Harvard University SIMR (Summer Introduction to Mathematical Research)
Network Flow

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7:45pm ET, Tuesday, July 5, 2022

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    - ICPC Head Coach at Columbia University
    - ICPC Judge for Mid-central and Greater New York regions in N.A.
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    - Research Software Engineer at Google Research
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    - Stanford: Math & CS (BS '11) and CS (MS '13)
    - MIT: Operations Research (PhD, started in 2013 but on an extended leave-of-absence since 2016)

Network Flow: Real-life Example

- We are given a directed graph, where each vertex represents a city and each directed edge represents a one-way road from one city to another.
- Suppose we want to send trucks from city s to city t and the **capacity** of each directed edge represents the number of trucks that can go on that road every hour.

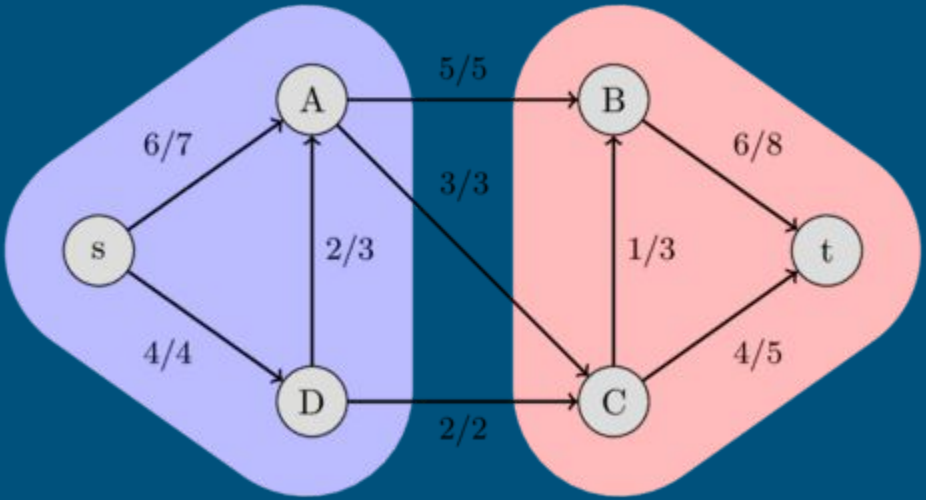
Photo Credit

Network Flow: Real-life Example (con't)

- We are given a directed graph, where each vertex represents a city and each directed edge represents a one-way road from one city to another.
- Suppose we want to send trucks from city s to city t and the **capacity** of each directed edge represents the number of trucks that can go on that road every hour.
- What is the **maximum number** of trucks that we can send from s to t every hour?

Lecture: Overview

- Flow Network
- Maximum Flow Problem
- Ford-Fulkerson Method
- Edmonds-Karp Algorithm
- Integral Flow Theorem
- s-t cut and its capacity
- Flow Value Lemma
- Flow Weak Duality
- Flow Certificate of Optimality
- Max-flow Min-cut Theorem
- Menger's Theorem
- Further Topics in Flows
- References
- Contact Information



Lecture: Overview

- Flow Network
- **Maximum Flow Problem**
- **Ford-Fulkerson Method**
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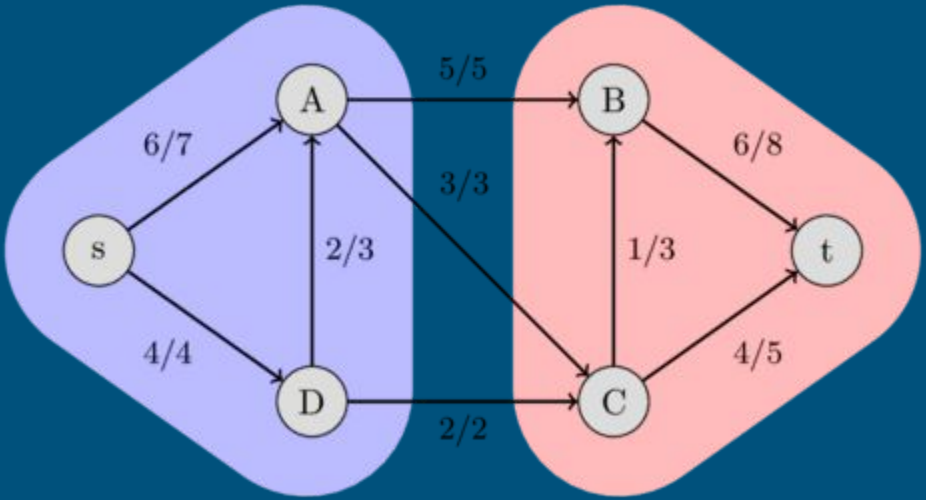


Flow Network

- A **network** is a directed graph G with vertices V and edges E combined with a function c, which assigns each edge e a non-negative integer value, the **capacity** of e.
- Such a network is called a **flow network**, if we additionally label two vertices, one as **source** and the other as **sink**.
- **Assumption**: We disallow a reverse edge $(v, u) \in E$ for each edge $(u, v) \in E$.

Flow Network (con't)

- A **flow** in a flow network is function f, that again assigns each edge e a non-negative integer value, namely the flow. The function has to fulfill the following conditions:
  - The flow of an edge cannot exceed the capacity: $f(e) \leq c(e)$.
  - The sum of the incoming flow of a vertex u has to be equal to the sum of the outgoing flow of u (except in the source and sink vertices): $\Sigma_v f((v, u)) = \Sigma_w f((u, w))$.
- The source vertex s only has outgoing flows.
- The sink vertex t only has incoming flows.
- $\Sigma_v f((v, t)) = \Sigma_w f((s, w))$.

Maximum Flow Problem

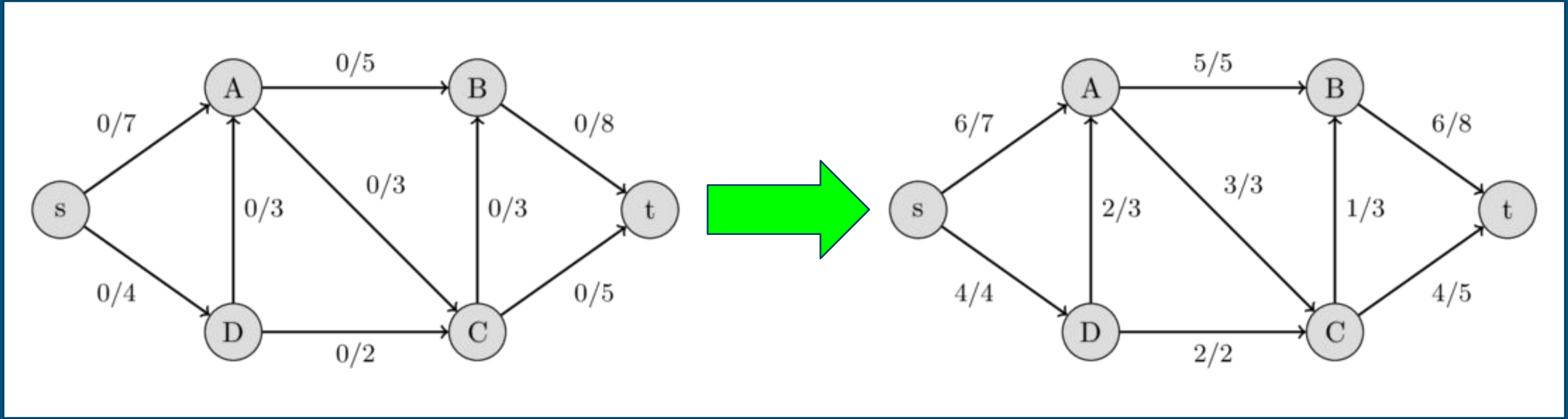
- The value of the flow of a network is the sum of all the flows that get produced in the source s, or equivalently to the sum of all the flows that are consumed by the sink t.
- Formally, val(f), the **value** of a flow f, is defined as:

$$\Sigma_{\text{e out of s}} f(e) - \Sigma_{\text{e in to s}} f(e).$$

- A **maximum flow** is a flow with the maximum possible value of val(f).
- **Problem: Find this maximum flow of a flow network!**

Ford-Fulkerson Method: Example

- The first value of each edge represents the flow, which is initially 0, and the second value represents the capacity.



Ford-Fulkerson Method (1956)

- A **residual capacity** of an directed edge is the capacity minus the flow.
  - For each edge (u, v), we can create a reverse edge (v, u) with capacity 0 such that f((v, u)) = -f((u, v)).
  - This also defines the residual capacity for all the reversed edges.
- We can create a **residual network** from all these edges, which is just a network with the same vertices and edges, but we use the residual capacities as capacities.

Ford-Fulkerson Method (con't)

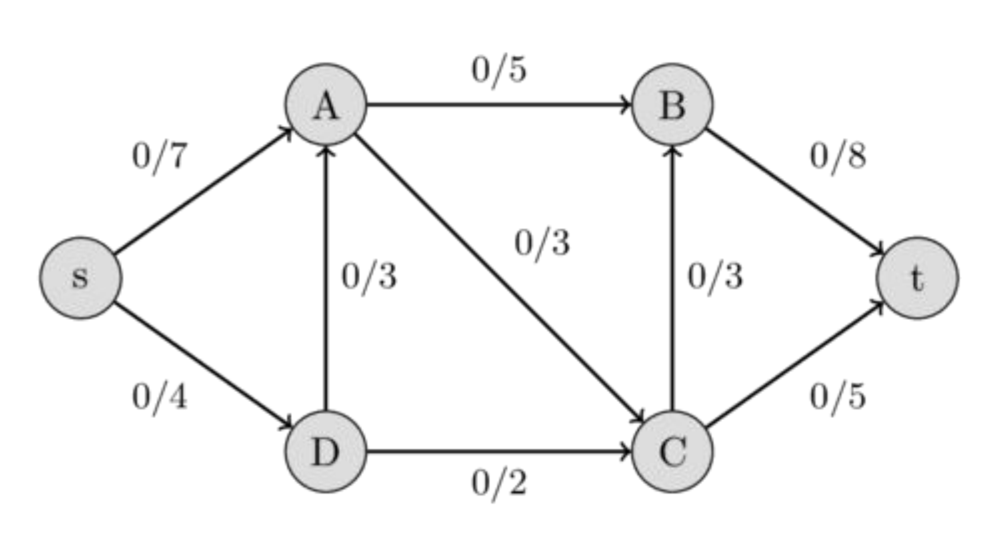
1. We set the flow of each edge to 0.
2. We look for an **augmenting path** from s to t.
   - An augmenting path is a simple path in the residual graph by always taking the edges whose residual capacity is **positive**.
3. If such a path is found then we can increase the flow along these edges.
   - Let C be the smallest residual capacity of the edges in the path. Then we increase the flow by updating f((u, v)) += C and f((v, u)) -= C for every edge (u, v) in the path.
4. Else, terminate! (A flow found is maximum).

Ford-Fulkerson Method: Augmenting Path?

- Ford-Fulkerson method **does not specify** how to find an augmenting path.
- Possible approaches are using *Depth-First Search* (DFS) or *Breadth-First Search* (BFS), both of which work in **O(E)**.
- **Integral** capacities
  - For each augmenting path, the flow of the network increases by at least 1. So, the complexity of Ford-Fulkerson is O(EF) where F is the maximum flow of the network (but, usually *much faster* in practice)!
- **Rational** capacities
  - The algorithm will terminate but the complexity is not bounded.
- **Irrational** capacities
  - The algorithm might never terminate and might not even converge to the maximum flow.

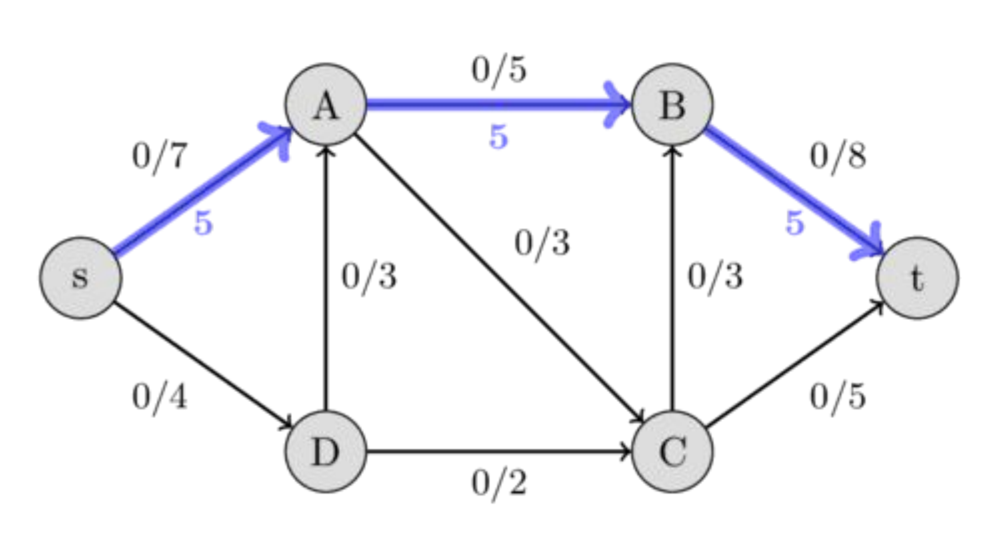
Ford-Fulkerson Method: Example

flow = 0



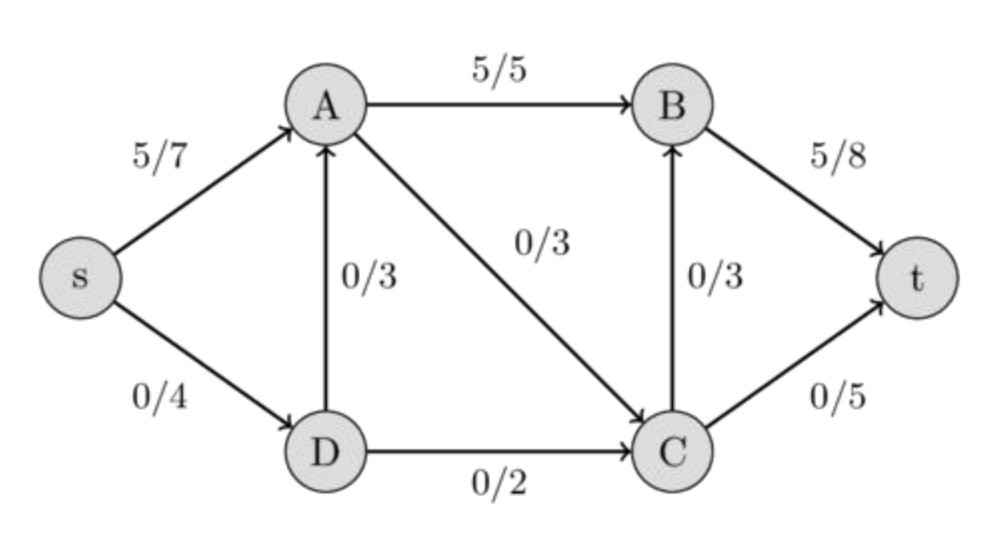
Ford-Fulkerson Method: Example

flow = 0



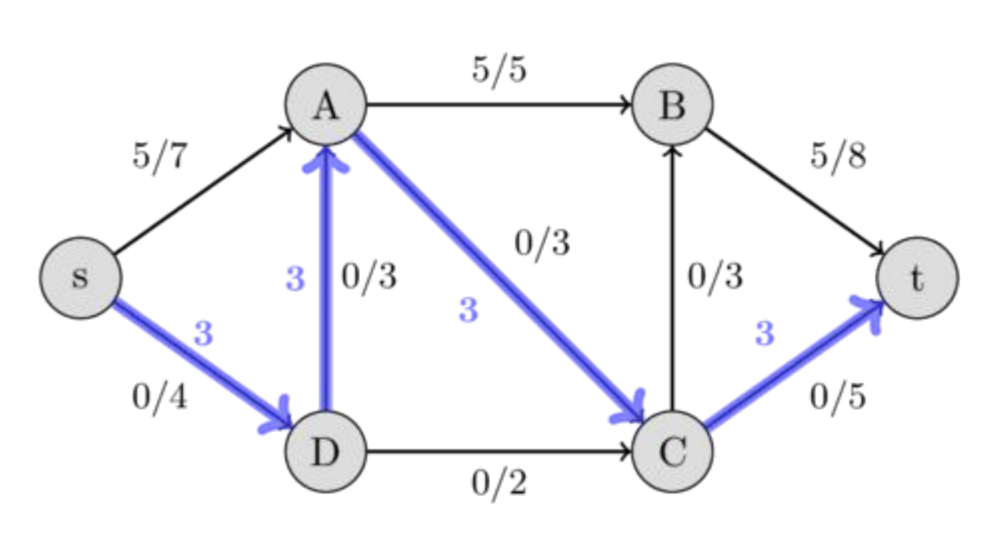
Ford-Fulkerson Method: Example

flow = 5



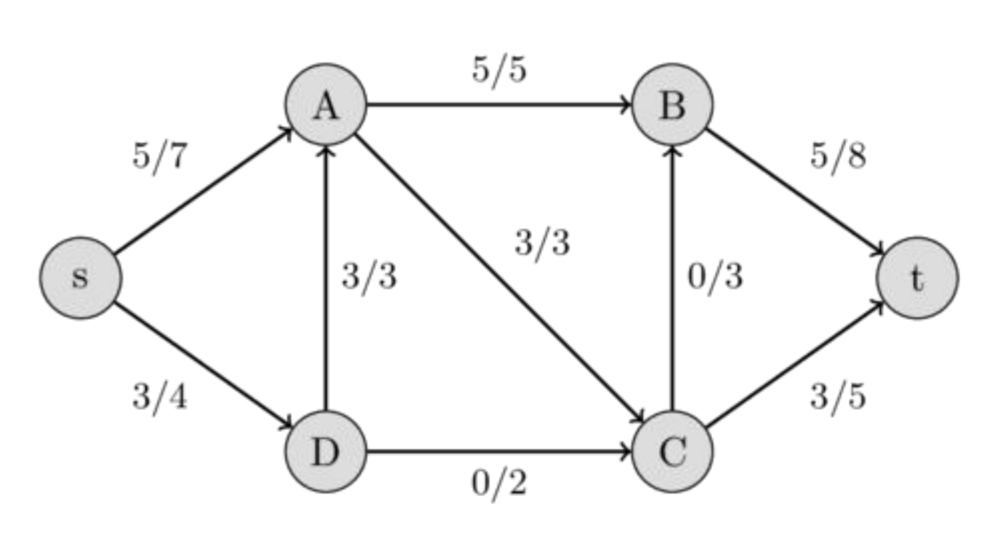
Ford-Fulkerson Method: Example

flow = 5



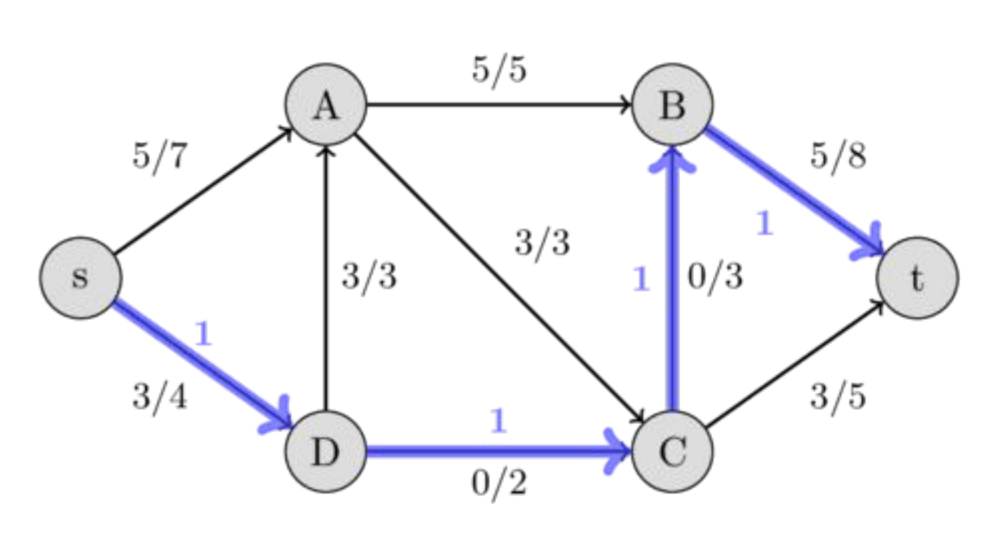
Ford-Fulkerson Method: Example

flow = 8



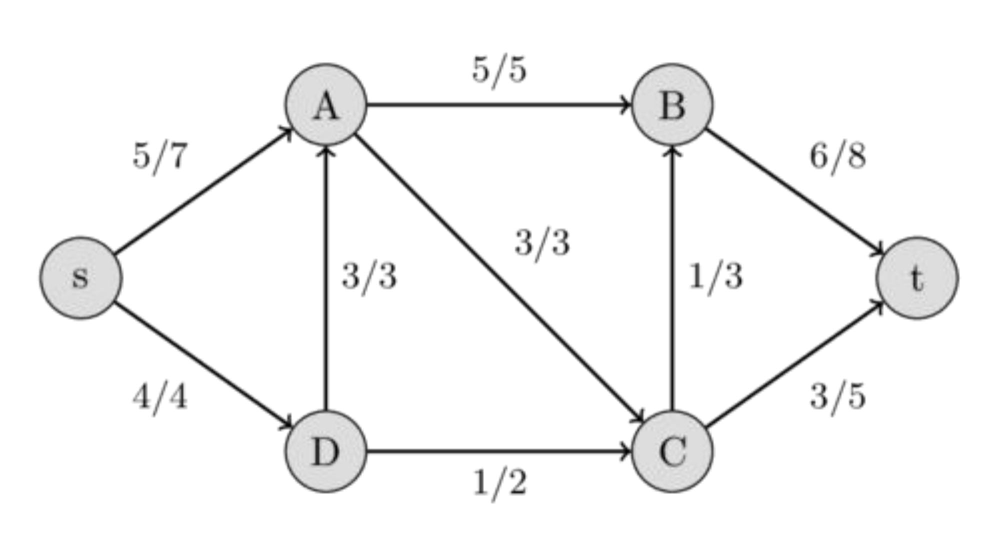
Ford-Fulkerson Method: Example

flow = 8



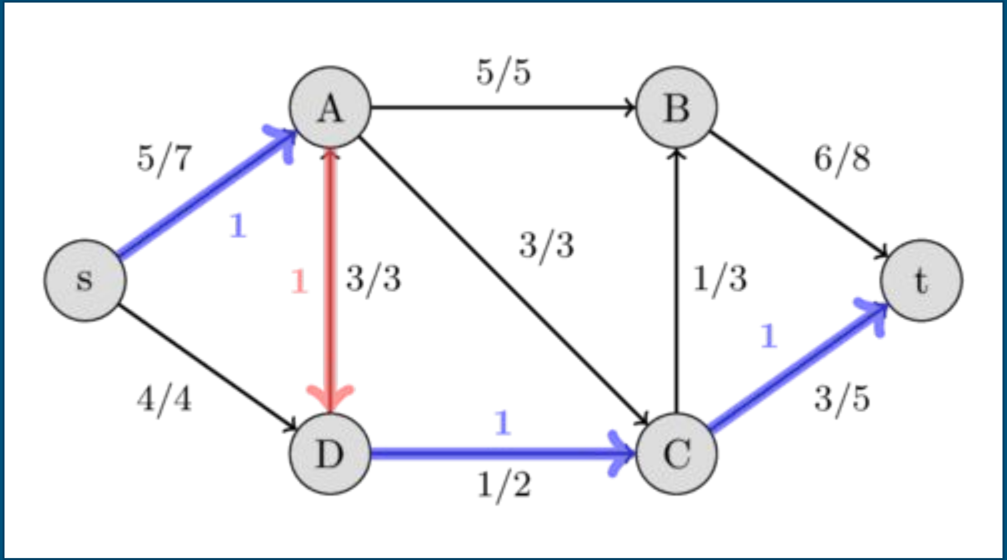
Ford-Fulkerson Method: Example

flow = 9



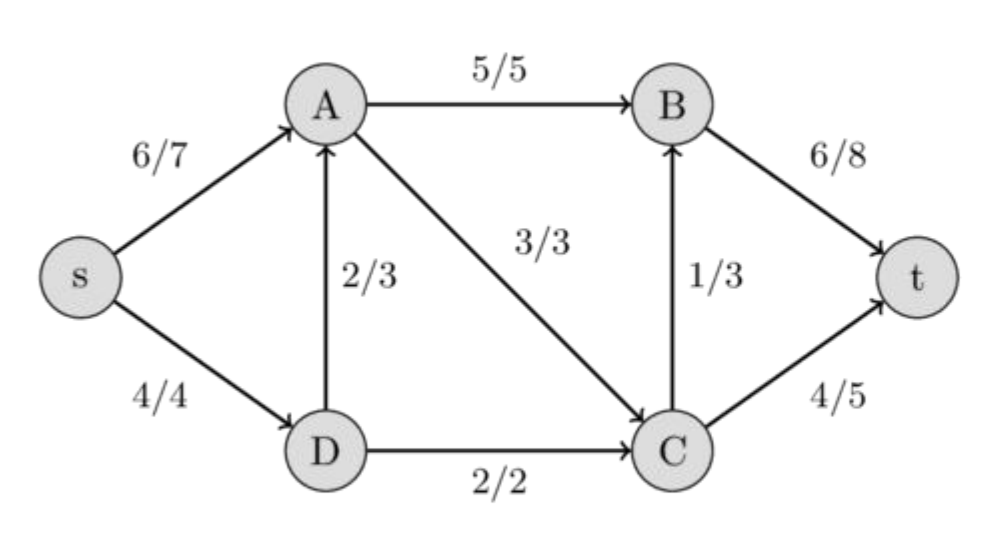
Ford-Fulkerson Method: Example

flow = 9



Ford-Fulkerson Method: Example

flow = **10**



Edmonds-Karp Algorithm (1972)

- **Edmonds-Karp algorithm** is just an implementation of the Ford-Fulkerson method that uses BFS for finding augmenting paths.
- First published by Yefim Dinitz in 1970; later, independently published by Jack Edmonds and Richard Karp in 1972.
- The complexity can be given *independently* of the maximum flow. The algorithm runs in **$O(VE^2)$** time (yes! even for irrational capacities).
- **Intuition**
  - Every time we find an augmenting path one of the edges becomes saturated and the distance from the edge to s will be longer if it appears again in an augmenting path later. The path length is bounded by V.

Integral Flow Theorem: Statement

- Suppose a given graph has each of its capacity integral. Then, there exists a maximum flow f where every flow value f(e) is an integer.

Integral Flow Theorem: Proof

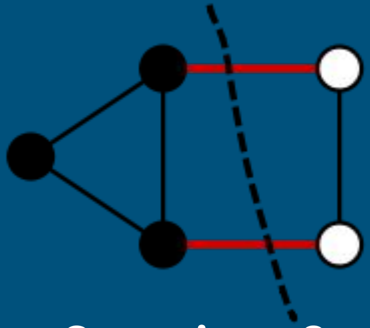
- Suppose a given graph has each of its capacity integral. Then, there exists a maximum flow f where every flow value f(e) is an integer.

- In each step of Edmonds-Karp algorithm, the augmenting path will change a flow value and each residual capacity by **integral** amount.
- Therefore, as long as the algorithm terminates in finite steps, a maximum flow would have flow values that are integral too!

s-t cut and its capacity

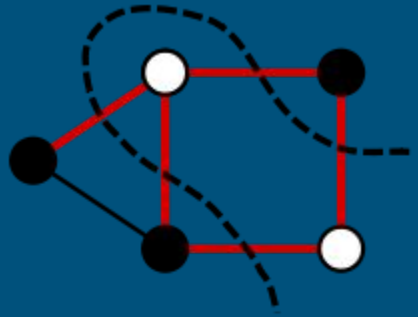
- An **s-t cut** is a partition (A, B) of the vertices with s ∈ A and t ∈ B.
- The capacity of a cut, **cap(A, B)**, is the sum of capacities of the edges from A to B.

Minimum Cut



Capacity = 2

Maximum Cut



Capacity = 5

Flow Value Lemma: Statement

- Let f be any flow and let (A, B) be any cut. Then,

$$val(f) = \Sigma_{\text{e out of A}} f(e) - \Sigma_{\text{e in to A}} f(e).$$

Flow Value Lemma: Proof

- Let f be any flow and let (A, B) be any cut. Then,

$$val(f) = \Sigma_{\text{e out of A}} f(e) - \Sigma_{\text{e in to A}} f(e).$$

$val(f) = \Sigma_{\text{e out of s}} f(e) - \Sigma_{\text{e in to s}} f(e)$	Definition of val(f)
$= \Sigma_{v \in A} (\Sigma_{\text{e out of v}} f(e) - \Sigma_{\text{e in to v}} f(e))$	All terms except v = s are 0
$= \Sigma_{\text{e out of A}} f(e) - \Sigma_{\text{e in to A}} f(e)$ //	Edges completely in A cancel

Flow Weak Duality Lemma: Statement

- Let f be any flow and let (A, B) be any cut. Then,

$$\mathrm{val}(f) \leq \mathrm{cap}(A, B).$$

Flow Weak Duality Lemma: Proof

- Let f be any flow and let (A, B) be any cut. Then,

$$\mathrm{val}(f) \le \mathrm{cap}(A, B).$$

$\mathrm{val}(f) = \Sigma_{e \text{ out of } A}\, f(e) - \Sigma_{e \text{ in to } A}\, f(e)$	Flow Value Lemma
$\le \Sigma_{e \text{ out of } A}\, f(e)$	Flows are non-negative
$\le \Sigma_{e \text{ out of } A}\, c(e)$	Capacity constraint
$= \mathrm{cap}(A, B)$ //	

Flow Certificate of Optimality: Statement

- Let f be a flow and let (A, B) be any cut. If val(f) = cap(A, B) then f is a maximum flow and (A, B) is a minimum cut.

Flow Certificate of Optimality: Proof

- Let f be a flow and let (A, B) be any cut. If val(f) = cap(A, B) then f is a maximum flow and (A, B) is a minimum cut.

- By Weak Duality Lemma, for any flow f', we have val(f') ≤ cap(A, B) = val(f).
- Similarly, for any cut (A', B'), we have cap(A', B') ≥ val(f) = cap(A, B) //

Max-flow Min-cut Theorem: Statement

- The maximum value of an s-t flow is equal to the minimum capacity over all s-t cuts.

Max-flow Min-cut Theorem: Proof Sketch

- Suffices to show the following three conditions are equivalent:
  - There exists a cut (A, B) such that cap(A, B) = val(f);
  - f is a maximum flow;
  - There is no augmenting path with respect to f;
- **(1 => 2)**
  - Weak duality!
- **(2 => 3)**
  - We prove the contrapositive. If there is an augmenting path then f is not a maximum flow. If f' is the flow achievable in an augmenting path then f + f' is a flow in G, which means f is not a maximum flow.
- **(3 => 1)**
  - Let f be a flow with no augmenting paths. Let A be the set of vertices reachable from s in the residual network. By the definition of A, $s \in A$. By the definition of flow f, $t \notin A$. So,

$$\mathrm{val}(f) = \Sigma_{e \text{ out of } A} f(e) - \Sigma_{e \text{ in to } A} f(e) = \Sigma_{e \text{ out of } A} f(e) - 0 = \mathrm{cap}(A, B).$$

Menger's Theorem

- Let G be a finite directed graph and let s and t be two nonadjacent vertices.
- The size of the minimum vertex cut for s and t (i.e., the minimum number of vertices, distinct from s and t, whose removal disconnects s and t) is equal to the maximum number of pairwise internally vertex-disjoint paths from s to t.

Menger's Theorem: Proof Sketch

- Let G be a finite directed graph and let s and t be two nonadjacent vertices.
- The size of the minimum vertex cut for s and t (i.e., the minimum number of vertices, distinct from s and t, whose removal disconnects s and t) is equal to the maximum number of pairwise internally vertex-disjoint paths from s to t.

- Consider the following flow network:
  - Split each vertex $v \in V$ into v_{in} and v_{out} and add the edge (v_{in}, v_{out}) with capacity 1.
  - Each edge $(u, v) \in E$ becomes (u_{out}, v_{in}) with capacity ∞.
- Use the maximum-flow minimum-cut theorem!

Further Topics in Flows

- Push-relabel algorithm
- Dinic's algorithm (most of the time, this algorithm is *sufficiently fast*!)
- Malhotra-Pramodh-Maheshwari (MPM) algorithm
- Flows with demands
- Minimum cost flow
- Assignment problem
- Nearly-linear(!) time algorithm (Li Chen, Rasmus Kyng, Yang P. Liu, Richard Peng, Maximilian Probst Gutenberg, Sushant Sachdeva)
- Weighted/Unweighted Bipartite/General Matching
- …
- Dynamic graphs! (Professor Richard Peng's lecture, scheduled on July 14)
- …

References

- https://cp-algorithms.com/graph/edmonds_karp.html
- https://faculty.cc.gatech.edu/~rpeng/CS7510_F19/Sep3MaxFlowMinCut.pdf
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- For those who are interested in **quantitative analysis in finance**, feel free to send me an email to schedule 1:1 zoom meeting.