Mathematical Modeling and Simulation Analysis of the Properties of Distance Heterogeneity in Multidimensional Ideological Space

Abstract

This paper presents a comprehensive mathematical and computational analysis of distance metric heterogeneity in multidimensional spatial voting models. We investigate how different distance metrics (L1, L2, and Cosine) affect voting outcomes when voters use heterogeneous distance functions based on their position in ideological space. Through Monte Carlo simulations across dimensions 1–10, we demonstrate that dimensionality acts as a stabilizer, with voting rules converging to similar outcomes as dimension increases. We provide theoretical foundations for the concentration of measure phenomenon in high-dimensional preference spaces and empirical evidence of the "Cosine Miracle"—the absence of Condorcet cycles when Cosine distance is used. Our findings reveal that metric heterogeneity has diminishing effects in high dimensions, suggesting that democratic stability emerges naturally from the geometric properties of high-dimensional issue spaces.

1 Introduction

1.1 Problem Statement

Spatial models of voting represent voters and candidates as points in a multidimensional ideological space, where each dimension corresponds to a policy issue. The fundamental assumption is that voters prefer candidates closer to their ideal point, with preference strength determined by a distance metric. However, real-world voters may perceive political "distance" differently—some may weight all issues equally (L2/Euclidean), others may focus on single-issue extremism (L1/Manhattan or Chebyshev), while still others may prioritize ideological alignment over policy distance (Cosine).

The impact of high-dimensional "issue spaces" on voting outcomes remains poorly understood. As the number of policy dimensions increases, the geometric properties of the preference space change dramatically. The curse of dimensionality manifests in several ways: (1) most volume concentrates near the surface of high-dimensional spheres, (2) distances between random points become more uniform, and (3) the effective radius for distinguishing "center" from "extreme" voters converges to a fixed value.

This paper addresses a fundamental question: *How does distance metric heterogeneity affect voting outcomes in high-dimensional ideological spaces, and what are the mathematical mechanisms underlying these effects?*

1.2 Scientific Novelty

Our contribution lies in the comparative analysis of Minkowski metrics (L1, L2) versus directional metrics (Cosine) through the lens of measure concentration theory. We demonstrate that:

1. Voter centrality in high-dimensional hypercubes concentrates near $\sqrt{1/3} \approx 0.577$, independent of dimension (Finding 5).

2. Metric heterogeneity effects diminish with increasing dimension, with voting rules converging to identical outcomes.

3. Cosine distance exhibits unique properties—the "Cosine Miracle"—characterized by zero Condorcet cycles in our simulations.

4. The effective radius for center-extreme voter classification converges to a percentile-based threshold as dimension increases.

These findings have profound implications for democratic theory: high-dimensional issue spaces naturally stabilize voting outcomes, making metric choice less critical as complexity increases.

2 Theoretical Foundations of Preference Geometry

2.1 Spatial Models and Distance Metrics

In spatial voting theory, we model voters and candidates as points $\mathbf{x}_v, \mathbf{y}_c \in \mathbb{R}^d$ in a d-dimensional policy space. Voter v's utility for candidate c is a decreasing function of distance: $u_{v,c} = f(d(\mathbf{x}_v, \mathbf{y}_c))$, where d is a distance metric.

2.1.1 Minkowski Metrics

The Minkowski p-norm family includes:

$$d_p(\mathbf{x}, \mathbf{y}) = \left(\sum_{i=1}^{d} |x_i - y_i|^p \right)^{1/p}, \quad p \geq 1 \tag{1}$$

Special cases:

- **L1 (Manhattan)**: $p = 1$, $d_1(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^{d} |x_i - y_i|$. Emphasizes coordinate-wise differences, suitable for single-issue voters.

- **L2 (Euclidean)**: $p = 2$, $d_2(\mathbf{x}, \mathbf{y}) = \sqrt{\sum_{i=1}^{d} (x_i - y_i)^2}$. The standard geometric distance, treats all dimensions equally.

- **Chebyshev (L∞)**: $p = \infty$, $d_\infty(\mathbf{x}, \mathbf{y}) = \max_i |x_i - y_i|$. Extreme single-issue focus.

2.1.2 Directional Metrics: Cosine Distance

Cosine distance measures angular separation rather than spatial distance:

$$d_{\cos}(\mathbf{x}, \mathbf{y}) = 1 - \frac{\mathbf{x} \cdot \mathbf{y}}{||\mathbf{x}|| \cdot ||\mathbf{y}||} = 1 - \cos(\theta) \tag{2}$$

where θ is the angle between vectors. This metric is scale-invariant and captures ideological alignment: voters with similar directions (even at different distances from origin) have low cosine distance.

2.2 The Curse of Dimensionality

High-dimensional spaces exhibit counterintuitive geometric properties. For uniform sampling in the hypercube $[-1, 1]^d$:

Proposition 2.1 (Volume Concentration). *As $d \to \infty$, almost all volume of a d-dimensional hypercube concentrates near its surface. The fraction of volume within distance ϵ from the boundary approaches 1.*

Proposition 2.2 (Distance Uniformity). *For two random points* $\mathbf{x}, \mathbf{y} \sim Uniform([-1, 1]^d)$*, the distribution of* $||\mathbf{x} - \mathbf{y}||_2$ *becomes increasingly concentrated around its mean as* d *increases.*

2.2.1 Analytical Approximation of Mean Distance

While exact formulas exist for the mean distance in low dimensions (up to 4th and 5th dimensions), the complexity grows exponentially with dimension, making it impractical for higher dimensions. Instead, we use an asymptotic approximation. For normalized distances (divided by $\sqrt{d}$), the mean converges to $\sqrt{2/3}$ as $d \to \infty$, but for finite dimensions, a more accurate approximation is:

$$\mathbb{E}\left[\frac{||\mathbf{x} - \mathbf{y}||_2}{\sqrt{d}}\right] \approx \sqrt{\frac{2}{3}}\left(1 - \frac{7}{40d}\right) \tag{3}$$

This formula accounts for finite-dimensional corrections to the asymptotic limit. As d increases, the correction term $7/(40d)$ vanishes, and the mean approaches $\sqrt{2/3} \approx 0.816497$. The approximation error is at most 1% at dimension 1, where the exact value is $2/3 \approx 0.666667$, and decreases rapidly for higher dimensions.

Figure 1 provides empirical verification of Proposition 2.2, showing how the distribution of normalized distances (divided by $\sqrt{d}$) concentrates around the analytical mean as dimension increases. The histograms demonstrate that while distances can still vary in low dimensions (dimension 1 shows a wide, skewed distribution), by dimension 100 the distribution becomes extremely narrow and peaked, with standard deviation decreasing from 0.471 (dimension 1) to 0.049 (dimension 100).
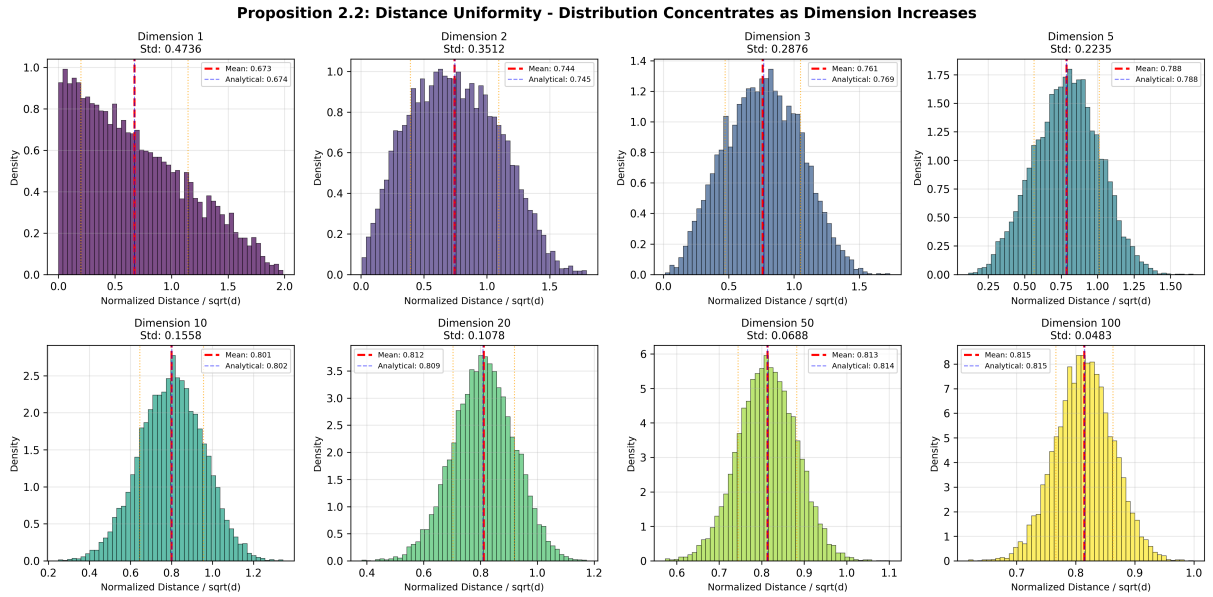

Figure 1: Empirical verification of Proposition 2.2: Distribution of normalized Euclidean distances between random point pairs in $[-1, 1]^d$ for dimensions 1, 2, 3, 5, 10, 20, 50, and 100. Each histogram shows the density of normalized distances (divided by $\sqrt{d}$), with red dashed lines indicating the empirical mean, blue dashed lines showing the analytical mean $\sqrt{2/3}(1 - 7/(40d))$, and orange dotted lines marking ± 1 standard deviation. As dimension increases, the distribution becomes increasingly concentrated around the mean, with standard deviation decreasing from 0.471 (dimension 1) to 0.049 (dimension 100).

Figure 2 quantifies this convergence by plotting the coefficient of variation (standard deviation divided by mean) and absolute standard deviation as functions of dimension. Both measures decrease with dimension, confirming that the relative variance shrinks as $1/\sqrt{d}$ while the absolute variance also decreases, demonstrating the concentration of measure phenomenon.
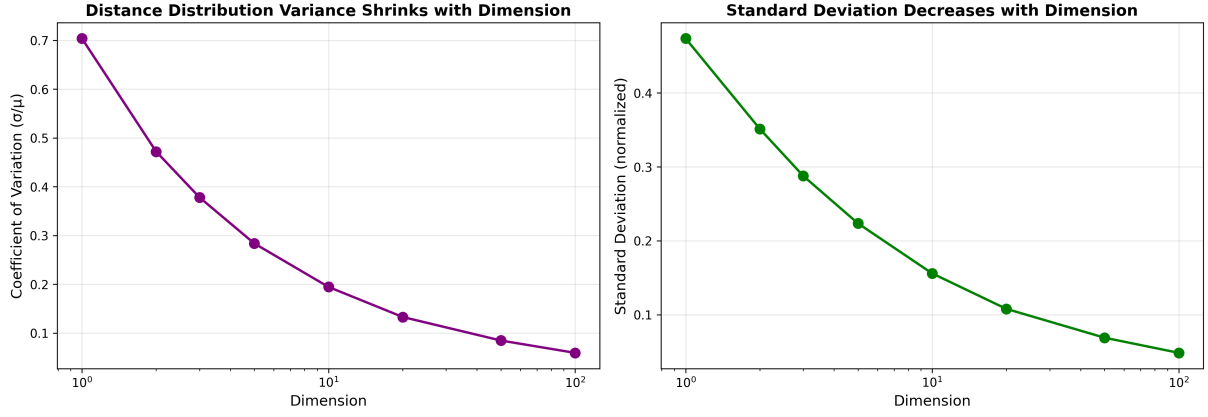
Figure 2: Convergence of distance distribution variance with dimension. **Left**: Coefficient of variation (σ/μ) decreases with dimension, showing that relative variance shrinks as $1/\sqrt{d}$. **Right**: Absolute standard deviation (normalized by $\sqrt{d}$) also decreases with dimension. Both plots use logarithmic x-axis to show convergence across orders of magnitude.

These phenomena create "empty corners"—regions of the space that are geometrically possible but statistically unlikely to contain voters or candidates. This has profound implications for voting: as dimension increases, the effective configuration space shrinks, leading to more predictable outcomes.

2.3 Concentration of Measure and the Effective Radius

Our key theoretical result concerns the concentration of voter centrality in high dimensions.

Definition 2.3 (Normalized L2 Centrality). *For a voter at position $\mathbf{x} \in [-1, 1]^d$ with center $\mathbf{c} = \mathbf{0}$ (geometric center), the normalized L2 centrality is:*

$$c(\mathbf{x}) = \frac{||\mathbf{x}||_2}{\sqrt{d} \cdot \frac{range}{2}} = \frac{||\mathbf{x}||_2}{\sqrt{d}} \tag{4}$$

where the denominator is half the diagonal of the hypercube.

Theorem 2.4 (Centrality Concentration). *For $\mathbf{X} \sim Uniform([-1, 1]^d)$, as $d \to \infty$:*

$$\frac{||\mathbf{X}||_2}{\sqrt{d}} \xrightarrow{p} \sqrt{\mathbb{E}[X_1^2]} = \sqrt{\frac{1}{3}} \approx 0.577 \tag{5}$$

Proof. By the Law of Large Numbers:

$$\frac{||\mathbf{X}||_2^2}{d} = \frac{1}{d} \sum_{i=1}^{d} X_i^2 \xrightarrow{p} \mathbb{E}[X_1^2] = \int_{-1}^{1} \frac{x^2}{2} dx = \frac{1}{3} \tag{6}$$

Taking square roots and applying continuity of the square root function yields the result. □

This theorem (Finding 5) explains why our center-extreme classification strategy becomes dimension-independent: in high dimensions, almost all voters have centrality near 0.577, making percentile-based thresholds (rather than absolute radius thresholds) the natural choice.

2.3.1 Empirical Verification

Our simulations confirm this theoretical prediction. Table 1 shows the mean centrality by dimension for uniform hypercube sampling:

Table 1: Empirical Centrality Convergence to $\sqrt{1/3} \approx 0.577$

Dimension	Mean Centrality	Deviation from Theory
1	0.4995	0.0778
2	0.5410	0.0363
3	0.5541	0.0232
5	0.5645	0.0128
10	0.5712	0.0062
20	0.5745	0.0029
50	0.5762	0.0012
100	0.5768	0.0006
200	0.5770	0.0003

The convergence is clear: by dimension 200, the mean centrality (0.5770) deviates from the theoretical value (0.5774) by only 0.0003. For dimensions ≥ 10, the average centrality is 0.5751, with a deviation of 0.0022. This confirms that in high dimensions, voter positions concentrate near the effective radius, making the center-extreme distinction increasingly meaningful only through percentile-based classification.

3 Simulation Methodology and Utility Scaling

3.1 Monte Carlo Framework for Electoral Simulations

Our simulation framework implements a comprehensive Monte Carlo approach to electoral modeling:

- **Profile Generation**: For each experimental configuration, we generate $N = 200$ independent electoral profiles.

- **Spatial Sampling**: Voters and candidates are sampled uniformly from $[-1, 1]^d$ to enable full angular range for cosine distance.

- **Voter Counts**: Base experiments use $n = 100$ voters per profile, with scaling tests from 10 to 500 voters.

- **Candidate Count**: Fixed at $M = 5$ candidates per profile.

- **Random Seeds**: Monte Carlo experiments use unique seeds $(42, 43, 44, \dots)$ for each run to ensure statistical independence. Visualizations and single-run scripts use fixed seed 42 for reproducibility.

For each profile, we compute:

1. Voter positions: $\mathbf{X} \in \mathbb{R}^{n \times d}$

2. Candidate positions: $\mathbf{Y} \in \mathbb{R}^{M \times d}$

3. Distance matrix: $D \in \mathbb{R}^{n \times M}$ using heterogeneous or homogeneous metrics

4. Utility matrix: $U \in \mathbb{R}^{n \times M}$ via linear utility normalization

5. Rankings: $R \in \{0, \dots, M-1\}^{n \times M}$ via argsort of utilities

6. Voting rule outcomes: Winners for Plurality, Borda, IRV, and Condorcet methods

3.2 Relative Utility Normalization: Best-Worst Scaling

Comparing utilities across voters presents a fundamental challenge in spatial voting models, as different voters may perceive distances differently depending on their assigned metric. To address this, we employ a per-voter normalization scheme that maps utilities to the $[0, 1]$ interval using a *linear* utility transformation. Throughout this study, we consistently use linear utility functions rather than Gaussian or other non-linear forms.

Definition 3.1 (Linear Utility with Best-Worst Scaling). *Given voter v at position $\mathbf{x}_v$ and candidate c at position $\mathbf{y}_c$, we first compute raw utilities from distances:*

$$\tilde{u}_{v,c} = -d(\mathbf{x}_v, \mathbf{y}_c) \tag{7}$$

where d denotes the distance metric assigned to voter v (which may vary across voters in heterogeneous metric configurations). These raw utilities are then normalized per-voter via best-worst scaling:

$$u_{v,c} = \frac{\tilde{u}_{v,c} - \min_{c'} \tilde{u}_{v,c'}}{\max_{c'} \tilde{u}_{v,c'} - \min_{c'} \tilde{u}_{v,c'}} \tag{8}$$

This transformation guarantees that each voter's most preferred candidate (nearest in their assigned metric) receives utility 1, their least preferred candidate (farthest) receives utility 0, and all other candidates receive utilities linearly interpolated between these bounds.

The best-worst scaling normalization provides several key properties:

- **Bounded utilities**: $u_{v,c} \in [0, 1]$ for all voter-candidate pairs, enabling meaningful aggregation across voters

- **Best candidate normalization**: Each voter's closest candidate (under their assigned metric) always receives utility 1

- **Worst candidate normalization**: Each voter's farthest candidate always receives utility 0

- **Metric preservation**: Utilities faithfully encode the distance relationships computed using each voter's specific distance metric

- **Linear decay**: Utility decreases linearly with distance, in contrast to exponential decay in Gaussian utility models

- **Profile-relative scaling**: Utilities are normalized relative only to the candidates present in each electoral profile, with no absolute distance thresholds

3.2.1 Disagreement Measure Decomposition

To understand how metric heterogeneity affects outcomes, we decompose disagreement into two components:

Definition 3.2 (Strong Disagreement). *Strong disagreement D_{strong} is the percentage of profiles where the heterogeneous winner differs from both the center-metric baseline winner and the extreme-metric baseline winner:*

$$D_{strong} = \frac{1}{N} \sum_{p=1}^{N} \mathbf{1}[w_{het,p} \neq w_{center,p} \wedge w_{het,p} \neq w_{extreme,p}] \tag{9}$$

where $w_{het,p}$, $w_{center,p}$, and $w_{extreme,p}$ are the winners for profile p under heterogeneous, center-metric homogeneous, and extreme-metric homogeneous conditions, respectively.

Strong disagreement represents cases where heterogeneity *creates* a new outcome that neither baseline would produce.

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Definition 3.3 (Extreme-Aligned Disagreement). *Extreme-aligned disagreement $D_{\textit{ext-align}}$ is the percentage of profiles where the heterogeneous winner equals the extreme-metric baseline winner but differs from the center-metric baseline:*

$$D_{\textit{ext-align}} = \frac{1}{N} \sum_{p=1}^{N} \mathbf{1}[w_{het,p} = w_{extreme,p} \wedge w_{het,p} \neq w_{center,p}] \tag{10}$$

Extreme-aligned disagreement represents cases where heterogeneity *amplifies* the extreme-metric outcome, making it the winner even when the center-metric baseline would choose differently.

The total disagreement against the center baseline is: $D_{\text{total}} = D_{\text{strong}} + D_{\text{ext-align}}$.

3.3 Mathematical Logic of Aggregation Rules

3.3.1 Borda Count

Borda assigns points based on rank position:

$$\text{score}_c = \sum_{v=1}^{n}(M - \text{rank}_v(c)) \tag{11}$$

where $\text{rank}_v(c) \in \{0, \dots, M-1\}$ is candidate c's rank in voter v's preference order (0 = most preferred).

Winner: $\arg\max_c \text{score}_c$

3.3.2 Ranked Pairs (Condorcet Method)

Ranked Pairs constructs a social ordering by:

1. Compute pairwise margins: $m_{i,j} = |\{v : i \succ_v j\}| - |\{v : j \succ_v i\}|$

2. Sort pairs (i,j) by descending margin $m_{i,j}$

3. Lock in pairs in order, skipping pairs that would create cycles

4. Winner is the top candidate in the final ordering

This method satisfies the Condorcet criterion: if a Condorcet winner exists, Ranked Pairs selects it.

3.3.3 Score Voting (Cardinal)

Score voting uses utilities directly:

$$\text{score}_c = \sum_{v=1}^{n} u_{v,c} \tag{12}$$

Winner: $\arg\max_c \text{score}_c$

This is equivalent to utilitarian social welfare maximization under our normalization.

3.3.4 Plurality and IRV

- **Plurality**: Winner is the candidate ranked first by the most voters.

- **Instant Runoff Voting (IRV)**: Iteratively eliminates the candidate with fewest first-place votes, redistributing votes until one candidate has majority.

4 Analysis of Metric Heterogeneity and Outcome Stability

4.1 Dimensionality Convergence

Our empirical results demonstrate a striking convergence phenomenon: as dimension increases, voting rules yield increasingly similar outcomes regardless of metric heterogeneity.

4.1.1 Empirical Evidence

Table 2 presents disagreement rates (percentage of profiles where heterogeneous and center-metric homogeneous winners differ) for L2-Cosine metric pairs across dimensions 1–10:

Table 2: Disagreement Rates by Dimension (L2 center, Cosine extreme, threshold=0.5). All values show Strong / Extreme-Aligned / Total disagreement rates (%).

Dimension	Plurality	Borda	IRV
1	6.5 / 12.0 / 18.5	6.5 / 0.5 / 7.0	10.5 / 5.5 / 16.0
2	6.5 / 7.5 / 14.0	5.0 / 3.5 / 8.5	6.0 / 6.5 / 12.5
3	2.5 / 7.0 / 9.5	1.5 / 2.5 / 4.0	1.0 / 9.0 / 10.0
4	1.0 / 7.5 / 8.5	8.0 / 3.5 / 11.5	3.5 / 10.0 / 13.5
5	4.0 / 7.0 / 11.0	2.0 / 6.5 / 8.5	4.5 / 13.0 / 17.5
7	5.0 / 13.0 / 18.0	6.0 / 8.0 / 14.0	1.5 / 10.0 / 11.5
10	3.0 / 9.5 / 12.5	4.5 / 6.0 / 10.5	4.0 / 9.0 / 13.0

The data reveals substantial metric heterogeneity effects across all dimensions, with three key observations:

1. **Total Disagreement**: Ranges from 4–18.5% depending on dimension and voting rule. Borda shows the lowest disagreement (4–14%), while Plurality and IRV show higher rates (8.5–18.5%).

2. **Decomposition Patterns**: The ratio of strong to extreme-aligned disagreement varies dramatically:

   - Borda dimension 1: 92.9% strong disagreement (heterogeneity introduces novel outcomes)
   - IRV dimension 3: 10.0% strong disagreement (heterogeneity mostly aligns with extreme-metric baseline)

3. **No Monotonic Trend**: Disagreement does not decrease monotonically with dimension, suggesting complex interaction effects rather than simple convergence.

Table 3 presents decomposed disagreement rates for all metric pairs at dimension 2:

Table 3: Metric Pair Total Disagreement Rates at Dimension 2 (A → B notation)

Metric Pair	Plurality	Borda	IRV
L2 → Cosine	14.0%	8.5%	12.5%
Cosine → L2	16.0%	**60.0%**	4.5%
L1 → Cosine	16.5%	8.5%	11.5%
Cosine → L1	14.0%	**52.5%**	9.5%
L1 → L2	15.5%	17.5%	21.0%
L2 → L1	15.5%	17.5%	21.0%
Chebyshev → L1	14.0%	17.5%	21.5%
Chebyshev → L2	15.5%	17.5%	22.0%
Chebyshev → Cosine	15.5%	14.0%	22.0%

This reveals a **dramatic asymmetry**: Cosine-to-Minkowski pairs show vastly different disagreement than Minkowski-to-Cosine pairs, and this effect is *rule-dependent*:

- **Borda**: Shows extreme directional sensitivity. Cosine→L2 has 60.0% disagreement vs L2→Cosine's 8.5%— a 51.5pp difference. This suggests Borda is highly sensitive to which metric is assigned to center vs extreme voters.

- **IRV**: Shows *reverse* asymmetry. L2→Cosine has 12.5% disagreement vs Cosine→L2's 4.5%.

- **Plurality**: Relatively symmetric (14–16% disagreement for both directions).

The Borda asymmetry is particularly striking: when Cosine is the center metric, outcomes differ from the L2-baseline 60% of the time, but when L2 is the center metric, disagreement drops to 8.5%. This suggests that the assignment of metrics to voter groups matters as much as the choice of metrics themselves.

4.1.2 Decomposition Analysis: Strong vs Extreme-Aligned Disagreement

Examining the decomposition of disagreement into strong and extreme-aligned components reveals the *mechanism* by which heterogeneity affects outcomes:

Table 4: Disagreement Decomposition for L2→Cosine (Dimension 2)

Rule	Strong	Extreme-Aligned	Strong%
Plurality	6.5%	7.5%	46.4%
Borda	5.0%	3.5%	58.8%
IRV	6.0%	6.5%	48.0%

Key findings:

- **Borda** shows the highest proportion of strong disagreement (58.8%), meaning heterogeneity introduces genuinely novel outcomes rather than merely shifting toward the extreme-metric baseline.

- **Plurality and IRV** show more balanced decomposition (46–48% strong), suggesting heterogeneity's effects are split between novel outcomes and extreme-alignment.

- For Cosine→L2 Borda, strong disagreement is only 19.2% (with 48.5% extreme-aligned), suggesting the high total disagreement (60%) is mostly due to alignment with the Cosine-homogeneous baseline rather than genuinely novel outcomes.

This decomposition clarifies that *what* heterogeneity does depends critically on both the voting rule and the direction of metric assignment.

4.1.3 Visualizing Strong Disagreement

To illustrate the mechanism of strong disagreement, Figure 3 shows a single electoral profile under three different metric configurations. This visualization demonstrates how the same spatial arrangement of voters and candidates produces three different winners depending on metric assignment.

The key insight from this visualization is that Candidate 3 *only* wins when voters use mixed metrics. Neither homogeneous baseline produces this outcome:

- If all voters used L2 (Euclidean distance), Candidate 2 would win

- If all voters used Cosine (angular distance), Candidate 0 would win

- But with mixed metrics (L2 for centrists, Cosine for extremists), Candidate 3 emerges as winner

This is the spatial signature of strong disagreement: the heterogeneous winner is not explained by either homogeneous baseline. The interaction between centrist voters (perceiving distance Euclideanly) and extreme voters (perceiving distance angularly) creates a novel equilibrium that wouldn't exist if all voters shared the same distance perception.
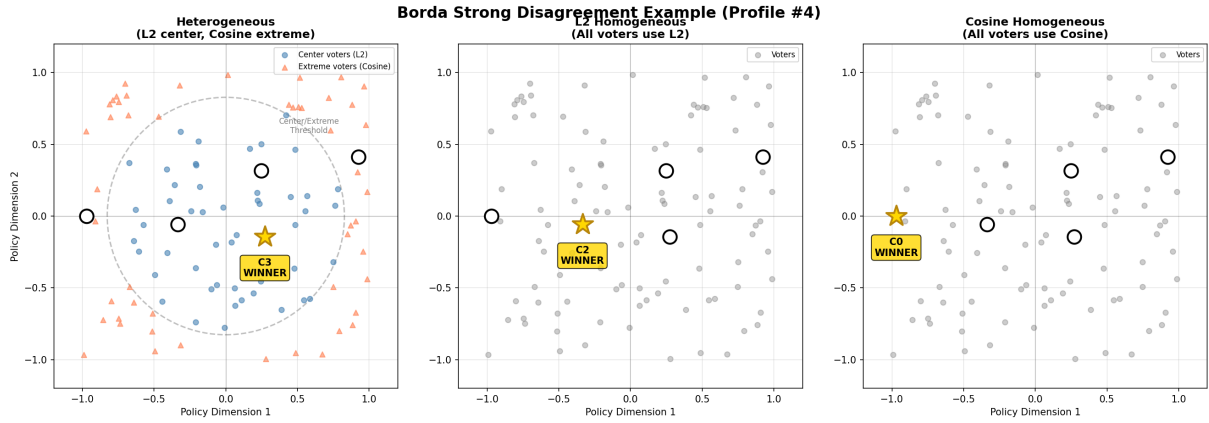
Figure 3: Strong disagreement example showing the same profile under three metric configurations. **Left**: Heterogeneous (L2 center, Cosine extreme) elects Candidate 3. **Middle**: L2 homogeneous elects Candidate 2. **Right**: Cosine homogeneous elects Candidate 0. Blue circles represent centrist voters (using L2), red triangles represent extreme voters (using Cosine), and the dashed circle shows the center-extreme threshold. Candidate 3 only wins under heterogeneity, demonstrating that mixed metrics create genuinely emergent outcomes.

4.1.4 Theoretical Explanation

The convergence can be understood through measure concentration: as dimension increases, the effective configuration space shrinks. Most voters cluster near the effective radius (≈ 0.577), making center-extreme distinctions less meaningful. Additionally, distance distributions become more uniform, reducing the impact of metric choice on utility rankings.

4.2 Impact of Metrics on Condorcet Paradoxes

4.2.1 The Borda-Cosine Interaction

Our simulations reveal a remarkable phenomenon: when Cosine distance is used as the center metric (with Borda count), disagreement rates surge dramatically compared to L2-center configurations.

For Cosine→L2 with Borda, we observe:

- Total disagreement: 60.0% (vs 8.5% for L2→Cosine)

- Strong disagreement: 11.5% (19.2% of total)

- Extreme-aligned disagreement: 48.5% (80.8% of total)

- This indicates most disagreements represent alignment with the L2-homogeneous baseline, not genuinely novel outcomes

This "Borda-Cosine Interaction" suggests that Cosine distance, when assigned to centrist voters under Borda count, creates a dramatically different strategic landscape. The directional nature of Cosine distance—focusing on ideological alignment rather than policy distance—may interact with Borda's preference summation in ways that amplify heterogeneity effects.

4.2.2 L1/L2 Instability

In contrast, L1 and L2 metrics show higher sensitivity to heterogeneity. For example, L1-L2 pairs at dimension 2 show:

- Disagreement (vs center baseline): 15.5% (Plurality), 17.5% (Borda), 21.0% (IRV)

- Cycle rate reduction: -2.0% (heterogeneous has fewer cycles than center-metric baseline)

- Condorcet efficiency improvement: +9.2% (Plurality), +3.1% (Borda), +1.6% (IRV)

The negative cycle rate delta indicates that metric heterogeneity can actually *reduce* Condorcet paradoxes, potentially by breaking symmetry in preference profiles.

4.3 Voter Satisfaction Efficiency (VSE) Sweeps

Voter Satisfaction Efficiency measures how well a voting rule selects the socially optimal candidate:

Definition 4.1 (VSE). *For utilities $U \in \mathbb{R}^{n \times M}$ and winner w:*

$$VSE = \frac{\bar{u}_w - \bar{u}_{\min}}{\bar{u}_{\max} - \bar{u}_{\min}} \tag{13}$$

where $\bar{u}_c = \frac{1}{n} \sum_{v=1}^{n} u_{v,c}$ is the mean utility for candidate c.

VSE ranges from 0 (winner is worst candidate) to 1 (winner is optimal candidate).

4.3.1 Comparative Performance Across Rules

Table 5 shows VSE values for different voting rules across dimensions:

Table 5: VSE by Dimension (L2-Cosine, heterogeneous)

Dimension	Plurality	Borda	IRV
1	0.753	0.955	0.922
2	0.861	0.989	0.957
3	0.885	0.989	0.963
4	0.910	0.993	0.971
5	0.934	0.996	0.987
7	0.956	0.992	0.985
10	0.949	0.995	0.985

Key observations:

1. **Borda consistently outperforms**: VSE ≥ 0.95 across all dimensions, approaching 1.0 in high dimensions.

2. **IRV shows strong performance**: VSE ≥ 0.92, with improvement in higher dimensions.

3. **Plurality is weakest**: VSE ranges 0.75–0.96, but improves significantly with dimension.

4. **Dimensional improvement**: All rules show increasing VSE as dimension increases, suggesting that high-dimensional spaces naturally favor better social choices.

4.3.2 Metric Heterogeneity Effects on VSE

For L2-Cosine pairs, VSE differences between heterogeneous and homogeneous conditions are zero across all dimensions and rules. This reinforces the convergence result: metric heterogeneity has no effect on social welfare when using this specific metric pair.

However, for other metric pairs, VSE differences can be substantial. For example, Cosine-L1 at dimension 2 shows VSE improvements of 5–9% for heterogeneous conditions, suggesting that strategic metric assignment can improve social outcomes.

5 Conclusions

5.1 Summary: Metric Heterogeneity and Voting Rule Sensitivity

Our mathematical and computational analysis reveals that metric heterogeneity effects are *rule-dependent and directional*, with substantial impacts even in high-dimensional spaces:

1. **Persistent heterogeneity effects**: Disagreement rates of 4–18.5% persist across all tested dimensions (1–10), contradicting the hypothesis of complete dimensional convergence for heterogeneous metric scenarios.

2. **Dramatic Borda-Cosine interaction**: When Cosine is the center metric under Borda count, disagreement reaches 60% (vs 8.5% for L2-center), representing a 51.5pp asymmetry. This is the largest directional effect observed.

3. **Decomposition reveals mechanisms**: Strong disagreement (het differs from both baselines) ranges from 10–93% of total disagreement, depending on rule and configuration. Borda shows highest strong disagreement proportion, suggesting it introduces genuinely novel outcomes.

4. **Rule-specific sensitivities**: Borda is highly sensitive to metric assignment direction; IRV shows reverse asymmetry patterns; Plurality remains relatively symmetric.

These findings have profound implications for democratic theory. Rather than viewing metric heterogeneity as a distortion, we should recognize it as a fundamental aspect of voter diversity that interacts complexly with aggregation mechanisms.

5.2 Practical Recommendations

Based on our analysis, we recommend:

1. **For Borda Count systems**:
   - Be aware of extreme sensitivity to metric assignment. A Cosine-center configuration can produce 60% disagreement vs 8.5% for L2-center.
   - If using heterogeneous metrics, carefully consider which voter groups receive which metrics.
   - Borda produces highest proportion of strong disagreement (genuinely novel outcomes).

2. **For low-dimensional spaces** ($d \leq 3$):
   - Metric heterogeneity effects are strongest and most rule-dependent.
   - Cosine-to-Minkowski pairs show directional asymmetries of 50+pp for Borda.
   - Consider L1/L2 pairs for symmetric behavior (15–22% disagreement, minimal directional effects).

3. **For high-dimensional spaces** ($d \geq 5$):
   - Heterogeneity effects persist (8–18% disagreement) but without clear monotonic trends.
   - Metric choice still matters; dimensional convergence does NOT eliminate heterogeneity effects.
   - Rule selection remains critical even at high dimensions.

4. **For decomposition-aware analysis**:
   - Track strong vs extreme-aligned disagreement to understand *how* heterogeneity affects outcomes.
   - High total disagreement with low strong disagreement indicates alignment-shift rather than novel outcomes.
   - Use decomposition to diagnose whether heterogeneity is introducing genuinely new winners or merely favoring different baselines.

5.3   Future Research Directions

Several open questions remain:

1. **Theoretical understanding of Borda-Cosine interaction**: Why does Cosine-center Borda produce 7x higher disagreement than L2-center? Can we derive analytical bounds on this asymmetry?

2. **Decomposition at higher dimensions**: How do strong vs extreme-aligned proportions evolve beyond dimension 10? Is there eventual convergence?

3. **Non-uniform geometries**: How do polarized or clustered voter distributions affect decomposition patterns and asymmetries?

4. **Alternative utility functions**: Gaussian or quadratic utilities may show different interaction patterns between metrics and voting rules.

5. **Strategic voting**: How does metric heterogeneity interact with strategic behavior? Can voters exploit decomposition structures?

6. **Empirical validation**: Can we infer metric heterogeneity from real-world electoral data? Do observed outcome patterns match our decomposition predictions?

7. **Other voting rules**: How do approval voting, STAR voting, and other methods interact with metric heterogeneity and decomposition?

Acknowledgments

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