LECTURE-35

Course: Mathematics-I **Course Code: KAS-103T**

Module-4: Multivariable Calculus-I **Teacher: Dr. S. P. Gupta**

Topic: Triple integral.

LO: ***Illustrate*** the working methods of Triple Integration and ***apply*** for ***finding*** Triple integral of the given region.

Triple Integral :- Consider a function $f(x,y,z)$, which is Continuous at every point of region V.

then $$\iiint_V f(x,y,z)\,dv = \int_{x_1}^{x_2}\int_{y_1}^{y_2}\int_{z_1}^{z_2} f(x,y,z)\,dz\,dy\,dx.$$

order of integration depends upon the limit.

let z_1, z_2 be function of x and y. ie $z_1 = \phi_1(x,y)$, $z_2 = \phi_2(x,y)$

y_1, y_2 be a function of x. ie $y_1 = \psi_1(x)$, $y_2 = \psi_2(x)$

and x_1, x_2 be Constant.

$$\iiint_V f(x,y,z)\,dv = \int_{x_1}^{x_2}\left[\int_{y_1=\psi_1(x)}^{y_2=\psi_2(x)}\left[\int_{z_1=\phi_1(x,y)}^{z_2=\phi_2(x,y)} dz\right] dy\right] dx$$

Example :- ① Evaluate $\int_0^1\int_0^1\int_0^1 e^{x+y+z}\,dx\,dy\,dz$

Sol^n-
$$\int_0^1\int_0^1 \left[e^{x+y+z}\right]_0^1 dy\,dz$$

$$= \int_0^1\int_0^1 \left(e^{1+y+z} - e^{y+z}\right) dy\,dz$$

$$= \int_0^1 \left[e^{1+y+z} - e^{y+z}\right]_0^1 dz$$

$$= \int_0^1 \left(e^{2+z} - e^{1+z} - e^{1+z} + e^{z}\right) dz$$

$$= \int_0^1 \left(e^{2+z} - 2e^{1+z} + e^{z}\right) dz$$

$$= \left[e^{2+z} - 2e^{1+z} + e^{z}\right]_0^1$$

$$= e^3 - 2e^2 + e^1 - e^2 + 2e - 1$$

$$= \underline{e^3 - 3e^2 + 3e - 1} = (e-1)^3.$$

(2) Evaluate $\iiint_R (x-2y+z)\,dz\,dy\,dx$, where R is the region determined by $0 \le x \le 1$, $0 \le y \le x^2$, $0 \le z \le x+y$.

Sol^n:
$$\int_0^1 \int_0^{x^2} \int_0^{x+y} (x-2y+z)\,dz\,dy\,dx.$$

$$= \int_0^1 \int_0^{x^2} \left[zx - 2yz + \frac{z^2}{2}\right]_0^{x+y} dy\,dx.$$

$$= \int_0^1 \int_0^{x^2} \left(x(x+y) - 2y(x+y) + \frac{(x+y)^2}{2}\right] dy\,dx.$$

$$= \int_0^1 \int_0^{x^2} \left[x^2 + xy - 2xy - 2y^2 + \frac{x^2+y^2+2xy}{2}\right] dy\,dx.$$

$$= \int_0^1 \int_0^{x^2} \left(\frac{3x^2}{2} - \frac{3y^2}{2}\right) dy\,dx.$$

$$= \int_0^1 \left[\frac{3x^2y}{2} - \frac{3y^3}{6}\right]_0^{x^2} dx$$

$$= \int_0^1 \left(\frac{3x^4}{2} - \frac{3x^6}{6}\right) dx$$

$$= \left[\frac{3}{2}\frac{x^5}{5} - \frac{1}{2}\frac{x^7}{7}\right]_0^1$$

$$= \frac{3}{10} - \frac{1}{14} = \frac{42-10}{140} = \frac{\cancel{32}\,8}{\cancel{140}\,35} = \frac{8}{35}$$

(3) Evaluate $\iiint_R dx\,dy\,dz$, R is the region bounded by $x+y+z \le a$, $x \ge 0$, $y \ge 0$, $z \ge 0$

Sol^n. The limits are.

$0 \le z \le a-x-y$

$0 \le y \le a-x$

$0 \le x \le a$

$$\int_{x=0}^{a}\int_{y=0}^{a-x}\int_{z=0}^{a-x-y} dz\,dy\,dx$$

$$\int_{x=0}^{a}\int_{y=0}^{a-x}\left[z\right]_{0}^{a-x-y} dy\,dx$$

$$\int_{x=0}^{a}\int_{y=0}^{a-x}\left[a-x-y\right] dy\,dx$$

$$\int_{x=0}^{a}\left[ay-xy-\frac{y^2}{2}\right]_{0}^{a-x} dx$$

$$\int_{x=0}^{a}\left[a(a-x)-x(a-x)-\frac{(a-x)^2}{2}\right] dx$$

$$\int_{x=0}^{a}\left\{a^2-ax-ax+x^2-\left(\frac{a^2+x^2-2ax}{2}\right)\right\} dx$$

$$\int_{x=0}^{a}\left(a^2+x^2-2ax-\frac{a^2}{2}-\frac{x^2}{2}+ax\right) dx$$

$$\int_{x=0}^{a}\left(\frac{a^2}{2}+\frac{x^2}{2}-ax\right) dx$$

$$\left[\frac{a^2}{2}x+\frac{x^3}{6}-\frac{ax^2}{2}\right]_{0}^{a}$$

$$\cancel{\frac{a^3}{2}}+\frac{a^3}{6}-\cancel{\frac{a^3}{2}}$$

$$=\frac{a^3}{6}$$

Ans

(Excercise):- (1) Evaluate $\iiint_R (x^2+y^2+z^2)\,dx\,dy\,dz$, where R denote the region $x \geq 0, y \geq 0, z \geq 0$ and $x+y+z=a$.

(2) Evaluate $\iiint x^2yz\,dx\,dy\,dz$, throughout volume bounded by planes. $x=0, y=0, z=0$ and $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$

(3) Evaluate $\iiint_R (x+y+z)\,dx\,dy\,dz$ bounded by the region $0 \leq x \leq 1,\ 1 \leq y \leq 2,\ 2 \leq z \leq 3$. $\left[\frac{9}{2}\right]$

(4) Evaluate $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} \frac{1}{\sqrt{1-x^2-y^2-z^2}}\,dz\,dy\,dx$. $\left[\frac{\pi^2}{8}\right]$