Code Jam 2008 - Round 1A

Analysis: Minimum Scalar Product

In spite of the geometric flavor in the name of this problem, it is truly a story of two arrays.

There are two permutations involved. However, after you fix the permutation for V_1, you have complete freedom in choosing the permutation for V_2. It is clear that the first permutation really does not matter. The important thing is which x_i gets to be matched to which y_j.

To make thinking easier, we may assume that the first permutation has

$$x_1 \le x_2 \le \ldots \le x_n. \quad (1)$$

And our task is to match the y's to the x's so that the scalar product is as small as possible.

At this point, if you need a little exercise in the middle: Think about the case n = 2, and just use a concrete example. What you will surely discover is that, in order to achieve the minimum scalar product, you always want to match the smaller x_i with the bigger y_j.

What we would like to prove is: under condition (1), one of the optimal solutions is of the form

$$y_1 \ge y_2 \ge \ldots \ge y_n. \quad (2)$$

The rigorous proof of (2) is given at the end of this analysis. For an easy reading, we point out that the key step is really the n = 2 case. If x < x' and y < y', then

$$(xy + x'y') - (xy' + x'y) = (x - x')(y - y') > 0. \quad (*)$$

So we prefer to match bigger y's with smaller x's.

Therefore, this problem is solved by the following simple algorithm.

```
sort(v1.begin(), v1.end());
sort(v2.begin(), v2.end(), greater<int>());
long long ret = 0;
for (int i = 0; i < n; i++)
  ret += (long long)(v1[i]) * v2[i];
```

Proof of (2). We prove that any permutation that does not satisfy (2) can be transformed into one satisfying (2), and in each step we do not increase the scalar product.
Indeed, any unsorted array can be transformed to a sorted one by only interchanging adjacent elements that are out of order. (For more rigorous readers: prove this, maybe by counting the number of flipped pairs in the array.)
At each step, some $x = x_i$ is matched to some y, and $x' = x_{i+1}$ is matched to some y' so that $y \le y'$. We interchange y and y' in this step. The inequality similar to (*), with > replaced by ≥, tells us that the scalar product is not increased in this step. ◊

More information:

[The Scalar product]