LECTURE-25

Course: Mathematics-I **Course Code: KAS-103T**
Module-3: Differential Calculus-II **Teacher: Dr. S. P. Gupta**

Topic: Euler's Theorem for homogeneous functions

LO-1: ***Define*** Homogeneous function.

LO-2: ***State*** and ***Derive*** Euler's theorem for first and second order results.

Homogeneous function :- A function $f(x,y)$ is said to be Homogeneous of degree (or order) n in the variables x and y if it can be expressed in the form $x^n \phi\left(\frac{y}{x}\right)$ or $y^n \phi\left(\frac{x}{y}\right)$.

An Alternative test for a function $f(x,y)$ to be Homogeneous of degree n is that

$$f(tx, ty) = t^n f(x,y)$$

example :- If $f(x,y) = \dfrac{x+y}{\sqrt{x}+\sqrt{y}} = \dfrac{x\left(1+\frac{y}{x}\right)}{\sqrt{x}\left(1+\sqrt{\frac{y}{x}}\right)} = x^{1/2}\phi\left(\frac{y}{x}\right)$

$\Rightarrow f(x,y)$ is a Homogeneous function of degree $\frac{1}{2}$ in x and y.

Euler's Theorem :-

① **Statement :-** If u is a Homogeneous function of degree n in x and y

then $$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu$$

Pf :- Since u is a Homogeneous function, then

$$u = x^n f\left(\frac{y}{x}\right)$$

$$\frac{\partial u}{\partial x} = nx^{n-1} f\left(\frac{y}{x}\right) + x^n f'\left(\frac{y}{x}\right)\left(-\frac{y}{x^2}\right)$$

$$x\frac{\partial u}{\partial x} = nx^n f\left(\frac{y}{x}\right) - x^{n-1}\cdot y\, f'\left(\frac{y}{x}\right) \quad \text{—— ①}$$

Also $$\frac{\partial u}{\partial y} = x^n f'\left(\frac{y}{x}\right)\cdot\frac{1}{x}$$

$$y\frac{\partial u}{\partial y} = x^{n-1} y\, f'\left(\frac{y}{x}\right) \quad \text{—— ②}$$

adding eqⁿ ① and ②, we get

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nx^n f\left(\frac{y}{x}\right) - \cancel{x^{n-1} y f'\left(\frac{y}{x}\right)} + \cancel{x^{n-1} y f'\left(\frac{y}{x}\right)}$$

$$= nx^n f\left(\frac{y}{x}\right)$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu$$

Proved

(2) Statement:- If u is a Homogeneous function of degree n in x and y, then.

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=n(n-1)u.$$

Pf:- we know that Euler's theorem is.

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=nu \quad \text{—} (1)$$

diffⁿ eqⁿ (1) partially w r to x, we get

$$x\frac{\partial^2 u}{\partial x^2}+\frac{\partial u}{\partial x}\cdot 1+y\frac{\partial^2 u}{\partial x\partial y}=n\frac{\partial u}{\partial x} \quad \text{—} (2)$$

again diffⁿ eqⁿ (1) partially w r to y, we get.

$$x\frac{\partial^2 u}{\partial y\partial x}+y\frac{\partial^2 u}{\partial y^2}+\frac{\partial u}{\partial y}\cdot 1=n\frac{\partial u}{\partial y} \quad \text{—} (3)$$

Multiplying eqⁿ (2) by x and eqⁿ (3) by y, adding them.

$$x^2\frac{\partial^2 u}{\partial x^2}+x\frac{\partial u}{\partial x}+xy\frac{\partial^2 u}{\partial x\partial y}+xy\frac{\partial^2 u}{\partial y\partial x}+y^2\frac{\partial^2 u}{\partial y^2}+y\frac{\partial u}{\partial y}=nx\frac{\partial u}{\partial x}+ny\frac{\partial u}{\partial y}$$

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}+nu=n^2u$$

$$\left[\frac{\partial^2 u}{\partial x\partial y}=\frac{\partial^2 u}{\partial y\partial x}\right]$$

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=n(n-1)u.$$

Proved

Deduction from Euler's Theorem :-

(1) Statement:- If $f(u)=v(x,y,z)$, where v is a Homogeneous function in x,y,z of degree n, then

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=n\frac{f(u)}{f'(u)}$$

Pf- we know that the Euler's theorem for first derivative is.

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=nu \quad \text{—} (1)$$

Since $u=f(u)$ is a Homogeneous function, therefore

from (1)

$$x\frac{\partial}{\partial x}(f(u))+y\frac{\partial}{\partial y}(f(u))=nf(u)$$

$$xf'(u)\frac{\partial u}{\partial x}+yf'(u)\frac{\partial u}{\partial y}=nf(u)$$

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=n\frac{f(u)}{f'(u)}$$

Proved

② **Statement** :- If $f(u)$ is a function of $u(x,y,z)$ then

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \phi(u)[\phi'(u)-1]$$

where $\phi(u) = n\dfrac{f(u)}{f'(u)}$

Pf- let $\dfrac{nf(u)}{f'(u)} = \phi(u)$

from Deduction ①, we have

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \phi(u) \quad \text{——— ①}$$

differentiate eqⁿ ① partially w r to x and y. , we get

$$x\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} + y\frac{\partial^2 u}{\partial x\partial y} = \phi'(u)\frac{\partial u}{\partial x}$$

$$x\frac{\partial^2 u}{\partial x^2} + y\frac{\partial^2 u}{\partial x\partial y} = [\phi'(u)-1]\frac{\partial u}{\partial x} \quad \text{— ②}$$

and.

$$x\frac{\partial^2 u}{\partial y\partial x} + y\frac{\partial^2 u}{\partial y^2} + \frac{\partial u}{\partial y} = \phi'(u)\frac{\partial u}{\partial y}$$

$$x\frac{\partial^2 u}{\partial y\partial x} + y\frac{\partial^2 u}{\partial y^2} = [\phi'(u)-1]\frac{\partial u}{\partial y} \quad \text{——— ③}$$

Multiplying eqⁿ ② by x and eqⁿ ③ by y and adding them,

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = [\phi'(u)-1]\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right)$$

$$= \phi(u)[\phi'(u)-1] \quad [\text{from ①}]$$

(Example) :-

① If $u = (x^{1/4}+y^{1/4})(x^{1/5}+y^{1/5})$. Apply Euler's theorem to find the value of

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}.$$

Solⁿ-

given $u = (x^{1/4} + y^{1/4})(x^{1/5} + y^{1/5})$

$$u = x^{1/4}\left[1+\left(\frac{y}{x}\right)^{1/4}\right] x^{1/5}\left[1+\left(\frac{y}{x}\right)^{1/5}\right]$$

$$= x^{\frac{1}{4}+\frac{1}{5}}\left[1+\left(\frac{y}{x}\right)^{1/4}\right]\left[1+\left(\frac{y}{x}\right)^{1/5}\right]$$

$$= x^{9/20}\,\phi\left(\frac{y}{x}\right)$$

$$n = \frac{9}{20}$$

$\therefore$ u is a Homogeneous function of degree $\frac{9}{20}$

Hence By Euler's theorem.

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu$$

$$= \frac{9}{20}u$$

Ans

(Problems Based on Euler's Theorem)

① If $u = x^4 y^2 \sin^{-1}\left(\frac{x}{y}\right)$

$v = \log x - \log y$

then find the value of $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y}$.

Solⁿ:- Here $u = x^4 y^2 \sin^{-1}\left(\frac{x}{y}\right)$

$$u(tx, ty) = (tx)^4 (ty)^2 \sin^{-1}\left(\frac{tx}{ty}\right)$$

$$= t^6 x^4 y^2 \sin^{-1}\left(\frac{x}{y}\right)$$

$$u(tx,ty) = t^6 u(x,y)$$

$\therefore$ By Alternative Test u is a Homogeneous function of degree 6.

Now $v = \log x - \log y$.

$$v(tx,ty) = \log\left(\frac{tx}{ty}\right)$$

$$= t^0 v(x,y)$$

$\therefore$ By Alternative Test v is a Homogeneous function of degree 0.

By Euler's Theorem.

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 6u \quad \text{— ①}$$

$$x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = 0 \quad \text{— ②}$$

adding ① and ②

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 6u$$

$$= 6x^4y^2 \sin^{-1}\left(\frac{x}{y}\right)$$

Ans.

② If $u = \cos\left(\frac{xy + yz + zx}{x^2+y^2+z^2}\right)$, Prove that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0.$$

Sol^n: given that $u(x,y,z) = \cos\left(\frac{xy+yz+zx}{x^2+y^2+z^2}\right)$

$$u(tx,ty,tz) = \cos\left(\frac{\cancel{t^2}(xy+yz+zx)}{\cancel{t^2}(x^2+y^2+z^2)}\right)$$

$$= t^0 \cos\left(\frac{xy+yz+zx}{x^2+y^2+z^2}\right)$$

$$= t^0\, u(x,y,z)$$

Hence u is a Homogeneous function of degree 0.

therefore by Eulers Theorem

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = nu$$

$$= \underline{\underline{0}} \text{ Ans}$$

③ Show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 2\tan u$

where $u = \sin^{-1}\left(\frac{x^3+y^3+z^3}{ax+by+cz}\right)$

Sol^n:-

$$\sin u = \frac{x^3+y^3+z^3}{ax+by+cz} = f(x,y,z)$$

$$f(tx,ty,tz) = \frac{t^3(x^3+y^3+z^3)}{t(ax+by+cz)}$$

$$= t^2 f(x,y,z)$$

$\therefore$ $f(x,y,z)$ is a Homogeneous function of degree 2.

$\therefore$ By Euler's theorem.

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + z\frac{\partial f}{\partial z} = nf$$

$$x\frac{\partial}{\partial x}(\sin u) + y\frac{\partial}{\partial y}(\sin u) + z\frac{\partial}{\partial z}(\sin u) = 2\sin u$$

$$x\cos u\frac{\partial u}{\partial x} + y\cos u\frac{\partial u}{\partial y} + z\cos u\frac{\partial u}{\partial z} = 2\sin u$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 2\tan u$$

Pa.

④ Verify Euler's theorem for $u = \log\left(\frac{x^4+y^4}{x+y}\right)$.

Solⁿ:

$$e^u = \frac{x^4+y^4}{x+y} = f(x,y)$$

$$\because \quad f(tx,ty) = \frac{t^4(x^4+y^4)}{t(x+y)} = t^3 f(x,y)$$

$\therefore$ $f(x,y)$ is a Homogeneous function of degree 3.

$\therefore$ By Euler's Theorem

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf$$

$$x\frac{\partial}{\partial x}e^u + y\frac{\partial}{\partial y}e^u = ne^u$$

$$e^u\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right) = ne^u$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 3\frac{e^u}{e^u} = 3.$$

To verify - $\quad u = \log(x^4+y^4) - \log(x+y)$

$$\frac{\partial u}{\partial x} = \frac{1}{x^4+y^4}\cdot 4x^3 - \frac{1}{x+y}.$$

$$\frac{\partial u}{\partial y} = \frac{1}{x^4+y^4}\cdot 4y^3 - \frac{1}{x+y}.$$

$$\therefore \quad x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{4x^4}{x^4+y^4} - \frac{x}{x+y} + \frac{4y^4}{x^4+y^4} - \frac{y}{x+y}$$

$$= \frac{4(x^4+y^4)}{x^4+y^4} - \frac{(x+y)}{(x+y)}$$

$$= 4-1 = \underline{\underline{3}}$$

(Excercize):-

① Verify Euler's Theorem for $u = \dfrac{x+y+z}{\sqrt{x}+\sqrt{y}+\sqrt{z}}$.

② Using Euler's Theorem to show that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 2u \log u$

where $u = e^{x^2+y^2}$

③ If $u = \tan^{-1}\left(\dfrac{x^3+y^3}{x-y}\right)$ then prove that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \sin 2u$$ and

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \sin 4u - \sin 2u$$

$$= 2\cos 3u . \sin u.$$

④ If $u = \sin^{-1}\left(\dfrac{x^{1/3}+y^{1/3}}{x^{1/2}-y^{1/2}}\right)$, prove that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = -\frac{1}{12}\tan u.$$