Problems on Asymptotic notations:-

1 **P.T** if $f(x) = O(g(x))$ and $g(x) = O(f(x))$ then $f(x) = \Theta(g(x))$

If $f(x)$ is $O(g(x))$ then there are two positive constant c_2 and n_0', such that,

$$0 \leq f(x) \leq c_2 g(x) \text{ for all } n \geq n_0' \longrightarrow (i)$$

Similarly if $g(x)$ is $O(f(x))$ then there are two positive constant c_1' and n_0'', such that,

$$0 \leq g(x) \leq c_1' f(x) \text{ for all } n \geq n_0'' \longrightarrow (ii)$$

equation (ii) is divided by c_1', then we get,

$$0 \leq \frac{1}{c_1'} g(x) \leq \frac{c_1'}{c_1'} f(x) \text{ for all } n \geq n_0''$$

$$0 \leq \frac{1}{c_1'} \leq g(x) \leq f(x) \text{ for all } n \geq n_0''$$

if we set, $\frac{1}{c_1'} = c_1$ and n_0 is $\max(n_0', n_0'')$ then we have,

$$0 \leq c_1 \leq g(x) \leq f(x) \leq c_2 g(x) \text{ for all } n \geq n_0.$$

Then $\boxed{f(x) = \Theta(g(x))}$.

2 Let $f(x) = O(g(x))$ and $g(x) = O(h(x))$,

S.T $f(x) = O(h(x))$.

If $f(x)$ is $O(g(x))$ then there are two positive constants c_1 and n_0' such that,

$$0 \le f(x) \le c_1 g(x) \text{ for all } n \geq n_0' \longrightarrow \text{(i)}$$

and if $g(x)$ is $O(h(x))$ then there are positive constants c_2 and n_0'' such that,

$$0 \le g(x) \le c_2 h(x) \text{ for all } n \geq n_0'' \longrightarrow \text{(ii)}$$

equ^n (ii) is multiplied by c_1 ~~and~~ we get,

$$0 \le c_1 g(x) \le c_1 c_2 h(x)$$

Now, we set, $c_1 c_2 = c_3$ and n_0 is $\max(n_0', n_0'')$

Hence,

$$0 \le f(x) \le c_1 g(x) \le c_3 h(x) \text{ for } n \geq n_0$$

$$\therefore 0 \le f(x) \le c_3 h(x) \text{ for all } n \geq n_0.$$

Thus $\boxed{f(x) = O(h(x))}$

Scanned by CamScanner

3 Find tight bound on —

$$f(x) = x^8+7x^7-10x^5-2x^4+3x^2-17.$$

First it is clear that, when $x>0$.

$$x^8+7x^7-10x^5-2x^4+3x^2-17 \leq x^8+7x^7+3x^2$$

Next, it is also clear that for $x \geq 1$

$$x^8+7x^7+3x^2 \leq x^8+7x^8+3x^8 = 11x^8$$

$$\therefore x^8+7x^7+3x^2 \leq 11x^8.$$

Hence we can write,

$$x^8+7x^7-10x^5-2x^4+3x^2-17 \leq 11x^8 \text{ for all } x \geq 1$$

$\therefore$ $\underline{f(x) = O(x^8)}$ for all $x \geq 1$, $c_1 = 11$

Again for $x \geq 0$.

$$x^8+7x^7-10x^5-2x^4+3x^2-17 \geq x^8-10x^5-2x^4-17$$

next for $x \geq 1$

$$x^8-10x^5-2x^4-17 \geq x^8-10x^7-2x^7-17x^7$$

ie, $x^8-10x^5-2x^4-17 \geq x^8-29x^7$

we can write, $x^8-29x^7 \geq cx^8$

$$\Rightarrow (1-c)x^8 \geq 29x^7 \longrightarrow (i)$$

when $x \geq 1$, we can divide, equ^n (i) by x^7 and get,

$$(1-c)x \geq 29. \Rightarrow x - cx \geq 29.$$

hence $c \leq 1 - \frac{29}{x}$

for $x \geq 58$, $c = \frac{1}{2}$,

Then we have just shown that, if $x \geq 58$ then

$$f(x) = x^8 + 7x^7 - 10x^5 - 2x^4 + 3x^2 - 17 \geq \frac{1}{2} x^8.$$

Thus, $\underline{f(x) = \Omega(x^8)}$ $\left[f(x) \geq \frac{1}{2} x^8 \right]$

Since we have, $f(x) = O(x^8)$ and $f(x) = \Omega(x^8)$,

therefore $\boxed{f(x) = \theta(x^8)}$

4. S.T $\log x = O(x)$

for, $x \geq 1$, $\log x \leq 1.x$

Hence $\log(x)$ is $O(x)$ for $x \geq 1$

5. S.T $n! = O(n^n)$.

Proof, for, $n \geq 1$,

$$0 \leq n! = 1.2.3.\ldots.n \leq n.n.n.\ldots.n = n^n$$

Therefore $0 \leq n! \leq 1.n^n$ for $n \geq 1$ and $c = 1$

$\therefore$ $n!$ is $O(n^n)$.

6. S.T $\log n! = O(n \log n)$.

We know from above problem,

$$n! \leq n^n.$$

taking log on both side,

$$\log n! \leq \log n^n = n \log n.$$

$\therefore$ $0 \leq \log n! \leq n \log n$ for all $n \geq 1$

Hence, $\log n!$ is $O(n \log n)$.

7 Find a good upper bound on

$n \log(n^2+1) + n^2 \log n$

if $n > 1$

$\log(n^2+1) \leq \log(n^2+n^2) = \log 2n^2 = \log 2 + \log n^2$

$\therefore \log(n^2+1) \leq \log 2 + \log n^2 \leq \log n + 2\log n = 3\log n.$

$\therefore \log(n^2+1) \leq 3\log n.$

Now, $n\log(n^2+1) \leq n.3\log n.$

$\therefore n\log(n^2+1) + n^2\log n \leq n \cdot 3\log n + n^2\log n$

$\leq 3n^2\log n + n^2\log n$

$\therefore n\log(n^2+1) + n^2\log n \leq 4n^2\log n.$

Then, $n\log(n^2+1) + n^2\log n = O(n^2\log n).$

8 S.T, $2^x = O(3^x).$

If $x \geq 1$, then it is clear that, $\left(\frac{3}{2}\right)^x \geq 1.$

So, $2^x \leq 2^x \cdot \left(\frac{3}{2}\right)^x = \left(\frac{2 \cdot 3}{2}\right)^x = 3^x$

$\therefore 2^x \leq 3^x$ for $x \geq 1$

hence 2^x is $O(3^x).$

9. Show that,

$(n+a)^b = \Theta(n^b)$, a and b are any real constant where $b>0$.

Note that,

$n+a \leq n+|a|$

$\Rightarrow n+a \leq n+n$ when $|a| \leq n$.

$\Rightarrow n+a \leq 2n$ when $|a| \leq n$.

and, $n+a \geq n-|a|$

$\geq \frac{1}{2}n$ when $|a| \leq \frac{1}{2}n$

Thus for, $n \geq 2|a|$

$0 \leq \frac{1}{2}n \leq n+a \leq 2n$.

~~where~~ Since $b>0$, the above inequality holds when all parts are raised to power b:

$\therefore 0 \leq \left(\frac{1}{2}n\right)^b \leq (n+a)^b \leq (2n)^b$.

$0 \leq \left(\frac{1}{2}\right)^b n^b \leq (n+a)^b \leq 2^b n^b$. for $n \geq 2|a|$

Thus $c_1 = \left(\frac{1}{2}\right)^b$, $c_2 = 2^b$ and $n_0 = 2|a|$, ~~satisfy~~

satisfy the definition —

$(n+a)^b = \Theta(n^b)$.

Scanned by CamScanner