\alpha$, $i = 1, \ldots, m$, such that $x_1 + \cdots + x_m = 1$. A fuzzy mixed strategy $\tilde{y}$, for Player II is similarly defined. Let $X(Y)$, be the set of all fuzzy mixed strategies for Player I (Player II), respectively. The fuzzy expected payoff, $\tilde{E}$, of the fuzzy game is determined by its α-cuts $[\tilde{E}(\tilde{x}, \tilde{y})]^\alpha = \{\sum_{i=1}^n \sum_{j=1}^n x_i a_{ij} y_j / (x, y) \in \mathrm{S}\}$, where S is the set of pairs (x, y) such that $x_i \in [\tilde{x}_i]^\alpha$, $i = 1, \ldots, m$, $y_j \in [\tilde{y}_j]^\alpha$, $j = 1, \ldots, n$ and $a_{ij} \in [\tilde{a}_{ij}]^\alpha$ for all i and j. The value V_I for Player I is defined as follows. First, define the function $V_I(\tilde{x}) = \mathrm{Min}\,\{\tilde{E}(\tilde{x}, \tilde{y}),\ \tilde{y} \in Y\}$, then $V_I = \mathrm{Max}\,\{V_I(\tilde{x}),\ \tilde{x} \in X\}$. The value for the Player II is defined similarly. Then fuzzy Monte Carlo method is employed to produce approximate solutions for the fuzzy values V_I for Player I and V_{II} for Player II, in an example of fuzzy game with no (fuzzy) saddle point; and also approximate solutions for the optimal fuzzy mixed strategies for both players. It is also showed that $V_I \leqslant V_{II}$. It remains an open question whether or not $V_I = V_{II}$ for these fuzzy games. Note that Buckley and Jowers' approach differs from the existing approaches in the sense that fuzzy mixed strategies are considered.

4.1.7. Xu et al. approach

In [72] Xu et al. consider a matrix game in which the payoffs are characterized as fuzzy variables. Based on possibility measure, credibility measure, and fuzzy expected value operators, three types of concept of minmax equilibrium strategies are defined as follows.

Assume that in the game (16) the entries a_{ij} of the fuzzy matrix $\tilde{A}$ are fuzzy independent variables.

Definition 4.9. A mixed strategy $(x^*, y^*) \in S^m \times S^n$ is called an expected minmax equilibrium strategy to the game if

$$E(x^T \tilde{A} y^*) \leqslant E(x^{*T} \tilde{A} y^*) \leqslant E(x^{*T} \tilde{A} y) \quad \text{for all } (x, y) \in S^m \times S^n.$$

Definition 4.10. Let r be a crisp predetermined level of the payoffs. Then $(x^*, y^*) \in S^m \times S^n$ is called a r-possible minmax equilibrium strategy to the game if

$$Poss(x^T \tilde{A} y^* \geqslant r) \leqslant Poss(x^{*T} \tilde{A} y^* \geqslant r) \leqslant Poss(x^{*T} \tilde{A} y \geqslant r) \quad \text{for all } (x, y) \in S^m \times S^n.$$

Definition 4.11. Let r be a crisp predetermined level of the payoffs. Then $(x^*, y^*) \in S^m \times S^n$ is called a r-credible minmax equilibrium strategy to the game

$$Cr(x^T \tilde{A} y^* \geqslant r) \leqslant Cr(x^{*T} \tilde{A} y^* \geqslant r) \leqslant Cr(x^{*T} \tilde{A} y \geqslant r) \quad \text{for all } (x, y) \in S^m \times S^n.$$

An iterative algorithm based on fuzzy simulation is designed to find the equilibrium strategies. Note that this approach differs from the existing approaches by the fact that the fuzzy payoffs are fuzzy variables not fixed fuzzy numbers.

Remark 4.2. Nicolas and Grabisch [53] argue that decision makers do not always follow the minmax strategy under situations of risk. They propose an extension of traditional matrix games which are based on minimax strategies to other kinds of strategies based on t-norms and t-conorms. Examples of matrix games where t-norm an t-conorm are suitable are provided. In [29] a model for crisply and fuzzily determined interval valued matrix games using an appropriate fuzzy interval comparison operator is proposed. The notion of optimal mixed strategies for interval valued games is also introduced.

Remark 4.3. In [1,2] two algorithms for solving fuzzy matrix games are presented. In the first algorithm [2] a decomposition of a bilinear programming model into a series of linear programming models is proposed. The second algorithm [1] is based on a coevolutionary approach.

4.2. Bimatrix games with fuzzy payoffs

In this section we present two approaches for solving bimatrix games with fuzzy payoffs. The first one is an approach based on ranking fuzzy numbers [67]; in the second an ordering of fuzzy numbers by possibility measure is used [49].

4.2.1. Duality and ranking function approach: Vijay et al. model

Using a ranking of fuzzy numbers approach, Vijay et al. [67] have studied the following bimatrix game with fuzzy payoffs:

$$G_{10} = (S^m,\ S^n,\ \tilde{A}, \tilde{B}). \tag{28}$$

Definition 4.12. Let $\tilde{v}$ and $\tilde{w}$ be two real fuzzy numbers. Then $(\tilde{v}, \tilde{w})$ is called a reasonable solution to the game (28) if there exists a pair of mixed strategies $(\overline{p}, \overline{q})$ such that

$$p^T \tilde{A}\overline{q} \leqslant_{\tilde{t}} \tilde{v}, \quad \forall p \in S^m,$$

$$\overline{p}^T \tilde{B}q \leqslant_{\tilde{k}} \tilde{w}, \quad \forall q \in S^n,$$

$$\overline{p}^T \tilde{A}\overline{q} \geqslant_{\tilde{t}'} \tilde{v},$$

$$\overline{p}^T \tilde{B}\overline{q} \geqslant_{\tilde{k}'} \tilde{w},$$

where $\tilde{t}$, $\tilde{t}'$, $\tilde{k}$ and $\tilde{k}'$ are adequacies (see Sections 4.1.1 and 4.1.3).

Definition 4.13. Let T_1 and T_2 be the sets of reasonable values for Players I and II, respectively. Let there exists a pair of values $(\tilde{v}^*, \tilde{w}^*) \in T_1 \times T_2$ such that

$$F(\tilde{v}^*) \geqslant F(\tilde{v}), \quad \forall \tilde{v} \in T_1 \quad \text{and} \quad F(\tilde{w}^*) \geqslant F(\tilde{w}), \ \forall \tilde{w} \in T_2,$$

where $F(.)$ is the chosen ranking function. Then the corresponding pair of mixed strategies $(\overline{p}, \overline{q})$ is called equilibrium solution for the game (28).

Equilibrium solutions can be found by solving the following fuzzy non linear programming problem:

$$\begin{array}{ll} \text{Max} & F(\tilde{v}) + F(\tilde{w}) \\ \text{s.t.} & p^T \tilde{A}q \leqslant_{\tilde{t}} \tilde{v}, \quad \forall p \in S^m, \\ & p^T \tilde{B}q \leqslant_{\tilde{k}} \tilde{w}, \quad \forall q \in S^n, \\ & p^T \tilde{A}q \geqslant_{\tilde{t}'} \tilde{v}, \\ & p^T \tilde{B}q \geqslant_{\tilde{k}'} \tilde{w}, \quad \forall p \in S^m, \ \forall q \in S^n. \end{array}$$

Here, $\tilde{t}, \tilde{t}', \tilde{k}$ and $\tilde{k}'$ are adequacies, $\tilde{v}$, $\tilde{w}$ are fuzzy numbers. Using the same method as in Section 4.1.3, the last fuzzy optimization problem is transformed into the following crisp nonlinear programming problem:

$$\begin{array}{ll}
\text{Max} & F(\tilde{v}) + F(\tilde{w}) \\
\text{s.t.} & \displaystyle\sum_{j=1}^{n} F(\tilde{a}_{ij}) q_j \leqslant F(\tilde{v}) + (1-\lambda) F(\tilde{t}), \quad i = 1, \ldots, m, \\
& \displaystyle\sum_{i=1}^{m} F(\tilde{b}_{ij}) p_i \leqslant F(\tilde{w}) + (1-\eta) F(\tilde{k}), \quad j = 1, \ldots, n, \\
& F(p^T \tilde{A} q) \geqslant F(\tilde{v}) - (1-\lambda) F(\tilde{t}'), \\
& F(p^T \tilde{B} q) \geqslant F(\tilde{w}) - (1-\eta) F(\tilde{k}'), \\
& p \in S^m, \quad q \in S^n, \quad \lambda, \eta \in [0,1] \text{ and } \tilde{v}, \tilde{w} \text{ are fuzzy numbers.}
\end{array}$$

4.2.2. Possibility measure approach: Maeda's model

In order to define a solution to the fuzzy bimatrix game (28), Maeda [49] uses the following ordering of fuzzy numbers by possibility measure. Let $\tilde{a}$ and $\tilde{b}$ be two fuzzy numbers

$$Poss(\tilde{a} \geqslant \tilde{b}) = \sup_{x \geqslant y} (\min(\mu_{\tilde{a}}(x), \mu_{\tilde{b}}(y))).$$

Maeda defines a solution for the game when the entries of matrices $\tilde{A}$ and $\tilde{B}$ are TFNs. He also assumes that a real number v can be represented as a TFN by $\tilde{v} = (v, v, v)$.

Definition 4.14. Let $v, w \in R$. An element $(\overline{p}, \overline{q}) \in S^m \times S^n$ is called a (v, w)-possible Nash equilibrium strategy if

$$Poss(\overline{p}^T \tilde{A} \overline{q} \geqslant \tilde{v}) \geqslant Poss(p^T \tilde{A} \overline{q} \geqslant \tilde{v}), \quad \forall p \in S^m,$$

$$Poss(\overline{p}^T \tilde{B} \overline{q} \geqslant \tilde{w}) \geqslant Poss(\overline{p}^T \tilde{B} q \geqslant \tilde{w}), \quad \forall q \in S^n,$$

where $\tilde{v} = (v, v, v)$ and $\tilde{w} = (w, w, w)$.

Theorem 4.2. *For every $v, w \in R$ the fuzzy bimatrix game (28) has at least one (v, w)-possible Nash equilibrium strategy.*

The following theorem provides a tool for the computation of (v, w)-possible Nash equilibrium strategy.

Theorem 4.3. *Let $v, w \in R$, and let $(\overline{p}, \overline{q}) \in S^m \times S^n$ be a (v, w)-possible Nash equilibrium strategy for the game (28). Then $(\overline{p}, \overline{q}, \overline{\alpha}, \overline{\beta})$ is an optimal solution of the following problem*:

$$\begin{array}{ll}
\text{Max} & Poss(p^T \tilde{A} \overline{q} \geqslant \tilde{v}) + Poss(\overline{p}^T \tilde{B} q) \geqslant \tilde{w}) - \alpha - \beta \\
\text{s.t.} & Poss(\tilde{A} q \geqslant e\tilde{v}) \leqslant \alpha, \\
& Poss(\tilde{B}^T p \geqslant e\tilde{w}) \leqslant \beta, \\
& e^T p = 1, \quad e^T q = 1, \quad p, q \geqslant 0 \text{ and } \alpha, \beta \in [0,1],
\end{array}$$

with $\overline{\alpha} = Poss(\overline{p}^T \tilde{A} \overline{q} \geqslant \tilde{v})$ and $\overline{\beta} = Poss(\overline{p}^T \tilde{B} \overline{q} \geqslant \tilde{w})$.

Conversely, let $(\overline{p}, \overline{q}, \overline{\alpha}, \overline{\beta})$ be a solution of this problem. Then $(\overline{p}, \overline{q})$ is a (v, w)-possible Nash equilibrium strategy of the game (28).

4.3. Two-person finite games with fuzzy goals and fuzzy payoffs

In this section, we present three approaches for solving two-person finite games with fuzzy goals and fuzzy payoffs.

4.3.1. Nishizaki and Sakawa's approach

In a series of works [54,55], Nishizaki and Sakawa have studied single and multiobjective two-person finite games with fuzzy goals and fuzzy payoffs. Because of shortage of space, we present only the case of multiobjective two-person zero-sum game with fuzzy payoffs and fuzzy goals. In such a game each player has r objectives, the following multiple fuzzy matrices represent a multiobjective two-person zero-sum game with fuzzy payoffs:

$$\tilde{A}^1 = (\tilde{a}^1_{ij})_{\substack{1\leqslant i\leqslant m\\ 1\leqslant j\leqslant n}} = \begin{pmatrix} \tilde{a}^1_{11} & \cdots\cdots\cdots & \tilde{a}^1_{1n} \\ \vdots & \ddots & \vdots \\ \tilde{a}^1_{m1} & \cdots\cdots\cdots & \tilde{a}^1_{mn} \end{pmatrix}, \ldots.., \quad \tilde{A}^r = (\tilde{a}^r_{ij})_{\substack{1\leqslant i\leqslant m\\ 1\leqslant j\leqslant n}} = \begin{pmatrix} \tilde{a}^r_{11} & \cdots\cdots\cdots & \tilde{a}^r_{1n} \\ \vdots & \ddots & \vdots \\ \tilde{a}^r_{m1} & \cdots\cdots\cdots & \tilde{a}^r_{mn} \end{pmatrix},$$

where the entries $(\tilde{a}^k_{ij})$ are assumed to be L–R fuzzy numbers i.e. $\tilde{a}^k_{ij} = (\tilde{a}^k_{ij}, \tilde{a}'^k_{ij}, \tilde{a}^k_{ij})_{LR}$.

Definition 4.15. For any pair of mixed strategies (p, q) , the k-th fuzzy expected payoff of Player I is defined as the fuzzy number

$$p^T \tilde{A}^k q = \left(\sum_{i=1}^{m}\sum_{j=1}^{n} \tilde{a}^k_{ij} p_i q_j, \sum_{i=1}^{m}\sum_{j=1}^{n} \tilde{a}'^k_{ij} p_i q_j, \sum_{i=1}^{m}\sum_{j=1}^{n} \tilde{a}^k_{ij} p_i q_j \right)_{LR},$$

characterized by the membership function $\mu_{p^T \tilde{A}^k q} : D^k \to [0, 1]$, where $D^k \subset R$ is the domain of the k-th expected payoff function of Player I. The fuzzy goal $\tilde{g}_k$ with respect to the k-th payoff function of Player I is a fuzzy set on D^k characterized by the membership function $\mu_{g_k} : D^k \to [0, 1]$.

Definition 4.16. The state of attainment of the k-th fuzzy goal $\tilde{g}_k$ of Player I is defined by the membership function

$$\mu^k_{a(p,q)}(t) = \mathrm{Min}\{\mu_{g_k}(t),\ \mu_{p^T \tilde{A}^k q}(t)\}, \quad (p, q) \in S^m \times S^n,\ t \in D^k.$$

The degree of attainment of the k-th fuzzy goal $\tilde{g}_k$ of Player I is defined by

$$\mu^k_{a(p,q)}(t^{*k}) = \underset{t\in D^k}{\mathrm{Max}}\, \mu^k_{a(p,q)}(t).$$

The degree of attainment of the aggregated fuzzy goal of the Player I is defined by

$$\mu^{(t^*)}_{a(p,q)} = \underset{k}{\mathrm{Min}}\, \mu^k_{a(p,q)}(t^{*k}) = \underset{k}{\mathrm{Min}}\, \underset{t\in D^k}{\mathrm{Max}}\, \mu^k_{a(p,q)}(t).$$

Definition 4.17. The Player I's maxmin value with the respect to the degree of attainment of the aggregated goal is

$$\underset{p\in S^m}{\mathrm{Max}}\ \underset{q\in S^n}{\mathrm{Min}}\, \mu_{a(p,q)}(t^*).$$

The mixed strategy p for which this value is reached is called the maxmin solution with respect to the degree of attainment of the aggregate fuzzy goal.

Theorem 4.4. *In case the membership functions of the fuzzy goals and the entries $\tilde{a}^k_{ij}$ of $\tilde{A}^k$ are linear, a Player I's maxmin solution is the optimal solution of the non linear programming problem*

$$\underset{p,\sigma}{Max}\ \sigma$$

$$\frac{\sum_{i=1}^{n}\sum_{j=1}^{m} (a^k_{ij} + a^k_{ij}) p_i q_j - \underline{a}^k}{\sum_{i=1}^{n}\sum_{j=1}^{m} a^k_{ij} p_i q_j + \overline{a}^k - \underline{a}^k} \geqslant \sigma, \quad \forall q \in S^n,\ \ p \in S^m,\ \ k = 1, \ldots, r,$$

if the optimal value σ^* satisfies $0 < \sigma^* < 1$. Here $\overline{a}^k$ and $\underline{a}^k$ are the payoffs giving the best degree and the worst degree of satisfaction to the k-th objective of Player I, respectively.

4.3.2. Duality and ranking function approach: Vijay et al. model

In [66,67] Vijay et al. studied two-person finite games with fuzzy goals and fuzzy payoffs using fuzzy duality theory. We present only the case of matrix games with fuzzy goals and fuzzy payoffs [66]. The following matrix game is considered:

$$G_{11} = (S^n, S^m, \tilde{A}, \tilde{v}, \tilde{\geqslant}, \tilde{t}, \tilde{w}, \tilde{\leqslant}, \tilde{k}), \tag{29}$$

where $\tilde{v}$ and $\tilde{w}$ are the aspiration levels of Players I and II, respectively; $\tilde{t}$ and $\tilde{k}$ are the adequacies or tolerances of Players I and II, respectively; $\tilde{\geqslant}$ and $\tilde{\leqslant}$ are as defined in Section 4.1.3.

Definition 4.18. A point $(\overline{p}, \overline{q}) \in S^m \times S^n$ is called a solution of the game (29) if

$$\overline{p}^T \tilde{A} q \,\tilde{\geqslant}_{\tilde{t}} \tilde{v}, \quad \forall q \in S^n,$$

$$p^T \tilde{A} \overline{q} \,\tilde{\leqslant}_{\tilde{k}} \tilde{w}, \quad \forall p \in S^m,$$

where $\overline{p}$ and $\overline{q}$ are called optimal strategy of Players I and II, respectively. Using both Yager's method [73] and Zimmermann's approach [76], the problem of finding optimal strategies for Players I and II is transformed into the following primal-dual pair of linear programming problems in the fuzzy sense:

$$\begin{aligned} \text{Max} \quad & \lambda \\ \text{s.t.} \quad & p^T F(\tilde{A})_j \geqslant F(\tilde{v}) - F(\tilde{t})(1-\lambda), \quad j = 1, \ldots, n, \\ & e^T p = 1, \quad \lambda \leqslant 1 \text{ and } p, \lambda \geqslant 0, \end{aligned}$$

and

$$\begin{aligned} \text{Max} \quad & \eta \\ \text{s.t.} \quad & F(\tilde{A})_i q \leqslant F(\tilde{w}) + F(\tilde{k})(1-\eta), \quad i = 1, \ldots, m, \\ & e^T q = 1, \quad \eta \leqslant 1 \text{ and } q, \eta \geqslant 0 \end{aligned}$$

where $F()$ is the chosen ranking function, $F(\tilde{A})_j$ is the j-th column of $F(\tilde{A})$ and $F(\tilde{A})_i$ is the i-th row of $F(\tilde{A})$.

4.3.3. Fuzzy relation approach: Vijay et al. model

Recently, Vijay et al. [68] introduced an approach based on fuzzy relations. The fuzzy relation approach unifies the existing theories on fuzzy matrix games. In this approach, the fuzzy matrix game is transformed into a pair of fuzzy optimization problems where constraints are expressed by fuzzy relations. Further, this pair is formulated as a pair of semi-infinite programming problems. Matrix games with fuzzy goals and matrix games with possibility and necessity relations have been studied as special cases.

Suppose Players I and II prescribe their aspiration levels as fuzzy numbers represented by $\tilde{v}$ and $\tilde{w}$, respectively. Then a generalized model for a *matrix game with fuzzy goals and fuzzy payoffs* is defined as follows:

$$G_{12} = (S^m, S^n, \tilde{A}, \tilde{v}, \tilde{P}^T, \tilde{w}, \tilde{P}_S),$$

where the fuzzy relations $\tilde{P}^T$ and $\tilde{P}_S$ are fuzzy extensions of the classical binary relation $\leqslant$. Also $\tilde{P}^T$ and $\tilde{P}_S$ are dual fuzzy relations to each other. Then a *solution* of G_{12} using these fuzzy relations is defined as follows.

Definition 4.19 (*Solution of* G_{12}). $(\overline{x}, \overline{y}, \tilde{v}, \tilde{w}) \in S^m \times S^n \times N(R) \times N(R)$ is called solution of G_{12} if
(i) $\tilde{v} \tilde{P}^T \overline{x}^T \tilde{A} y, \forall y \in S^n$,
(ii) $x^T \tilde{A} \overline{y} P_S \tilde{w}, \forall x \in S^m$.
Here, $\overline{x}$ and $\overline{y}$ are called optimal strategies for Players I and II, respectively.

Using Bellman and Zadeh's principle [10], the problem of computing a solution of G_{12} is transformed into the following pair of crisp optimization problems with an infinite number of constraints:

$$\begin{aligned} Max \quad & \lambda \\ \text{s.t.} \quad & \lambda \leqslant \mu_{\tilde{P}^T}(\tilde{v}, \tilde{P}^T x^T \tilde{A} y), \quad \forall y \in S^n, \ \forall x \in S^m \end{aligned}$$

and

$$\begin{array}{ll} Max & \eta \\ \text{s.t.} & \eta \leqslant \mu_{\tilde{P}_S}(x^T \tilde{A} y P_S, \tilde{w}), \quad \forall x \in S^m, \ \forall y \in S^n, \end{array}$$

where $\mu_{\tilde{P}^T}$ and $\mu_{\tilde{P}_S}$ are membership functions for the constraints of the first and the second problem, respectively. For some special cases this pair of problems reduces to a pair of finite linear programming problems.

Remark 4.4. The results obtained so far in games with fuzzy evaluation of payoffs are limited to finite two-person games: matrix games, bimatrix games, muliobjective finite two-person zero-sum (or non-zero sum) games. In addition to this, these results focus more on numerical aspects than on game theory aspects as beliefs of players, accuracy of prediction … etc. As far as we know, there are no results in literature concerning the general case (15), especially, when the sets of strategies are infinite or/and the payoff functions are nonlinear. This type of games seems to be difficult and challenging. It may be useful for real-world applications. It is worth noting that Gao [30] has extended the work of Arsany [34] to games involving uncertainty of fuzzy type.

Note that, since fixed point theorems are intensively used in game theory as a tool for proving the existence of solutions, many researchers have studied the generalization of Kakutani's fixed point theorem to the case of fuzzy set-valued functions [3,16,36,58]. The fixed point approach addresses the existence problem only, it does not provide a method for computation of Nash equilibrium. It is well known that finding a fixed point of a crisp set-valued function is a difficult task. Needless to say, the problem of computing a fixed point of a fuzzy set-valued function would be more difficult and challenging. To our knowledge, there is no work in literature on this interesting problem.

5. Non cooperative games with fuzzy parameters

In this section, we present our contribution to the theory of non cooperative normal form games with fuzzy evaluation of payoffs (15). We present a new class of such games, the games with fuzzy parameters. This class of games is an extension of the traditional theory of games against nature [51]

$$G_{13} = (X_i, f_i(x, \tilde{y}))_{i \in I}, \tag{30}$$

where I, X_i, $i = 1, \dots, n$ and x are defined as in the game (1); $\tilde{y} = (\tilde{y}_1, \tilde{y}_2, \dots, \tilde{y}_m)$ is a fuzzy parameter such that each of its components $\tilde{y}_s$, is a real fuzzy numbers as defined by Dubois and Prade [28], denoted by, $\tilde{y}_s = (\mu_{\tilde{y}_s}(y), R)$, $s = 1, \dots, m$; $f_i : X \times R^m \to R$ is the payoff function of the i-th player, where $X = \prod_{i \in I} X_i$ is defined as in the game (1). We make also the following assumptions:

(i) The players have no control on the parameters $\tilde{y}_s$, $s = 1, \dots, m$.
(ii) The players know all the data of the game including the membership functions of the parameters $\tilde{y}_s$, $s = 1, \dots, m$.

A parameter $\tilde{y}_s$ may represent the price of Raw material (e.g. oil), the rate of exchange between currencies (e.g. Euro/USD), a risk, an interest rate, an imprecision in measurement or evaluation, the behavior of a competitor or player not in I … etc.

The game takes place as follows. Once each player has chosen his strategy, we get a strategy profile $x = (x_1, x_2, \dots, x_n) \in X = \prod_{i \in I} X_i$, then depending on the realizations y_s, $s = 1, \dots, m$ of the parameters $\tilde{y}_s$, $s = 1, \dots, m$ in the universe R^m, players get the payoffs $f_i(x, y)$, $i = 1, \dots, n$, respectively. When the players choose their strategies they have to take into account both the behavior of the other players and the possible realizations of the fuzzy parameter $\tilde{y}$. Hence, this game involves two aspects: the game aspect and a decision making under uncertainty aspect [45]. Thus, a rational solution for this game should at least reflect these two aspects.

5.1. Solution concept and its existence

In [37], we propose a solution that takes into account the two aspects involved in the game (30). It is based on the approach developed in [62] for multiobjective problems with fuzzy parameters. First, using α-cuts of the fuzzy parameters $\tilde{y}_s$, $s = 1, \dots, m$, we associate a crisp game to the initial game (30), then we define a solution for it.

Each player i is asked to provide his confidence level $\alpha_i \in [0, 1]$ on the possible realizations of the fuzzy parameters $\tilde{y}_s$, $s = 1, \ldots, m$, we get α_i, $i = 1, \ldots, n$. Let $\alpha^{\max} = \mathrm{Max}_{i \in I}\ \alpha_i$, then construct the α-cuts $[\tilde{y}_s]^{\alpha^{\max}} = \{y_s / y_s \in R,\ \mu_{\tilde{y}_s}(y_s) \geqslant \alpha^{\max}\}$, $s = 1, \ldots, m$, next, define

$$[\tilde{y}]^{\alpha^{\max}} = \prod_{s=1}^{m} [y_s]^{\alpha^{\max}}. \tag{31}$$

Note that in order to take into account the chosen cut levels α_i, $i = 1, \ldots, n$ by all players, it is natural to choose $\alpha^{\max} = \mathrm{Max}_{i \in I}\ \alpha_i$. Indeed, we have

$$\alpha \leqslant \alpha' \Rightarrow [\tilde{y}_s]^{\alpha'} \subset [\tilde{y}_s]^{\alpha}.$$

Using (31), we associate the following crisp game to the game (30):

$$G_{14} = (X_i, f_i(x, y))_{i \in I}, \tag{32}$$

where $y \in [\tilde{y}]^{\alpha^{\max}}$ is a crisp parameter. In the game (32), the parameter y is just an unknown crisp parameter that takes its values in the crisp set $[\tilde{y}]^{\alpha^{\max}}$.

Definition 5.1. A strategy profile $x^0 \in X$ is an $\alpha^{\max}$-equilibrium of the game (30) if there exists a value $y^0 \in [\tilde{y}]^{\alpha^{\max}}$ such that the pair (x^0, y^0) verifies the following conditions:

(1) $f_i(x^0//x_i, y^0) \leqslant f_i(x^0, y^0)$, for all $x_i \in X_i$ and $i \in I$.
(2) There does not exist any realization $y \in [\tilde{y}]^{\alpha^{\max}}$ such that $f_i(x^0, y) < f_i(x^0, y^0)$, for all $i \in I$.

The definition 5.1 can be interpreted as follows. The condition (1) means that x^0 is a Nash equilibrium of the game (32) when the parameter y takes the value y^0. If any player unilaterally deviates from x^0, he will not be better off. The condition (2) means, when players choose x^0, if the parameter y takes a value different from y^0 then the payoffs of all players cannot be decreased at the same time i.e. if some of them decrease, some of the others will increase or remain at the same level. That is, y^0 is weak Pareto optimal when players choose x^0.

Theorem 5.1. *Assume that the following conditions are verified*

(1) *X_i, is a compact and convex subset of R^{n_i}, $i = 1, \ldots, n$.*
(2) *The function $t_i \to f_i(x//t_i, y)$ is continuous and concave over X_i, for all $x \in X$ and $y \in [\tilde{y}]^{\alpha^{\max}}$, $i = 1, \ldots, n$.*
(3) *The function $y \to \sum_{i=1}^{n} f_i(x, y)$ is convex over $[\tilde{y}]^{\alpha^{\max}}$, for all $x \in X$.*

Then the game (30) has a $\alpha^{\max}$-Nash equilibrium.

5.2. Computation of $\alpha^{\max}$-equilibrium

Let us introduce the following real valued function $\Phi : Z \times Z \to R$ with $Z = X \times [\tilde{y}]^{\alpha^{\max}}$, defined by:

$$\Phi(z, t) = \sum_{i=1}^{n} f_i(x, y) - f_i(x, y') + \sum_{i=1}^{n} f_i(x//x'_i, y) - f_i(x, y'), \quad z = (x, y), \ \ t = (x', y') \in Z.$$

Proposition 5.1. *If $z^0 = (x^0, y^0) \in Z$ verifies*

$$\Phi(z^0, t) \leqslant 0, \quad \forall t \in Z, \tag{33}$$

then x^0 is an $\alpha^{\max}$-Nash equilibrium of the game (30) with y^0 as corresponding value of the parameter y. The converse is also true, if $z^0 = (x^0, y^0) \in Z$ is a solution of (33) then x^0 is an $\alpha^{\max}$-Nash equilibrium with y^0 as the corresponding value of the parameter y.

Procedure 5.1. Assume that the conditions of Theorem 5.1 are verified.

Step 1: Ask the players to provide their α-cut levels. Then construct the α-cut $[\tilde{y}]^{\alpha^{\max}}$.

Step 2. Solve the inequality (33). According to Theorem 5.1, it has a solution.

Example 5.1. Assume that $I = \{1, 2\}$ and $X_i = [0, 1]$, $i = 1, 2$,

$$\tilde{y} = (\tilde{y}_1, \tilde{y}_2), \quad f_1(x, \tilde{y}) = x_1^2 - x_2^2 + \tilde{y}_1^2 + \tilde{y}_2^2, \quad f_2(x, \tilde{y}) = -x_1^2 + x_2^2 + \tilde{y}_1 + \tilde{y}_2^2$$

and

$$\mu_{\tilde{y}_1}(y_1) = \begin{cases} 0 & \text{if } y_1 \leqslant 3, \\ y_1 - 3 & \text{if } 3 < y_1 \leqslant 4, \\ 1 & \text{if } 4 < y_1 \leqslant 5, \\ 6 - y_1 & \text{if } 5 < y_1 \leqslant 6, \\ 0 & \text{if } 6 < y_1, \end{cases} \qquad \mu_{\tilde{y}_2}(y_2) = \begin{cases} 0 & \text{if } y_2 \leqslant 2, \\ y_2 - 2 & \text{if } 2 < y_2 \leqslant 3, \\ 1 & \text{if } 3 < y_2 \leqslant 5, \\ 6 - y_2 & \text{if } 5 < y_2 \leqslant 6, \\ 0 & \text{if } 6 < y_2. \end{cases}$$

Step 1: Assume that both players have chosen $\alpha_1 = \alpha_2 = \frac{1}{2}$ then $\alpha^{\max} = \frac{1}{2}$.

We have $[\tilde{y}]^{\alpha^{\max}} = [\tilde{y}]^{1/2} = [\frac{7}{2}, \frac{11}{2}] \times [\frac{5}{2}, \frac{11}{2}]$ and the function

$$y \to f_1(x, y) + f_2(x, y) = y_1^2 - y_1 + y_2^2 + y_2,$$

is convex over R^m, for all$x \in X$. The other conditions of Theorem 5.1 are also verified.

Step 2: We have

$$\Phi(z^0, t) = y_1^2 - y_1 - (y_1'^2 - y_1') + y_2^2 + y_2 - (y_2'^2 + y_2') + (x_1'^2 - x_1^2) + (x_2'^2 - x_2^2) \leqslant 0,$$

for all $t = (x', y') \in Z$ with $z^0 = (x^0, y^0) = ((1, 1), (\frac{7}{2}, \frac{5}{2}))$ then $x^0 = (1, 1)$ is an $\frac{1}{2}$-Nash equilibrium of the given game and the corresponding value of the parameter y is $y^0 = (\frac{7}{2}, \frac{5}{2})$.

Remark 5.1. In [37], we have presented just one possible solution to the game (30), other solutions could be proposed depending on the principle of decision making under uncertainty [45] used to deal with the fuzzy parameters. For particular classes of the game (30), as matrix and bimatrix games, interesting results can be obtained. In [38], we have investigated the case when in the game (30) the players have fuzzy goals. The generalization of the game (30) to the case of multiobjective games is being investigated. In [37,38], it is assumed that the fuzzy parameters $\tilde{y}_s$, $s = 1, \ldots, m$ are given and constant, the general case when the parameters $\tilde{y}_s$ are variable real fuzzy numbers is a challenging and worthy direction of research.

6. Applications of fuzzy non-cooperative games in normal form

In our survey of FNCGNF, we did not find many publications pertaining to their application. This is due to the fact that the theory of FNCGNF is still at its beginning stage. However, the few publications that have appeared on application on FNCGNF show that this theory has a promising future. Let us elaborate on this statement.

Much of what counts as the game-theoretic approach to the study of social, and economic phenomena is based on Aristotelian or Boolean logic which admits two truth values, true and false (or, equivalently,1 and 0) and rational behavior. The formulation of games in fuzzy set/fuzzy logic framework will certainly enlarge the scope of applications of game theory and help to solve many of its dilemmas and difficulties rooted in Boolean logic and rational behavior assumption. Moreover, fuzzyfying a crisp game may generate new and more stable solutions. Let us illustrate this on one of the most important game theory dilemmas—Prisoner's Dilemma game [7]

According to Raman [59], the importance of finding ways to resolve Prisoner's Dilemma, hereafter referred to as PD, can hardly be overstated, given its far-ranging applications in fields spanning economics, marketing, political science, production management, biology, artificial intelligence and engineering. The proposed ways of moving towards the cooperative outcome include infinitely repeated interaction among the players or finitely repeated interaction with uncertain date for the last play of the game, fictitious play, tit-for-tat strategy, the formation of coalitions and the use of credible threats to induce cooperation. Each of these solutions has well-known limitations. These limitations stem from the rigorous micro-economic resolutions of the PD that are based on strict binary logic and fully rational behavior.

Raman justifies a fuzzy approach to Prisoner's Dilemma game based on fuzzy controller, which is a very similar to Garazic and Cruz's (see Section 3.4), as follows. The assumption of binary logic implies that players classify signals in a dichotomous manner. Interpreted purely in terms of binary or Aristotelian logic, a signal is either seen as a threat or it is not. However, binary classification is unduly restrictive and somewhat artificial, for in the real world, both empirical observations and common sense inform us that signals are perceived as threats to varying degrees. Fuzzy logic accommodates such a perspective whereas classical binary logic does not. In the fuzzy framework, when a company invests in additional capacity, the degree to which a competitor perceives it as a threat is an increasing function of the amount of investment. In the binary framework, a competitor either perceives the investment as a threat or does not. Surely, the fuzzy view makes both conceptual and pragmatic sense. The usual binary analysis of the PD restricts players to two possible actions: cooperate or defect. In the fuzzy case, the action set can be represented more generally as a fuzzy set C. Without loss of generality, we can assume that the cooperation space is the unit interval [0, 1] and that the outcome's cooperative nature increases with the signal x. Thus, 0 indicates total noncooperation and 1 indicates total cooperation. Full membership in C ($C(x) = 1$) indicates cooperative behavior, the signal is perceived as a threat. Zero membership in C ($C(x) = 0$) indicates non-cooperative behavior, the signal is perceived to be non threatening. Intermediate values of $C(x)$ indicate varying degrees of cooperative behavior.

The classical economic paradigm of humans as fully rational utility-maximizing beings has been challenged by the Nobel-prize winning economist Simon [63], who proposed that humans rely upon an alternative process that he termed "bounded rationality" in making their choices. In the "bounded rationality" paradigm, agents are assumed to be rational within limits and to satisfy rather than optimize. Seminal research by Kahneman and Tversky's [39] shows that people often use logical shortcuts called heuristics. Lindsay and Norman [44] define heuristics as "rules of thumb" or general action plans that people use in problem-solving. Such heuristics are generally operationalized as production systems featuring "If–Then–Else" rules. Heuristics are simple rules easy to apply, and they reduce cognitive effort by simplifying decision making in the face of cognitive complexity.

In the context of the PD, a player observes the signal, classifies the extent to which it is a threat and makes a decision according to "If–Then" rules of following type:

"If the threat is low, then do not cooperate".

"If the threat is high, then cooperate".

In the next level of sophistication, players judge both the level of the threat and its credibility and use that information to guide their action. This leads to heuristics of the following type:

"If the threat is low and its credibility is high, then do not cooperate".

"If the threat is low and its credibility is low, then do not cooperate".

Arfi [4] applies his linguistic fuzzy logic game (see Section 3.6) to the PD game and a social game of cooperation with n players. He finds that there is always an optimum strong Nash equilibrium which is Pareto optimal, thereby lifting many of the dilemmas that emerge in crisp game theory and social games.

Borges et al. [15] propose an approach to the PD game, where the players' decisions are taken with the guidance of fuzzy expert systems. The main purpose of the work is to investigate a model of the PD game in which the players' decisions are taken by means of a qualitative reasoning system. Mathew and Kaimal [50] introduce a fuzzy approach to the PD game using fuzzy expected value models. The expected value is computed based on the notion of credibility measure introduced in [47].

Let us now present some other applications of FNCGNF. Butnariu [20] explains how his model (see Section 3.1) can be used to model real-game phenomena. As an example he shows how to model a political situation involving two players in a conflicting state. The players wish to negotiate the simultaneous reduction of their military investments.

Garazic and Cruz [31] illustrate their approach (see Section 3.4) based on fuzzy controller by an application in military conflicts. They consider a situation in the context of planning and conducting a military operation. Due to shortage of space we do not present this interesting but lengthy application. We refer the reader to the above reference for details.

Aristidou and Sarangi [5] illustrate their approach by an application to duopoly. A basic duopoly model is set up and the implication of its fuzzy game formulation is discussed. A single period homogenous product Cournot duopoly is considered. The inverse demand function in this market is given by the standard linear formulation

$$p = a - bQ, \quad Q = q_1 + q_2 \quad \text{with } a, b > 0.$$

It is also assumed that both firms have identical constant marginal cost functions given by $C(q_i) = cq_i$, $i = 1, 2$. The profit function can be written as

$$\prod_i (q_1, q_2) = (a - bQ - c)q_i, \quad i = 1, 2.$$

In the fuzzy version of this game the constraint set is assumed to be a crisp set. Thus, each firm considers all its strategies equally feasible, i.e. $\mu_i(q_i) = 1$ for all q_i and for $i = 1, 2$. In order to make things simple, it is assumed that the strategy set is compact and defined by $q_i \in [0, a/b]$. The goal function however is fuzzy and each firm believes that the collusive outcome is the best possible outcome. Hence, the membership function is single peaked such that $\gamma_i((a - c)/4b, (a - c)/4b) = 1$, for $i = 1, 2$. An example of a membership function with this property is

$$\gamma_i(q_1, q_2) = \exp\left(-\left(q_1 - \frac{a-c}{4b}\right) - \left(q_2 - \frac{a-c}{4b}\right)\right).$$

Proposition 6.1. *The fuzzy duopoly game as defined above has a Cournot–Nash equilibrium.*

Since the two firms are symmetric in all respects, $(Q/4, Q/4)$, where $Q/4 = (a - c)/4b$ is indeed an equilibrium. Hence the collusive outcome can easily be supported as an equilibrium in the fuzzy game.

Ponsard [58] establishes the equilibrium existence conditions in a spatial duopoly. Two fuzzy approaches to oligopolistic competition are proposed in [32,33].

Artificial intelligence is another field where FNCGNF are applied. Wu and Soo [71] formulate a multi-agent coordination problem as a matrix game with fuzzy utilities. They show that a fuzzy strategy can perform better than a mixed strategy in traditional game theory in dealing with more than one Nash equilibrium games.

E-commerce is one of the potential areas of application of FNCGNF. Indeed, web site visitors have budgets of time, attention, and patience that also have a generalized utility value. Uniquely, internet factors frequently become paramount such as visual aesthetics, sensory-motor interactivity, and social interplay as well as affective, habit-based and loss-prospect-avoidance determiners of their competitive and strategic e-consumer behaviors. These factors which determine user persistence and web business success are quite unlike the traditional rational summations in games. In [60] a fuzzy game-theoretic approach is explored that begins to deal with some of the above e-commerce peculiarities.

Chen and Larbani [26] establish a relation between multi-attribute decision making with fuzzy evaluations and fuzzy matrix games.

In [57] Peldschus and Zavadskas use fuzzy matrix games in engineering problems associated with optimal alternative selection. An application for selecting the variants of water-supply systems is provided. Li et al. [41] develop two fuzzy approaches to the well known game of chicken. The first approach uses a fuzzy multicriteria decision analysis method to obtain optimal strategies for the players. It incorporates subjective factors into the decision-makers' objectives and aggregates objectives by using a weight vector. The second approach applies the theory of fuzzy moves (TFM) to the game of chicken. De Wilde [27] presents a framework for fuzzy decision making, using techniques from fuzzy logic, game theory and micro-economics. Then fuzzy Cournot adjustment is implemented, equilibria are defined and their stability is studied. Bogataj and Usenik [14] present a fuzzy approach to build up a model of spatial hierarchy based on spatial games. The results of this fuzzy approach in logistic games are compared with analytical results. Such fuzzy model is used in solving strategic problems of a supply chain that are connected to the question of location and capacity of individual activity cells to achieve the best supply chain coordination.

7. Conclusion and some directions of research

After reviewing most of the works on FNCGNF and their applications, we came to the following conclusions:

(i) The theory of FNCGNF is at its beginning for most of the available results are limited to finite two-person games, especially, games with fuzzy payoffs.

(ii) The theory of FNCGNF is a considerable departure from traditional game theory at least in two aspects. The first is that fuzzy logic allows fuzzy game theory to handle situations where binary (Aristolian) logic fails e.g. Prisoner's Dilemma game [4]. The second is that fuzzy set theory is an adequate tool for modeling vagueness of human judgment and uncertainty in the data of the game, while in crisp game theory, the data of the game are assumed to be known precisely or involve uncertainty of probabilistic type. FNCGNF will certainly enlarge the scope of applications of game theory in social sciences, artificial intelligence and many other areas.

(iii) In FNCGNF, it is possible to define several solutions to the same game (see Sections 3 and 4) depending on how players consider the fuzziness involved in the game. Fuzzyfying a crisp game offers possibilities for exploring new and/or more stable solutions (e.g. fuzzyfying the Prisoner Dilemma game [4]). Sometimes fuzzyfying a crisp game without solution generates a game that has a solution.

(iv) The application of FNCGNF is also at its beginning. It is not in the mainstream literature of social sciences. However, the few published works show that FNGCNF is increasingly approaching social sciences and has a promising future.

(v) Almost all published works on FNCGNF focus on Nash equilibrium as a solution concept.

Based on these conclusions, we suggest the following directions for future research:

(a) Introduce new concepts of fuzzy solutions based on solutions of crisp normal form games other than Nash equilibrium. Introduce new types of fuzzy games.
(b) Investigate the general form of fuzzy games (15). Propose concepts of solutions for this game and study their properties, existence and computation.
(c) Investigate the fuzzy game (30) when the parameter $\tilde{y}$ is variable.
(d) Investigate fuzzy extensive form games and fuzzy differential games in normal form.
(e) Investigate multiple criteria fuzzy games in normal form.
(f) Apply the existing approaches (see Sections 3–5) for the resolution of FNCGNF to social sciences, artificial intelligence and other areas.
(g) Develop effective numerical methods for computing solutions of FNCGNF.

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