Code Jam 2008 - EMEA Semifinal

Analysis: Scaled Triangle

In this problem we are given a triangle and another triangle obtained by applying an affine transformation on the first one -- translation, rotation and scaling. We are asked to find a fixed point for this transformation.

This is actually a particular case of "Bannach's fixed point theorem" which guarantees the existence and uniqueness of fixed points of certain self maps of metric spaces, though knowing the theorem was not a requirement.

To solve this problem, we need to find the parameters of the transformation.

Such transformations are familiar to anyone who has ever played with computer graphics. To view the transformation as a linear operator, it is convenient to use homogeneous coordinates, where our plane is embedded as the plane z=1 in 3D space. i.e., consider each point (x, y) as (x, y, 1). (*Note*: As usual, all the vectors corresponding to points are considered column vectors, in spite of the horizontal way we write them here.) In this setting, rotating a point by an angle alpha around the point (0, 0) corresponds to multiplying the vector v = (x, y, 1) by the matrix R = [[cos alpha sin alpha 0] [-sin alpha cos alpha 0] [0 0 1]]. Translating a point x, y by (dx, dy) corresponds to multiplying v by T = [[1 0 dx] [0 1 dy] [0 0 1]], and a scaling transform centered at 0 corresponds to multiplying a point v = (x, y, 1) by the matrix S = [[S 0 0] [0 S 0] [0 0 1]]. The total transform looks like this

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v' = T R S v = M v.
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Note that above we focused on the effect of the transformation on the plane z=1. The interested reader can verify that it maps every horizontal plane $z=z_0$ to itself.

To get the matrix **M**, we may solve separately for the matrices T, R, and S. There is an easier way. From the input constraints, we know that point A is mapped to A', B to B' and C to C'. View each point as a vector in the plane z=1, so that A, B, C are linearly independent. So the 3 by 3 matrix [A B C] is invertible. Therefore,

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M [A B C] = [A' B' C']
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has a unique solution for **M**.

From here, there are still two solutions to our problem. One observation is that if we apply the transformation and we have a point that doesn't change, we can apply it again on the resulting triangle and the point will remain the same. So we apply this transformation until the triangle becomes very small. The fixed point will still be inside the triangle so we can stop when the side lengths of the current triangle get smaller than the needed precision.

The more algebraic solution looks at the equation again. We know there is a unique point v = (x, y, 1) such that **M** v = v. From here we know that (i) 1 must be an eigen-value for the matrix M; (ii) the space of the eigen-vectors corresponding to 1 must be one-dimensional.
So we can just solve the equation [M - I] v = 0, and find the intersection of the solution space (a line) and the plane z=1.

More information

[Banach fixed point theorem] - [Homogeneous coordinate] - [Eigen value, eigenvectors, and more]