Topic - 5.2

Cauchy - Gaursat Theorem or Cauchy's Integral Theorem :-

Statement :- If a function $f(z)$ is analytic and its derivative $f'(z)$ continuous at all points inside and on a simple closed curve c, then $\int_c f(z)dz = 0$.

Pf- Let the region enclosed by the curve c be R and let $f(z) = u+iv$, $z = x+iy$, $dz = dx+idy$.

$$\therefore \int_c f(z)dz = \int_c (u+iv)(dx+idy) = \int_c (udx - vdy) + i\int_c (vdx+udy)$$

We know that the Green's theorem is.

$$\int_c Mdx + Ndy = \iint_S \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy.$$

$$\Rightarrow \iint_S \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dxdy + i\iint_S \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dxdy.$$

Replacing $-\frac{\partial v}{\partial x}$ by $\frac{\partial u}{\partial y}$ and $\frac{\partial v}{\partial y}$ by $\frac{\partial u}{\partial x}$ by C-R eqn, we get

$$\int_c f(z)dz = \iint_R \left(\frac{\partial u}{\partial y} - \frac{\partial u}{\partial y}\right) dxdy + i\iint_R \left(\frac{\partial u}{\partial x} - \frac{\partial u}{\partial x}\right) dxdy.$$

$|z-a| = r$ is called equation of the circle.

$$\Rightarrow 0 + i0 = 0 \text{ Proved}$$

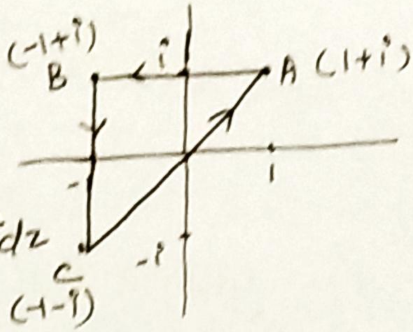
Poles of $f(z)$:- Poles are given by equating to zero the denominator of $f(z)$.

NOTE ① If there is no pole inside and on the contour then the value of the integral of the function is zero.

② → given c always compare with $|z-a| = r$, where 'a' is centre and 'r' is radius.

Ex ①- Verify Cauchy's Theorem for the function $f(z) = e^{iz}$ along the boundary of the triangle with vertices at the points $1+i$, $-1+i$ and $-1-i$.

Integration of e^{iz} = Integration along the line AB, BC and CA.



$$= \underbrace{\int_{AB} e^{iz}\,dz}_{I_1} + \underbrace{\int_{BC} e^{iz}\,dz}_{I_2} + \underbrace{\int_{CA} e^{iz}\,dz}_{I_3} \quad \text{--- (1)}$$

$$I_1 = \int_{1+i}^{-1+i} e^{iz}\,dz = \left[\frac{e^{iz}}{i}\right]_{1+i}^{-1+i} = \frac{1}{i}\left[e^{i(-1+i)} - e^{i(1+i)}\right]$$
$$= \frac{1}{i}\left[e^{-i-1} - e^{i-1}\right]$$

$$I_2 = \int_{-1+i}^{-1-i} e^{iz}\,dz = \left[\frac{e^{iz}}{i}\right]_{-1+i}^{-1-i} = \frac{1}{i}\left[e^{i(-1-i)} - e^{i(-1+i)}\right]$$
$$= \frac{1}{i}\left[e^{-i+1} - e^{-i-1}\right]$$

$$I_3 = \int_{-1-i}^{1+i} e^{iz}\,dz = \left[\frac{e^{iz}}{i}\right]_{-1-i}^{1+i} = \frac{1}{i}\left[e^{i(1+i)} - e^{i(-1-i)}\right]$$
$$= \frac{1}{i}\left[e^{i-1} - e^{-i+1}\right]$$

$$\therefore \int_{ABC} e^{iz}\,dz = I_1 + I_2 + I_3$$
$$\Rightarrow \frac{1}{i}\left[\cancel{e^{-i-1}} - \cancel{e^{i-1}} + \cancel{e^{-i+1}} - \cancel{e^{-i-1}} + \cancel{e^{i-1}} - \cancel{e^{-i+1}}\right]$$

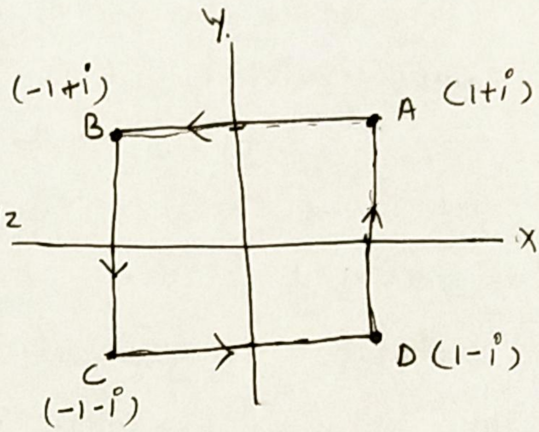
$$\boxed{\int_{ABC} e^{iz}\,dz = 0} \quad \text{--- (1)}$$

If a function $f(z)$ is analytic and its derivative $f'(z)$ continuous at all points inside and on a simple closed curve C,

then $\int_C f(z)\,dz = 0$ --- (2)

Hence from (2) & (3) Cauchy's theorem verified.

Ex-2:- Verify Cauchy's theorem for the function $f(z) = 3z^2 + iz - 4$ along the parameter of square with vertices $1 \pm i$, $-1 \pm i$.



$$\int_{ABCD} (3z^2 + iz - 4)\, dz =$$

$$\underbrace{\int_{AB} (3z^2 + iz - 4)\, dz}_{I_1} + \underbrace{\int_{BC} (3z^2 + iz - 4)\, dz}_{I_2}$$

$$+ \underbrace{\int_{CD} (3z^2 + iz - 4)\, dz}_{I_3} +$$

$$\underbrace{\int_{DA} (3z^2 + iz - 4)\, dz}_{I_4}$$

$$I_1 = \int_{1+i}^{-1+i} (3z^2 + iz - 4)\, dz = \left[z^3 + i\frac{z^2}{2} - 4z \right]_{1+i}^{-1+i}$$

$$= \left\{ (-1+i)^3 + i\frac{(-1+i)^2}{2} - 4(-1+i) \right\} - \left\{ (1+i)^3 + i\frac{(1+i)^2}{2} - 4(1+i) \right\} \quad \text{— (1)}$$

Similarly.

$$I_2 = \int_{-1+i}^{-1-i} (3z^2 + iz - 4)\, dz$$

$$= \left\{ (-1-i)^3 + \frac{i}{2}(-1-i)^2 - 4(-1-i) \right\} - \left\{ (-1+i)^3 + \frac{i}{2}(-1+i)^2 - 4(-1+i) \right\} \quad \text{— (2)}$$

$$I_3 = \int_{-1-i}^{1-i} (3z^2 + iz - 4)\, dz$$

$$= \left\{ (1-i)^3 + \frac{i}{2}(1-i)^2 - 4(1-i) \right\} - \left\{ (-1-i)^3 + \frac{i}{2}(-1-i)^2 - 4(-1-i) \right\} \quad \text{— (3)}$$

$$I_4 = \int_{1-i}^{1+i} (3z^2 + iz - 4)\, dz$$

$$= \left\{ (1+i)^3 + \frac{i}{2}(1+i)^2 - 4(1+i) \right\} - \left\{ (1-i)^3 + \frac{i}{2}(1-i)^2 - 4(1-i) \right\} \quad \text{— (4)}$$

Adding (1), (2), (3), 4.

$$I_1 + I_2 + I_3 + I_4 = 0$$

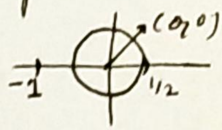
In the square ABCD there is no pole, so by Cauchy ~~the~~ theorem

$$\int_{ABCD} (3z^2 + iz - 4)\, dz = 0$$

Hence Cauchy Goursat theorem satisfied

Ex-③- Find the integral $\int_C \frac{3z^2+7z+1}{z+1}\,dz$, where C is the circle $|z| = \frac{1}{2}$.

Sol: Poles of the integrand are given by putting the denominator equal to zero.



$$z+1 = 0 \Rightarrow z = -1$$

the given circle $|z| = \frac{1}{2}$ with centre at $z=0$ and radius $\frac{1}{2}$ $\boxed{|z-a| = r}$

does not enclose any singularity of the given function

Hence $\int_C \frac{3z^2+7z+1}{z+1}\,dz = 0$ (by Cauchy Goursat theorem)

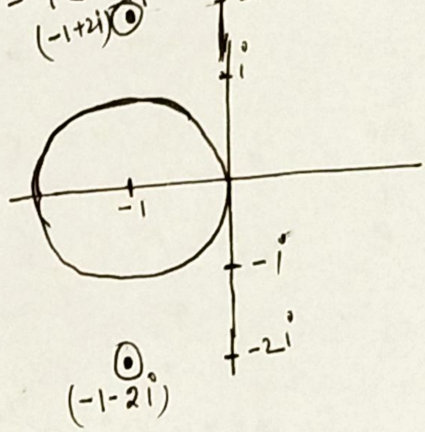
Ex-④- Find the value of $\int_C \frac{z+4}{z^2+2z+5}\,dz$, if C is the circle $|z+1| = 1$

Sol: Poles of the integrand are given by putting the denominator equal to zero.

$$z^2+2z+5 = 0$$

$$z = \frac{-2 \pm \sqrt{4-20}}{2} = \frac{-2 \pm 4i}{2} = -1 \pm 2i$$

The given circle $|z+1| = 1$ with centre at $z=-1$ and radius is 1. $\rightarrow |z-a| = r$



Clearly the given circle does not enclose any singularity of the function. $\frac{z+4}{z^2+2z+5}$

$\therefore$ By Cauchy Goursat theorem

$$\int_C \frac{z+4}{z^2+2z+5}\,dz = 0$$

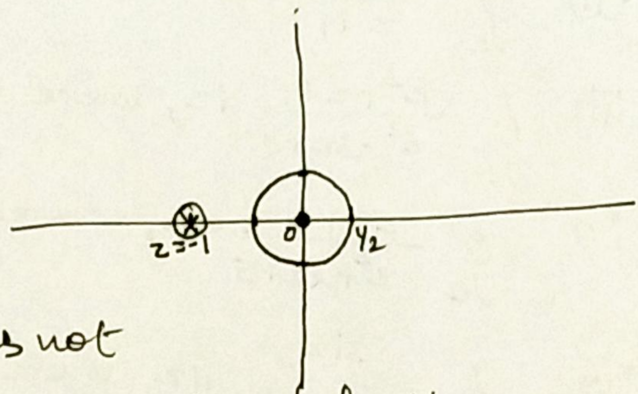
Ex-5 :- Evaluate $\int_C \frac{e^{-z}}{z+1}\,dz$, where C is the circle $|z| = \frac{1}{2}$.

Sol^n: Poles of Integrand are given by putting the denominator equal to zero.

i.e. $z+1=0 \Rightarrow \boxed{z=-1} \rightarrow$ poles.

The given circle $|z| = \frac{1}{2}$ have

Centre 0 and radius is $\frac{1}{2}$.



The given pole does not enclose any singularities of the function $\frac{e^{-z}}{z+1}$.

$$\therefore \boxed{\int_C \frac{e^{-z}}{z+1}\,dz = 0.}$$

Ans

Exercise for Cauchy's Integral Theorem

① Verify Cauchy's theorem by Integrating the function (z^3+iz) along the boundary of the rectangle with vertices $+1, -1, i, -i$. [ANS- 0 in all cases]

② Verify Cauchy's theorem by integrating z^3, along the boundary of a square with vertices at $1+i$, $1-i$, $-1+i$, $-1-i$. [ANS-0]

③ Evaluate :- (i) $\oint_C \frac{z^2+5}{z-3}\,dz$, where C is the circle $|z|=1$

(ii) $\oint_C \frac{3z^2+7z+1}{z+1}\,dz$, where C is the circle $|z+i|=1$

(iii) $\oint_C \frac{z^3+z+1}{z^2-3z+2}\,dz$, where C is the ellipse $4x^2+9y^2=1$

(iv) $\oint_C \frac{z+4}{z^2+2z+5}\,dz$, where C is the circle $|z+1|=1$

(v) $\oint_C \frac{e^{3iz}}{(z+\pi)^3}\,dz$, where C is the circle $|z-\pi|=3.2$.