Duality

Ex- VII(A) pg-276

(12) maximize $z = 3x_1 + x_2 + x_3 - x_4$

Subject to
$$x_1 + 5x_2 + 3x_3 + 4x_4 \le 5$$
$$x_1 + x_2 = -1$$
$$x_3 - x_4 \le -5$$
$$x_1, x_2, x_3, x_4 \geq 0$$

Since the problem is of maximization, we make all the constrains '$\le$' type and we have

$$\max Z = 3x_1 + x_2 + x_3 - x_4$$

s.t
$$x_1 + 5x_2 + 3x_3 + 4x_4 \le 5$$
$$x_1 + x_2 \le -1$$
$$-x_1 - x_2 \le 1$$
$$x_3 - x_4 \le -5$$
$$x_1, x_2, x_3, x_4 \geq 0$$

It can be written in matrix notation as

$$\max Z = CX$$
$$\text{s.t } AX \le B$$

where $C = [3 \; 1 \; 1 \; -1]$

$$A = \begin{bmatrix} 1 & 5 & 3 & 4 \\ 1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}; \quad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}; \quad B = \begin{bmatrix} 5 \\ -1 \\ 1 \\ -5 \end{bmatrix}$$

Here have 4 constrains, we assume 4 new variables v_1, v_2', v_2'', v_3 all ≥ 0 and let $V = \begin{bmatrix} v_1 \\ v_2' \\ v_2'' \\ v_3 \end{bmatrix}$; Then the dual problem is

$$\min W = B'V$$
$$\text{s.t } A'V \geq C'$$

where '' indicates transpose of the matrices

or $\min W = [5 \; -1 \; 1 \; -5] \begin{bmatrix} v_1 \\ v_2' \\ v_2'' \\ v_3 \end{bmatrix}$

s.t
$$\begin{bmatrix} 1 & 1 & -1 & 0 \\ 5 & 1 & -1 & 0 \\ 3 & 0 & 0 & 1 \\ 4 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2' \\ v_2'' \\ v_3 \end{bmatrix} \geq \begin{bmatrix} 3 \\ 1 \\ 1 \\ -1 \end{bmatrix}$$

or, $\min W = 5v_1 - v_2' + v_2'' - 5v_3$

s.t, $v_1 + v_2' - v_2'' \geq 3$

$5v_1 + v_2' - v_2'' \geq 1$

$3v_1 \quad + v_3 \geq 1$ let, $v_2' - v_2'' = v_2$

$4v_1 \quad - v_3 \geq -1$

Then $\text{Min } w = 5v_1 - v_2 - 5v_3$

s.t $v_1 + v_2 \geq 3$ $v_1, v_3 \geq 0$

$5v_1 + v_2 \geq 1$

$3v_1 + v_3 \geq 1$ v_2 is unrestricted in sign

$4v_1 - v_3 \geq -1$

(3). Minimize $z = x_1 - 3x_2 + 2x_3$

s.t $3x_1 - x_2 + 2x_3 \leq 7$

$-2x_1 + 4x_2 \leq 12$

$-4x_1 + 3x_2 + 8x_3 \leq 10$

$x_1, x_2, x_3 \geq 0$

Since it is a problem of minimization, we make all the constrains '$\geq$' 0 type and we have

$\text{Min } z = x_1 - 3x_2 + 2x_3$

s.t $-3x_1 + x_2 - 2x_3 \geq -7$

$2x_1 - 4x_2 \geq -12$

$4x_1 - 3x_2 - 8x_3 \geq -10$

$x_1, x_2, x_3 \geq 0$

It can be written in matrix notation as

$\text{min } z = CX$ where $C = \begin{bmatrix} 1 & -3 & 2 \end{bmatrix}$

s.t $AX \geq B$ $A = \begin{bmatrix} -3 & 1 & -2 \\ 2 & -4 & 0 \\ 4 & -3 & -8 \end{bmatrix}$; $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$; $B = \begin{bmatrix} -7 \\ -12 \\ -10 \end{bmatrix}$

Here have three constrains so we assume three variables v_1, v_2, v_3 all ≥ 0 and let $V = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ and we have the dual pro as

$\text{Max } w = B'V$ where '$'$' indicates transpose of the

s.t $A'V \leq C'$ matrices

or $\text{Max } w = \begin{bmatrix} -7 & -12 & -10 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$

s.t

$$\begin{bmatrix} -3 & 2 & 4 \\ 1 & -4 & -3 \\ -2 & 0 & -8 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \leq \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}$$

or Max $w = -7v_1 - 12v_2 - 10v_3$

Subject to,

$$-3v_1 + 2v_2 + 4v_3 \leq 1$$
$$v_1 - 4v_2 - 3v_3 \leq -3$$
$$-2v_1 \quad - 8v_3 \leq 2$$
$$v_1, v_2, v_3 \geq 0$$ Ans.

(4) Minimize $z = x_3 + x_4 + x_5$

Subject to
$$x_1 - x_3 + x_4 - x_5 = -2$$
$$x_2 - x_3 - x_4 + x_5 = 1$$
$$x_j \geq 0 \; j = 1, 2, \ldots, 5$$

Since it's the problem of minimization, we make all the constrains '$\geq$' type and we have

minimize $z = x_3 + x_4 + x_5$

s.t.
$$x_1 - x_3 + x_4 - x_5 \geq -2$$
$$-x_1 + x_3 - x_4 + x_5 \geq 2$$
$$x_2 - x_3 - x_4 + x_5 \geq 1$$
$$-x_2 + x_3 + x_4 - x_5 \geq -1$$
$$x_j \geq 0 \; j = 1, 2, \ldots, 5$$

It can be written in matrix notation as

Min $z = CX$ ~~where $c = [1 1 1]$~~ $\begin{bmatrix} x_3 \\ x_4 \\ \cancel{x_5} \end{bmatrix}$

s.t. $AX \geq B$

where $C = \begin{bmatrix} 0 & 0 & 1 & 1 & 1 \end{bmatrix}$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}; \; B = \begin{bmatrix} -2 \\ 2 \\ 1 \\ -1 \end{bmatrix} \qquad A = \begin{bmatrix} 1 & 0 & -1 & 1 & -1 \\ -1 & 0 & 1 & -1 & 1 \\ 0 & 1 & -1 & -1 & 1 \\ 0 & -1 & 1 & 1 & -1 \end{bmatrix}$$

Here we have four constrains. So we assume four variables

v_1', v_1'', v_2', v_2'' all ≥ 0 and let $V = \begin{bmatrix} v_1' \\ v_1'' \\ v_2' \\ v_2'' \end{bmatrix}$ Then the dual problem is

$$\text{Max } W = B'V$$

s.t $A'V \leq C'$ where ' indicates transpose of the matrices

or, $$\text{Max } W = \begin{bmatrix} -2 & 2 & 1 & -1 \end{bmatrix} \begin{bmatrix} v_1' \\ v_1'' \\ v_2' \\ v_2'' \end{bmatrix}$$

s.t $$\begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ -1 & 1 & -1 & 1 \\ 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} v_1' \\ v_1'' \\ v_2' \\ v_2'' \end{bmatrix} \leq \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

or $$\text{Max } W = -2v_1' + 2v_1'' + v_2' - v_2''$$

s.t

$v_1' - v_1'' \leq 0$ let $v_1 = v_1' - v_2''$

$v_2' - v_2'' \leq 0$ $v_2 = v_2' - v_2''$

$-v_1' + v_1'' - v_2' + v_2'' \leq 1$

$v_1' - v_1'' - v_2' + v_2'' \leq 1$

$-v_1' + v_1'' + v_2' - v_2'' \leq 1$

$\therefore$ $$\text{Max } W = -2v_1 + v_2$$

s.t $-v_1 - v_2 \leq 1$ [illegible]

$v_1 - v_2 \leq 1$

$-v_1 + v_2 \leq 1$

v_1, v_2 are unrestricted in sign.

(15) $\text{Max } Z = 6x_1 + 4x_2 + 6x_3 + x_4$

s.t $4x_1 + 4x_2 + 4x_3 + 8x_4 = 21$

$3x_1 + 17x_2 + 80x_3 + 2x_4 \leq 48$

$x_1, x_2 \geq 0$

x_3, x_4 are unrestricted in sign

$\therefore$ let $x_3 = x_3' - x_3''$ where x_3', x_3'', x_4', x_4'' all ≥ 0

$x_4 = x_4' - x_4''$

since, the problem is of maximization, we make all the constrains '$\leq$' type and we have

$$Max\ Z = 6x_1 + 4x_2 + 6x_3' - 6x_3'' + x_4' - x_4''$$

s.t. $4x_1 + 4x_2 + 4x_3' - 4x_3'' + 8x_4' - 8x_4'' \leq 21$

$-4x_1 - 4x_2 - 4x_3' + 4x_3'' - 8x_4' + 8x_4'' \leq -21$

$3x_1 + 17x_2 + 80x_3' - 80x_3'' + 2x_4' - 2x_4'' \leq 48$

$x_1, x_2, x_3', x_3'', x_4', x_4'' \geq 0$

It can be written in matrix notation as

$Max\ Z = CX$ where $C = \begin{bmatrix} 6 & 4 & 6 & -6 & 1 & -1 \end{bmatrix}$ $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3' \\ x_3'' \\ x_4' \\ x_4'' \end{bmatrix}$

s.t $AX \leq B$ $A = \begin{bmatrix} 4 & 4 & 4 & -4 & 8 & -8 \\ -4 & -4 & -4 & 4 & -8 & 8 \\ 3 & 17 & 80 & -80 & 2 & -2 \end{bmatrix}$; $B = \begin{bmatrix} 21 \\ -21 \\ 48 \end{bmatrix}$

Here have three constrains. So we introduce three new variables v_1', v_1'', v_2 all ≥ 0 and let $V = \begin{bmatrix} v_1' \\ v_1'' \\ v_2 \end{bmatrix}$

Then the dual problem is

min $W = B'V$

s.t $A'V \geq C'$ where '$'$' indicates transpose of the matrices

or, min $W = \begin{bmatrix} 21 & -21 & 48 \end{bmatrix} \begin{bmatrix} v_1' \\ v_1'' \\ v_2 \end{bmatrix}$

s.t $\begin{bmatrix} 4 & -4 & 3 \\ 4 & -4 & 17 \\ 4 & -4 & 80 \\ -4 & 4 & -80 \\ 8 & -8 & 2 \\ -8 & 8 & -2 \end{bmatrix} \begin{bmatrix} v_1' \\ v_1'' \\ v_2 \end{bmatrix} \geq \begin{bmatrix} 6 \\ 4 \\ 6 \\ -6 \\ 1 \\ -1 \end{bmatrix}$

or, min $W = 21v_1' - 21v_1'' + 48v_2$

s.t $4v_1' - 4v_1'' + 3v_2 \geq 6$

$4v_1' - 4v_1'' + 17v_2 \geq 4$

$4v_1' - 4v_1'' + 80v_2 \geq 6$

$-4v_1' + 4v_1'' - 80v_2 \geq -6$ let, $v_1' - v_1'' = v_1$

$8v_1' - 8v_1'' + 2v_2 \geq 1$

$-8v_1' + 8v_1'' - 2v_2 \geq -1$

or, min $W = 21v_1 + 48v_2$

s.t, $4v_1+3v_2 \geq 6$ $\quad$ ~~$4v_1+80v_2=6$~~ $\quad 4v_1+80v_2=6$

$4v_1+17v_2 \geq 4$ $\quad\quad 8v_1+2v_2=1$

$v_2 \geq 0$, v_1 is unrestricted in sign.

(16) $\text{Max } z = x_1 - x_2 + 3x_3 + 2x_4$

s.t $\quad x_1 + x_2 \geq -1$

$x_1 - 3x_2 - x_3 \leq 7$

$x_1 + x_3 - 3x_4 = -2$

$x_1, x_4 \geq 0$ $\quad x_2, x_3$ unrestricted in sign

let, $x_2 = x_2' - x_2''$ $\quad$ where

$x_3 = x_3' - x_3''$ $\quad x_2', x_2'', x_3', x_3'' \geq 0$

Since the problem is of maximization we make the constrains '$\leq$' type and we have

$\text{max } z = x_1 - x_2' + x_2'' + 3x_3' - 3x_3'' + 2x_4$

s.t $\quad$ ~~$x_1 + x_2' - x_2'' \geq 0$~~

$-x_1 - x_2' + x_2'' \leq 1$

$x_1 - 3x_2' + 3x_2'' - x_3' + x_3'' \leq 7$

$x_1 + x_3' - x_3'' - 3x_4 \leq -2$

$-x_1 - x_3' + x_3'' + 3x_4 \leq 2$

$x_1, x_2', x_2'', x_3', x_3'', x_4 \geq 0$

It can be written in matrix notation as

$\text{max } z = CX$. $\quad$ where $C = \begin{bmatrix} 1 & -1 & 1 & 3 & -3 & 2 \end{bmatrix}$ $\quad B = \begin{bmatrix} 1 \\ 7 \\ -2 \\ 2 \end{bmatrix}$

s.t $AX \leq B$

$$A = \begin{bmatrix} -1 & -1 & 1 & 0 & 0 & 0 \\ 1 & -3 & 3 & -1 & 1 & 0 \\ 1 & 0 & 0 & 1 & -1 & -3 \\ -1 & 0 & 0 & -1 & 1 & 3 \end{bmatrix} \quad X = \begin{bmatrix} x_1 \\ x_2' \\ x_2'' \\ x_3' \\ x_3'' \\ x_4 \end{bmatrix}$$

Here have 4 constrains $\therefore$ we introduce 4 new variables v_1, v_2, v_3', v_3'' all ≥ 0 and let $V = \begin{bmatrix} v_1 \\ v_2 \\ v_3' \\ v_3'' \end{bmatrix}$

Then the dual problem is

$\text{Min } w = B'V$ $\quad$ where '$'$' indicates transpose of the matrices

s.t $A'V \geq C'$

or, $\text{Min } w = \begin{bmatrix} 1 & 7 & -2 & 2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3' \\ v_3'' \end{bmatrix}$

s.t

$$\begin{bmatrix} -1 & 1 & 1 & -1 \\ -1 & -3 & 0 & 0 \\ 1 & 3 & 0 & 0 \\ 0 & -1 & 1 & -1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & -3 & 3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3' \\ v_3'' \end{bmatrix} \geq \begin{bmatrix} 1 \\ -1 \\ 1 \\ 3 \\ -3 \\ 2 \end{bmatrix}$$

or, $\text{Min } w = v_1 + 7v_2 - 2v_3' + 2v_3''$

s.t $-v_1 + v_2 + v_3' - v_3'' \geq 1$

$-v_1 - 3v_2 + \cancel{v_3' - v_3''} \geq -1$

$v_1 + 3v_2 - \cancel{v_3' + v_3''} \geq 1$

$\cancel{+v_3 \geq 3}$

$\cancel{v_2 \geq -3}$

$\cancel{-3v_3' + 3v_3'' \geq 2}$

$-v_2 + v_3' - v_3'' \geq 3$ $\quad -3v_3' + 3v_3'' \geq 2$

$v_2 - v_3' + v_3'' \geq -3$

let, $v_3 = v_3' - v_3''$

where $v_3', v_3'' \geq 0$

$\therefore \text{Min } w = v_1 + 7v_2 - 2v_3$

s.t $-v_1 + v_2 + v_3 \geq 1$, $v_1 + 3v_2 = 1$

$-v_2 + v_3 = 3$

$-3v_2 \geq 2$ or $3v_2 \leq -2$

$v_1, v_2 \geq 0$, v_3 is unrestricted in sign.

Ex-VIII(B)

Pg-301

① Max $z = 3x_1 + 2x_2$

s.t $2x_1 + x_2 \leq 5$

$x_1 + x_2 \leq 3$

$x_1, x_2 \geq 0$

Since the problem is of maximization and all the constrains are '$\leq$' type the problem can be written in matrix notation as

Max $z = CX$

s.t $AX \leq B$

where $C = \begin{bmatrix} 3 & 2 \end{bmatrix}$; $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$; $B = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$.

Here have two constrains so we assume two new variables

① ~~Maximize $z = 2x_1 + x_2$~~

~~s.t $x_1 + 2x_2 \le 10$~~

~~$x_1 + x_2 \le 6$~~

~~$x_1 - x_2$~~

③ $\min z = 15x_1 + 10x_2$

s.t $3x_1 + 5x_2 \geq 5$

$5x_1 + 2x_2 \geq 3$

$x_1, x_2 \geq 0$

Here the problem is of minimization and all the constrains are of '$\geq$' type so the problem can be written in matrix notation as

$\min z = CX$

s.t $AX \geq B$

where $C = \begin{bmatrix} 15 & 10 \end{bmatrix}$; $B = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$; $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$A = \begin{bmatrix} 3 & 5 \\ 5 & 2 \end{bmatrix}$

Here have two constrains. So we introduce two new variables $v_1, v_2 \geq 0$ and let $V = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ Then the dual problem is

$\max W = B'V$

s.t $A'V \leq C'$

where '′' indicates transpose of the matrices

or, $\max W = \begin{bmatrix} 5 & 3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$

s.t $\begin{bmatrix} 3 & 5 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \leq \begin{bmatrix} 15 \\ 10 \end{bmatrix}$

or, $\max W = 5v_1 + 3v_2$

s.t, $3v_1 + 5v_2 \leq 15$

$5v_1 + 2v_2 \leq 10$

$v_1, v_2 \geq 0$

Introducing slack ~~and artificial~~ variables we rewrite the problem in standard form as

$\max W = 5v_1 + 3v_2 + 0.v_3 + 0.v_4$

s.t $3v_1 + 5v_2 + v_3 + 0.v_4 = 15$

$5v_1 + 2v_2 + 0.v_3 + v_4 = 10$

$v_j \geq 0$ $j = 1, 2, 3, 4$

Here (a_3, a_4) forms initial basis

x_B = Basis variable; B = Bases Vector, C_B = coefficient of basis variable in objective fn.

$a_3 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $a_4 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

			c_j	5	3	0	0	Ratio	Operation	Remarks
C_B	B	X_B	b	a_1	a_2	a_3	a_4	(min)		
0	a_3	x_3	15	3	5	1	0	5		$a_1 \to$ entering, $a_4 \to$ departing, $5 \to$ key element
0	a_4	x_4	10	$\boxed{5}$	2	0	1	2 (min)		
$z_j - c_j$				-5	-3	0	0			
0	a_3	x_3	9	0	$\boxed{19/5}$	1	$-3/5$	$\frac{45}{19}$ (min)	$R_2' = \frac{1}{5}R_2$, $R_1' = R_1 - 3R_2'$	$a_2 \to$ entering, $a_3 \to$ departing, $\frac{19}{5} \to$ key element
5	a_1	x_1	2	1	$2/5$	0	$1/5$	5		
$z_j - c_j$				0	-1	0	1			
3	a_2	x_2	$\frac{45}{19}$	0	1	$\frac{5}{19}$	$-\frac{3}{19}$		$R_1' = \frac{5}{19}R_1$	
5	a_1	x_1	$\frac{20}{19}$	1	0	$-\frac{2}{19}$	$5/19$		$R_2' = R_2 - \frac{2}{5}R_1'$	
$z_j - c_j$				0	0	$5/19$	$16/19$			

Here all $z_j - c_j \geq 0$ and thus the optimal condition is achived

$\therefore$ The optimal soln of dual problem is $u_1 = \frac{20}{19}$, $u_2 = \frac{45}{19}$

$\therefore \text{Max } \omega = 5 \cdot \frac{20}{19} + 3 \cdot \frac{45}{19}$

$= \frac{100 + 135}{19} = \frac{235}{19}$

The optimal soln of primal problem is

$x_1 = \frac{5}{19}$, $x_2 = \frac{16}{19}$

$\min Z = 15 \cdot \frac{5}{19} + 10 \cdot \frac{16}{19} = \frac{160 + 75}{19} = \frac{235}{19}$ Ans.

(4) $\min Z = 10x_1 + 6x_2 + 2x_3$

s.t. $-x_1 + x_2 + x_3 \geq 1$

$3x_1 + x_2 - x_3 \geq 2$

$x_1, x_2, x_3 \geq 0$

Since it is the problem of minimization, and the constrains are of '$\geq$' type we write the problem in matrix notation as

$\min Z = CX$ where $C = [10 \ 6 \ 2]$ $B = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$AX \geq B$

$A = \begin{bmatrix} -1 & 1 & 1 \\ 3 & 1 & -1 \end{bmatrix}$ Here have two constrains, so we introduce two new variables v_1, v_2 all ≥ 0 and let

$V = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$, then the dual problem is

Max $W = B'V$ where ',' indicates transpose of
s.t $A'V \leq C'$
the matrices

$\therefore$ Max $W = \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$

s.t $\begin{bmatrix} -1 & 3 \\ 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \leq \begin{bmatrix} 10 \\ 6 \\ 2 \end{bmatrix}$

or, Max $W = v_1 + 2v_2$

s.t, $-v_1 + 3v_2 \leq 10$

$v_1 + v_2 \leq 6$ $\quad v_1, v_2 \geq 0$ introducing slack ~~and~~

$v_1 - v_2 \leq 2$

~~artificial~~ variables we rewrite the problem in standard form as

Max $W = v_1 + 2v_2 + 0.v_3 + 0.v_4 + 0.v_5$

s.t $-v_1 + 3v_2 + v_3 + 0.v_4 + 0.v_5 = 10$

$v_1 + v_2 + 0.v_3 + v_4 + 0.v_5 = 6$

$v_1 - v_2 + 0.v_3 + 0.v_4 + v_5 = 2$

$v_j \geq 0, j = 1, 2, \ldots, 5$

Here (a_3, a_4, a_5) forms initial basis
x_B = Basis variable
B = Basis vector
c_B = coeff of basis variable in objective f^n

			c_j	1	2	0	0	0	Ratio	Operation	Remarks
c_B	B	x_B	b	a_1	a_2	a_3	a_4	a_5	min		
$\leftarrow 0$	a_3	x_3	10	-1	$\boxed{3}$	1	0	0	$\frac{10}{3}$		$a_2 \to$ entering
0	a_4	x_4	6	1	1	0	1	0	6		$a_3 \to$ departing
0	a_5	x_5	2	1	-1 $\uparrow$	0	0	1	—		$3 \to$ key element
$z_j - c_j$				-1	-2	0	0	0			
2	a_2	x_2	$\frac{10}{3}$	$-1/3$	1	$1/3$	0	0	—	$R_1' = \frac{1}{3} R_1$	$a_1 \to$ entering
$\leftarrow 0$	a_4	x_4	$8/3$	$\boxed{4/3}$	0	$-1/3$	1	0	2	$R_2' = R_2 - R_1'$	$a_4 \to$ departing
0	a_5	x_5	$16/3$	$2/3$ $\uparrow$	0	$1/3$	0	1	8	$R_3' = R_3 + R_1'$	$4/3 \to$ key element
$z_j - c_j$				$-1/3$	0	$2/3$	0	0			
2	a_2	x_2	4	0	1	$1/4$	$1/4$	0		$R_2' = \frac{3}{4} R_2$	
1	a_1	x_1	2	1	0	$-1/4$	$3/4$	0		$R_1' = R_1 + \frac{1}{3} R_2'$	
0	a_5	x_5	4	0	0	$1/2$	$-1/2$	1		$R_3' = R_3 - \frac{2}{3} R_2'$	
				0	0	$1/4$	$5/4$	0			

Here all $z_j - c_j \geq 0$ and the optimal condition is achieved

$\therefore$ The optimal soln. of the dual problem is

$v_1 = 2, \quad v_2 = 4 \quad ; \cancel{v_3 \geq 0}$

$\therefore \quad \text{Max } w = 2 + 8 = 10$

The optimal soln. of the primal problem is

$x_1 = \frac{1}{4}, \; x_2 = \frac{5}{4}, \; x_3 = 0$

$Z_{min} = 10$ Ans.

(5) $\text{Max } Z = 3x_1 + 2x_2$

s.t $x_1 + x_2 \geq 1$

$x_1 + x_2 \leq 7$

$x_1 + 2x_2 \leq 10$

$x_2 \leq 3$

$x_1, x_2 \geq 0$

Since, the problem is of maximizing we make all the constraints '$\leq$' type and we have

$\text{Max } Z = 3x_1 + 2x_2$

s.t $-x_1 - x_2 \leq -1$

$x_1 + x_2 \leq 7$

$x_1 + 2x_2 \leq 10$

$x_2 \leq 3$

$x_1, x_2 \geq 0$ It can be written in matrix notation as

$\text{Max } Z = CX$

s.t $AX \leq B$ where $C = \begin{bmatrix} 3 & 2 \end{bmatrix}$; $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$A = \begin{bmatrix} -1 & -1 \\ 1 & 1 \\ 1 & 2 \\ 0 & 1 \end{bmatrix}$; $B = \begin{bmatrix} -1 \\ 7 \\ 10 \\ 3 \end{bmatrix}$

Here have 4 constraints. so we assume 4 new variables v_1, v_2, v_3, v_4 all ≥ 0 and let $V = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix}$ Then the dual problem is

$\text{Min } w = B'V$

s.t $A'V \geq C'$ where ' indicates transpose of the matrices

or, $\text{Min } w = \begin{bmatrix} -1 & 7 & 10 & 3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix}$

s.t $\begin{bmatrix} -1 & 1 & 1 & 0 \\ -1 & 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} \geq \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

or, $\text{Max}(-w) = v_1 - 7v_2 - 10v_3 - 3v_4$

or, $\text{Min } w = -v_1 + 7v_2 + 10v_3 + 3v_4$

st, $-v_1 + v_2 + v_3 \geq 3 ; -v_1 + v_2 + 2v_3 + v_4 \geq 2 \quad v_i \geq 0$

$i = 1, 2, 3, 4$