Code Jam 2008 - Round 2

Analysis: Star Wars

This problem offers a nice excursion into some basic pictures in algebra and geometry, and certainly some programming basics.

The problem is to find the smallest power Y_0, such that there is a position where power Y_0 is good enough to reach all of the ships. Clearly, any power bigger than Y_0 is big enough, while any power smaller than Y_0 is not. So, the first step towards the solution is to use the binary search. Reduce the problem of finding the smallest Y to a sequence of easier problems of deciding whether a given Y is big enough. Below we can focus on the decision problem for a given Y instead of the original optimization problem.

For a given Y, we have a requirement that each ship i satisfies

$$(1) \qquad (|x_i - x| + |y_i - y| + |z_i - z|) \le p_i Y$$

Geometrically, this means that the point (x, y, z) for the cruiser must be in the octahedron centered at (x_i, y_i, z_i). Each of the N ships gives one octahedron, and a good position for the cruiser exists if and only if all these N octahedra intersect.

Algebraically, (1) is equivalent to the following set of inequalities (prove it!)

$$\begin{aligned}
x + y + z &\le x_i + y_i + z_i + p_i Y \\
x + y + z &\ge x_i + y_i + z_i - p_i Y \\
x + y - z &\le x_i + y_i - z_i + p_i Y \\
x + y - z &\ge x_i + y_i - z_i - p_i Y \\
x - y + z &\le x_i - y_i + z_i + p_i Y \\
x - y + z &\ge x_i - y_i + z_i - p_i Y \\
-x + y + z &\le -x_i + y_i + z_i + p_i Y \\
-x + y + z &\ge -x_i + y_i + z_i - p_i Y
\end{aligned}$$

For the geometrically inclined, each octahedron is associated with one of the four directions given by the vectors (1, 1, 1), (1, 1, -1), (1, -1, 1) and (-1, 1, 1). Each pair of inequalities states that the projection (the inner product) of (x, y, z) on a given direction vector must be in a certain range.

Now we have the problem of solving a set of inequalities of the form

$$\begin{aligned}
A &\le x + y + z \le B \\
C &\le x + y - z \le D \\
E &\le x - y + z \le F \\
G &\le -x + y + z \le H
\end{aligned}$$

where A, B, C, D, E, F, G and H are given. In general, this is a linear program. But it is such a trivial one that we do not need to pull out any serious linear programming algorithms.

Certainly, for the solution to exist, we must have $A \le B$, $C \le D$, $E \le F$, and $G \le H$. But these conditions are not enough. The inequalities can be rewritten as

$$\begin{aligned}
A - x &\le y + z \le B - x \\
G + x &\le y + z \le H + x
\end{aligned}$$

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C - x ≤ y - z ≤ D - x
-F + x ≤ y - z ≤ -E + x
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As long as y + z and y - z have solutions, we can get y and z. We want to see whether there is an x such that the range [A - x, B - x] intersects [G + x, H + x], and the range [C - x, D - x] intersects [-F + x, -E + x].

It is easy to see that in order for the first two ranges to intersect, we must have

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(2)     x in [(A - H) / 2, (B - G) / 2].
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And for the other two ranges, we must have

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(3)     x in [(C + E) / 2, (D + F) / 2].
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The last step of our solution is simply to decide whether the two intervals in (2) and (3) have a non-empty intersection.