Solution of Fuzzy Matrix Games: An Application of the Extension Principle

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Conventional game theory is concerned with how rational individuals make decisions when they are faced with known payoffs. This article develops a solution method for the two-person zero-sum game where the payoffs are only approximately known and can be represented by fuzzy numbers. Because the payoffs are fuzzy, the value of the game is fuzzy as well. Based on the extension principle, a pair of two-level mathematical programs is formulated to obtain the upper bound and lower bound of the value of the game at possibility level α. By applying a dual formulation and a variable substitution technique, the pair of two-level mathematical programs is transformed to a pair of ordinary one-level linear programs so they can be manipulated. From different values of α, the membership function of the fuzzy value of the game is constructed. It is shown that the two players have the same fuzzy value of the game. An example illustrates the whole idea of a fuzzy matrix game. © 2007 Wiley Periodicals, Inc.

1. INTRODUCTION

The general problem of how to make decisions in a competitive environment is a common and important one. In the real world, competitive situations arise not only in business, but also in other facets of human activity, including military battles, sport competitions, and different types of contest. In all of these situations, the results achieved depend on the actions of the competitor. The conventional decision problems treated are those of a single decision maker. Game theory, on the other hand, is concerned with how individuals make decisions when they are aware that their actions affect each other and when each individual takes this into account. It is the interaction among individual decision makers that makes strategic decisions different from other deci-

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sions. There are many features common to both simple games and complicated conflicts in business and industry. For this reason, knowledge of the theory of games should be helpful to the decision makers who continually face competitive situations.

Games arise in many different contexts that have many different features that are being studied and discussed.[1,2] For the studying purpose, games have been classified by the number of players, the number of strategies, the nature of the payoff function, the nature of preplay negotiation, and so forth. A typical game consists of a list of players, the strategies available to each player, the payoffs of each player for all possible combinations of strategies pursued by all players, and the assumption that each player is a rational maximizer. In the case of the two-person finite game, each player has a finite number of strategies. Associated with each pair of strategies is a payoff that one player pays to the other. This type of game is known as two-person zero-sum game because the gain by one player is equal to the loss to the other. In describing such kinds of games, it suffices to summarize the payoffs of one player to the other by a matrix. The actions of one player form the rows and the actions of the other compose the columns of the matrix. If the row player uses strategy i and the column player uses strategy j, then the row player receives a reward of a_{ij} and the column player loses the amount of a_{ij} accordingly.

Traditionally, the payoffs a_{ij} are represented by crisp values, indicating that they are precisely known. This is reasonable for clearly defined games. In the real world, however, there are cases in which the payoffs are not known and have to be estimated. There are also cases in which the payoffs are only approximately known. An example is one in which different advertising strategies of two competing companies lead to different market shares and the market shares must be estimated. Another example is in an election, where the candidates may select different campaign issues on which to focus. Different issues may bring different votes, and the number of votes can only be estimated. To deal quantitatively with imprecise information in making decisions, Bellman and Zadeh[3] and Zadeh[4] introduce the notion of fuzziness.

There have been several studies in the literature discussing the topic of fuzzy games. Campos[5] proposes solution methods for solving fuzzy matrix games; however, only crisp solutions are provided. Maeda[6] defines matrix equilibrium to fuzzy matrix games based on the fuzzy max order. Again, only crisp solutions are provided; moreover, the classical minimax theorems are not utilized. One theoretically sound property of game theory is that the mathematical model of the matrix game as formulated from the standpoints of the two players is a pair of two linear programs that are the dual of each other. Hence, solving either of the linear programs obtains the strategies of the two players by applying the duality theorem. Motivated by Zimmermann,[7] Bector et al.,[8] and Nishizaki and Sakawa[9] propose fuzzy linear programming models to solve fuzzy matrix games. Bector et al.[10] and Vijay et al.[11] employ Zimmermann's approach[7] and the Yager index[12] to formulate a primal–dual pair of fuzzy linear programs to solve matrix games. Similar to the case of Campos[5] and Maeda,[6] only crisp solutions are provided in these methods. There are also studies discussing cooperative games under fuzzy environment.[13–15]

Intuitively, when the payoffs are uncertain and are specified by convex fuzzy sets, the value of the game calculated from the fuzzy payoffs should be fuzzy as well. In this article we solve the fuzzy matrix games based on Zadeh's extension principle.[4,16] A pair of two-level mathematical programming models is formulated to calculate the upper bound and lower bound of the α-cut of the fuzzy value of the game. We also show that the two players have the same fuzzy value of the game. The rest of this article is organized as follows. In the next section we formulate the matrix game with fuzzy payoffs in the form of two-level mathematical programs. We then transform the two-level programs to the conventional one-level linear programs to be able to solve. An example that illustrates the whole idea is provided. Finally, some conclusions are drawn from the discussion.

2. PROBLEM FORMULATION

Consider a matrix game of two players, where player 1 has m strategies to select and player 2 has n strategies to select in each play of the game. If in a play of the game, player 1 chooses strategy i and player 2 chooses strategy j, then player 1 gains a_{ij} units of reward and player 2 loses a_{ij}. The matrix $A = \|a_{ij}\|$ is known as the payoff matrix of the game.

The standard criterion proposed by game theory for selecting a strategy is the minimax criterion, in that each player should play in such a way as to minimize his maximum losses. In terms of the payoff matrix, it implies that player 1 should choose the strategy whose minimum payoff is largest, whereas player 2 should select the strategy whose maximum payoff to player 1 is the smallest. When the same entry in the payoff matrix yields both the maximin and minimax values, then a stable solution exists. Both players will stick on that specific strategy, called pure strategy, in every play. Unfortunately, for most cases, the game is unstable, where the entry of the maximin value does not coincide with the minimax value. In this case, the players will choose their strategies in a probabilistic manner.

Let x_i be the probability that player 1 will use strategy i, for $i = 1, \dots, m$, and y_j be the probability that player 2 will use strategy j, for $j = 1, \dots, n$. The probability vectors $\boldsymbol{x} = (x_1, \dots, x_m)$ and $\boldsymbol{y} = (y_1, \dots, y_n)$ are known as the mixed strategies of player 1 and player 2, respectively. Player 1 is interested in finding a mixed strategy $\boldsymbol{x}$ that produces the largest expected reward. When player 2 uses strategy j, the expected reward for player 1 is $a_{1j}x_1 + \cdots + a_{mj}x_m$. Because player 2 can use any strategy j, player 1 attempts to assure a minimum expected reward v via the constraints $a_{1j}x_1 + \cdots + a_{mj}x_m \geq v$ for all $j = 1, \dots, n$. According to the minimax criterion that a player should select the mixed strategy that minimizes the maximum expected loss (or equivalently that a player should select the mixed strategy that maximizes the minimum expected gain), the optimal strategy for player 1 can be modeled by the following linear program:

$$Z = \text{Max } v$$

$$\text{s.t.} \quad \sum_{i=1}^{m} a_{ij} x_i \geq v, \quad j = 1, \ldots, n$$

$$\sum_{i=1}^{m} x_i = 1$$

$$x_i \geq 0, i = 1,.., m, \quad v \text{ unrestricted in sign} \tag{1}$$

From the viewpoint of player 2, his expected loss is $a_{i1} y_1 + \cdots + a_{in} y_n$ when player 1 uses strategy i. Because player 1 can use any strategy i, player 2 tries to minimize the expected loss u via the constraint $a_{i1} y_1 + \cdots + a_{in} y_n \leq u$ for $i = 1, \ldots, m$. Consequently, the linear program for player 2 to find the optimal strategy $\boldsymbol{y}$ and the corresponding game value u is

$$Z = \text{Min } u$$

$$\text{s.t.} \sum_{j=1}^{n} a_{ij} y_j \leq u, \quad i = 1, \ldots, m$$

$$\sum_{j=1}^{n} y_j = 1$$

$$y_j \geq 0, \quad j = 1, \ldots, n, \quad u \text{ unrestricted in sign} \tag{2}$$

Clearly, this model is the dual of Model (1); hence, the optimal strategies for both players can be solved from either program.

Suppose the payoff a_{ij} is approximately known and can be represented by the convex fuzzy set $\tilde{A}_{ij}$. Let $\mu_{\tilde{A}_{ij}}$ denote its membership function; we have

$$\tilde{A}_{ij} = \{(a_{ij}, \mu_{\tilde{A}_{ij}}(a_{ij})) | a_{ij} \in S(\tilde{A}_{ij})\} \tag{3}$$

where $S(\tilde{A}_{ij})$ is the support of $\tilde{A}_{ij}$, which denotes the universal set of the payoff. Without loss of generality, all the payoffs in this research are assumed to be fuzzy numbers, as crisp values can be represented by degenerated membership functions that only have one value in their domains. When the payoffs are fuzzy numbers, the left-hand side of the constraint $\sum_{i=1}^{m} a_{ij} x_i \geq v$ in Model (1) becomes $\sum_{i=1}^{m} \tilde{A}_{ij} x_i$, which is a fuzzy number. A different value of a_{ij} in the fuzzy set $\tilde{A}_{ij}$ leads to a different objective value. Accordingly, the objective value becomes a fuzzy number $\tilde{Z}$, instead of a crisp number. The purpose of this article is to derive the membership function of the value of the game, $\tilde{Z}$.

Because $\tilde{Z}$ is a fuzzy number, it is a function that cannot be maximized directly. To tackle this problem, one can rely on Zadeh's extension principle[4,16] of the following form:

$$\mu_{\tilde{Z}}(z) = \operatorname*{Sup}_{a_{ij}} \min\{\mu_{\tilde{A}_{ij}}(a_{ij}), \quad \forall i, j | z = Z(\mathbf{a})\} \tag{4}$$

where $Z(\mathbf{a})$ is defined in Model (1). The application of the extension principle to $\tilde{Z}$ may be viewed as the application of this extension principle to the α-cuts of $\tilde{Z}$.[17] In Equation (4), several membership functions are involved. To derive $\mu_{\tilde{Z}}$ from $\mu_{\tilde{A}_{ij}}$ directly in closed form is hardly possible. According to (4), $\mu_{\tilde{Z}}(z)$ is the minimum of those $\mu_{\tilde{A}_{ij}}$, $\forall i, j$, such that $z = Z(\mathbf{a})$. At a specific possibility level α, we need $\mu_{\tilde{A}_{ij}}(a_{ij}) \geq \alpha$ and at least one $\mu_{\tilde{A}_{ij}}(a_{ij})$, $\forall i, j$, equal to α such that $z = Z(\mathbf{a})$ to satisfy $\mu_{\tilde{Z}}(z) = \alpha$. To find the membership function $\mu_{\tilde{Z}}$, it suffices to find the right shape function and left shape function of $\mu_{\tilde{Z}}$. This is equivalent to finding the upper bound Z_α^{U} and lower bound Z_α^{L} of the α-cuts of $\tilde{Z}$. Denote the α-cut of $\tilde{A}_{ij}$ as

$$\begin{aligned}(A_{ij})_\alpha &= [(A_{ij})_\alpha^{\mathrm{L}}, (A_{ij})_\alpha^{\mathrm{U}}] \\ &= \left[\min_{a_{ij}}\{a_{ij} \in S(\tilde{A}_{ij}) \mid \mu_{\tilde{A}_{ij}}(a_{ij}) \geq \alpha\}, \max_{a_{ij}}\{a_{ij} \in S(\tilde{A}_{ij}) \mid \mu_{\tilde{A}_{ij}}(a_{ij}) \geq \alpha\}\right]\end{aligned} \tag{5}$$

Because Z_α^{U} is the maximum of $Z(\mathbf{a})$ and Z_α^{L} is the minimum of $Z(\mathbf{a})$, they can be expressed as

$$Z_\alpha^{\mathrm{U}} = \max\{Z(\mathbf{a}) \mid (A_{ij})_\alpha^{\mathrm{L}} \leq a_{ij} \leq (A_{ij})_\alpha^{\mathrm{U}}, \quad \forall i, j\} \tag{6a}$$

$$Z_\alpha^{\mathrm{L}} = \min\{Z(\mathbf{a}) \mid (A_{ij})_\alpha^{\mathrm{L}} \leq a_{ij} \leq (A_{ij})_\alpha^{\mathrm{U}}, \quad \forall i, j\} \tag{6b}$$

The value of a_{ij} that attains the largest value for v can be determined from the following two-level mathematical programming model:

$$Z_\alpha^{\mathrm{U}} = \underset{\substack{(A_{ij})_\alpha^{\mathrm{L}} \leq a_{ij} \leq (A_{ij})_\alpha^{\mathrm{U}} \\ \forall i, j}}{\mathrm{Max}} \left\{ \begin{array}{ll} \underset{x}{\mathrm{Max}}\, v & \\ \text{s.t.} \displaystyle\sum_{i=1}^{m} a_{ij} x_i \geq v, & j = 1, \ldots, n \\ \quad \displaystyle\sum_{i=1}^{m} x_i = 1 & \\ \quad x_i \geq 0, & \text{for } i = 1, \ldots, m, \quad v \text{ unrestricted in sign} \end{array} \right. \tag{7}$$

At least one a_{ij} must hit the boundary of its α-cut to satisfy $\mu_{\tilde{Z}}(z) = \alpha$. In Model (7), the inner program calculates the expected reward for each a_{ij} specified by the outer program, whereas the outer program determines the value of a_{ij} that generates the largest reward. The objective value Z_α^{U} is the upper bound of the expected reward of the α-level cut for player 1.

By the same token, to find the value of a_{ij} that produces the smallest reward, a two-level mathematical program is formulated by simply replacing the outer program of Model (7) from “Max” to “Min”:

$$Z_\alpha^{\mathrm{L}} = \underset{\substack{(A_{ij})_\alpha^{\mathrm{L}} \le a_{ij} \le (A_{ij})_\alpha^{\mathrm{U}} \\ \forall i,j}}{\mathrm{Min}} \begin{cases} \underset{x}{\mathrm{Max}}\ v \\ \text{s.t.} \sum_{i=1}^{m} a_{ij} x_i \ge v, \quad j = 1, \ldots, n \\ \quad \sum_{i=1}^{m} x_i = 1 \\ \quad x_i \ge 0, \qquad \text{for } i = 1, \ldots, m, \quad v \text{ unrestricted in sign} \end{cases} \tag{8}$$

In this model the inner program calculates the expected reward for each given value of a_{ij}, whereas the outer program determines the value of a_{ij} that produces the smallest reward. The objective value Z_α^{L} is the lower bound of the expected reward of the α-level cut for player 1. When the fuzzy payoff $\tilde{A}_{ij}$ degenerates to the point payoff a_{ij}, the outer program of Models (7) and (8) vanishes. In this case, Models (7) and (8) degenerate to the conventional model of matrix games. This property shows that the fuzzy-valued model developed here is a generalization of the conventional point-valued model.

We can also formulate the upper bound and lower bound of the expected reward for player 2 based on Model (2):

$$Z_\alpha^{\mathrm{U}} = \underset{\substack{(A_{ij})_\alpha^{\mathrm{L}} \le a_{ij} \le (A_{ij})_\alpha^{\mathrm{U}} \\ \forall i,j}}{\mathrm{Max}} \begin{cases} \underset{y}{\mathrm{Min}}\ u \\ \text{s.t.} \sum_{j=1}^{n} a_{ij} y_j \le u, \quad i = 1, \ldots, m \\ \quad \sum_{j=1}^{n} y_j = 1 \\ \quad y_j \ge 0, \qquad \text{for } j = 1, \ldots, n, \quad u \text{ unrestricted in sign} \end{cases} \tag{9}$$

$$Z_\alpha^{\mathrm{L}} = \underset{\substack{(A_{ij})_\alpha^{\mathrm{L}} \le a_{ij} \le (A_{ij})_\alpha^{\mathrm{U}} \\ \forall i,j}}{\mathrm{Min}} \begin{cases} \underset{y}{\mathrm{Min}}\ u \\ \text{s.t.} \sum_{j=1}^{n} a_{ij} y_j \le u, \quad i = 1, \ldots, m \\ \quad \sum_{j=1}^{n} y_j = 1 \\ \quad y_j \ge 0, \qquad \text{for } j = 1, \ldots, n, \quad u \text{ unrestricted in sign} \end{cases} \tag{10}$$

Although the two-level model clearly expresses the problem, it cannot be manipulated to find a solution. A transformation is required. In the next section, we shall transform the two-level mathematical program to the conventional one-level linear program from which the fuzzy value of the fuzzy game can be derived.

3. ONE-LEVEL TRANSFORMATION

Models (7) and (8) are a pair of mathematical programs for calculating the upper and lower bounds of the α-cut of the fuzzy value of the matrix game from the standpoint of player 1, whereas Models (9) and (10) are from the standpoint of player 2. Because both the inner program and outer program of (7) have the same maximization operation, they can be combined into one level with the constraints of the two programs considered at the same time. That is,

$$
\begin{aligned}
Z_\alpha^{\mathrm{U}} = \operatorname{Max} v & \\
\text{s.t.} \sum_{i=1}^{m} a_{ij} x_i \geq v, &\quad j = 1, \ldots, n \\
\sum_{i=1}^{m} x_i = 1 & \\
(A_{ij})_\alpha^{\mathrm{L}} \leq a_{ij} \leq (A_{ij})_\alpha^{\mathrm{U}}, &\quad i = 1, \ldots, m, \quad j = 1, \ldots, n \\
x_i \geq 0, \quad \text{for } i = 1, \ldots, m, &\quad v \text{ unrestricted in sign}
\end{aligned} \tag{11}
$$

This model is nonlinear due to the nonlinear terms $a_{ij} x_i$. A variable substitution of $p_{ij} = a_{ij} x_i$ can be applied to transform the nonlinear program to a linear one:

$$
\begin{aligned}
Z_\alpha^{\mathrm{U}} = \operatorname{Max} v & \\
\text{s.t.} \sum_{i=1}^{m} p_{ij} \geq v, &\quad j = 1, \ldots, n \\
\sum_{i=1}^{m} x_i = 1 & \\
(A_{ij})_\alpha^{\mathrm{L}} x_i \leq p_{ij} \leq (A_{ij}) \leq (A_{ij})_\alpha^{\mathrm{U}} x_i, &\quad i = 1, \ldots, m, \quad j = 1, \ldots, n \\
x_i \geq 0, \quad p_{ij}, \quad v \text{ unrestricted in sign}, &\quad i = 1, \ldots, m, \quad j = 1, \ldots, n
\end{aligned} \tag{12}
$$

The third set of constraints is a result of multiplying each term by x_i and replacing $a_{ij} x_i$ by p_{ij}. Solving this linear program obtains the upper bound of the α-cut of the fuzzy value of the fuzzy game. After the optimal solution p_{ij}^* and x_i^* are solved, the optimal value of a_{ij}^* is calculated as $a_{ij}^* = p_{ij}^*/x_i^*$. If x_i^* is 0, then a_{ij}^* can be any value in the range of $(A_{ij})_\alpha^{\mathrm{L}}$ and $(A_{ij})_\alpha^{\mathrm{U}}$.

To solve Model (8) is not so straightforward as Model (7) because its outer program and inner program have different directions for optimization, one for minimization and one for maximization. A reverse of the program is required to make a solution obtainable. Based on duality theorem, a dual has the same optimal objective value as its primal when an optimal solution exists. Therefore, we can replace the inner program of Model (8) by its dual to make it become a minimization problem. After this replacement, both the inner and outer programs have the same direction for optimization:

$$Z_\alpha^{\mathrm{L}} = \underset{\substack{(A_{ij})_\alpha^{\mathrm{L}} \le a_{ij} \le (A_{ij})_\alpha^{\mathrm{U}} \\ \forall i,j}}{\mathrm{Max}} \left\{ \begin{array}{ll} \underset{y}{\mathrm{Min}}\, u & \\ \text{s.t.} \sum_{j=1}^{n} a_{ij} y_j \le u, & i = 1, \ldots, m \\ \sum_{j=1}^{n} y_j = 1 & \\ y_j \ge 0, & \text{for } j = 1, \ldots, n, \quad u \text{ unrestricted in sign} \end{array} \right. \tag{13}$$

Now that both the inner program and outer program have the same minimization operation, they can be merged into a one-level program with the constraints at the two levels considered at the same time. The nonlinear terms $a_{ij} y_j$ can be linearized similarly as in Model (11) by substituting by q_{ij} and the constraints $A_{ij}^{\mathrm{L}} \le a_{ij} \le A_{ij}^{\mathrm{U}}$ multiplied by y_j, accordingly. The resulting linear program is

$$\begin{aligned} Z_\alpha^{\mathrm{L}} = {} & \mathrm{Min}\, u \\ & \text{s.t.} \sum_{j=1}^{n} q_{ij} \le u, i = 1, \ldots, m \\ & \sum_{j=1}^{n} y_j = 1 \\ & (A_{ij})_\alpha^{\mathrm{L}} y_j \le q_{ij} \le (A_{ij})_\alpha^{\mathrm{U}} y_j, \quad i = 1, \ldots, m, \quad j = 1, \ldots, n \\ & y_j \ge 0, \quad q_{ij}, \quad u \text{ unrestricted in sign}, \quad i = 1, \ldots, m, \quad j = 1, \ldots, n \end{aligned} \tag{14}$$

When the optimal values of q_{ij}^* and y_j^* are solved, the optimal value of a_{ij}^* is calculated as $a_{ij}^* = q_{ij}^*/y_j^*$. The objective value Z_α^{L} together with Z_α^{U} solved from Model (12), $[Z_\alpha^{\mathrm{L}}, Z_\alpha^{\mathrm{U}}]$, constitutes the α-cut that the value of the fuzzy game lies at the specific α level.

A similar transformation can be applied to Models (9) and (10), which are formulated from the standpoint of player 2. Specifically, a dual formulation of the linear program transforms Model (9) to a model that is exactly the same as Model (7), and Model (10) in its current form is exactly the same as Model (13). Note that Models (7) and (13) have been transformed to Models (12) and (14), respectively, to calculate the upper and lower bounds of the α-cut of the fuzzy value of the game. This derivation shows that the fuzzy value of the game calculated from the two players is the same.

For two possibility levels α_1 and α_2 such that $0 < \alpha_2 < \alpha_1 \leq 1$, the feasible regions defined by α_1 in Models (7), (8), (9), and (10) are smaller than those defined by α_2, respectively. Consequently, we have $Z^{\mathrm{U}}_{\alpha_1} \leq Z^{\mathrm{U}}_{\alpha_2}$ and $Z^{\mathrm{L}}_{\alpha_1} \geq Z^{\mathrm{L}}_{\alpha_2}$; in other words, the right shape function is nonincreasing and the left shape function is nondecreasing. This property, based on the definition of "convex fuzzy set,"[16] assures the convexity of $\tilde{Z}$. If both Z^{U}_{α} and Z^{L}_{α} are invertible with respect to α, then a right shape function of $R(z) = (Z^{\mathrm{U}}_{\alpha})^{-1}$ and a left shape function of $L(z) = (Z^{\mathrm{L}}_{\alpha})^{-1}$ can be obtained. From $R(z)$ and $L(z)$, the membership function $\mu_{\tilde{Z}}$ is constructed as

$$\mu_{\tilde{Z}} = \begin{cases} L(z), & Z^{\mathrm{L}}_{\alpha=0} \leq z \leq Z^{\mathrm{L}}_{\alpha=1} \\ 1, & Z^{\mathrm{L}}_{\alpha=1} \leq z \leq Z^{\mathrm{U}}_{\alpha=1} \\ R(z), & Z^{\mathrm{U}}_{\alpha=1} \leq z \leq Z^{\mathrm{U}}_{\alpha=0} \end{cases} \tag{15}$$

In most cases, the values of Z^{U}_{α} and may not be solved analytically. However, the numerical solutions for Z^{U}_{α} and Z^{L}_{α} at a different possibility level α can be collected to approximate the shapes of $R(z)$ and $L(z)$, respectively.

4. AN EXAMPLE

Consider a matrix game in which both players have four strategies to choose from. Of the 16 payoffs shown in Table I, 8 are fuzzy numbers. The ones with three components are triangular and the ones with four components are trapezoidal. Conceptually, the upper bound Z^{U}_{α} and lower bound Z^{L}_{α} of the fuzzy value of the game at a specific α-level can be formulated as

$$\begin{aligned} Z^{\mathrm{U}}_{\alpha} = {} & \operatorname{Max} v \\ & \text{s.t. } (26,27,28,30)x_1 - 28x_2 + (-27,-25,-22)x_3 + 30x_4 \geq v \\ & -10x_1 + 20x_2 + (10,11,12,14)x_3 + (-38,-36,-35,-34)x_4 \geq v \\ & -20x_1 - 10x_2 + 20x_3 + (-34,-32,-30)x_4 \geq v \\ & (-3,0,3)x_1 + (32,34,35,37)x_2 - 30x_3 + (32,33,34,36)x_4 \geq v \\ & x_1 + x_2 + x_3 + x_4 = 1 \\ & x_1, x_2, x_3, x_4 \geq 0, \quad v \text{ unrestricted in sign} \end{aligned}$$

Table I. Payoffs of the example.

		Player 2			
	Strategy	1	2	3	4
Player 1	1	(26, 27, 28, 30)	−10	−20	(−3, 0, 3)
	2	−28	20	−10	(32, 34, 35, 37)
	3	(−27, −25, −22)	(10, 11, 12, 14)	20	−30
	4	30	(−38, −36, −35, −34)	(−34, −32, −30)	(32, 33, 34, 36)

$$
\begin{aligned}
Z_\alpha^{\mathrm{L}} = \mathrm{Min}\, u \\
\text{s.t. } & (26, 27, 28, 30)y_1 - 10y_2 - 20y_3 + (-3, 0, 3)y_4 \le u \\
& -28y_1 + 20y_2 - 10y_3 + (32, 34, 35, 37)y_4 \le u \\
& (-27, -25, -22)y_1 + (10, 11, 12, 14)y_2 + 20y_3 - 30y_4 \le u \\
& 30y_1 + (-38, -36, -35, -34)y_2 + (-34, -32, -30)y_3 \\
& \qquad + (32, 33, 34, 36)y_4 \le u \\
& y_1 + y_2 + y_3 + y_4 = 1 \\
& y_1, y_2, y_3, y_4 \ge 0, \quad u \text{ unrestricted in sign}
\end{aligned}
$$

Based on Models (12) and (14), the upper bound Z_α^{U} and lower bound Z_α^{L} at different α levels are calculated, respectively. Table II lists the α-cuts of the value of the game at 11 distinct α values: 0, 0.1, 0.2, ... ,1.0. The α-cut of $\tilde{Z}$ represents the possibility that the value of the game will appear in the associated range. The larger the α value, the lower the degree of uncertainty is. Because the fuzzy value of the game lies in a range, different α-cuts show different intervals and the uncertainty level of the value of the game. Specifically, $\alpha = 0$ has the widest interval yet the lowest possibility, indicating that the value of the game will never fall outside of this range. At the other extreme of $\alpha = 1$, its associated interval is the most likely value of the game. In this example, although the value of the game is fuzzy, its most likely value falls between −2.8213 and −2.6405, and it is impossible for its value to fall outside the range of −3.6105 and −1.5959. The membership function of $\tilde{Z}$, as constructed from 50 α-cuts, is depicted in Figure 1. Visually, $\tilde{Z}$ looks like a trapezoidal fuzzy number. In fact, it is not.

Table II. The α-cuts of the fuzzy value of the game at 11 α values for the example.

α	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
Z_α^{U}	−1.5959	−1.7009	−1.8058	−1.9105	−2.0151	−2.1196	−2.2240	−2.3283	−2.4325	−2.5365	−2.6405
Z_α^{L}	−3.6105	−3.5317	−3.4530	−3.3742	−3.2953	−3.2164	−3.1375	−3.0585	−2.9795	−2.9005	−2.8213

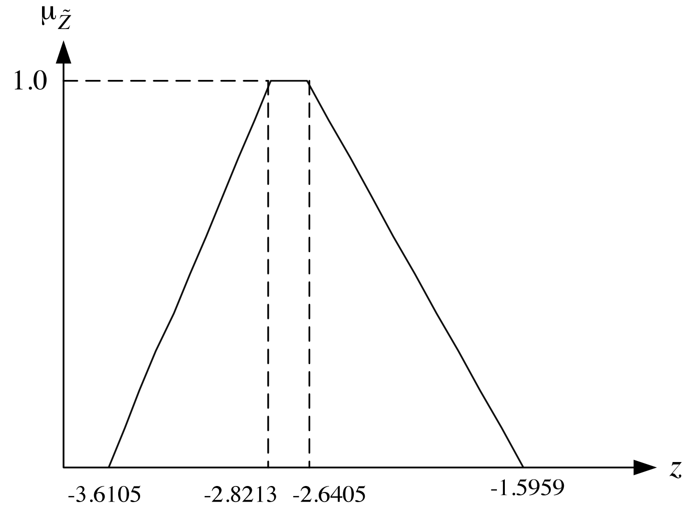

Figure 1. The membership function of $\tilde{Z}$ of the example.

At α-cut 0, the upper bound $Z^{\mathrm{U}}_{\alpha=0}$ solved is -1.5959, occurring at $x_1^* = 0.1331$, $x_2^* = 0.0925$, $x_3^* = 0.5045$, $x_4^* = 0.2699$. For the eight fuzzy numbers, the values that achieve $Z^{\mathrm{U}}_{\alpha=0} = -1.5959$ are $a_{11} = 30$, $a_{14} = 3$, $a_{24} = 37$, $a_{31} = -22$, $a_{32} = 14$, $a_{42} = -34$, $a_{43} = -30$, $a_{44} = 36$. The corresponding strategy for player 2 is $y_1^* = 0.2514$, $y_2^* = 0.1047$, $y_3^* = 0.4357$, $y_4^* = 0.2082$. The lower bound $Z^{\mathrm{L}}_{\alpha=0}$ is solved as -3.6105, which occurs at $y_1^* = 0.2621$, $y_2^* = 0.0750$, $y_3^* = 0.4521$, $y_4^* = 0.2108$. The eight fuzzy numbers have the value of $a_{11} = 26$, $a_{14} = -3$, $a_{24} = 32$, $a_{31} = -27$, $a_{32} = 10$, $a_{42} = -38$, $a_{43} = -34$, $a_{44} = 32$. The corresponding strategy for player 1 is $x_1^* = 0.2045$, $x_2^* = 0.1144$, $x_3^* = 0.4589$, $x_4^* = 0.2222$. At α-cut 1, the upper bound of $Z^{U}_{\alpha=1} = -2.6405$ occurs at $x_1^* = 0.1690$, $x_2^* = 0.0977$, $x_3^* = 0.4843$, $x_4^* = 0.2490$. The values for the eight fuzzy numbers are $a_{11} = 28$, $a_{14} = 0$, $a_{24} = 35$, $a_{31} = -25$, $a_{32} = 12$, $a_{42} = -35$, $a_{43} = -32$, $a_{44} = 34$. The corresponding strategy for player 2 is $y_1^* = 0.2568$, $y_2^* = 0.0880$, $y_3^* = 0.4476$, $y_4^* = 0.2076$. The lower bound $Z^{\mathrm{L}}_{\alpha=1} = -2.8213$ occurs at $y_1^* = 0.2580$, $y_2^* = 0.00887$, $y_3^* = 0.4451$, $y_4^* = 0.2082$. The eight fuzzy numbers are $a_{11} = 27$, $a_{14} = 0$, $a_{24} = 34$, $a_{31} = -25$, $a_{32} = 11$, $a_{42} = -36$, $a_{43} = -32$, $a_{44} = 33$. The corresponding strategy for player 1 is $x_1^* = 0.1890$, $x_2^* = 0.1092$, $x_3^* = 0.4713$, $x_4^* = 0.2305$. One interesting finding is that, in Model (12), the associated optimal solution for a_{ij} always occurs at the upper bound A^{U}_{ij}, whereas for Model (14) it always occurs at the lower bound A^{L}_{ij}. This is intuitively reasonable. Consider a game in which the payoff a_{ij} is allowed to vary between A^{L}_{ij} and A^{U}_{ij}. The largest value of the game obviously occurs at the upper bound A^{U}_{ij} and the smallest at the lower bound A^{L}_{ij}. This finding relieves the computational effort very much in the solution process. It also assures the condition required by the extension principle that at least one a_{ij} must hit the boundary of its α-cut to satisfy $\mu_{\tilde{Z}}(z) = \alpha$.

5. CONCLUSIONS

Game theory provides a basic conceptual framework for formulating and analyzing the problem that the decision of one person depends on the decision of his opponents. Of the different types of game, the two-person zero-sum game, which is also called the matrix game, with crisp payoffs has been widely discussed. Some solution methods have been devised. In the real world, there are cases in which the payoffs are not exactly known. This article uses fuzzy numbers to represent the uncertain data and develops a solution method for the matrix game.

Based on Zadeh's extension principle, this article employs a two-level mathematical programming technique to find the α-cuts of the fuzzy value of the game. By applying a dual formulation and a variable substitution technique, the two-level mathematical programs are transformed to ordinary one-level linear programs to be able to solve. The α-cuts of different possibility levels can be used to approximate the membership function. The more α-cuts one calculates, the better approximation one gets. When the fuzzy data degenerate to point data, the pair of two-level mathematical programs degenerates to the conventional model of matrix games. In other words, the model developed in this article is a generalization of the conventional model for taking care of uncertain cases. From the theoretical side, this article shows that under fuzzy environment, the two players have the same fuzzy value of the game. Moreover, when the payoffs vary in ranges, the upper bound of the value of the game always occurs at the upper bounds of the payoffs and the lower bound of the value of the game always occurs at the lower bounds of the payoffs.

Usually, when some data are only approximately known, the averages or the most likely values are used to find a point solution. Because only one point value is obtained, much valuable information is lost. Besides, the decision maker will be overconfident with the actually uncertain result. In this article, the value of the game is expressed by the membership function rather than by a point value; more information is provided for making better decisions. The idea of this article is readily applicable to other games such as nonzero-sum games, n-person games, and cooperative games.

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