Module - 4. Complex Variable - I (Differentiation)

"Topic 4.1 :- Introduction - Complex Number & Funtion of Complex Variable, and Limit of a function of a Complex variable"

Introduction :- Complex Number :-

A Complex number z is an ordered pair (x,y) of real numbers and is written as $z = x+iy$ —① , where $i=\sqrt{-1}$.

The real numbers x and y are called the real and imaginary parts of z.

In polar form z can be expressed as

$$z = r(\cos\theta + i\sin\theta) \quad \text{—②}$$

and in exponential form it can be expressed as

$$z = re^{i\theta} \quad \text{—③}$$

If $z = x+iy$, then the complex number $z = x-iy$ is called the conjugate of the complex number z and is denoted by $\bar{z}$.

$$|\bar{z}| = |z|, \quad |z|^2 = z\bar{z},$$

(Real Value of z) -:

$$\left[Re(z) = \frac{x+iy+x-iy}{2} = \frac{2x}{2} = x \right] \quad Re(z) = \frac{z+\bar{z}}{2} = x, \quad Im(z) = \frac{z-\bar{z}}{2i} = y$$

(Imaginary value of z)

$$\left[Im(z) = \frac{x+iy-x+iy}{2i} = \frac{2iy}{2i} = y \right]$$

Function of a Complex Variable :-

If x and y are real variables, then $z = x+iy$ is called a complex variable. If corresponding to each value of complex variable $z\,(=x+iy)$ in a given region R, there correspond one or more values of another complex variable $w\,(=u+iv)$, then w is called a function of the complex variable z and is denoted by

$$w = f(z) = u+iv$$

ex:- if $w = z^2$, where $z = x+iy$ and $w = f(z) = u+iv$

then $u+iv = (x+iy)^2 = (x^2-y^2) + i(2xy)$

$\Rightarrow u = x^2-y^2$, and $v = 2xy$.

Thus u and v are the real and imaginary parts of w, are functions of the real variables x and y.

$\therefore \quad w = f(z) = u(x,y) + i\,v(x,y)$

If to each value of z, there corresponds one and only one value of w, then w is called a <u>single-valued function</u> of z. If to each value of z, there correspond more than one values of w, then w is called a <u>multi-valued function</u> of z.

<u>Limit of a function of a Complex Variable :-</u>

Let $f(z)$ be a single valued function defined at all points in some nbd of point z_0, then $f(z)$ is said to have the limit l as z approaches z_0 along any path if given an arbitrary real number $\epsilon > 0$, however small there exists a real number $\delta > 0$, such that

$$|f(z) - l| < \epsilon \quad \text{, whenever} \quad 0 < |z - z_0| < \delta$$

ie.

$$l - \epsilon < f(z) < l + \epsilon \quad \text{, whenever} \quad z_0 - \delta < z < z_0 + \delta$$
$$z \neq z_0.$$

and we write $\boxed{\lim_{z \to z_0} f(z) = l.}$

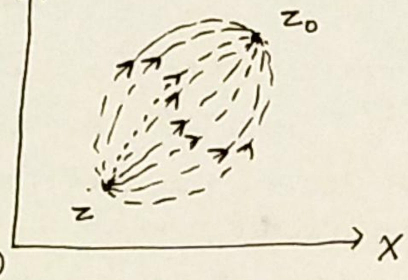
<u>NOTE-</u> ① δ usually depends upon ϵ.

② In real variables, $x \to x_0$ implies that x approaches x_0 along the number line, either from left or from right.
In Complex Variables, $z \to z_0$ implies that z approaches z_0 along any path, straight or curved, since the two points representing z and z_0 in a complex plane can be joined by an infinite number of curves.

③ If we get two different limits as $z \to z_0$ along two different paths then limits does not exist.

<u>Path.</u>

origin (i) taking First $x = 0$
(ii) taking First $y = 0$.
line (iii) Taking $y = mx$, and $x \to 0$.
Curve (iv) Taking $y = x^2$, and $x \to 0$.



Ex ① - Prove that $\lim_{z \to 1-i} \frac{(z^2+4z+3)}{z+1} = 4-i$

LHS.

$$\lim_{z \to 1-i} \frac{(z^2+4z+3)}{(z+1)} = \lim_{z \to 1-i} \frac{(z^2+3z+z+3)}{(z+1)}$$

$$\Rightarrow \lim_{z \to 1-i} \frac{z(z+3)+1(z+3)}{(z+1)}$$

$$\Rightarrow \lim_{z \to 1-i} \frac{(z+1)(z+3)}{(z+1)}$$

$$\Rightarrow \lim_{z \to 1-i} (z+3) \Rightarrow 1-i+3 \Rightarrow 4-i \text{ R.H.S.}$$

Ex ② - Show that $\lim_{z \to 0} \frac{z}{|z|}$ does not exist

$$\lim_{z \to 0} \frac{z}{|z|} = \lim_{\substack{x \to 0 \\ y \to 0}} \frac{x+iy}{\sqrt{x^2+y^2}}$$

let $y = mx$

$$\Rightarrow \lim_{x \to 0} \frac{x+imx}{\sqrt{x^2+m^2x^2}} \Rightarrow \lim_{x \to 0} \frac{\cancel{x}(1+im)}{\cancel{x}\sqrt{1+m^2}}$$

$$\Rightarrow \frac{1+im}{\sqrt{1+m^2}}$$

The Value of $\frac{1+im}{\sqrt{1+m^2}}$ are different for different Values of m.

Hence limit of the function does not exist.

Ex ③ - Prove that $\lim_{z \to 0} \frac{z}{\bar{z}}$ does not exist.

Case I - $\lim_{z \to 0} \frac{z}{\bar{z}} = \lim_{\substack{x \to 0 \\ y \to 0}} \frac{x+iy}{x-iy} = \lim_{x \to 0} \left[\lim_{y \to 0} \frac{x+iy}{x-iy} \right]$

$$\Rightarrow \lim_{x \to 0} \frac{x}{x} = 1$$ (Here the path is $y \to 0$ and then $x \to 0$)

Case II - Again $\lim_{z \to 0} \frac{z}{\bar{z}} = \lim_{\substack{x \to 0 \\ y \to 0}} \frac{x+iy}{x-iy} = \lim_{y \to 0} \left[\lim_{x \to 0} \frac{x+iy}{x-iy} \right]$

$$\Rightarrow \lim_{y \to 0} \frac{iy}{-iy} = -1$$ (Here the path is $x \to 0$ and then $y \to 0$)

As $z \to 0$ along two different paths we get different limits.
Hence limit does not exist.

Ex-④- Find the limit of the following $\lim_{z\to\infty} \frac{iz^3+iz-1}{(2z+3i)(z-i)^2}$

$$\Rightarrow \lim_{z\to\infty} \frac{iz^3+iz-1}{(2z+3i)(z-i)^2}$$

$$\Rightarrow \lim_{z\to\infty} \frac{\cancel{z^3}\left[i+\frac{i}{z^2}-\frac{1}{z^3}\right]}{\cancel{z}\left(2+\frac{3i}{z}\right)\cancel{z^2}\left(1-\frac{i}{z}\right)^2}$$

$$\Rightarrow \frac{i}{2} \text{ Ans}$$

Ex-⑤- Find the limit of the following $\lim_{z\to 1+i} \frac{z^2-z+1-i}{z^2-2z+2}$

$$\lim_{z\to 1+i} \frac{(z^2+1)-z-i}{z^2-2z+2} = \lim_{z\to 1+i} \frac{(z+i)(z-i)-(z+i)}{(z-1-i)(z-1+i)}$$

$$\Rightarrow \lim_{z\to 1+i} \frac{(z+i)(z-1-i)}{(z-1-i)(z-1+i)} = \lim_{z\to 1+i} \frac{z+i}{(z-1+i)}$$

$$\Rightarrow \frac{1+2i}{2i} = \frac{(1+2i)(-i)}{(2i)(-i)} \Rightarrow \frac{-i+2}{2} = \frac{2-i}{2}$$

$$\Rightarrow 1-\frac{i}{2} \text{ Ans}$$

Excercise-

① Show that the following limits do not exist

(i) $\lim_{z\to 0} \frac{\mathrm{Im}(z)^3}{\mathrm{Re}(z)^3}$, (ii) $\lim_{z\to -i} \frac{z^2}{z+i}$, (iii) $\lim_{z\to 0} \frac{\mathrm{Re}(z)^2}{\mathrm{Im}(z)}$

(iv) $\lim_{z\to 0} \frac{z}{(\bar{z})^2}$

② Find the limits of the following-

(i) $\lim_{z\to 0} \frac{\mathrm{Re}(z)^2}{|z|}$, (ii) $\lim_{z\to 1+i} \frac{2z^3}{\mathrm{Im}(z)^2}$, (iii) $\lim_{z\to 0} \frac{z^2+6z+3}{z^2+2z+2}$

(ANS - 0) (ANS - $2(-1+i)$) (ANS - $\frac{3}{2}$)