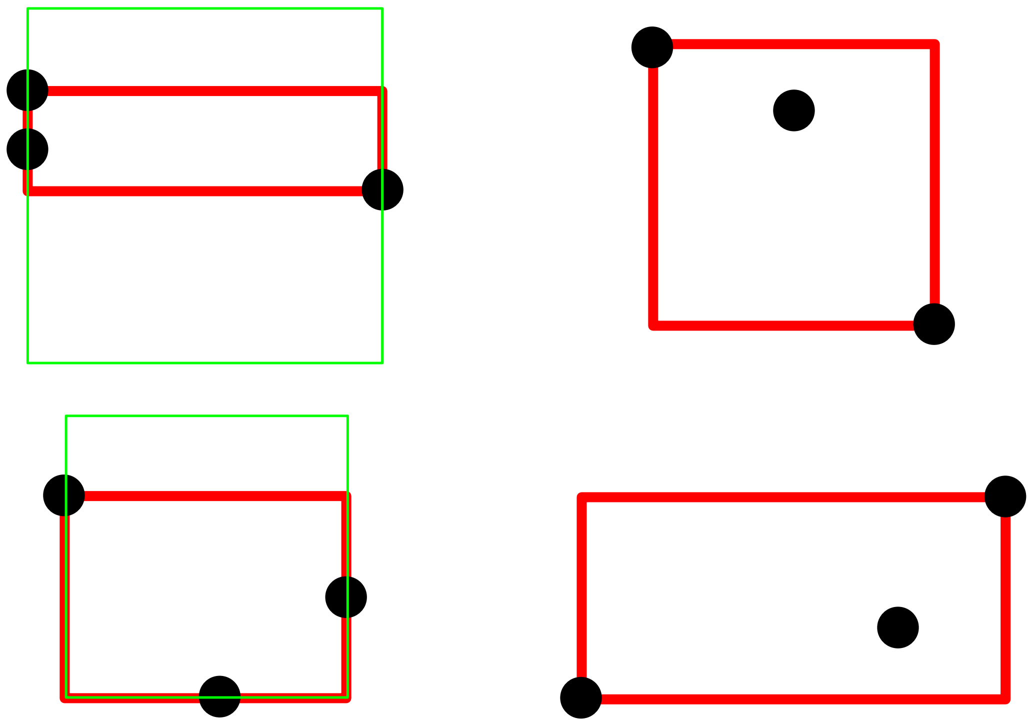
Kick Start 2018 - Round B

Analysis: King's Circle

Let's set the story aside and consider the underlying geometry problem: given **N** points in the 2D plane, how many triplets of points have an axis-aligned square that goes through them?

It is natural to first tackle this subproblem: given three points in the 2D plane, is there an axis-aligned square that goes through those three points?

After trying different sets of points, it is not too difficult to come up with the key restriction. Consider the axis-aligned bounding box (that is, the rectangle which contains all the points which is as small as possible) for the three points. An axis-aligned square that goes through the three points exists if and only if all three points lie somewhere on the edges of the bounding box (see below).



Square exists

Square does not exist

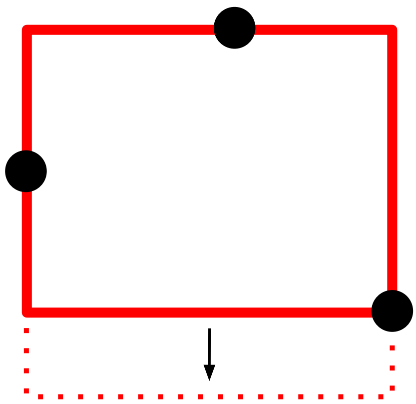
This idea is simple enough to start working with even without a thorough proof (which we will get back to later).

A bounding box around two points will always have those points at opposite corners. The bounding box might have zero area (if both points are horizontally or vertically aligned), but that's fine. This gives us enough information to solve the Small dataset: If two points are chosen as the corners of a bounding box, then these two points will form a valid triplet with any point that is **not** strictly inside the bounding box. That is, any point outside the bounding box or along its edges will form a valid triplet with these two points. This gives us an O($\mathbf{N}^2$) solution: For every pair of points, count the number of points that are strictly inside their bounding box, which can be done in O(1) using a 2D cumulative sum array, taking advantage of the small bounds on the coordinates in the small test set. Since every invalid triplet is counted exactly once (for any invalid triplet, exactly one of the points will be inside the bounding box defined by the other two points).

For the Large dataset, we need to turn around our thinking. Instead of fixing the bounding box and counting the number of points inside it, we will fix a point as the "inside" point, and count how many different bounding boxes contain it. Let's say we fix an inside point. Let **A** be the set of points above and to the right of our fixed point, and **B** be points below and to the left. Then there are **A** × **B** bounding boxes that contain our fixed inside point: every point in **A** forms a bounding box containing our fixed inside point with any of the points in **B**. The same can be said for the points above and to the right combined with those below and to the left.

This gives us an O(**N** log **N**) solution: for each point, count the number of bounding boxes that would contain it by finding the number of points above and to the left, above and to the right, below and to the left and below and to the right, and multiplying accordingly. Counting these points can be done ine O(log **N**) using a linear sweep with a range tree or with a self balancing binary search tree (among other ways).

To wrap things up, we need to prove our original assertion: An axis-aligned square that goes through the three points exists if and only if all three points lie somewhere on the edges of the bounding box. The negative case is easy. If there is a point not on the edges of the bounding box, then there is no axis-aligned *rectangle* that goes through the three points, let alone a square. The affirmative case is more difficult. We will start with the bounding box and morph it into a square. If the bounding box is already a square, then we are done. Otherwise, one pair of parallel sides is longer. If either of these sides does not have a point on it *excluding the corners*, then that side can be extended until the shape is a square (see below).



So what do we do if both long sides have a point on them (not at a corner)? This is actually impossible. For a contradiction, suppose that both long sides do have a point on them, not at a corner. It doesn't matter where the third point is, there is one side with no points on it at all (not even at the corners)! This means the bounding box can be shrunk, which contradicts our definition of the bounding box being as small as possible.