Coding Competitions Farewell Rounds - Round C

Analysis: The Decades of Coding Competitions

Let's start by treating the city as a graph, where nodes represent bus stops, edges represent bus routes, and a label on each edge represents the club cheered on by the bus route driver.

There is a key observation for this problem: if two nodes $\mathbf{P_j}$ and $\mathbf{C_j}$ are connected, we would be able to traverse all the edges in the connected component in the path from $\mathbf{P_j}$ to $\mathbf{C_j}$. The proof for this observation is provided at the end of this analysis. With this observation, we can find that if the total number of distinct edge clubs in this connected component is odd, there is always a path from $\mathbf{P_j}$ to $\mathbf{C_j}$ that can walk on an odd number of clubs by traversing all the edges in the connected component at least once.

Therefore, for each query, we can check if there exists a subgraph that satisfies:

- $\mathbf{P_j}$ and $\mathbf{C_j}$ are connected.
- There is an odd number of distinct edge clubs in the connected component that contains $\mathbf{P_j}$ and $\mathbf{C_j}$.

Test Set 1

Since the number of distinct clubs $\mathbf{K}$ is small in Test Set 1, we can enumerate all combinations of an odd number of distinct clubs. For each combination, build a new graph G' by removing edges with clubs not in the combination. Afterwards, for each query we can check if $\mathbf{P_j}$ and $\mathbf{C_j}$ are connected in any G', and if there is an odd number of edge clubs in the corresponding connected component by iterating through all the edges in the connected component.

We can use Disjoint Set Union (DSU) to find the connected componnets in G' and all the nodes and edges in them. A connected component has edge club c if and only if u and v are in this component (in the same set in DSU) and the edge (u, v) has club c. We can prebuild the DSU for all G's and find the number of distinct clubs for each of the connected component in them. For each query, we can iterate through all prebuilt DSUs and check if $\mathbf{P_j}$ and $\mathbf{C_j}$ are connected in a component with odd number of edge clubs in it.

Space complexity:

There are at most $2^{\mathbf{K}}$ G's, and each graph has at most $\mathbf{N}$ connected components:

- **Disjoint Set Unions(DSUs)**: $O(2^{\mathbf{K}} \times \mathbf{N})$
- **Number of distinct clubs in connected components in graphs**: $O(2^{\mathbf{K}} \times \mathbf{N})$

Time complexity:

- **Build DSUs and component clubs set (number of distinct clubs)**: $O(2^{\mathbf{K}} \times (\mathbf{N} + \mathbf{M}))$
- **Per query**: $O(2^{\mathbf{K}})$ for checking all DSUs and number of distinct clubs in the corresponding connected component.
- **Overall**: $O(2^{\mathbf{K}} \times (\mathbf{N} + \mathbf{M}) + \mathbf{Q} \times 2^{\mathbf{K}}) = O(2^{\mathbf{K}} \times (\mathbf{N} + \mathbf{M} + \mathbf{Q}))$

Test Set 2 and Test Set 3

We can improve our solution by reducing the club combinations to check: we just need to check the original graph G and all G' built by removing edges with one club. We can prove this statement by considering all possible cases for query $\mathbf{P_j}$ and $\mathbf{C_j}$:

Let $\mathrm{Clubs}(G, \mathbf{P_j}, \mathbf{C_j})$ be the set of distinct clubs in the connected component containing $\mathbf{P_j}$ and $\mathbf{C_j}$ in graph G, and $|\mathrm{Clubs}(G, \mathbf{P_j}, \mathbf{C_j})|$ be the number of distinct clubs:

- If $\mathbf{P_j}$ and $\mathbf{C_j}$ are not connected in G, there is no valid path for this query.
- If $\mathrm{Clubs}(G, \mathbf{P_j}, \mathbf{C_j})$ has an odd number of distinct clubs ($|\mathrm{Clubs}(G, \mathbf{P_j}, \mathbf{C_j})| = 2n + 1$), there is always a valid path walking on an odd number of distinct clubs.
- Otherwise, if there is a path from $\mathbf{P_j}$ to $\mathbf{C_j}$ composed of edges with odd number $(2n + 1)$ of distinct clubs, $\mathbf{P_j}$ and $\mathbf{C_j}$ will always be connected after removing edges with those $\mathbf{K} - (2n + 1)$ clubs not in that path from graph G. Then, let $G_{minimal}$ be graph after removing those edges, we can find another path with an odd number of clubs by traversing additional connected edges with two extra clubs c_1 and c_2 where $c_1, c_2 \notin \mathrm{Clubs}(G_{minimal}, \mathbf{P_j}, \mathbf{C_j})$ and $c_1, c_2 \in \mathrm{Clubs}(G, \mathbf{P_j}, \mathbf{C_j})$. Recursively, we can find a valid path in G' by removing only one club from $\mathrm{Clubs}(G, \mathbf{P_j}, \mathbf{C_j})$. It means if there is a valid path, there is always a G' where $|\mathrm{Clubs}(G', \mathbf{P_j}, \mathbf{C_j})|$ is odd and built by removing edges of one club.

Therefore, we just need to build the DSUs and connected components clubs sets for the $\mathbf{K} + 1$ graphs that result from removing 0 or 1 club from the original graph.

Space complexity:

There are $\mathbf{K} + 1$ G's, and each graph has at most $\mathbf{N}$ connected components:

- **Disjoint Set Unions(DSUs)**: $O((\mathbf{K} + 1) \times \mathbf{N}) = O(\mathbf{K} \times \mathbf{N})$
- **Number of distinct clubs in connected components in graphs**: $O(\mathbf{K} \times \mathbf{N})$

Time complexity:

- **Build DSUs and component clubs set (number of distinct clubs)**: $O((\mathbf{K} + 1) \times (\mathbf{N} + \mathbf{M})) = O(\mathbf{K} \times (\mathbf{N} + \mathbf{M}))$
- **Per query**: $O(\mathbf{K})$ for checking all DSUs and number of distinct clubs in the corresponding connected component.
- **Overall**: $O(\mathbf{K} \times (\mathbf{N} + \mathbf{M} + \mathbf{Q}))$

Test Set 2 does not require prebuilding and sharing the DSU and clubs sets across queries. Solutions with time complexity $O(\mathbf{Q} \times (\mathbf{K} \times (\mathbf{N} + \mathbf{M})))$ that build DSUs and clubs sets for each of the query separately are acceptable.

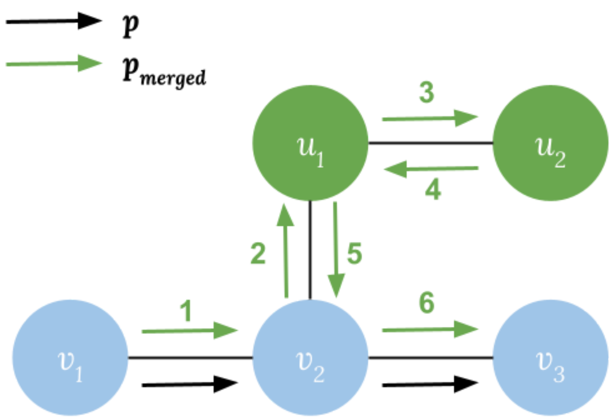
Proof: Path traversing all edges in the connect component always exists

Suppose there is a path from v_1 to v_n where $p = (v_1, v_2, \ldots, v_{n-1}, v_n)$, and let $\mathrm{Nodes}(p) = \{v_1, v_2, \ldots, v_n\}$ to be the set of nodes visited at least once in path p:

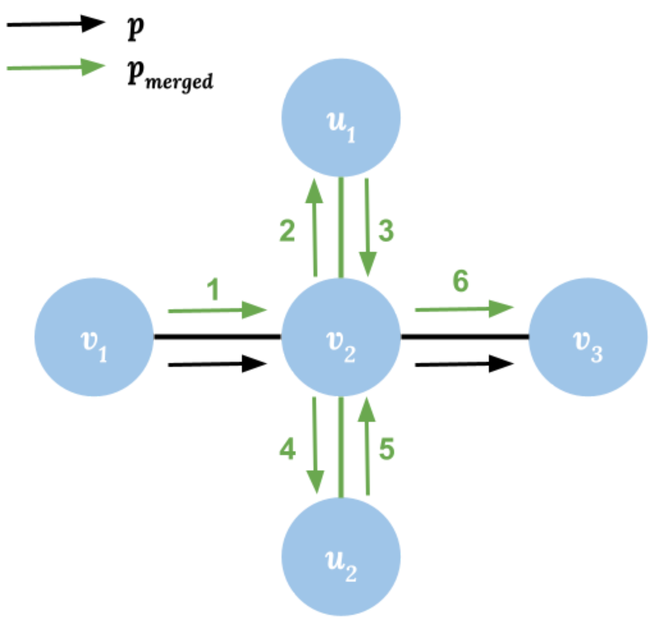
1. **Node Reachability:** For any node $v_i \in \mathrm{Nodes}(p)$, if there is a node u_m and a path from v_i to u_m where $p' = (v_i, u_1, u_2, \ldots, u_m)$ in the graph, there is always a path p_{merged} from v_1 to v_n where $\mathrm{Nodes}(p_{merged}) = \mathrm{Nodes}(p) \cup \mathrm{Nodes}(p') \supseteq \mathrm{Nodes}(p) \cup \{u_m\}$. A valid path would be $p_{merged} = (v_1, \ldots, v_i, u_1, \ldots, u_{m-1}, u_m, u_{m-1}, \ldots, u_1, v_i, v_{i+1}, \ldots, v_n)$. Notice that we can visit same node or edge multiple times.
2. **Edge Reachability:** For any node $v_i \in \mathrm{Nodes}(p)$, and for edges $(v_i, u_1), (v_i, u_2), \ldots, (v_i, u_m)$ directly connect to v_i, there always exists a path p'_{merged}

from v_1 to v_n which traverses all these edges at least once. A valid path would be $p'_{merged} = (v_1, \ldots, v_i, u_1, v_i, u_2, \ldots, v_i, u_m, v_i, v_{i+1}, \ldots, v_n)$.

3. From (1), we can derive that if nodes $\mathbf{P}$ and $\mathbf{C}$ are connected, there exist a path from $\mathbf{P}$ to $\mathbf{C}$ which visits all reachable nodes from $\mathbf{P}$ at least once, which are all nodes in the connected component.
4. From (2) and (3), we can derive that if nodes $\mathbf{P}$ and $\mathbf{C}$ are connected, there exists a path which traverses all the edges in the connected component at least once.



Node Reachability: p_{merged} visits additional nodes u_1 and u_2, as there exists a path $p' = (v_2, u_1, u_2)$.



Edge Reachability: p_{merged} visits all edges connected to v_2 — (v_2, u_1) and (v_2, u_2).