Analysis: Pixelated Circle

Let C and C_w be the set of pixels colored by `draw_circle_filled`$(\mathbf{R})$ and `draw_circle_filled_wrong`$(\mathbf{R})$, the number of pixels that have different colors in these pictures would be the cardinality (size) of the symmetric difference of C and C_w, that is: $|C \Delta C_w| = |(C \setminus C_w) \cup (C_w \setminus C)|$.

Test Set 1

For test set 1, $\mathbf{R}$ is small enough to build the sets of colored pixels from `draw_circle_filled`$(\mathbf{R})$ and `draw_circle_filled_wrong`$(\mathbf{R})$ with hash set of tuples. By implementing the pseudocode in the problem statement, the time complexity would be $O(\mathbf{R}^2)$ to get all colored pixels and $O(\mathbf{R}^2)$ to compute the symmetric difference of the two sets.

Test Set 2

The key observation for optimizing the solution is that for any $\mathbf{R}$, every pixel colored by `draw_circle_filled_wrong`$(\mathbf{R})$ is also colored by `draw_circle_filled`$(\mathbf{R})$, that is $C_w \subseteq C$. Therefore, we can simplify the size of symmetric difference to:

$$\begin{aligned} |C \Delta C_w| &= |(C \setminus C_w) \cup (C_w \setminus C)| \\ &= |(C \setminus C_w) \cup \emptyset| \\ &= |C| - |C \cap C_w| \\ &= |C| - |C_w| \end{aligned}$$

which means we can count the number of pixels colored by `draw_circle_filled`$(\mathbf{R})$ and `draw_circle_filled_wrong`$(\mathbf{R})$ separately, and the answer would be the difference between these two numbers. The proof of this observation is given at the end of this analysis.

Count pixels colored by **`draw_circle_filled`**$(\mathbf{R})$

To get the number of pixels colored by `draw_circle_filled`$(\mathbf{R})$, we need to iterate through all possible values of x and find y_{min} and y_{max} for each x which satisfy $\mathbf{round}(\sqrt{x^2+y^2}) \leq \mathbf{R}$ for all $y_{min} \leq y \leq y_{max}$. A solution for them is $y_{max} = \mathbf{floor}(\sqrt{\mathbf{R}+0.5}^2 - x^2)$ and $y_{min} = -y_{max}$. Therefore, we can get the number of colored pixels with a for-loop for the following equation:

$$|C| = \sum_{x=-\mathbf{R}}^{\mathbf{R}} \mathbf{floor}(\sqrt{(\mathbf{R}+0.5)^2 - x^2}) \times 2 + 1$$

Time complexity: $O(\mathbf{R})$

Count pixels colored by **`draw_circle_filled_wrong`**$(\mathbf{R})$

`draw_circle_filled_wrong`$(\mathbf{R})$ is composed of `draw_circle_perimeter`(r) calls with r from 0 to $\mathbf{R}$. Notice that pixels colored by `draw_circle_perimeter`(r_1) and `draw_circle_perimeter`(r_2) never overlap if $r_1 \neq r_2$. Based on this observation, we can count the number of pixels colored by each `draw_circle_perimeter`(r) separately and sum them up to get the total number of colored pixels. The proof of this observation is given at the end of this analysis.

Looking into function `draw_circle_perimeter(r)`, we can break the colored pixels into 4 quadrants and count them separately. Since the colored pattern is symmetric to both x-axis and y-axis, we just need to count the pixels in Quadrant 1 (Q1), and the total count excluding the origin pixel would be that number times 4, and plus 1 to include the origin pixel.

For $r \geq 1$, the colored pixels in Q1 are symmetric to the line $x = y$, and there are exactly x_t colored pixels between y-axis and (x_t, y_t), the closest point to line $x = y$ (above or on the line, $x_t \geq y_t$). Since $x = y$ is at 45° to the x-axis, the integer x_t would be either $\mathbf{ceil}(r/\mathbf{cos}(45^\circ))$ or $\mathbf{floor}(r/\mathbf{cos}(45^\circ))$. We can compute the corresponding $y_t = \mathbf{round}(\sqrt{r^2 - x_t^2})$ and choose the closer one above or on the line $x = y$. Afterwards, the number of colored pixels in Q1 including x-axis would be $2 \times x_t + 1$, and minus 1 if (x_t, y_t) lies on the line $x = y$ since it is not mirrored in this case.

Time complexity: $O(1)$ for counting pixels colored by `draw_circle_perimeter(r)`, and $O(\mathbf{R})$ for all colored pixels.

Proof: $C_w \subseteq C$

For every positive $\mathbf{R}$, r and x such that $0 \leq r \leq \mathbf{R}$ and $-r \leq x \leq r$, we want to prove that the following inequality always satisfies:

$$\begin{aligned}
& \mathbf{round}(\sqrt{x^2 + \mathbf{round}(\sqrt{r^2 - x^2})^2}) \leq r \\
\iff & \sqrt{x^2 + \mathbf{round}(\sqrt{r^2 - x^2})^2} - 0.5 \leq r \\
\iff & \sqrt{x^2 + \mathbf{round}(\sqrt{r^2 - x^2})^2} \leq r + 0.5 \\
\iff & x^2 + \mathbf{round}(\sqrt{r^2 - x^2})^2 \leq r^2 + r + 0.25 \\
\iff & x^2 + (\sqrt{r^2 - x^2} + 0.5)^2 \leq r^2 + r + 0.25 \\
\iff & r^2 + \sqrt{r^2 - x^2} + 0.25 \leq r^2 + r + 0.25
\end{aligned}$$

which always holds since $\sqrt{r^2 - x^2} \leq \sqrt{r^2} = r$ when $|x| \leq r$ and $r \geq 0$.

Secondly, $y = \mathbf{round}(\sqrt{r^2 - x^2}) \leq \mathbf{round}(\sqrt{r^2}) \leq \mathbf{R}$. Therefore $-\mathbf{R} \leq y \leq \mathbf{R}$ always holds.

The proof above shows that pixels colored by `draw_circle_filled_wrong`(r) with $0 \leq r \leq \mathbf{R}$ also satisfy the coloring condition in `draw_circle_filled`($\mathbf{R}$), which implies $C_w \subseteq C$.

Proof: `draw_circle_perimeter(`r_1`)` and `draw_circle_perimeter(`r_2`)` never overlap

For the first coloring statement in `draw_circle_perimeter(r)`, we want to prove that given a fixed x, the inequality $y_1 = \mathbf{round}(\sqrt{r_1^2 - x^2}) \neq y_2 = \mathbf{round}(\sqrt{r_2^2 - x^2})$ for any pair of integers r_1 and r_2 such that $r_1 > r_2 \geq 0$ and $|x| \leq r_2$ is always true:

$$
\begin{aligned}
& |\sqrt{r_1^2 - x^2} - \sqrt{r_2^2 - x^2}| \\
= & \sqrt{r_1^2 - x^2} - \sqrt{r_2^2 - x^2} && r_1 > r_2 \\
\geq & \sqrt{r_1^2 - x^2} - \sqrt{(r_1 - 1)^2 - x^2} && r_1 \texttt{ and } r_2 \texttt{ are integers} \\
\geq & (\sqrt{r_1^2} - \sqrt{x^2}) - (\sqrt{(r_1 - 1)^2} - \sqrt{x^2}) && \texttt{for any } |x| \leq r_1 - 1 \\
= & r_1 - x - (r_1 - 1) + x \\
= & 1
\end{aligned}
$$

Based on the proof above, we can further get:

$$
\begin{aligned}
& \mathbf{round}(\sqrt{r_1^2 - x^2}) - \mathbf{round}(\sqrt{r_2^2 - x^2}) \\
\geq & \mathbf{round}(\sqrt{r_1^2 - x^2}) - \mathbf{round}(\sqrt{(r_1 - 1)^2 - x^2}) \\
\geq & \mathbf{round}(\sqrt{r_1^2 - x^2}) - \mathbf{round}(\sqrt{r_1^2 - x^2} - 1) \\
\geq & 1
\end{aligned}
$$

Therefore $y_1 \neq y_2$ always holds for all $r_1 > r_2 \geq 0$ given a fixed x such that $|x| \leq r_2$. For the cases that $r_2 < |x| \leq r_1$, the pixels will nevery satisfy the coloring condition in `draw_circle_perimeter(`r_2`)`.

We can further extend the proof above for the 2nd, 3rd, and 4th coloring statement in `draw_circle_perimeter(`r`)` and prove that pixels colored by `draw_circle_perimeter(`r_1`)` and `draw_circle_perimeter(`r_2`)` never overlap.