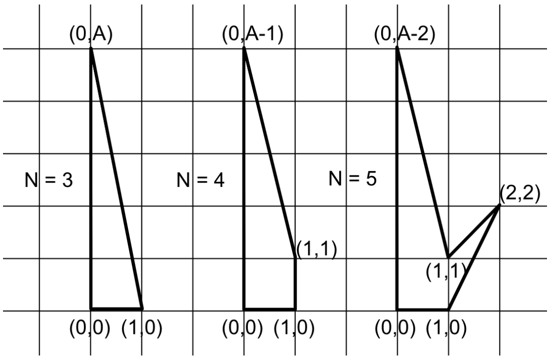
Kick Start 2021 - Round G

Analysis: Simple Polygon

According to Pick's theorem, the area of a simple polygon having integer vertex coordinates is $Area = i + \frac{b}{2} - 1$, where i is the number of integer points inside the polygon and b is the number of integer points on its border. If we double this area, as in our problem statement, it follows that $\mathbf{A} = 2 \times Area = 2i + b - 2$. Since $i \geq 0$ and $b \geq \mathbf{N}$, a lower bound on the 'doubled-area' $\mathbf{A}$ of a polygon with $\mathbf{N}$ vertices is $\mathbf{A} = 2i + b - 2 \geq \mathbf{N} - 2$. Therefore, if $\mathbf{A} < \mathbf{N} - 2$, the answer is `IMPOSSIBLE`. In what follows, we will show that this is a tight lower bound by constructing an $\mathbf{N}$ vertex simple polygon having a 'doubled-area' $\mathbf{A}$ for any given $\mathbf{A} \geq \mathbf{N} - 2$.

Test Set 1

There are many ways to construct the necessary polygons. The following drawing shows possible constructions for $3 \leq \mathbf{N} \leq 5$.

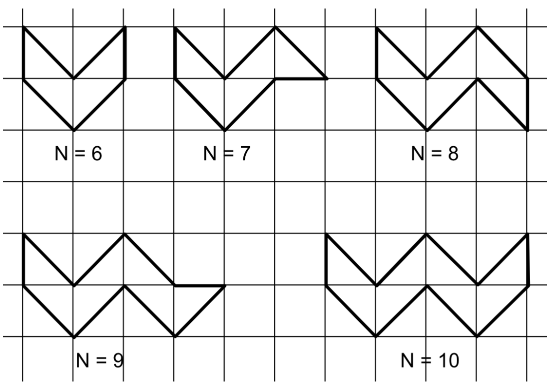


These polygons have no internal integer points, therefore, by Pick's theorem, their 'doubled-area' is $b - 2$. For example, for $\mathbf{N} = 5$, we can verify that $b = \mathbf{A} + 2$ by counting the integer points on the border. Therefore, the 'doubled-area' is $b - 2 = \mathbf{A} + 2 - 2 = \mathbf{A}$, which validates our construction. Similarly, it can be verified that we have achieved the desired area for $\mathbf{N} = 3$ and $\mathbf{N} = 4$ as well.

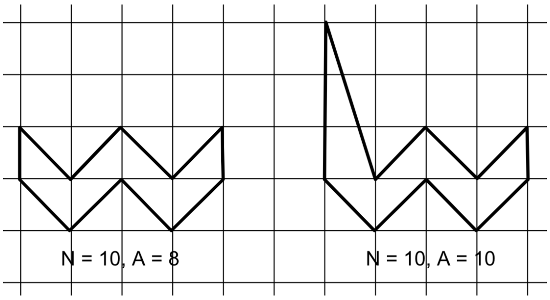
The time complexity of the construction is $O(1)$.

Test Set 2

For $\mathbf{N} > 5$, the construction is a little more involved. Let us start with the base case, where the 'doubled-area' of the polygon is the smallest possible, namely, $\mathbf{N} - 2$. The following drawing illustrates the construction for $6 \leq \mathbf{N} \leq 10$, but it can be generalized for arbitrary $\mathbf{N}$ by extending the zig-zag shape to the right.



The base polygon has $\mathbf{N}$ integer points on the border and no internal integer points, therefore, its 'doubled-area' is $\mathbf{N}-2$. If $\mathbf{A}>\mathbf{N}-2$, we just need to introduce $\mathbf{A}-\mathbf{N}+2$ more points on the border by say, lifting the top-left vertex up $\mathbf{A}-\mathbf{N}+2$ units as shown in the following drawing for $\mathbf{N}=10$ and $\mathbf{A}=10$.



The time complexity of the construction is $O(\mathbf{N})$.