Module - III, Sequence and Series

Sequence :- A sequence is a succession of number or terms formed according to some definite rule. The n^{th} term in a sequence is denoted by u_n.

eg - $u_n = 2n+1$

by giving different values of n in u_n, we get different terms of the sequence.

Thus $u_1 = 3$, $u_2 = 5$, $u_3 = 7$, - - - - -

A sequence having unlimited number of terms is known as an infinite ~~seri~~ Sequence.

Limit of sequence :- If a sequence tends to a limit l, then we write

$$\boxed{\lim_{n \to \infty} (u_n) = l}$$

Increasing and Decreasing Sequence :-

Increasing Sequence :- The sequence $\sum_{n=0}^{m} a_n$ is said to be increasing if $a_{n+1} \geq a_n$, for each $n \leq m$.

Decreasing Sequence :- The sequence $\sum_{n=0}^{m} a_n$ is said to be decreasing if $a_n \geq a_{n+1}$, for each $n \leq m$.

Example- Show that the sequence $\left\{\frac{3}{n+3}\right\}$ is a decreasing sequence.

Sol: We know that the sequence $\sum_{n=0}^{m} a_n$ is decreasing if $a_n \geq a_{n+1}$ for each $n \leq m$.

We have $a_n = \frac{3}{n+3}$

$$a_{n+1} = \frac{3}{n+4} < \frac{3}{n+3}$$

Thus we obseame that $a_n \geq a_{n+1}$

Hence the sequence $\left\{\frac{3}{n+3}\right\}$ is a decreasing sequence.

Convergent Sequence :- find $\lim_{n\to\infty}(u_n)$.

(i) If the limit of a sequence is finite, then the sequence is "Convergent".

(ii) If the limit of a sequence does not tend to a finite number, then the sequence is "Divergent".

(iii) If the limit does not tend to a unique, finite or infinite number, then the sequence is "Oscillatory".

Bounded Sequence :- $u_1, u_2, u_3, \ldots u_n, \ldots$ is a bounded sequence if $u_n < \underset{(\text{finite})}{K}$ for every n.

Monotonic Sequence :- The sequence is either increasing or decreasing, such sequences are called "Monotonic."

eg- (i) $1, 4, 7, 10, \ldots$ is a Monotonic (increasing) sequence.

(ii) $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4} \ldots$ is also Monotonic (decreasing) sequence.

(iii) $1, -1, 1, -1, 1 \ldots$ is not a Monotonic Sequence.

A sequence which is Monotonic and bounded is a Convergent Sequence.

Eg- ① prove that $1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \ldots, \frac{1}{n^2}, \ldots$ is a Convergent sequence.

we have $u_n = \frac{1}{n^2}$

$$\lim_{n\to\infty}(u_n) = \lim_{n\to\infty} \frac{1}{n^2} \Rightarrow \frac{1}{\infty} = \underline{\underline{0}} \text{ (finite)}$$

∴ the given Monotonic (decreasing) sequence is "Convergent".

② Prove that $3, 5, 7, \ldots (2n+1) \ldots$ is a Convergent Sequence.

we have $u_n = 2n+1$

$$\lim_{n\to\infty}(u_n) = \lim_{n\to\infty}(2n+1) = \infty \text{ (infinite)}$$

∴ The given Monotonic (increasing) sequence is "divergent"

Exercise – ① Find the general Term (Σ term) of the following sequence and prove they are Convergent or divergent.

(i) $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \cdots \frac{1}{2^n} + \cdots$

(ii) $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5} \cdots \frac{n}{n+1} + \cdots$

(iii) $1, -1, 1, -1, \cdots \begin{cases} 0, & n \text{ is even} \\ 1, & n \text{ is odd.} \end{cases}$

(iv) $\frac{1^2}{1!}, \frac{2^2}{2!}, \frac{3^2}{3!}, \frac{4^2}{4!}, \cdots \frac{n^2}{n!} \cdots$

② prove that the following Sequence are Convergent or divergent.

(i) $u_n = \frac{n+1}{n}$, (ii) $u_n = 3n$, (iii) $u_n = n^2$, (iv) $u_n = \frac{1}{n}$.

Remember the following limits :-

(i) $\lim_{n\to\infty} x^n = 0$, if $x < 1$ and $\lim_{n\to\infty} x^n = \infty$ if $x > 1$

(ii) $\lim_{n\to\infty} \frac{x^n}{n!} = 0$ for all values of x.

(iii) $\lim_{n\to\infty} \frac{\log n}{n} = 0$

(iv) $\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e$

(v) $\lim_{n\to\infty} (n)^{1/n} = 1$

(vi) $\lim_{n\to\infty} \left[\frac{(n!)}{n}\right]^{1/n} = \frac{1}{e}$

(vii) $\lim_{n\to\infty} nx^n = 0$ if $x < 1$

(viii) $\lim_{n\to\infty} n^a = \infty$

(ix) $\lim_{n\to\infty} \frac{1}{n^a} = 0$

(x) $\lim_{x\to\infty} \left[\frac{a^x - 1}{x}\right] = \log a$

(xi) $\lim_{n\to\infty} \frac{a^{1/n} - 1}{1/n} = \log a$

(xii) $\lim_{x\to 0} \frac{\sin x}{x} = 1$

(xiii) $\lim_{x\to 0} \frac{\tan x}{x} = 1$.