∴ coordinate of A ([illegible]) ; $z =$ [illegible]

" B $(30, 0)$; $z = 120$

" C $(3, 18)$; $z = 12 + 36 = 48$

" D $(0, 27)$; $z = 54$

Since, it is a problem of minimization, we move the objective line over the region towards origin and lastly it touches at D$(3, 18)$ in the feasible region

∴ min $z = 48$, optimal soln is $x = 3$, $y = 18$ Ans.

(7) Minimize: $z = 3x_1 + x_2$

S.t $2x_1 + 3x_2 \le 6$

$x_1 + x_2 \ge 1$

$x_1, x_2 \ge 0$

$2x_1 + 3x_2 = 6$

$\frac{x_1}{3} + \frac{x_2}{2} = 1$ —— (1)

$\frac{x_1}{1} + \frac{x_2}{1} = 1$ —— (2)

Considering the constrains as equation, we draw the graph. Now, giving the direction of inequalities and non negativity restriction we get the common region ABCDA shaded i.e called feasible region. Coordinate of A $(1, 0)$, B$(3, 0)$ C$(0, 2)$, D$(0, 1)$

Now we draw the objective line by suitable z here $z = 12$ over the region

Since the problem is of minimization, we move it towards the origin and it touches lastly D$(0, 1)$ in the feasible region

∴ At A ; $z = 3$

B ; $z = 9$

C ; $z = 2$

D ; $z = 1$

∴ Min $z = 1$

optimal soln. is

$x_1 = 0$

$x_2 = 1$

✓

Coordinate of C is (3, 18)

∴ coordinate of A (12, 9) ; $Z = 48 + 18 = 66$

" B (30, 0) ; $Z = 120$

" C (3, 18) ; $Z = 12 + 36 = 48$

" D (0, 27) ; $Z = 54$

Since it is a problem of minimization, we move the objective line over the region towards origin and lastly it touches at C (3, 18) in the feasible region

∴ Min $Z = 48$, optimal soln is $x = 3, y = 18$ Ans.

(7.) Minimize $Z = 3x_1 + x_2$

S.t $2x_1 + 3x_2 \leq 6$

$x_1 + x_2 \geq 1$

$x_1, x_2 \geq 0$

$2x_1 + 3x_2 = 6$

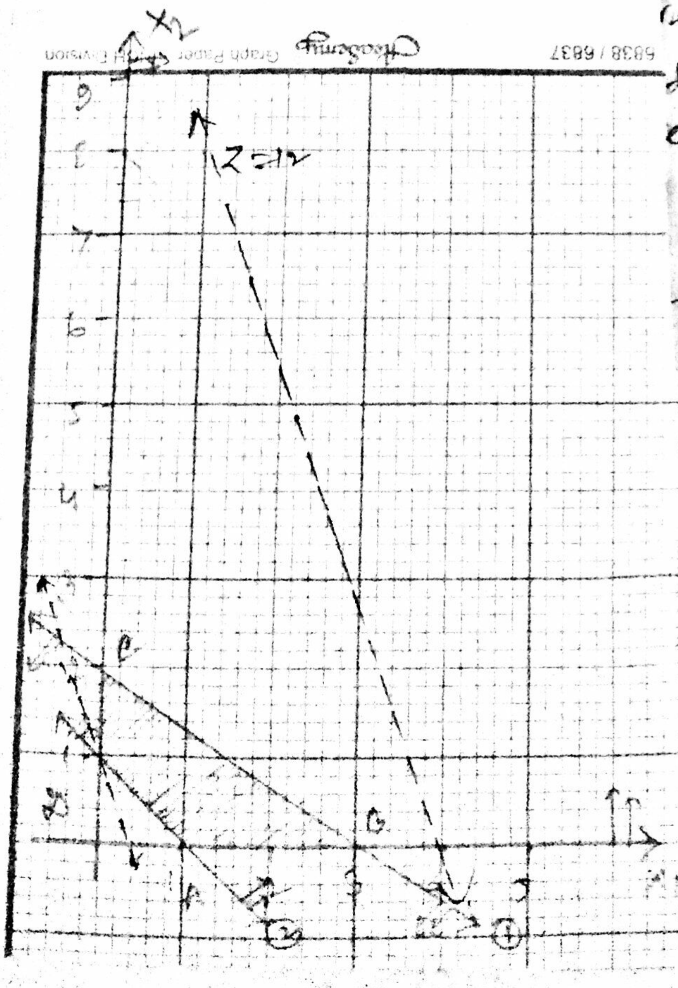
$\frac{x_1}{3} + \frac{x_2}{2} = 1$ —— (1)

$\frac{x_1}{1} + \frac{x_2}{1} = 1$ —— (2)

... equation, we draw the ...ection of inequalities and ...t the common region ABCDA ...egion. Coordinate of A (1, 0),

... line by suitable Z over the region

...nization, we move it towards ...tly D (0, 1) in the feasible region



Min Z = 1

...ptimal soln. is

$x_1 = 0$

$x_2 = 1$

Since, it is a problem of maximization we move the objective line far away from origin over the region and it touches lastly the point C is the feasible region

$\therefore$ At O, $z=0$

At A, $z=8$

At B, $z=18$

At C, $z=21$

At D, $z=15$

$\therefore$ Max $z=21$

and the feasible sol^n is

$x=3, y=3$.

(8) Minimize $z=4x+2y$

subject to $3x+y \geq 27$

$-x-y \leq -21$

$x+2y \geq 30$

$x, y \geq 0$

$3x+y=27$

or $\frac{x}{9}+\frac{y}{27}=1$ ... (A)

$x+y \geq 21$

or $\frac{x}{21}+\frac{y}{21}=1$ ... (B)

$x+2y=30$

or $\frac{x}{30}+\frac{y}{15}=1$ ... (C)

Considering the constrains as equations we draw the graph. Now giving the direction of inequalities and non-negativity restriction of decision variable, we see that the common region is unbounded and i.e. ABCD

let, us draw the objective line by suitable z

here $z=144$

$\therefore 4x+2y=144$

$\therefore \frac{x}{36}+\frac{y}{72}=1$

solving (B), (C)

$$\begin{array}{r} x+y=21 \\ x+2y=30 \\ (-)\ \ (-)\ \ (-) \\ \hline -y=-9 \end{array}$$

$\therefore y=9$ $\therefore x=21-9$
$=12$

$\therefore$ coordinate of A is $(12,9)$

solving (A), (B)

$$\begin{array}{r} x+y=21 \\ 3x+y=27 \\ (-)\ \ (-)\ \ (-) \\ \hline -2x=-6 \end{array}$$

$\therefore x=3$

$\therefore y=21-3$
$=18$

Since, it is a maximizing problem we draw the objective line far from origin and lastly it touch point B and passout from feasible region.

$\therefore$ At O, $Z = 0$

At A, $Z = \frac{15}{2} = 7.5$

At B, $Z = \frac{15}{2} + \frac{7}{2} = 11$

At C, $Z = \frac{20}{5} + \frac{42}{5} = \frac{62}{5} = 12\frac{2}{5}$

At D, $Z = \frac{21}{2} = 10.5$

Hence $Max\ Z = \frac{62}{5}$ and the optimal soln is $x_1 = 4/5$, $x_2 = \frac{6}{5}$

(a)

1) Max $Z = 2x + 5y$

Subject to $0 \le x \le 4$

$0 \le y \le 3$

$x + y \le 6.$

$x + y = 6$

$x = 4, x = 0$

$y = 3, y = 0$

(a)

Considering the constraints as equations, we draw the graphs of the lines. Now giving the direction of inequalities and nonnegativity restrictions of decision variables, we have a common region OABCDO (shaded) called feasible region with extreme points O, A, B, C, D

$x + y = 6$ —— ①

$x = 4$ —— ②

$y = 2$

$x + y = 6$ —— ①

$y = 3$

$\therefore x = 3$

$\therefore$ Coordinate of O is (0,0)

" " A is (4,0)

" " B is (4,2)

" " C is (3,3)

" " D is (0,3)

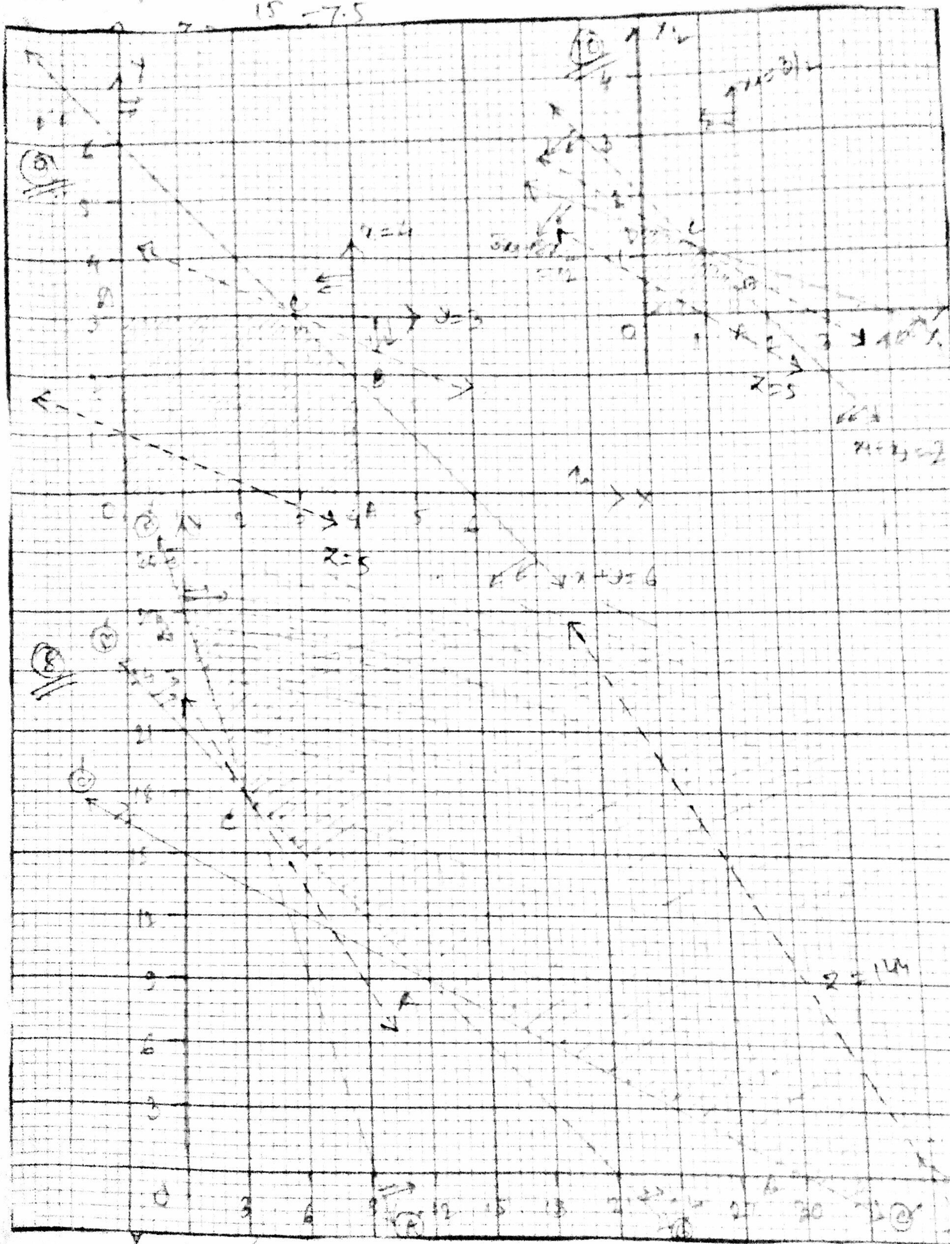
Now we draw the objective line by suitable Z, here $Z = 5$

$\therefore 2x + 5y = 5$

or, $\frac{x}{2.5} + \frac{y}{1} = 1$

Since, it is a maximizing problem we draw the objective line far from origin and lastly it touch point B and passout from feasible region.

$\therefore$ At O, $Z = 0$



$\therefore$ here $Z = 5$

$\therefore 2x + 5y = 5$

or, $\frac{x}{2.5} + \frac{y}{1} = 1$

(10) Maximize $z = 5x_1 + 7x_2$

Subject to $3x_1 + 8x_2 \le 12$

$x_1 + x_2 \le 2$

$2x_1 \le 3$

$x_1, x_2 \geq 0$

$3x_1 + 8x_2 = 12$

or, $\frac{x_1}{4} + \frac{x_2}{1.5} = 1$

$\frac{x_1}{2} + \frac{x_2}{2} = 1$

$x_1 = 3/2$

Considering the constraints as equations, we draw the graph. Now, giving the directions of inequalities and non-negativity restriction of decision variable, we have a common region with feasible region called region OABCDO (shaded)

extreme point ~~of~~ O, A, B, C, D

$x_1 + x_2 = 2$ — (1)

$2x_1 = 3$ — (2)

$2x_1 + 2x_2 = 4$

or $3 + 2x_2 = 4$

$\therefore x_2 = \frac{1}{2}$

$\therefore$ Coordinate of B $(3/2, 1/2)$

$3x_1 + 3x_2 = 6$

$3x_1 + 8x_2 = 12$

$(-) \quad (-) \quad (-)$

$-5x_2 = -6$

$\therefore x_2 = \frac{6}{5}$,

$\therefore x_1 = 2 - x_2$

$= 2 - \frac{6}{5} = \frac{4}{5}$

coordinate of C $\left(\frac{4}{5}, \frac{6}{5}\right)$

$\therefore$ Coordinate of O is $(0, 0)$

" " A is $(3/2, 0)$

" " B is $(3/2, 1/2)$

" " C is $(4/5, 6/5)$

" " D is $(0, 3/2)$

Now we draw the objective line by suitable z,

here $z = 5$ $\therefore 5x_1 + 7x_2 = 5$

$7) 5.0 (.7$
49

$\therefore \frac{x_1}{1} + \frac{x_2}{5/7} = 1$

or, $\frac{x_1}{1} + \frac{x_2}{0.71} = 1$

(17) Maximize

$Z = 9x + 8y$

Subject to $4x + 3y \le 30$

$2x + 3y \le 18$

$2x + y \le 10$

$x, y \ge 0$

$4x + 3y = 30$

or $\frac{x}{7.5} + \frac{y}{10} = 1$

$2x + 3y = 18$

or $\frac{x}{9} + \frac{y}{6} = 1$

$2x + y = 10$

or $\frac{x}{5} + \frac{y}{10} = 1$

$2x + 3y = 18$ ——— (1)

$2x + y = 10$ ——— (2)

Solving (1) and (2), $2y = 8$

$\therefore y = 4 \quad \therefore x = 3$

$\therefore (x, y) = (3, 4)$

Considering the constrains as equations, we draw the graphs of the lines. Now giving the directions of inequalities & non-negativity restrictions of the decision variable, we see that the feasible region is OABC (shaded region) with the extreme points $O(0,0)$; $A(5,0)$; $B(3,4)$; $C(0,6)$.

Now, we draw the objective line by suitable Z

here $Z = 36$

$\therefore 9x + 8y = 36$

or $\frac{x}{4} + \frac{y}{4.5} = 1$

Since it is a maximising problem, we move the objective line far away from origin and we have

at O, $Z = 0$

At A, $Z = 45$

At B, $Z = 59$

At C, $Z = 48$

Hence Max $Z = 59$

and the optimal soln is

$x = 3$

$y = 4$ Ans.

(1) Maximize $Z = 9x + 8y$

subject to $4x + 3y \le 30$

$2x + 3y \le 18$

$x, y \ge 0$

Let $4x + 3y = 30$

or, $\frac{x}{7.5} + \frac{y}{10} = 1$

$\frac{x}{9} + \frac{y}{6} = 1$

Considering the constrains as equations, we draw the graphs of the lines. By giving the direction of inequalities and non-negativity restriction of the decision variable, we see that feasible solⁿ is at OABCO (shaded region).

with extreme points

$O(0,0)$, $A(7.5,0)$, $B(6,2)$, $C(0,6)$

$$\begin{aligned} 4x + 3y &= 30 \\ 4x + 6y &= 36 \\ \hline -3y &= -6 \\ \therefore y &= 2 \end{aligned}$$

$\therefore 2x = 18 - 6 = 12$

$\therefore x = 6.$

$$\begin{aligned} 4x + 3y &= 30 \\ 2x + 3y &= 18 \\ \hline 2x &= 12 \\ x &= 6 \end{aligned}$$

$\therefore y = \frac{30 - 24}{3} = 2$

$Z = 9x + 8y$

Now, we draw the objective line by suitable Z

here $Z = 45$

$\therefore 9x + 8y = 45$

or, $\frac{x}{5} + \frac{y}{\frac{45}{8}} = 1$,

~~or, $\frac{x}{6} + \frac{y}{6.75} = 1$~~

or, $\frac{x}{5} + \frac{y}{5.625} = 1$

Since it is a problem of maximization, we move it faraway from origin over the region and it lastly touch and B and left the feasible region.

$\therefore$ At O, $Z = 0$

At A, $Z = 67.5$

At $B(6,2)$, $Z = 54 + 16 = 70$

Hence Max $Z = 70$

The optimal solⁿ is $x = 6, y = 2$

(17) Maximize

$Z = 9x + 8y$

Subject to $4x + 3y \le 30$

$2x + 3y \le 18$

$2x + y \le 10$

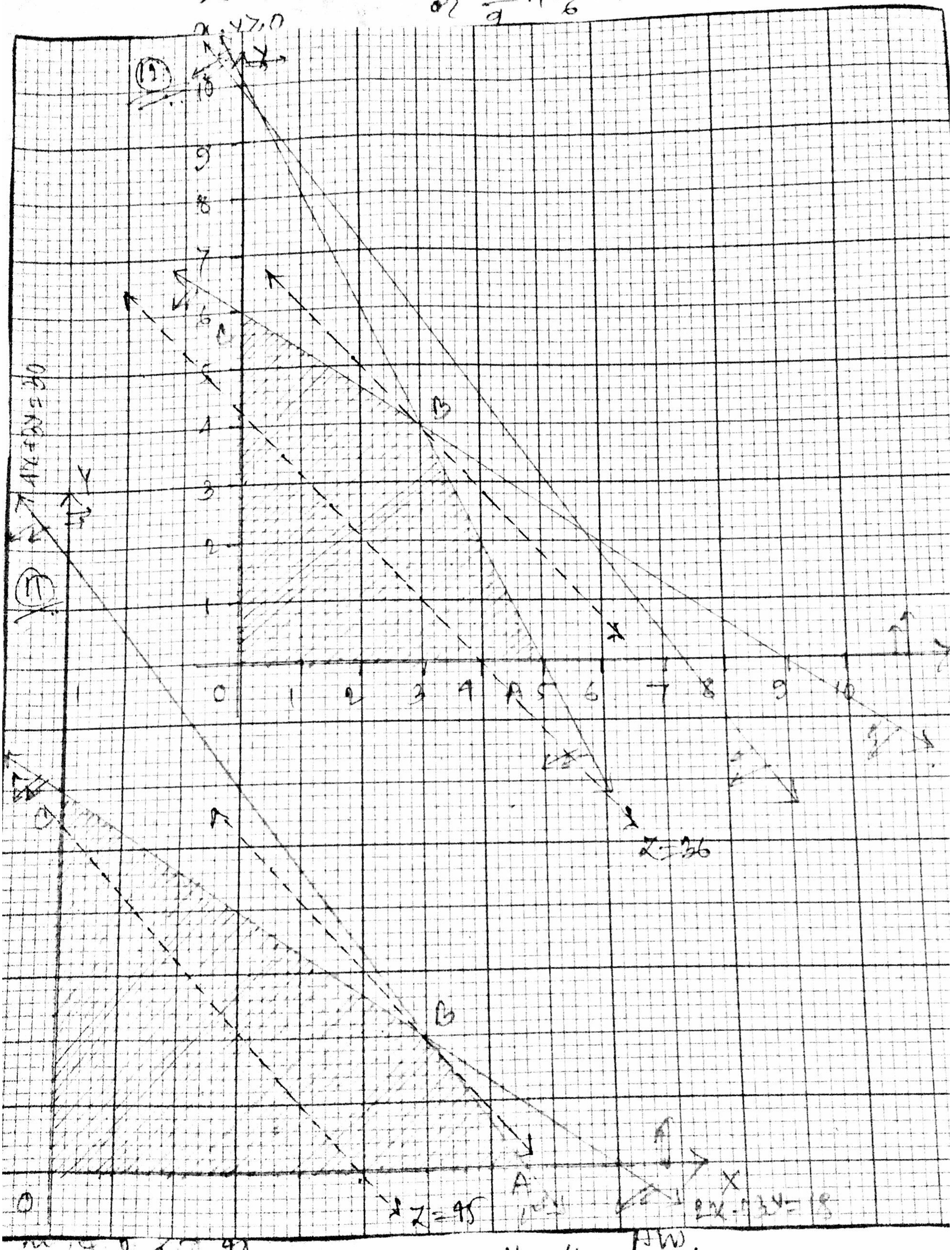
$x, y \ge 0$

$4x + 3y = 30$

or $\frac{x}{7.5} + \frac{y}{10} = 1$

$2x + 3y = 18$

or $\frac{x}{9} + \frac{y}{6} = 1$



$x = 4$) $Z = 40$

$y = 4$ Ans.

so that

$$X = \{ \mathbf{x} : \mathbf{x} = \sum_{i=1}^{n} \mu_i \mathbf{x}_i, \text{ for all } \mu_i \geq 0 \text{ and } \sum_{i=1}^{n} \mu_i = 1 \}.$$

We are to show that the set X is convex.

Let $\mathbf{u}$ and $\mathbf{v}$ be any two points such that

$$\mathbf{u} = \sum_{i=1}^{n} \mu_i' \mathbf{x}_i, \ \mu_i' \geq 0 \text{ and } \sum_{i} \mu_i' = 1$$

and $$\mathbf{v} = \sum_{i=1}^{n} \mu_i'' \mathbf{x}_i, \ \mu_i'' \geq 0 \text{ and } \sum_{i} \mu_i'' = 1.$$

If, for any $\lambda \ (0 \leq \lambda \leq 1)$,

$$\lambda \mathbf{u} + (1 - \lambda) \mathbf{v}$$

be in the set, then the set will be convex.

Now, $$\lambda \mathbf{u} + (1 - \lambda) \mathbf{v} = \sum_{i=1}^{n} \{ \lambda \mu_i' + (1 - \lambda) \mu_i'' \} \mathbf{x}_i$$

in which $\lambda \mu_i' + (1 - \lambda) \mu_i'' \geq 0$, for $\mu_i', \mu_i'' \geq 0$ and $0 \leq \lambda \leq 1$

and $$\sum_{i=1}^{n} \{ \lambda \mu_i' + (1 - \lambda) \mu_i'' \} = \lambda \sum_{i=1}^{n} \mu_i' + (1 - \lambda) \sum_{i=1}^{n} \mu_i'' = 1.$$

Hence $\lambda \mathbf{v} + (1 - \lambda) \mathbf{u}$

is also a convex combination of $\mathbf{x}_i$, $(i = 1, 2, \ldots\ldots, n)$ and the set X is thus convex.

3.4 EXTREME POINT.

A point $\mathbf{x}$ of a convex set X is an extreme point if it connot be expressed as a convex combination of two other points in X. Geometrically, a point $\mathbf{x}$ in a convex set is said to be an *extreme point*, if it does not lie on the line segment joining any two points other than $\mathbf{x}$ in the set. For example, any point on the circumference of a circle is an extreme point of the convex set of points within and on the circle. The vertices of a polygon are the extreme points of the convex set of points within and on the boundaries of the polygon.

The vertices are the only extreme points of the convex set formed by a triangle and its interior.

Note. With similar reasoning it can be shown that a half space, open or closed, is a convex set.

(b) *Intersection of two convex sets is also a convex set.*

Let X_1 and X_2 be two convex sets and the set X be the intersection of X_1 and X_2 such that

$$X = X_1 \cap X_2 .$$

Let $\mathbf{x}_1$ and $\mathbf{x}_2$ be any two points in X.

Then $\lambda \mathbf{x}_2 + (1-\lambda)\mathbf{x}_1 \in X_1$, for $0 \le \lambda \le 1$

and $\lambda \mathbf{x}_2 + (1-\lambda)\mathbf{x}_1 \in X_2$, for $0 \le \lambda \le 1$.

Hence $\lambda \mathbf{x}_2 + (1-\lambda)\mathbf{x}_1 \in X_1 \cap X_2 = X$.

Thus, if $\mathbf{x}_1$ and $\mathbf{x}_2 \in X$, then any convex combination of $\mathbf{x}_1$ and $\mathbf{x}_2$ also will belong to X hence X, that is,

$$X_1 \cap X_2 \text{ is a convex set.}$$

Generalising this result we can show that the *intersection of a finite number of convex sets is also a convex set.*

It should be noted here that the union (*i.e.* $X_1 \cup X_2$) or the difference (*i.e.* $X_1 \sim X_2$) of two convex sets may not be convex.

Furthermore, if the sets X_1 and X_2 be closed sets, then their intersection X can also be shown to be closed. This result can also be generalised to a finite number of closed sets.

Note. We have seen earlier that hyperplanes or half-spaces are convex sets. In a linear programming problem the set of feasible solutions is given by

$$\mathbf{A}\mathbf{x} \,(\le = \ge)\, \mathbf{b}, \ \mathbf{x} \ge \mathbf{0}$$

which will be nothing but the intersection of a finite number of hyperplanes or half-spaces or both as given by the constraints. Now the intersection of a finite number of hyperplanes or closed half-spaces or both is a closed convex set. Hence the set of feasible solutions to a linear programming problem (if it exists) is also a closed convex set. We shall prove this later in the form of a theorem.

(c) *The set of all convex combinations of a finite number of*

But the figures given below are not convex sets as in all the cases the line segment joining the points P and Q does not lie wholly in it.

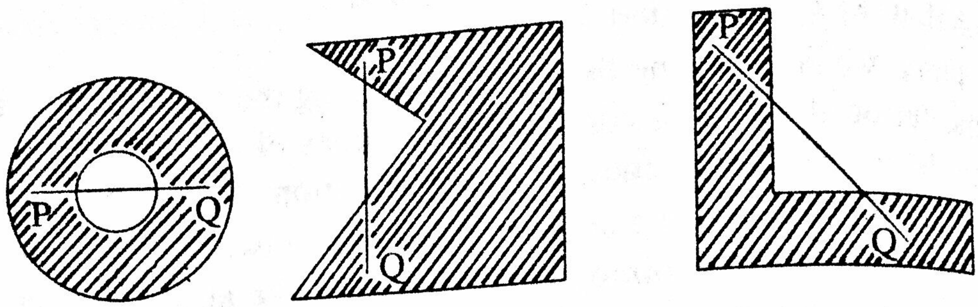


Fig. 14

A convex set may be bounded or unbounded. The first quadrant of the two dimensional plane is an unbounded convex set while the circle is a bounded convex set.

3.3 A FEW IMPORTANT RESULTS.

We give with proof a few results associated with convex sets.

(a) *A hyperplane is a convex set.*

Let us consider the hyperplane $X = \{ \mathbf{x} : \mathbf{cx} = z \}$.

Let $\mathbf{x}_1$ and $\mathbf{x}_2$ be two points in X ; then

$$\mathbf{cx}_1 = z \text{ and } \mathbf{cx}_2 = z. \quad \ldots \quad (1)$$

Now let the point $\mathbf{x}_3$ be given by the convex combination of $\mathbf{x}_1$ and $\mathbf{x}_2$ as

$$\mathbf{x}_3 = \lambda \mathbf{x}_1 + (1 - \lambda) \mathbf{x}_2, \quad 0 \leq \lambda \leq 1.$$

Then

$$\begin{aligned} \mathbf{cx}_3 &= \mathbf{c} \{ \lambda \mathbf{x}_1 + (1 - \lambda) \mathbf{x}_2 \} \\ &= \lambda \mathbf{cx}_1 + (1 - \lambda) \mathbf{cx}_2 \\ &= \lambda z + (1 - \lambda) z, \quad \text{by (1)} \\ &= z, \end{aligned}$$

so that $\mathbf{x}_3$ satisfies $\mathbf{cx} = z$. Thus $\mathbf{x}_3$ is in X and it being the convex combination of $\mathbf{x}_1$ and $\mathbf{x}_2$ in X, X is a convex set.

Thus the hyperplane $\mathbf{cx} = z$ is a convex set.

then $\mathbf{x}$ is said to be the *convex combination* of the points

$$\mathbf{x}_1, \mathbf{x}_2, \ldots\ldots, \mathbf{x}_p .$$

If we consider two points $\mathbf{x}_1$ and $\mathbf{x}_2$, then their convex combination is a point $\mathbf{x}$ given by $\mathbf{x} = \lambda_1 \mathbf{x}_1 + \lambda_2 \mathbf{x}_2$ such that $\lambda_1 + \lambda_2 = 1$ and $\lambda_1, \lambda_2 \geq 0$.

Thus we see that the line segment joining the two points $\mathbf{x}_1$ and $\mathbf{x}_2$ is the set of all possible convex combinations of those two points. In a triangle, the centroid is the convex combination of the vertices.

A set X is said to be a *convex set* if, for any two points $\mathbf{x}_1$, $\mathbf{x}_2$ in the set, the line segment joining these two points be also in the set. Thus, for a convex set, the convex combinations of any two points in the set also lie in the set.

This implies that if X be a convex set, then every point

$$\mathbf{x} = \lambda \mathbf{x}_2 + (1 - \lambda)\mathbf{x}_1,\ 0 \leq \lambda \leq 1$$

where $\mathbf{x}_1, \mathbf{x}_2 \in X$, must also be in the set.

A circle and a triangle in $\mathbf{E}^2$ and a sphere and a cube in $\mathbf{E}^3$ are the examples of convex set.

By convention, we say that a set containing only one point is a convex set.

The set given by $X = \{ (x_1, x_2) : x_1^2 + x_2^2 \leq 4 \}$

is convex. A circle and its interior costitute a convex set. But the set of points that forms the boundary of a circle is not a convex set.

A convex set, according to the definition, cannot have holes or re-entrants in it.

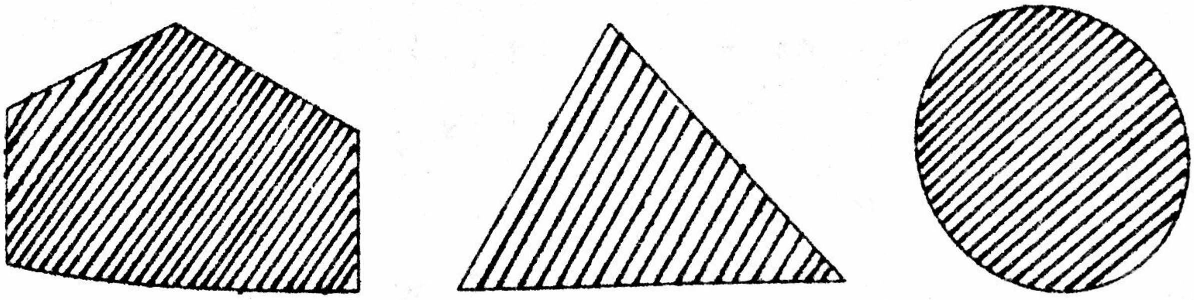


Fig. 13

this line is constrained to lie within ... the n-dimensional Euclidean space as giv
$\mathbf{x}_2$. Thus a set of points in the n-dimensional Euclidean space as given

$$X = \{ \mathbf{x} : \mathbf{x} = \lambda \mathbf{x}_2 + (1 - \lambda)\mathbf{x}_1, 0 \le \lambda \le 1 \}$$

is defined to be the *line segment* joining the points $\mathbf{x}_1$ and $\mathbf{x}_2$.

Consider a set of points

$$X = \{ \mathbf{x} : |\mathbf{x} - \mathbf{a}| = \varepsilon > 0 \}.$$

This set of points forms a *hypersphere* in $\mathbf{E}^n$ with centre at radius equal to ε.

If $n = 2$, then it is a circle in $\mathbf{E}^2$ and if $n = 3$, then it is a sphere

An ε-*neighbourhood* about the point $\mathbf{a}$ is defined to be the points inside the hypersphere with centre $\mathbf{a}$ and radius $\varepsilon >$ assumed to be very small. Thus $|\mathbf{x} - \mathbf{a}| < \varepsilon$.

A point $\mathbf{a}$ is said to be an *interior point* of the set X ε-neighbourhood about the point $\mathbf{a}$ contains only points of the set

An interior point of X must be an element of X.

A point $\mathbf{w}$ is a *boundary point* of a set X if every ε-neighbou about $\mathbf{w}$ contains points of the set and also points not of th According to the definition, it is clear that a boundary point m may not belong to the set, but an interior point must belong to t A set is said to be *closed* if it contains all the boundary points set. On the other hand, an *open set* contains only the interior po the set.

Thus $X = \{ (x_1, x_2) : x_1^2 + x_2^2 \le 4 \}$ is a closed set

and $X = \{ (x_1, x_2) : x_1^2 + x_2^2 < 4 \}$ is an open set.

A set is said to be strictly *bounded* if there exists a positive n r such that for every $\mathbf{x} \in X$, $|\mathbf{x}| < r$. If each component of point of a set has a lower limit, then the set is bounded from belo

3.2 CONVEX COMBINATION AND CONVEX SET.

If a point $\mathbf{x}$ can be expressed as

$$\mathbf{x} = \lambda_1 \mathbf{x}_1 + \lambda_2 \mathbf{x}_2 + \cdots\cdots + \lambda_p \mathbf{x}_p, \ \lambda_i \ge 0,$$

where $\mathbf{x}_i$'s are a finite number of points in $\mathbf{E}^n$, for all

$$i = 1, 2, \ldots\ldots, p$$

and

$$\Sigma\lambda_i = \lambda_1 + \lambda_2 + \cdots\cdots + \lambda_p = 1,$$

The two vectors

$$\frac{\pm \mathbf{c}}{|\mathbf{c}|}$$

are unit normals to the hyperplane.

The hyperplanes having the same unit normals are said to be *parallel*. Moving a hyperplane $\mathbf{cx} = z$ parallel to itself is accomplished by increasing or decreasing the value of z.

The hyperplane $\mathbf{cx} = z$ in $\mathbf{E}^n$ divides whole of $\mathbf{E}^n$ into three mutually disjoint sets as

$$X_1 = \{ \mathbf{x} : \mathbf{cx} < z \},$$
$$X_2 = \{ \mathbf{x} : \mathbf{cx} = z \}$$
$$\text{and} \quad X_3 = \{ \mathbf{x} : \mathbf{cx} > z \}.$$

X_1 and X_3 as defined above are called *open half spaces* while sets like

$$X_4 = \{ \mathbf{x} : \mathbf{cx} \leq z \} \text{ and } X_5 = \{ \mathbf{x} : \mathbf{cx} \geq z \}$$

are called *closed half spaces.*

It is easily seen that the point $x = [1, 2, 3, 4]$ lies in the open half space of the type $\mathbf{cx} > z$ generated by the hyperplane

$$2x_1 + 3x_2 + 4x_3 + 5x_4 = 7,$$

since $\quad 2.(1) + 3.(2) + 4.(3) + 5.(4) = 40 > 7.$

But $\quad x = [1, 2, 3, -4]$ lies in the space $\mathbf{cx} < z$.

It should be noted that in a linear programming problem

Optimize $z = \mathbf{cx}$

subject to $\mathbf{Ax} (\leq = \geq) \mathbf{b}, \mathbf{x} \geq \mathbf{0}$,

the objective function as also the constraints with the equality sign represent hyperplanes. The constraints with signs $\leq$ or $\geq$ are the half spaces produced by the hyperplanes with the sign of equality only.

A *line* in the n-dimensional Euclidean space, passing through the points $\mathbf{x}_1$ and $\mathbf{x}_2$ ($\mathbf{x}_1 \neq \mathbf{x}_2$) is defined to be the set of points

$$X = \{ \mathbf{x} : \mathbf{x} = \lambda \mathbf{x}_2 + (1 - \lambda) \mathbf{x}_1, \ \lambda \text{ is real} \}.$$

Similarly, in three dimensions, a linear equation in x_1, x_2, x_3 of the form

$$c_1 x_1 + c_2 x_2 + c_3 x_3 = z$$

represents a plane.

Thus a line is a set of points in $\mathbf{E}^2$ satisfying

$$c_1 x_1 + c_2 x_2 = z$$

and a plane is a set of points in $\mathbf{E}^3$ satisfying

$$c_1 x_1 + c_2 x_2 + c_3 x_3 = z .$$

The above two equations can be written in a compact form as

$$\mathbf{cx} = z$$

where $\mathbf{c} = (c_1, c_2)$ or (c_1, c_2, c_3) and $\mathbf{x} = [x_1, x_2]$ or $[x_1, x_2, x_3]$ in two or three dimensions respectively.

Generalising the idea of dimensions, we say that a set of points in n-dimensional space whose co-ordinates satisfy the linear equation of the form

$$c_1 x_1 + c_2 x_2 + \cdots\cdots + c_n x_n = z$$

is called a *hyperplane* for fixed values of z and c_i, $i = 1, 2, \cdots\cdots, n$.

The equation of the hyperplane can be put in short as $\mathbf{cx} = z$ where $\mathbf{c} = (c_1, c_2, \cdots\cdots, c_n)$ and $\mathbf{x} = [x_1, x_2, \cdots\cdots, x_n]$ and in which all the c_i's are constants but not zero simultaneously. For different values of z, we get different hyperplanes. In the notation of sets,

$$H = \{ \mathbf{x} : \mathbf{cx} = z \}$$

is a hyperplane whose equation is $\mathbf{cx} = z$.

If $z = 0$, $\mathbf{cx} = 0$, so that the hyperplane passes through the origin. From this, we see that the vector $\mathbf{c}$ is orthogonal to every vector $\mathbf{x}$ on the hyperplane and this $\mathbf{c}$ is called the *normal* to the hyperplane.

If $z \neq 0$ and $\mathbf{x}_1, \mathbf{x}_2$ be two distinct points on the hyperplane $\mathbf{cx} = z$, then

$$\mathbf{c}(\mathbf{x}_1 - \mathbf{x}_2) = \mathbf{cx}_1 - \mathbf{cx}_2 = z - z = 0 .$$

Thus $\mathbf{c}$ is orthogonal to any vector $(\mathbf{x}_1 - \mathbf{x}_2)$ on the hyperplane.

CHAPTER III — Characteristics of an optimal solution

3.1 DEFINITIONS.

Point sets are the sets whose elements are points or vectors in n-dimensional Euclidean space E^n. Thus the set

$$X = \{(x_1, x_2) : x_1^2 + x_2^2 < 1\}$$

represents set of points in E^2 lying inside a circle of unit radius with centre at the origin.

In two dimensions, a linear equation in x_1, x_2 of the form

$$c_1 x_1 + c_2 x_2 = z$$

represents a straight line.