Techsoc Freshie Roadmaps

MATHEMATICS CLUB X PROGRAMMING CLUB

6 August 2024

Instructions

- Welcome to the first session of the Freshie Roadmaps! The goal is for you to have as much fun as possible.

- All 8 problems are worth 10 marks and the bonus problem is worth 5 marks.

- Try all questions even if you can't solve them completely. Write your answers clearly and concisely.

- Don't use unfair means like the internet because all of you are here to learn and enjoy.

Problem 1: An Army of Unfair Dice

a) A blindfolded Navin throws two identical six-sided dice which have been carefully constructed by Aditi in such a way that $P(X = 1) = P(X = 3) = P(X = 5) = 2/9$, and $P(X = 2) = P(X = 4) = P(X = 6) = 1/9$ (where X is the number that shows up on the top face). Aditi calculates the sum of the numbers that appear and asks Navin to guess the sum. What should Navin say to maximize his chances of being right?

b) Now, Navin throws 50 such dice together and painstakingly calculates the sum of the numbers and asks a blindfolded Aditi to find the remainder on dividing the sum by 4. What would be Aditi's best guess?

Problem 2: A Tale of Two Cities

In Mathland, a country of 101 cities live a few coordinators of the Mathematics Club. Their cities are connected by one-way roads.

a) There are 50 roads going into and coming out of each city. What's the minimum number of roads one of them would have to travel to reach a fellow coordinator's city?

b) The coordinators are lazy. They don't like travelling more than 3 roads to reach each other. In order to optimize this, the heads decide that n roads in and out of each city are sufficient to make them meet and get them to do work. Help Atreya and Pradyumnan find n.

Problem 3: Stumbling Across a Grid

Pranjal starts at the point $(4, 4)$ on a grid of lattice points and moves until he reaches the coordinate axes for the 1st time. At any point Pranjal can move either left, down or left-down with equal chance, independent of his previous moves. What is the probability that he reaches the coordinate axes at $(0, 0)$? Let the probability be $\dfrac{a}{3^b}$, where a and b re positive integers, and 3 does not divide a. Find $a + b$.

Problem 4: The Circle of Lif- Polynomials

Standing at the edge of the circle of life, Prasanna and KK are pondering over the intricacies of reflection and polynomials. As such, they set up an experiment where they stand on the boundary of the circle of life and shoot out lasers.
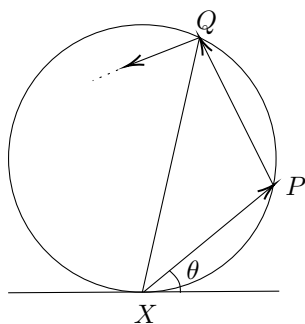
While KK shoots out laser over $\theta \in [0, \pi]$, Prasanna measures the distances $p = |XP|$, $q = |XQ|$. Now they construct a polynomial

$$f(z) = pz^2 + qz + 1$$

What is the probability that this polynomial has real roots? Keep in mind that a wise man once said the circle of life has a radius of $1/2$.

Problem 5: **Skill Issues**

For both parts of this question, you have to find the error(s) that the algorithm can run into, and also suggest a way to fix them.

a) Parijat gave Yukash a sequence of n positive integers $a_1, a_2, ..., a_n$. Since Yukash likes to think backwards, he would like to know if the sequence is a palindrome† or not. He wrote the following logic to solve this problem, but since he suffers from skill issues there is(are) some mistake(s). Help Yukash fix his problems.

†A sequence is called a palindrome if it reads the same forwards and backwards. For example, $(1, 2, 3, 2, 1)$ is a palindrome but $(1, 2, 3, 2)$ is not.

- **Step 1:** Start the algorithm.

- **Step 2:** Set x to n.

- **Step 3:** If $x > 0$, go to **Step 4**. Otherwise, go to **Step 7**.

- **Step 4:** If $a_x \neq a_{n-x}$, go to **Step 5**. Otherwise, go to **Step 6**.

- **Step 5:** Output that "The given sequence is not a palindrome". Go to **Step 8**.

- **Step 6:** Increase x by 1. Go to **Step 3**.

- **Step 7:** Output that "The given sequence is a palindrome".

- **Step 8:** End the algorithm.

b) To take revenge, Yukash gave Parijat a whole number n and asked him to check whether it's a perfect square or not. Parijat was very confident but he got infected with Yukash's skill issues and made some error(s) himself. Help him fix them.

- **Step 1:** Start the algorithm.

- **Step 2:** Set x to 0.

- **Step 3:** If $x^2 = n$, go to **Step 4**. Otherwise, go to **Step 5**.

- **Step 4:** Output that "The given number is a perfect square". Go to **Step 7**.

- **Step 5:** Increase x by 1. Go to **Step 3**.

- **Step 6:** Output that "The given number is not a perfect square".

- **Step 7:** End the algorithm.

Problem 6: Financial Foresight: Broke

Tushar and Parva are quite rich at the moment. As they have no care for money, they decide to play a game with their entire worth at stake. Initially, let's say Tushar has a coins and Parva has b coins in their wallets. Each person takes turns to play, with Tushar starting first.

In each turn, a player performs both the following steps in order:

1. Exchange wallets with the other player, or decide to keep his current wallet.

2. Throw away 1 coin from his wallet. A player loses if he is unable to remove any coins.

If both players play optimally (which means that Tushar and Parva are both strategic geniuses), when would Tushar win and when would Parva win?

Problem 7: Euler Cameo

Arya wants to place his home on one of the vertices of the following graph so that he can roam around all day on different roads (edges) and still come back home. But right now he isn't able to do that. To help him, add exactly one edge to the following graph such that after adding it, you will be able to draw all the edges of the graph *without lifting up the pen from the paper or drawing any edge multiple times*, and *starting and ending at the same point on the paper*. To answer, write down the two distinct vertices joined by the new edge.
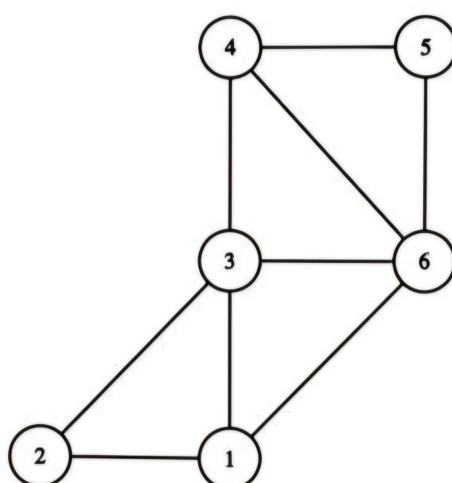
Problem 8: Conflict Management

Karthik has 10 intervals on the number line: $(6, 9), (1, 3), (3, 7), (2, 6), (7, 9), (2, 5), (5, 7), (4, 6), (6, 8), (1, 10)$. He wants to choose as many intervals as possible, such that no point on the number line belongs to more than 2 intervals. Let k be the maximum number of such intervals. Help Karthik by giving any k intervals satisfying the condition. For a few extra points, try to explain why you think your value of k is the highest!

Example: $(1, 3), (2, 4), (3, 5)$ satisfies the condition as no point lies in more than 2 intervals (note that 3 lies in only $(2, 4)$), but $(1, 4), (2, 5), (3, 6)$ doesn't satisfy the condition as 3.5 lies in three intervals.

Bonus Problem! Thinkception

Answer any positive integer between 1 and 100. Let's call your answer x. We will calculate $\bar{x}$, the average of all x values answered by everyone giving this paper right now. If x is between $\frac{\bar{x}}{2} \pm 5\%$, you will get points for this question. Have fun :)

Answer Sheet

Team Name:

Participant 1 Name:

Participant 1 Roll Number:

Participant 1 Contact Number:

Participant 2 Name:

Participant 2 Roll Number:

Participant 2 Contact Number:

1. a)

   b)

2. a)

   b)

3.

4.

5. a)

   b)

6.

7.

8.

Bonus: