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PROBABILITY, STATISTICS AND NUMERICAL METHODS  
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**Unit-4 Solutions Of A System Of Linear Equations & Roots Of Algebraic And  
Transcendental Equations**

**Iterative methods:**

The direct methods lead to exact solutions in many cases but are subject to errors due to round off and other factors. In the iterative method, an approximation to the true solution is assumed initially to start the method. By applying the method repeatedly, better and better approximations are obtained. For large systems, iterative methods are faster than direct methods and round-off errors are also smaller. Any error made at any stage of computation gets automatically corrected in the subsequent steps.

We will discuss two iterative methods:

- (i) Gauss-Jacobi method
- (ii) Gauss-Seidel method

**DIAGONALLY DOMINANT PROPERTY (SUFFICIENT CONDITION FOR CONVERGENCE):**

A matrix is said to be diagonally dominant if for every row of the matrix, the magnitude of the diagonal entry in a row is larger than or equal to the sum of the magnitudes of all the other (non-diagonal) entries in that row. More precisely, the matrix A is diagonally dominant if

$$|a_{ii}| \geq \sum_{j \neq i} |a_{ij}| \quad \text{for all } i$$

where  $a_{ij}$  denotes the entry in the  $i^{\text{th}}$  row and  $j^{\text{th}}$  column.

Note that this definition uses a weak inequality, and is therefore sometimes called weak diagonal dominance. If a strict inequality ( $>$ ) is used, this is called strict diagonal dominance. The unqualified term diagonal dominance can mean both strict and weak diagonal dominance.

**(i) Gauss-Jacobi method:**

Given a general set of  $n$  equations and  $n$  unknowns, we have

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots + a_{1n}x_n = c_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \dots + a_{2n}x_n = c_2$$

$$\cdot \quad \quad \cdot$$

$$\cdot \quad \quad \cdot$$

$$\cdot \quad \quad \cdot$$

$$a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \dots + a_{nn}x_n = c_n$$

If the diagonal elements are non-zero, each equation is rewritten for the corresponding unknown, that is, the first equation is rewritten with  $x_1$  on the left hand side, the second equation is rewritten with  $x_2$  on the left hand side and so on as follows

$$x_1 = \frac{c_1 - a_{12}x_2 - a_{13}x_3 \dots \dots - a_{1n}x_n}{a_{11}}$$

$$x_2 = \frac{c_2 - a_{21}x_1 - a_{23}x_3 \dots \dots - a_{2n}x_n}{a_{22}}$$

$$\vdots$$

$$\vdots$$

$$x_{n-1} = \frac{c_{n-1} - a_{n-1,1}x_1 - a_{n-1,2}x_2 \dots \dots - a_{n-1,n-2}x_{n-2} - a_{n-1,n}x_n}{a_{n-1,n-1}}$$

$$x_n = \frac{c_n - a_{n1}x_1 - a_{n2}x_2 - \dots \dots - a_{n,n-1}x_{n-1}}{a_{nn}}$$

These equations can be rewritten in a summation form as

$$x_1 = \frac{c_1 - \sum_{\substack{j=1 \\ j \neq 1}}^n a_{1j}x_j}{a_{11}}$$

$$x_2 = \frac{c_2 - \sum_{\substack{j=1 \\ j \neq 2}}^n a_{2j} x_j}{a_{22}}$$

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$$x_{n-1} = \frac{c_{n-1} - \sum_{\substack{j=1 \\ j \neq n-1}}^n a_{n-1,j} x_j}{a_{n-1,n-1}}$$

$$x_n = \frac{c_n - \sum_{\substack{j=1 \\ j \neq n}}^n a_{nj} x_j}{a_{nn}}$$

Hence for any row  $i$ ,

$$x_i = \frac{c_i - \sum_{\substack{j=1 \\ j \neq i}}^n a_{ij} x_j}{a_{ii}}, i = 1, 2, \dots, n.$$

Now to find  $x_i$ 's, one assumes an initial guess for the  $x_i$ 's and then uses the rewritten equations to calculate the new estimates. Remember, one always uses the most recent estimates to calculate the next estimates,  $x_i$ .

The above iteration process is continued until two successive approximations are nearly equal.

### **Working Rule:**

(i) Arrange the equations in such a manner that the leading elements are large in magnitude in their respective rows satisfying the conditions

$$\begin{aligned} |a_{11}| &> |a_{12}| + |a_{13}| \\ |a_{22}| &> |a_{21}| + |a_{23}| \\ |a_{33}| &> |a_{31}| + |a_{32}| \end{aligned}$$

(ii) Express the variables having large coefficients in terms of other variables.

(iii) Start the iteration 1 by assuming the initial values of  $(x, y, z)$  as  $(x_0, y_0, z_0)$  and obtain

$(x_1, y_1, z_1)$  using equations

$$\begin{aligned}
 x_1 &= \frac{c_1 - a_{12}x_2 - a_{13}x_3 \dots - a_{1n}x_n}{a_{11}} \\
 x_2 &= \frac{c_2 - a_{21}x_1 - a_{23}x_3 \dots - a_{2n}x_n}{a_{22}} \\
 &\vdots \\
 &\vdots \\
 x_{n-1} &= \frac{c_{n-1} - a_{n-1,1}x_1 - a_{n-1,2}x_2 \dots - a_{n-1,n-2}x_{n-2} - a_{n-1,n}x_n}{a_{n-1,n-1}} \\
 x_n &= \frac{c_n - a_{n1}x_1 - a_{n2}x_2 - \dots - a_{nn}x_{n-1}}{a_{nn}}
 \end{aligned}$$

1.

(iv) Start the iteration 2 by putting the values of  $(x, y, z)$  as  $(x_1, y_1, z_1)$  and obtain

$$(x_2, y_2, z_2).$$

(v) The above iteration process is continued until two successive approximations are nearly equal.

**Example:** Solve the system of linear equations by Gauss- Jacobi method correct up to 2-

decimal places.

$$6x + 2y - z = 4, \quad x + 5y + z = 3, \quad 2x + y + 4z = 27$$

**Solution:**

Re-writing the equations,

$$\left. \begin{aligned}
 x &= \frac{1}{6}(4 - 2y + z) \\
 y &= \frac{1}{5}(3 - x - z) \\
 z &= \frac{1}{4}(27 - 2x - y)
 \end{aligned} \right\} \dots \dots \dots (1)$$

Iteration 1: Assuming  $x_0 = 0$ ,  $y_0 = 0$  and  $z_0 = 0$  as initial approximation and putting in Eq.1

$$x_1 = \frac{2}{3} = 0.67$$

$$y_1 = \frac{1}{5}(3) = 0.6$$

$$z_1 = \frac{1}{4}(27) = 6.75$$

Iteration 2: Putting  $x_1$ ,  $y_1$  and  $z_1$  in Eq.1

$$x_2 = \frac{1}{6}(4 - 2(0.6) + 6.75) = 1.59$$

$$y_2 = \frac{1}{5}(3 - 0.67 - 6.75) = -0.884$$

$$z_2 = \frac{1}{4}(27 - 2(0.67) - 0.6) = 6.265$$

Continuing in this way, we have the solution as

$$x_5 = 2.00$$

$$y_5 = -1.00$$

$$z_5 = 6.00$$

Which is the required solution.

## (ii) Gauss-Seidel method:

The system of linear equations are same as given in Gauss-Jacobi method.

### **Working Rule:**

(i) Arrange the equations in such a manner that the leading elements are large in magnitude in their respective rows satisfying the conditions

$$|a_{11}| > |a_{12}| + |a_{13}|$$

$$|a_{22}| > |a_{21}| + |a_{23}|$$

$$|a_{33}| > |a_{31}| + |a_{32}|$$

(ii) Express the variables having large coefficients in terms of other variables.

(iii) Start the iteration 1 by assuming the initial values of  $(x, y, z)$  as  $(x_0, y_0, z_0)$

(iv) In the iteration 1, put  $y = y_0, z = z_0$  in the equation of  $x$  to obtain  $x_1$ , put  $x = x_1$  and  $z = z_0$  in the eq. of  $y$  to obtain  $y_1$ , put  $x = x_1, y = y_1$  in the eq. of  $z$  to obtain  $z_1$ .

(v) The above iteration process is continued until two successive approximations are nearly equal.

**Example:** Find the solution to the following system of equations using the Gauss-Seidel method.

$$12x_1 + 3x_2 - 5x_3 = 1$$

$$x_1 + 5x_2 + 3x_3 = 28$$

$$3x_1 + 7x_2 + 13x_3 = 76$$

Use  $x_1 = 1, x_2 = 0$  and  $x_3 = 1$  as the initial guess .

**Solution:**

The given system is diagonally dominant as

$$|a_{11}| = |12| = 12 \geq |a_{12}| + |a_{13}| = |3| + |-5| = 8$$

$$|a_{22}| = |5| = 5 \geq |a_{21}| + |a_{23}| = |1| + |3| = 4$$

$$|a_{33}| = |13| = 13 \geq |a_{31}| + |a_{32}| = |3| + |7| = 10$$

and the inequality is strictly greater than for at least one row. Hence, the solution should converge using the Gauss-Seidel method.

Rewriting the equations, we get

$$x_1 = \frac{1 - 3x_2 + 5x_3}{12}$$

$$x_2 = \frac{28 - x_1 - 3x_3}{5}$$

$$x_3 = \frac{76 - 3x_1 - 7x_2}{13}$$

Assuming an initial guess of  $x_1 = 1, x_2 = 0$  and  $x_3 = 1$

**Iteration 1**

$$x_1^1 = \frac{1 - 3(0) + 5(1)}{12}$$

$$= 0.50000$$

$$x_2^1 = \frac{28 - (0.50000) - 3(1)}{5}$$

$$= 4.9000$$

$$x_3^1 = \frac{76 - 3(0.50000) - 7(4.9000)}{13}$$

$$= 3.0923$$

## Iteration 2

$$x_1^2 = \frac{1 - 3(4.9000) + 5(3.0923)}{12}$$

$$= 0.14679$$

$$x_2^2 = \frac{28 - (0.14679) - 3(3.0923)}{5}$$

$$= 3.7153$$

$$x_3^2 = \frac{76 - 3(0.14679) - 7(3.7153)}{13}$$

$$= 3.8118$$

as you conduct more iterations, the solution converges as follows.

Iteratio n	$x_1$	$x_2$	$x_3$
1	0.50000	4.9000	3.0923
2	0.14679	3.7153	3.8118
3	0.74275	3.1644	3.9708
4	0.94675	3.0281	3.9971
5	0.99177	3.0034	4.0001
6	0.99919	3.0001	4.0001

## Examples: Solve

(1)  $5x + y - z = 10$ ;  $2x + 4y + z = 14$ ;  $x + y + 8z = 20$  by using Gauss-Seidel method correct upto three decimal places. ( $x = 2$ ;  $y = 2$ ;  $z = 2$ )

(2)  $6x + y + z = 105$ ;  $4x + 8y + 3z = 155$ ;  $5x + 4y - 10z = 65$  by using Gauss-Seidel method correct upto three decimal places. ( $x = 15$ ;  $y = 10$ ;  $z = 5$ )

(3)  $20x + y - 2z = 17$ ;  $3x + 20y - z = -18$ ;  $2x - 3y + 20z = 25$  by using Gauss-Jacobi method correct upto three decimal places. ( $x = 1$ ;  $y = -1$ ;  $z = 1$ )

(4)  $28x + 4y - z = 32; 2x + 17y + 4z = 35; x + 3y + 10z = 24$  by using Gauss-Jacobi method to correct upto two decimal places. ( $x = 0.99; y = 1.51; z = 1.85$ )

## **ROOTS OF ALGEBRAIC AND TRANSCENDENTAL EQUATIONS**

We all are aware about algebraic equation, so now we will discuss about transcendental functions and equations.

### **TRANSCENDENTAL FUNCTIONS AND EQUATION:**

A **transcendental function** "transcends" algebra in that it cannot be expressed in terms of a finite sequence of the algebraic operations of addition, multiplication, and root extraction. Examples of transcendental functions include the exponential function, the logarithm, and the trigonometric functions.

( $x^\pi, c^x, x^x, \log_c x, \sin x$  etc.)

An equation which includes transcendental functions is known as **Transcendental Equation**.

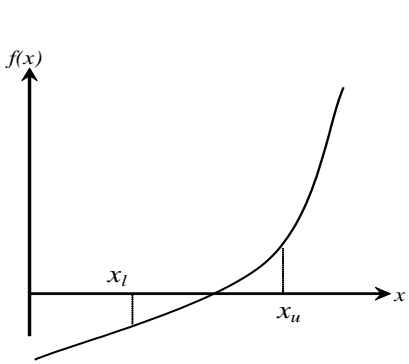
**To solve these types of equations we use the following methods:**

- 1) Bisection Method
- 2) Regula-Falsi Method(False-Position Method)
- 3) Newton-Raphson Method

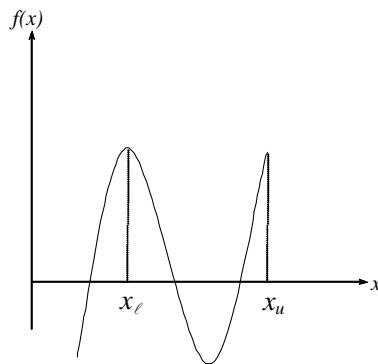
**1) Bisection Method:**One of the first numerical methods developed to find the root of a nonlinear equation  $f(x) = 0$  was the bisection method (also called Binary-Search method). The method is based on the following theorem:

**Theorem:**An equation  $f(x) = 0$ , where  $f(x)$  is a real continuous function, has at least one root between  $x_\ell$  and  $x_u$  if  $f(x_\ell)f(x_u) < 0$  ( Figure .1).Note that if  $f(x_\ell)f(x_u) > 0$ , there may or may not be any root between  $x_\ell$  and  $x_u$  (Figures 2 and 3). If  $f(x_\ell)f(x_u) < 0$ , then there may be more than one root between  $x_\ell$  and  $x_u$  (Figure 4). So the theorem only guarantees one root between  $x_\ell$  and  $x_u$ .





**Fig.1**



**Fig.2**

Fig.1 At least one root exists between two points if the function is real, continuous, and changes sign.

Fig.2 If function  $f(x)$  does not change sign between two points, roots of  $f(x) = 0$  may still exist between the two points.

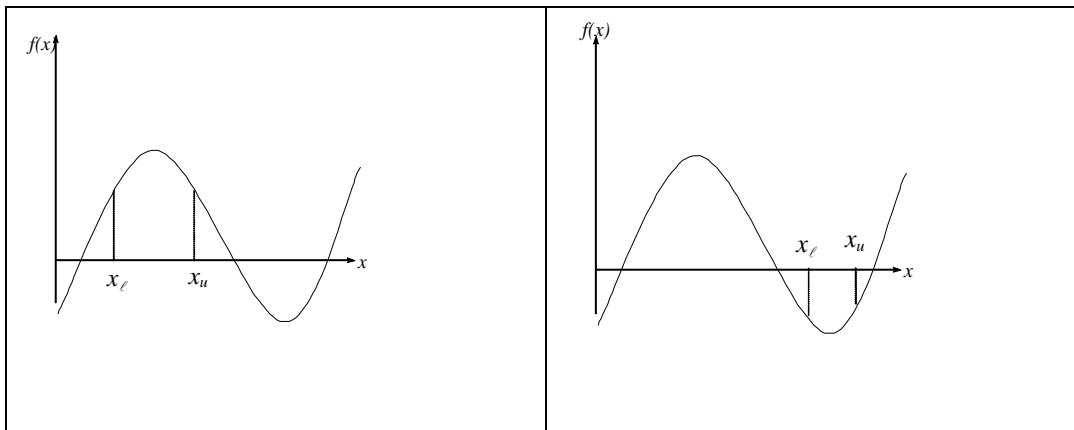


Fig.3 If the function  $f(x)$  does not change sign between two points, there may not be any roots  $f(x) = 0$  between the two points.

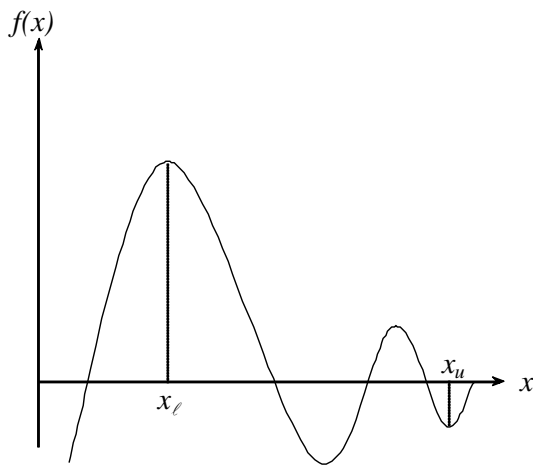


Fig .4 If the function  $f(x)$  changes sign between two points, more than one root for  $f(x) = 0$  may exist between the two points.

A general rule is:

- If the  $f(x_l)$  and  $f(x_u)$  have the same sign ( $f(x_l)f(x_u) > 0$ ):
  - There is no root between  $x_l$  and  $x_u$ .
  - There is an even number of roots between  $x_l$  and  $x_u$ .
- If the  $f(x_l)$  and  $f(x_u)$  have different signs ( $f(x_l)f(x_u) < 0$ ):
  - There is an odd number of roots.

### **Advantages of Bisection Method:**

- a) The bisection method is always convergent. Since the method brackets the root, the method is guaranteed to converge.
- b) As iterations are conducted, the interval gets halved. So one can guarantee the decrease in the error in the solution of the equation.

### **Drawbacks of Bisection Method:**

- a) The convergence of bisection method is slow as it is simply based on halving the interval.
- b) If one of the initial guesses is closer to the root, it will take larger number of iterations to reach the root.
- c) If a function  $f(x)$  is such that it just touches the x-axis such as

$$f(x) = x^2 = 0$$

it will be unable to find the lower guess,  $x_l$ , and upper guess,  $x_u$ , such that

$$f(x_l)f(x_u) < 0$$

## Working rule for Bisection Method:

The steps to apply the bisection method to find the root of the equation  $f(x) = 0$  are:

1. Choose  $x_l$  and  $x_u$  as two guesses for the root such that  $f(x_l)f(x_u) < 0$ , or in other words,  $f(x)$  changes sign between  $x_l$  and  $x_u$ .
2. Estimate the root,  $x_m$  of the equation  $f(x) = 0$  as the mid-point between  $x_l$  and  $x_u$  as

$$x_m = \frac{x_l + x_u}{2}$$

3. Now check the following
  - a. If  $f(x_l)f(x_m) < 0$ , then the root lies between  $x_l$  and  $x_m$ ; then  $x_l = x_l$  and  $x_u = x_m$ .
  - b. If  $f(x_l)f(x_m) > 0$ , then the root lies between  $x_m$  and  $x_u$ ; then  $x_l = x_m$  and  $x_u = x_u$ .
  - c. If  $f(x_l)f(x_m) = 0$ ; then the root is  $x_m$ . Stop the algorithm if this is true.
4. Find the new estimate of the root

$$x_m = \frac{x_l + x_u}{2}$$

Find the absolute approximate relative error as

$$|\varepsilon_a| = \left| \frac{x_m^{new} - x_m^{old}}{x_m^{new}} \right| \times 100$$

where

$x_m^{new}$  = estimated root from present iteration

$x_m^{old}$  = estimated root from previous iteration

5. Compare the absolute relative approximate error  $|\varepsilon_a|$  with the pre-specified relative error tolerance  $\varepsilon_s$ .  
If  $|\varepsilon_a| > \varepsilon_s$ , then go to Step 3, else stop the algorithm. Note one should also check whether the number of iterations is more than the maximum number of iterations allowed. If so, one needs to terminate the algorithm and notify the user about it.

**Example:** Find a root of the equation  $x^3 - 4x - 9 = 0$  using Bisection Method correct up to 3 decimal places.

**Sol:** Let  $f(x) = x^3 - 4x - 9$

Since  $f(-2)$  is  $-ve$  and  $f(3)$  is  $+ve$ , a root lies between 2 and 3.

Therefore, the first approximation to the root is  $x_1 = \frac{2+3}{2} = 2.5$

Thus  $f(2.5) = (2.5)^3 - 4(2.5) - 9 = -3.375$  i.e.,  $-ve$

Therefore, the root lies between  $x_1$  and 3.

Thus the second approximation to the root is  $x_2 = \frac{2.5+3}{2} = 2.75$

Thus,  $f(2.75) = (2.75)^3 - 4(2.75) - 9 = 0.7969$  i.e.,  $+ve$

$\therefore$  the root lies between  $x_1$  and  $x_2$ .

Thus the third approximation to the root is  $x_3 = \frac{2.5+2.75}{2} = 2.625$

Then  $f(2.625) = (2.625)^3 - 4(2.625) - 9 = -1.4121$  i.e.,  $-ve$

The root lies between  $x_2$  and  $x_3$  thus the fourth approximation to the root is

$$x_4 = \frac{2.625+2.75}{2} = 2.6875$$

**Repeating this process, the successive approximations are**

$$x_5 = 2.71875 \quad x_6 = 2.70313 \quad x_7 = 2.71094$$

$$x_8 = 2.70703 \quad x_9 = 2.70508 \quad x_{10} = 2.70605$$

$$x_{11} = 2.70654 \quad x_{12} = 2.70642$$

**Hence the root is 2.70642.**

**Example:** Find the real root of  $\cos \cos x - xe^x = 0$  by bisection method correct upto three decimal places.

**Solution:** Given equation  $f(x) = \cos \cos x - xe^x = 0$   
 $\therefore f(0) = 1$  and  $f(1) = -2.17$

Therefore, the root lies between 0 and 1. Hence by Bisection method:

$n$	$x_n$	$f(x_n)$	interval
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	$a = 0$	$1 > 0$	
	$b = 1$	$-2.17 < 0$	$[0, 1]$
1	0.5	0.053	$[x_1, b]$
2	0.75	-0.85	$[x_1, x_2]$
3	0.625	-0.3567	$[x_1, x_3]$
4	0.5625	-0.14	$[x_1, x_4]$
5	0.5313	-0.04	$[x_1, x_5]$
6	0.5157	0.0064	$[x_6, x_5]$
7	0.5235	-0.017	$[x_6, x_7]$
8	0.5196	-0.0056	$[x_6, x_8]$
9	0.5177	0.0003	$[x_9, x_8]$
10	0.5187	-0.0027	$[x_9, x_{10}]$
11	0.5182		

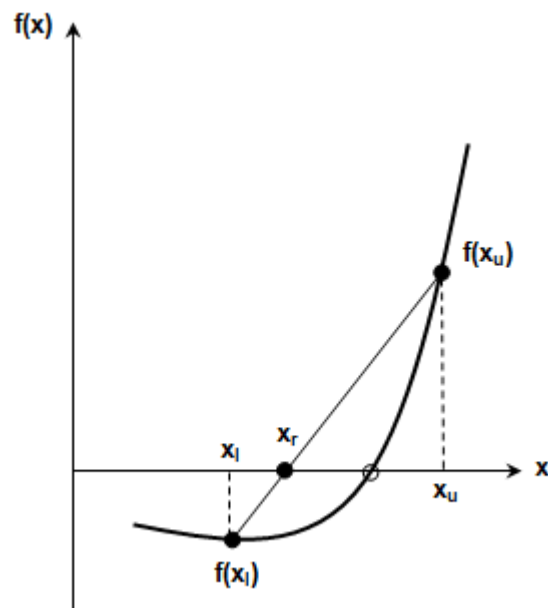
$\therefore$  The root correct to three decimal places is 0.5182

### Example:

(1) Find the real root of the equation  $3x = \cos x + 1$ , correct up to four decimal places. ( $x = 0.6071$ )

(2) Find the real root of the equation  $xe^x = 1$ , correct up to four decimal places. ( $x = 0.5671$ )

### 2) Regula-Falsi Method(False-Position Method):



**False-position method.**

A shortcoming of the bisection method is that in dividing the interval from  $x_l$  to  $x_u$  into equal halves, no account is taken of the magnitude of  $f(x_l)$  and  $f(x_u)$ . Indeed, if  $f(x_l)$  is close to zero, the root is more close to  $x_l$  than  $x_u$ .

The false position method uses this property:

A straight line joins  $f(x_l)$  and  $f(x_u)$ . The intersection of this line with the  $x$ -axis represents an improvement estimate of the root. This new root can be computed as:

$$\frac{f(x_l)}{x_r - x_l} = \frac{f(x_u)}{x_r - x_u}$$

$$x_r = x_u - \frac{f(x_u)(x_l - x_u)}{f(x_l) - f(x_u)} \quad \text{or} \quad x_r = \frac{af(b) - bf(a)}{f(b) - f(a)}$$

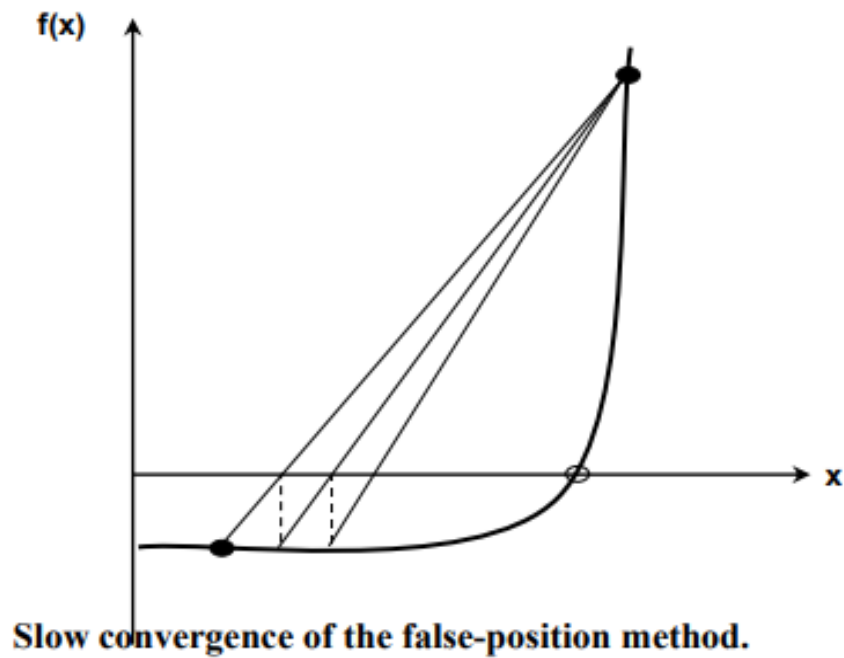
This is called the false-position formula

Then,  $x_r$  replaces the initial guess for which the function value has the same sign as  $f(x_r)$

The difference between bisection method and false-position method is that in bisection method, both limits of the interval have to change. This is not the case for false position method, where one limit may stay fixed throughout the computation while the other guess converges on the root.

### Working rule for False-Position Method (Regula-Falsi Method)

1. Define the first interval  $(a, b)$  such that solution exists between them. Check  $f(a)f(b) < 0$ .
2. Compute the first estimate of the numerical solution  $x_r$  using the above equation.
3. Find out whether the actual solution is between  $a$  and  $x_r$  or  $x_r$  and  $b$ . This is accomplished by checking the sign of the product  $f(a)f(x_r)$ 
  - If  $f(a)f(x_r) < 0$ , the solution is between  $a$  and  $x_r$ .
  - If  $f(a)f(x_r) > 0$ , the solution is between  $x_r$  and  $b$ .
4. Select the subinterval that contains the solution  $(a, x_r)$  or  $(x_r, b)$  is the new interval  $(a, b)$  and go to step no.2 and repeat the process. The method of False Position always converges to an answer, provided a root is initially bracketed in the interval  $(a, b)$ .



**Example Using the False Position method, find a root of the function**

$f(x) = e^x - 3x^2$  **to an accuracy of 5 digits. The root is known to lie between 0.5 and 1.0.**

**Sol:**

We apply the method of False Position with  $a = 0.5$  and  $b = 1.0$ . and the equation is

$$x_r = \frac{af(b) - bf(a)}{f(b) - f(a)}$$

The calculations based on the method of False Position are shown in the Table

n	A	b	$f(a)$	$f(b)$	$x_r$	$f(x_r)$
1	0.5	1	0.89872	-0.28172	0.88067	0.08577
2	0.88067	1	0.08577	-0.28172	0.90852	0.00441
3	0.90852	1	0.00441	-0.28172	0.90993	0.00022
4	0.90993	1	0.00022	-0.28172	0.91000	0.00001
5	0.91000	1	0.00001	-0.28172	0.91001	0

The root is 0.91 accurate to five digits.

**Example: Use False-position method to find a root of the function  $x^2 - x - 2 = 0$  in the range  $1 < x < 3$ , correct upto three decimal places.**

**Solution:** We have  $f(x) = x^2 - x - 2 = 0$ .

Given that  $x_0 = 1, x_1 = 3$ .  $\therefore f(x_0) = -2, f(x_1) = 4$

Using false position method, we have  $x_2 = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)}$

$n$	$x_n$	$f(x_n)$	$a$	$b$
0	1	-2		
1	3	4	1	3
2	1.6667	-0.8889	1.6667	3
3	1.9091	-0.2644	1.9091	3
4	1.9767	-0.0692	1.9767	3
5	1.9941	-0.0177	1.9941	3
6	1.9985	-0.0044	1.9985	3
7	1.9996	-0.0012	1.9996	3
8	1.9999	-0.0012		

**Example: (1)** Find a positive root of  $x^3 - 4x + 1$  correct up to three decimal places.  
( $x = 0.254$ )

**(2)** Find a positive root of  $x \log_{10} x = 1.2$  correct up to three decimal places.  
( $x = 2.740$ )

### 3) Newton-Raphson method:

Newton-Raphson method is based on the principle that if the initial guess of the root of  $f(x)=0$  is at  $x_i$ , then if one draws the tangent to the curve at  $f(x_i)$ , the point  $x_{i+1}$  where the tangent crosses the  $x$ -axis is an improved estimate of the root (Figure 3.12).

Using the definition of the slope of a function, at  $x = x_i$

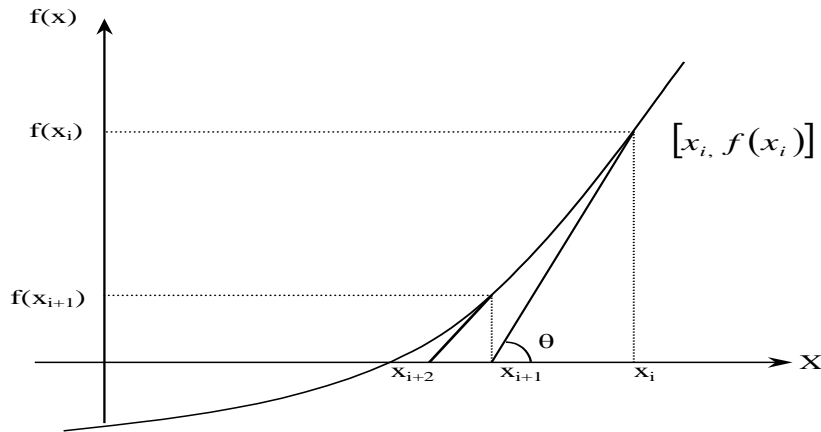
$$f'(x_i) = \frac{f(x_i) - 0}{x_i - x_{i+1}}$$

which gives

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

This equation is called the Newton-Raphson formula for solving nonlinear equations of the form  $f(x)=0$ . So starting with an initial guess,  $x_i$ , one can find the next guess,  $x_{i+1}$ , by using the above equation. One can repeat this process until one finds the root within a desirable tolerance.





**Figure. Geometrical illustration of the Newton-Raphson method.**

### **Working rule for Newton-Raphson method:**

The steps to apply using Newton-Raphson method to find the root of an equation  $f(x) = 0$  are

1. Evaluate  $f'(x)$  symbolically
2. Use an initial guess of the root,  $x_i$ , to estimate the new value of the root  $x_{i+1}$  as

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

3. Find the absolute relative approximate error,  $|\epsilon_a|$  as

$$|\epsilon_a| = \left| \frac{x_{i+1} - x_i}{x_{i+1}} \right| \times 100$$

4. Compare the absolute relative approximate error,  $|\epsilon_a|$  with the pre-specified relative error tolerance,  $\epsilon_s$ . If  $|\epsilon_a| > \epsilon_s$ , then go to step 2, else stop the algorithm. Also, check if the number of iterations has exceeded the maximum number of iterations.

### **Drawbacks of Newton-Raphson Method:**

**Divergence at inflection points:** If the selection of a guess or an iterated value turns out to be close to the inflection point of  $f(x)$ , that is, near where  $f'(x)=0$ , the roots start to diverge away from the true root.

**- Division by zero or near zero:** The formula of Newton-Raphson method is

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

Consequently if an iteration value,  $x_i$  is such that  $f'(x_i) \cong 0$ , then one can face division by zero or a near-zero number. This will give a large magnitude for the next value,  $x_{i+1}$ .

**- Root jumping:** In some case where the function  $f(x)$  is oscillating and has a number of roots, one may choose an initial guess close to a root. However, the guesses may jump and converge to some other root.

**Example: Find the root of the function  $x^3 + x - 1 = 0$  correct up to four decimal places.**

**Sol :**

$$\text{Let } f(x) = x^3 + x - 1$$

$$f(0) = -1 \text{ and } f(1) = 1$$

Since,  $f(0) < 0$  and  $f(1) > 0$ , the root lies between 0 and 1.

$$\text{Let } x_0 = 1$$

$$f'(x) = 3x^2 + 1$$

By the Newton-Raphson method ,  $x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$

$$f(x_0) = f(1) = 1$$

$$f'(x_0) = f'(1) = 4$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 1 - \frac{1}{4} = 0.75$$

$$\therefore f(x_1) = f(0.75) = 0.171875$$

$$\therefore f'(x_1) = f'(0.75) = 2.6875$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 0.75 - \frac{0.171875}{2.6875} = 0.68605$$

$$\therefore f(x_2) = f(0.68605) = 0.00894$$

$$\therefore f'(x_2) = f'(0.68605) = 2.41198$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 0.68605 - \frac{0.00894}{2.41198} = 0.68234$$

$$\therefore f(x_3) = f(0.68234) = 0.000028$$

$$\therefore f'(x_3) = f'(0.68234) = 2.39676$$

$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)} = 0.68234 - \frac{0.000028}{2.39676} = 0.68233$$

Since  $x_3$  and  $x_4$  are same up to four decimal places, the root is 0.6823.

**Example:** Find a real root of  $xe^x = 2$ , correct upto three decimal places, by using Newton-Raphson's method.

**Solution:** Let  $f(x) = xe^x - 2 = 0$

$$\therefore f(0) = -2, f(1) = 0.71$$

$\therefore$  The root lies between 0 and 1.

Using Newton-Raphson's method, we have

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, n = 0, 1, 2, \dots$$

Choose the initial approximation  $x_0 = 0.5$

Here,  $f'(x) = e^x + xe^x = e^x(1 + x)$

$n$	$x_n$	$f(x_n)$	$f'(x_n)$
0	0.5	-1.1756	2.4731
1	0.9754	0.5867	5.2392
2	0.8634	0.0473	4.4185
3	0.8527	0.0004	4.3464
4	0.8526		

$\therefore$  The approximate value to the root = 0.8526

**Example:**

(1) Find the root of the function  $x^4 - x - 10 = 0$  correct up to three decimal places.

( $x = 2.7406$ )

(2) Find the root of the function  $e^x = 3x$  correct up to three decimal places.

( $x = 1.512$ )

### • Some deduction from Newton-Raphson's Formula:

We can derive the following useful results from the Newton-Raphson's Formula

$$\frac{1}{N}$$

1) Iteration formula for  $\frac{1}{N}$  :

$$\text{Let } x = \frac{1}{N} \Rightarrow N = \frac{1}{x} \Rightarrow \frac{1}{x} - N = 0$$

Also let  $f(x) = \frac{1}{x} - N$  and  $f'(x) = -\frac{1}{x^2}$

By the Newton-Raphson method ,

$$\begin{aligned} x_{i+1} &= x_i - \frac{f(x_i)}{f'(x_i)} = x_i - \frac{\left(\frac{1}{x_i} - N\right)}{\left(-\frac{1}{x_i^2}\right)} \\ &= x_i + x_i^2 \left(\frac{1}{x_i} - N\right) \\ &= x_i(2 - Nx_i) \end{aligned}$$

**2) Iteration formula for  $\sqrt{N}$  :**

Let  $x = \sqrt{N} \Rightarrow N = x^2 \Rightarrow x^2 - N = 0$

Also let  $f(x) = x^2 - N$  and  $f'(x) = 2x$

By the Newton-Raphson method ,

$$\begin{aligned} x_{i+1} &= x_i - \frac{f(x_i)}{f'(x_i)} = x_i - \frac{(x_i^2 - N)}{(2x_i)} \\ &= x_i - \frac{1}{2} \left( x_i - \frac{N}{x_i} \right) \\ &= \frac{1}{2} \left( x_i + \frac{N}{x_i} \right) \end{aligned}$$

**3) Iteration formula for  $\frac{1}{\sqrt{N}}$  :**

Let  $x = \frac{1}{\sqrt{N}} \Rightarrow x^2 = \frac{1}{N} \Rightarrow x^2 - \frac{1}{N} = 0$

Also let  $f(x) = x^2 - \frac{1}{N}$  and  $f'(x) = 2x$

By the Newton-Raphson method ,

$$\begin{aligned}
 x_{i+1} &= x_i - \frac{f(x_i)}{f'(x_i)} = x_i - \frac{\left(x_i^2 - \frac{1}{N}\right)}{(2x_i)} \\
 &= \left( \frac{2x_i^2 - x_i^2 + \frac{1}{N}}{2x_i} \right) \\
 &= \frac{1}{2} \left( x_i + \frac{1}{x_i N} \right)
 \end{aligned}$$

**4) Iteration formula for  $\sqrt[k]{N}$  :**

$$\text{Let } x = \sqrt[k]{N} \Rightarrow N = x^k \Rightarrow x^k - N = 0$$

Also let  $f(x) = x^k - N$  and  $f'(x) = kx^{k-1}$

By the Newton-Raphson method ,

$$\begin{aligned}
 x_{i+1} &= x_i - \frac{f(x_i)}{f'(x_i)} = x_i - \frac{(x_i^k - N)}{(kx_i^{k-1})} \\
 &= \left( \frac{(x_i)(kx_i^{k-1}) - (x_i^k - N)}{kx_i^{k-1}} \right) \\
 &= \frac{1}{k} \left( (k-1)x_i + \frac{N}{x_i^{k-1}} \right)
 \end{aligned}$$

**Example: Evaluate the following (correct to four decimal places) by Newton's iteration method:**

$$(i) \frac{1}{31} \quad (ii) \sqrt{24} \quad (iii) 1/\sqrt{14} \quad (iv) \sqrt[3]{24}$$

**Sol:**  $(i) \frac{1}{31}$

Taking  $N=31$  in the equation  $x_{i+1} = x_i(2 - Nx_i)$

We have  $x_{i+1} = x_i(2 - 31x_i)$  , Since an approximate value of  $1/30=0.03$ , we take  $x_0 = 0.03$  .

Then

$$x_1 = x_0(2 - 31x_0) = 0.03(2 - 31(0.03)) = 0.0321$$

$$x_2 = x_1(2 - 31x_1) = 0.0321(2 - 31(0.0321)) = 0.032257$$

$$x_3 = x_2(2 - 31x_2) = 0.032257(2 - 31(0.032257)) = 0.03226$$

Since  $x_2 = x_3$  up to 4 decimal places, we have  $1/31 = 0.0323$

$$(ii) \sqrt{28}$$

Taking  $N = 24$  in the equation 
$$x_{i+1} = \frac{1}{2} \left( x_i + \frac{N}{x_i} \right)$$

We have  $x_{i+1} = \frac{1}{2} \left( x_i + \frac{28}{x_i} \right)$ , Since the value of  $\sqrt{28} = 5$ , we take  $x_0 = 5$ .

Then

$$x_1 = \frac{1}{2} \left( x_0 + \frac{28}{x_0} \right) = \frac{1}{2} \left( 5 + \frac{28}{5} \right) = 5.3$$

$$x_2 = \frac{1}{2} \left( x_1 + \frac{28}{x_1} \right) = \frac{1}{2} \left( 5.3 + \frac{28}{5.3} \right) = 5.29151$$

$$x_3 = \frac{1}{2} \left( x_2 + \frac{28}{x_2} \right) = \frac{1}{2} \left( 5.29151 + \frac{28}{5.29151} \right) = 5.29150$$

Since  $x_2 = x_3$  up to 4 decimal places, we have  $\sqrt{28} = 5.2915$ .

$$(iii) 1/\sqrt{14}$$

Taking  $N = 14$  in the equation 
$$x_{i+1} = \frac{1}{2} \left( x_i + \frac{1}{x_i N} \right)$$

We have  $x_{i+1} = \frac{1}{2} \left( x_i + \frac{1}{x_i 14} \right)$ , Since the value of  $1/\sqrt{14} = 1/4 = 0.25$ , we take  $x_0 = 0.25$ .

Then

$$x_1 = \frac{1}{2} \left( x_0 + \frac{1}{x_0 14} \right) = \frac{1}{2} \left( (0.25) + \frac{1}{(0.25)14} \right) = 0.26785$$

$$x_2 = \frac{1}{2} \left( x_1 + \frac{1}{x_1 14} \right) = \frac{1}{2} \left( (0.26785) + \frac{1}{(0.26785)14} \right) = 0.2672618$$

$$x_3 = \frac{1}{2} \left( x_2 + \frac{1}{x_2 14} \right) = \frac{1}{2} \left( (0.2672618) + \frac{1}{(0.2672618)14} \right) = 0.2672612$$

Since  $x_2 = x_3$  up to 4 decimal places, we have  $1/\sqrt{14} = 0.2673$ .

(iv)  $\sqrt[3]{24}$

Taking  $N = 24$  and  $k = 3$  in the equation  $x_{i+1} = \frac{1}{k} \left( (k-1)x_i + \frac{N}{x_i^{k-1}} \right)$

We have  $x_{i+1} = \frac{1}{3} \left( (2)x_i + \frac{24}{x_i^2} \right)$ , Since the value of  $27^{\frac{1}{3}} = 3$ , we take  $x_0 = 3$ .

Then

$$x_1 = \frac{1}{3} \left( (2)x_0 + \frac{24}{x_0^2} \right) = \frac{1}{3} \left( (2)(3) + \frac{24}{(3)^2} \right) = 2.8889$$

$$x_2 = \frac{1}{3} \left( (2)x_1 + \frac{24}{x_1^2} \right) = \frac{1}{3} \left( (2)(2.8889) + \frac{24}{(2.8889)^2} \right) = 2.88451$$

$$x_3 = \frac{1}{3} \left( (2)x_2 + \frac{24}{x_2^2} \right) = \frac{1}{3} \left( (2)(2.88451) + \frac{24}{(2.88451)^2} \right) = 2.8845$$

Since  $x_2 = x_3$  up to 4 decimal places, we have  $\sqrt[3]{24} = 2.8845$

Since  $x_2 = x_3$  up to 4 decimal places, we have  $\sqrt[3]{24} = 2.8845$