9

es:

$$S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

$$L(S) = \left\{ \lambda_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \lambda_2 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

$$= \left\{ \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \right\}$$

---

Express $(-1,2,4)$ as the linear combination of $\alpha = (-1,2,0)$, $\beta = (0,-1,1)$ and $\gamma = (3,-4,2)$

Let $\rightarrow$ $(1,2,4) = a(-1,2,0) + b(0,-1,1) + c(3,-4,2)$

$$\Rightarrow \left.\begin{aligned} -a+3c &= 1 \\ 2a-b-4c &= 2 \\ b+2c &= 4 \end{aligned}\right\}$$

solving $a = 4$, $b = 2$, $c = 1$

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Show that $(-1,2,1)$, $(3,0,-1)$ and $(-5,4,3)$ are linearly dependent

$$a(-1,2,1) + b(3,0,-1) + c(-5,4,3) = (0,0,0)$$

$$\left.\begin{aligned} -a+3b-5c &= 0 \\ 2a+0.b+4c &= 0 \\ a-b+3c &= 0 \end{aligned}\right\}$$

Let at least one $a_i$ ie $a_K \neq 0$

$$\Rightarrow \sum_{\substack{i=1 \\ i \neq k}}^{n} a_i \alpha_i + a_K \alpha_k = 0$$

$$\Rightarrow a_k \alpha_k = - \sum_{\substack{i=1 \\ i \neq k}}^{n} a_i \alpha_i$$

$$\Rightarrow \alpha_k = - \sum_{\substack{i=1 \\ i \neq k}}^{n} \frac{a_i}{a_k} \alpha_i$$

Now $a_K \neq 0 \Rightarrow a_k^{-1} \in F$

so $\alpha_k$ is the linear combination others.

## Linear span

Let V be a vector space over the field F and S be a nonempty subset of V. Then the linear span of S is defined as the set of all linear combinations of finite sets of elements of S.

denoted by $L(S)$

If $S = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$.

$a_i \in F$, $\alpha_i \in S$

$$L(S) = \{a_1\alpha_1 + a_2\alpha_2 + \cdots + a_n\alpha_n\}$$

7

## Linear dependence and independence

Let V be a vector space over F.

If $\alpha_1, \alpha_2, \ldots, \alpha_n \in V$ then any ~~vector~~ vector $\alpha$ is the linear combination of $\alpha_1, \alpha_2, \ldots, \alpha_n$ if $\alpha = a_1\alpha_1 + a_2\alpha_2 + \cdots + a_n\alpha_n$, $a_i \in F$.

$\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ is said to be linearly independent if $a_1\alpha_1 + a_2\alpha_2 + \cdots + a_n\alpha_n = 0$ $a_i \in F$.

$\Rightarrow a_1 = a_2 = \cdots = a_n = 0$.

$\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ is said to be linearly dependent if $\exists$ $a_1, a_2, \ldots, a_n$ not all zero such that $a_1\alpha_1 + a_2\alpha_2 + \cdots + a_n\alpha_n = 0$.

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Th: Let V be a vector space over the field F. Then the set S of all nonzero vectors $\alpha_1, \alpha_2, \ldots, \alpha_n \in V$ is linearly dependent if ~~any only if~~ some element of S be a linear combination of the others.

Let $S = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ is a set of linearly dependent vectors. Then $\exists$ scalars $a_i \in F$ not all zero. Such that $a_1\alpha_1 + a_2\alpha_2 + \cdots + a_n\alpha_n = 0$

6

**Ex** Let $V = \{(x, y, z) : x, y, z \in \mathbb{R}\}$ where $\mathbb{R}$ is the field of real nos

Show that if $W = \{(x, y, z) : x - 3y + 4z = 0\}$ then $W$ is a subspace of $V$ over $F$.

**Soln**

Let $\alpha, \beta \in W$

Then $\alpha = (3y_1 - 4z_1, y_1, z_1)$

$\beta = (3y_2 - 4z_2, y_2, z_2)$

If $a, b \in \mathbb{R}$ then,

$a\alpha + b\beta$

$= a(3y_1 - 4z_1, y_1, z_1) + b(3y_2 - 4z_2, y_2, z_2)$

$= (3ay_1 - 4az_1, ay_1, az_1) + (3by_2 - 4bz_2, by_2, bz_2)$

$= (3(ay_1 + by_2) - 4(az_1 + bz_2), ay_1 + by_2, az_1 + bz_2)$

Let $l = ay_1 + by_2$, $m = az_1 + bz_2$ $\in \mathbb{R}$.

$= (3l - 4m, l, m) \in W$

Satisfying ①. $3l - 4m - 3l + 4m = 0$

Q. No.

(5)



Result: If $W_1$ and $W_2$ are any two subspaces of a vector space V. then $W_1 \cap W_2$ is also a subspace of V in F.

Proof: Let $a, b \in F$ and $\alpha, \beta \in W_1 \cap W_2$

Now $\alpha \in W_1 \cap W_2 \Rightarrow \alpha \in W_1$ and $\alpha \in W_2$

and $\beta \in W_1 \cap W_2 \Rightarrow \beta \in W_1$ and $\beta \in W_2$

$\because W_1, W_2$ are subspaces of V

$\therefore \alpha \in W_1, \beta \in W_1 \Rightarrow a\alpha + b\beta \in W_1$

& $\alpha \in W_2, \beta \in W_2 \Rightarrow a\alpha + b\beta \in W_2$

Also, $a\alpha + b\beta \in W_1$ and $a\alpha + b\beta \in W_2$

$\Rightarrow a\alpha + b\beta \in W_1 \cap W_2$.

---

Show that the set W of ordered pairs $(a_1, a_2, 0)$ where $a_1, a_2 \in F$, a field, is a subspace of $V_3$ over F.

Let $\alpha = (a_1, a_2, 0)$, $\beta = (b_1, b_2, 0)$

where $a_1, a_2, b_1, b_2 \in F$.

Let $a, b \in F$

$\therefore a\alpha + b\beta = (aa_1, aa_2, 0) + (bb_1, bb_2, 0)$

$= (aa_1 + bb_1, aa_2 + bb_2, 0) \in W$

Q. No.



4

$$a(-\alpha) = -(a\alpha)$$

iv) $[a + (-a)]\alpha = a\alpha + (-a)\alpha$

$\Rightarrow 0\alpha = a\alpha + (-a)\alpha.$

$\Rightarrow 0 = a\alpha + (-a)\alpha$

$\Rightarrow (-a)\alpha = -(a\alpha)$

v) $a(\alpha - \beta) = a[\alpha + (-\beta)]$

$= a\alpha + a(-\beta)$

$= a\alpha + [-(a\beta)]$

$= a\alpha - a\beta.$

Q) Subspace: Let V be a vector space over the field F. A nonempty subset W of V is called a vector subspace if W is a vector space over a field F.

Th: The necessary condition for a nonempty subset W of a vector space V over F to be a subspace of V is

$a, b \in F, \alpha, \beta \in W$

$\Rightarrow ab + b\beta \in W$.

Q. No.

7

Results

i) $a.0 = 0 \quad \forall a \in F$

ii) $0\alpha = 0 \quad \forall \alpha \in V$

iii) $a(-\alpha) = -(a\alpha) \;\forall a \in F \;\&\; \alpha \in V$

iv) $(-a)\alpha = -(a\alpha) \;\forall a \in F \;\&\; \alpha \in V$

v) $a(\alpha - \beta) = a\alpha - a\beta \quad \forall a \in F, \alpha, \beta \in V$

~~vi) $a\alpha = 0 \Rightarrow a = 0.\ \forall\alpha$~~ $= 0$

i) $a0 = a(0+0) = a0 + a0$

$\because a0 \in V \Rightarrow 0 + a0 = a0$

$\Rightarrow 0 + a0 = a0 + a0$ (Right Cancellation)

$\Rightarrow \boxed{a0 = 0}$

ii) $0\alpha = (0+0)\alpha$ (Right Cancellation)

$\Rightarrow 0 + 0\alpha = 0\alpha + 0\alpha$

$\Rightarrow \boxed{0\alpha = 0}$

iii) $a[\alpha + (-\alpha)] = a\alpha + a(-\alpha)$

i.e, $a0 = a\alpha + a(-\alpha)$ $\quad \because \alpha + (-\alpha) = 0$

$\Rightarrow 0 = a\alpha + a(-\alpha)$ $\quad \because a0 = 0$

Therefore $a(-\alpha)$ is the additive inverse of $a\alpha$.

Q. No.



2

Defn Let $(F, +, \cdot)$ be a filed

$\exists$ an set V which is additively commutative group and multiplicatively semigroup.

Result:

Let V. be a vector space over F

1) If $a, b \in F$ and $\alpha$ be non null vector in V Then

$$a\alpha = b\alpha \Rightarrow a = b.$$

ii) If $\alpha, \beta \in V$ and a be nonzero element of F Then $a\alpha = a\beta$

$$\Rightarrow \alpha = \beta$$

1)

$$a\alpha = b\alpha$$

$\Rightarrow (a-b)\alpha = 0$ $\Rightarrow a - b = 0$ $\because \alpha \neq 0$

$\therefore \boxed{a = b}$

ii) $a\alpha = a\beta$

$\Rightarrow a(\alpha - \beta) = 0$ $\because a \neq 0$

$\Rightarrow \alpha - \beta = 0$

$\Rightarrow \boxed{\alpha = \beta}$

Q. No.



1

Consider V be a non empty set satisfying

i) $\alpha \in V, \beta \in V \Rightarrow \alpha + \beta \in V$.

ii) $(\alpha+\beta)+\gamma = \alpha + (\beta+\gamma) \;\forall\, \alpha, \beta, \gamma \in V$.

iii) $\exists$ null vector $0$ s.t

$\alpha + 0 = 0 + \alpha = \alpha \;\forall\, \alpha \in V$

iv) corresponding to $\alpha$, $\exists\, (-\alpha)$ s.t

$\alpha + (-\alpha) = (-\alpha) + \alpha = 0$, $\forall\, \alpha \in V$.

v) $\alpha + \beta = \beta + \alpha$, $\forall\, \alpha, \beta \in V$.

vi) $\exists$ an external composition in V over the field of scalars.

$a \in F, \alpha \in V \Rightarrow a\alpha \in V$

vii) $(a+b)\alpha = a\alpha + b\alpha$, $\alpha \in V$ and $a, b \in F$

viii) $a(\alpha+\beta) = a\alpha + a\beta$, $\alpha, \beta \in V, a \in F$

ix) $(ab)\alpha = a(b\alpha)$, $\alpha \in V$, $a, b \in F$

x) $1.\alpha = \alpha$ where $1$ is unity in the field F.