Module-2, Multivaride Calculus -

Improper Integral :-

Introduction :- Improper integral is the limit of a definite integral as an end point of the interval of integration approaches either a specified real number, ∞, $-\infty$ or in some instance.

Definition :- An Integral is an improper integral if either the interval of integration is not finite or if the function to integrate is not continuous (not bounded) in the interval of integration.

Definite integral is known as improper integral.

(i) If one or both limits of integration are infinite

eg. $\int_a^{\infty} f(x)\,dx$, $\int_{-\infty}^{b} f(x)\,dx$, $\int_{-\infty}^{\infty} f(x)\,dx$.

(ii) If the integrand is unbounded (discontinuous) at any point in the interval.

eg. $\int_0^2 \frac{1}{x^2}\,dx$, $\frac{1}{x^2}$ is a unbounded at $x=0$

$\int_{-1}^{4} \frac{1}{x-2}\,dx$, here $\frac{1}{x-2}$ is unbounded at $x=2$ in the interval $[-1,4]$

$\int_{-2}^{4} \frac{1}{x-4}\,dx$, Here $\frac{1}{x-4}$ is discontinuous at $x=4$.

There are three Types of Improper Integral.

① Improper Integral of First kind.

② Improper Integral of Second kind.

③ Improper Integral of Third kind.

* Improper Integral of First kind :-

If the range of integral is infinite then the integral is known as improper integral of first kind.

suppose $f(x)$ is bounded and integrable in (a,b), however x may be large, then we define

Type ① - $\int_a^\infty f(x)\,dx = \lim_{b\to\infty} \int_a^b f(x)\,dx$ —— ①

(i) If the value of ① exists (finite), then the integral is convergent.

(ii) If the value of ① is infinite, then the integral is divergent.

Type ② - $\int_{-\infty}^{b} f(x)\,dx = \lim_{x\to\infty} \int_{-x}^{b} f(x)\,dx$ —— ②

(i) If the value of ② exists (finite), integral is convergent.

(ii) If the value of ② infinite, integral is divergent.

(iii) If the value of ② does not exists, integral is oscillatory.

Example ① Test the convergence of $\int_0^\infty \frac{dx}{1+x^2}$

Sol^n

$$\int_0^\infty \frac{dx}{1+x^2} = \lim_{b\to\infty} \int_0^b \frac{dx}{1+x^2}$$

$$= \lim_{b\to\infty} \left[\tan^{-1}x\right]_0^b$$

$$= \lim_{b\to\infty} \left[\tan^{-1}b - \tan^{-1}0\right]$$

$$= \lim_{b\to\infty} \left[\tan^{-1}b\right] \Rightarrow \tan^{-1}\infty = \frac{\pi}{2} \text{ (which is finite)}$$

∴ Integral is convergent.

Example - ② - Test for Convergence or divergence $\int_{-\infty}^{\infty} \frac{dx}{a^2+x^2}$ and also evaluate it.

Sol^n - Since the limits are infinite and the integrand $\frac{1}{a^2+x^2}$ is not unbounded at any point in the integral $(-\infty,\infty)$

So the given integral is improper of first type.

$$\therefore \int_{-\infty}^{\infty} \frac{1}{a^2+x^2}\,dx = \lim_{t\to\infty} \int_{-t}^{t} \frac{1}{a^2+x^2}\,dx$$

$$= \lim_{t\to\infty} \left[\frac{1}{a}\tan^{-1}\frac{x}{a}\right]_{-t}^{t}$$

$$= \lim_{t\to\infty} \frac{1}{a}\left[\tan^{-1}\frac{t}{a} - \tan^{-1}\frac{-t}{a}\right]$$

$$\Rightarrow \lim_{t\to\infty} \frac{1}{a}\left[\tan^{-1}\frac{t}{a} + \tan^{-1}\frac{t}{a}\right]$$

$$\Rightarrow \frac{2}{a} \lim_{t\to\infty}\left(\tan^{-1}\frac{t}{a}\right)$$

$$\Rightarrow \frac{2}{a}\tan^{-1}\infty \Rightarrow \frac{2}{a}\times\frac{\pi}{2} \Rightarrow \frac{\pi}{a} \text{ (finite)}$$

Hence the integral is Convergent.

Example - ③ - Test the Convergence $\int_{0}^{\infty} \frac{dx}{1+x}$

Sol^n - $$\int_{0}^{\infty} \frac{dx}{1+x} = \lim_{b\to\infty} \int_{0}^{b} \frac{1}{1+x}\,dx$$

$$= \lim_{b\to\infty} \left[\log(1+x)\right]_{0}^{b}$$

$$\Rightarrow \lim_{b\to\infty} \left[\log(1+b) - \log 1\right]$$

$$\Rightarrow (\log\infty - 0) \Rightarrow \infty$$

Integral is divergence.

Example - ④ - Is $\int_{1}^{\infty} \frac{1}{x^{\sqrt{2}}}\,dx$ Convergent.

Sol^n - $$\int_{1}^{\infty} \frac{1}{x^{\sqrt{2}}}\,dx = \lim_{b\to\infty} \int_{1}^{b} \frac{1}{x^{\sqrt{2}}}\,dx = \lim_{b\to\infty} \int_{1}^{b} x^{-\sqrt{2}}\,dx$$

$$= \lim_{b\to\infty} \left[\frac{x^{1-\sqrt{2}}}{1-\sqrt{2}}\right]_{1}^{b} = \lim_{b\to\infty} \frac{1}{1-\sqrt{2}}\left[b^{1-\sqrt{2}} - 1\right]$$

$$\Rightarrow \lim_{b\to\infty} \frac{1}{1-\sqrt{2}}\left(b^{-0.41} - 1\right)$$

$\Rightarrow \frac{1}{1-\sqrt{2}}(0-1)$ as $b \to \infty$

$\Rightarrow \frac{1}{\sqrt{2}-1}$ (finite)

Hence the integral is convergent.

Example ⑤ - Investigate the convergence of $\int_{-2}^{2} \frac{1}{x^6}\,dx$.

Here $\frac{1}{x^6}$ is discontinuous (unbounded) at $x=0$.

$$\int_{-2}^{2} \frac{1}{x^6}\,dx = \int_{-2}^{0} \frac{1}{x^6}\,dx + \int_{0}^{2} \frac{1}{x^6}\,dx.$$

$$= \lim_{t\to 0} \int_{-2}^{t} \frac{1}{x^6}\,dx + \lim_{t\to 0} \int_{t}^{2} \frac{1}{x^6}\,dx.$$

$$= \lim_{t\to 0} \left[-\frac{1}{5x^5}\right]_{-2}^{t} + \lim_{t\to 0} \left[-\frac{1}{5x^5}\right]_{t}^{2}$$

$$= \lim_{t\to 0} -\frac{1}{5}\left(\frac{1}{t^5}+\frac{1}{32}\right) + \lim_{t\to\infty} -\frac{1}{5}\left[\frac{1}{32}-\frac{1}{t^5}\right]$$

$$\Rightarrow -\frac{1}{5}\left[\infty+\frac{1}{32}\right] - \frac{1}{5}\left[\frac{1}{32}-\infty\right] = \infty$$

$\therefore$ the integral is divergent.

Example ⑥ - Check the convergence of $\int_0^\infty \frac{dx}{\sqrt{9-x^2}}$

Sol^n:

$$\int_0^\infty \frac{dx}{\sqrt{9-x^2}} = \lim_{b\to\infty} \int_0^b \frac{dx}{\sqrt{9-x^2}}$$

$$= \lim_{b\to\infty} \left[\sin^{-1}\frac{x}{3}\right]_0^b$$

$$= \lim_{b\to\infty} \left[\sin^{-1}\frac{b}{3} - \sin^{-1} 0\right]$$

$$\Rightarrow \lim_{b\to\infty} \sin^{-1}\frac{b}{3}$$

$\Rightarrow$ finite

therefore integral is convergent.

Example - (7) Test the Convergence of the $\int_0^\infty \cos x \, dx$.

Sol:

$$\int_0^\infty \cos x \, dx = \lim_{b\to\infty} \int_0^b \cos x \, dx.$$

$$\Rightarrow \lim_{b\to\infty} \left[\sin x\right]_0^b$$

$$\Rightarrow \lim_{b\to\infty} (\sin b - \sin 0)$$

$$\Rightarrow \lim_{b\to\infty} \sin b \Rightarrow \sin\infty$$

$\Rightarrow$ does not exist.

Hence Integral is oscillatory.

Example - (8) - Evaluate the following improper Integral

$$\int_0^\infty \frac{dx}{(1+x^2)(1+\tan^{-1}x)}$$

Sol:

$$\int_0^\infty \frac{dx}{(1+x^2)(1+\tan^{-1}x)} = \lim_{b\to\infty} \int_0^b \frac{1}{1+x^2} \frac{dx}{(1+\tan^{-1}x)}$$

Put $\tan^{-1}x = z$

$\frac{1}{1+x^2} dx = dz$

Limits. when $x=0,\ z=0$

$x=b,\ z=\tan^{-1}b.$

$$\Rightarrow \lim_{b\to\infty} \int_{z=0}^{\tan^{-1}b} \frac{dz}{1+z}$$

$$= \lim_{b\to\infty} \left[\log(1+z)\right]_0^{\tan^{-1}b}$$

$$= \lim_{b\to\infty} \left[\log(1+\tan^{-1}b) - \log 1\right]$$

$$\Rightarrow \lim_{b\to\infty} \log(1+\tan^{-1}b)$$

$$\Rightarrow \log(1+\tan^{-1}\infty).$$

$$\Rightarrow \log(1+\pi/2) \quad (\text{finite})$$

Integral is convergent.