Code Jam 2022 - Round 1C

Analysis: Squary

The multinomial expansion for the power of 2 is the key to solving this problem. The expansion of the square of the sum of elements $X_1, X_2, \ldots, X_n$ in the list X looks like:

$$\begin{aligned}\text{square of sum} &= (X_1 + X_2 + X_3 + \ldots + X_N)^2 \\ &= X_1^2 + X_2^2 + X_3^2 + \ldots + X_N^2 + 2 \cdot X_1 \cdot X_2 + 2 \cdot X_2 \cdot X_3 + 2 \cdot X_1 \cdot X_3 + \ldots + 2 \cdot X_{N-1} \cdot X_N \\ &= \text{sum of squares} + 2 \cdot \text{sum of pairwise products}\end{aligned}$$

Let $S(X)$ be the sum of elements, $SQ(X)$ be the sum of squares of elements, and $SP(X)$ be the sum of all pairwise products of elements of the list X. We can now rewrite the above equation as:

$$S(X)^2 = SQ(X) + 2 \cdot SP(X)$$

We can also observe the following about how the above values change when an additional element n is added to the list X:

$$\begin{aligned}S(X + [n]) &= S(X) + n \\ SQ(X + [n]) &= SQ(X) + n^2 \\ SP(X + [n]) &= SP(X) + n \cdot S(X)\end{aligned}$$

Our task is to achieve $S(E')^2 = SQ(E')$, where E' is the extended list that we get by adding extra elements to $\mathbf{E}$. In other words, we want to make $SP(E') = 0$.

Test Set 1: $\mathbf{K} = 1$

If we are allowed only a single addition, we must choose an element n such that $SP(\mathbf{E} + [n]) = 0$.

$$\begin{aligned} SP(\mathbf{E} + [n]) &= 0 \\ \implies SP(\mathbf{E}) + n \cdot S(\mathbf{E}) &= 0 \\ \implies n \cdot S(\mathbf{E}) &= -SP(\mathbf{E})\end{aligned}$$

If $S(\mathbf{E}) \neq 0$, we can get a squary list whenever $-SP(\mathbf{E})/S(\mathbf{E})$ is an integer, which happens if and only if $S(\mathbf{E})$ divides $SP(\mathbf{E})$. In that case, $-SP(\mathbf{E})/S(\mathbf{E})$ is our answer.

If $S(\mathbf{E}) = 0$, then $S(\mathbf{E} + [n]) = n$. Since we want $S(\mathbf{E} + [n])^2 = SQ(\mathbf{E} + [n])$, we need $SQ(\mathbf{E} + [n]) = n^2$. This is possible only if $SQ(\mathbf{E}) = 0$, that is, if all elements in $\mathbf{E}$ are zeros. In this case, we can choose any value as our answer. But if any element in $\mathbf{E}$ is not zero, it is impossible to get a squary list with only one addition.

Test Set 2: $\mathbf{K} > 1$

At first, the search space might seem hopelessly broad here. But we can observe (or surmise and then confirm) that it is always possible to get a squary list by adding only two elements:

- $n_1 = 1 - S(\mathbf{E})$
- $n_2 = -SP(\mathbf{E} + [n_1])$

After adding n_1, we have

$$S(\mathbf{E} + [n_1]) = 1$$

After adding n_2, we have

$$\begin{aligned}SP(\mathbf{E} + [n_1, n_2]) &= SP(\mathbf{E} + [n_1]) + n_2 \cdot S(\mathbf{E} + [n_1]) \\ &= SP(\mathbf{E} + [n_1]) + (-SP(\mathbf{E} + [n_1])) \cdot 1 \\ &= 0\end{aligned}$$

Thus, the two numbers satisfy the condition $SP(E') = 0$. We can also see that since the numbers in the original list are each of magnitude no greater than 10^3, $|n_1| \le 10^6 + 1$, and $|n_2| \le 2 \cdot 10^{12}$, both well within the limits.