Exact Differential Equation

The necessary and sufficient condition for a differential equation of first degree ~~being exact~~ given by

$M\,dx + N\,dy = 0$ ...(1), where M and N are functions of x and y

being exact is

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

Proof: Let $u = c$ be its primitive ...(2)

ie $du = 0$

so $\dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y}\dfrac{dy}{dx} = 0$ ...(3)

Comparing (1) and (3) we get.

$$M = \frac{\partial u}{\partial x}, \quad N = \frac{\partial u}{\partial y}, \text{ so that}$$

$$\frac{\partial M}{\partial y} = \frac{\partial^2 u}{\partial y \partial x}, \quad \frac{\partial N}{\partial x} = \frac{\partial^2 u}{\partial x \partial y}.$$

Hence $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$.

The necessary condition has been proved.

Now to ~~find~~ prove sufficient condition.

If $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$, then we have to show that $M\,dx + N\,dy$ is an exact equation.

Let $\int M\,dx = U$, then $\dfrac{\partial U}{\partial x} = M$.

$$\therefore \frac{\partial^2 U}{\partial y \partial x} = \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \quad \because \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

ie $\dfrac{\partial N}{\partial x} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial U}{\partial y}\right)$

Integrating we get

$$N = \frac{\partial U}{\partial y} + f(y)$$

$$M + N\frac{dy}{dx} = \frac{\partial U}{\partial x} + \left(\frac{\partial U}{\partial y} + f(y)\right)\frac{dy}{dx}.$$

$$= \left(\frac{\partial U}{\partial x} + \frac{\partial U}{\partial y}\frac{dy}{dx}\right) + f(y)\frac{dy}{dx}.$$

$$= \frac{d}{dx}\left[U + \int \left(f(y)\frac{dy}{dx}\right)dx\right]$$

$$= \frac{d}{dx}\left[U + F(y)\right].$$

This shows that

$M + N\dfrac{dy}{dx} = 0$ is an exact

equation.

<u>Working Rule</u>

of the equation $Mdx + Ndy = 0$,

satisfies the condition

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x},$$

Then it is ~~exct~~ exact. To find the solution, proceed as follows.

Step 1 Integrate M with respect to x taking y as constant.

Step 2 Find out those terms in N which are free from x and integrate them with respect to y

Step 3 Add the two expressions so obtained and equate the sum to an arbitrary constant.

This gives the general solution of the given exact equation

Ex $(y^4 + 4x^3y + 3x)\,dx + (x^4 + 4xy^3 + y + 1)\,dy = 0$

Solution Here $M = y^4 + 4x^3y + 3x$,

and $N = x^4 + 4xy^3 + y + 1$

Now $\frac{\partial M}{\partial y} = 4y^3 + 4x^3$, $\frac{\partial N}{\partial x} = 4x^3 + 4y^3$.

$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

So the equation is exact

To find the solution of differential equation

Step 1 $\int M\,dx$ keeping y as constant.

$\int (y^4 + 4x^3y + 3x)\,dx = y^4x + x^4y + \frac{3}{2}x^2$

Step 2 Find out the terms in N which is free from x, here it is $y+1$

So on integrating we get

$$\int (y+1)\,dy = \frac{y^2}{2} + y.$$

The general solution is

$$y^4 x + x^4 y + \frac{3}{2}x^2 + \frac{1}{2}y^2 + y = c$$

p. 57 Ex. Pelerana.

Ex 1) Solve $x(x^2+y^2-a^2)\,dx + y(x^2-y^2-b^2)\,dy = 0$

2) Q. Solve $(x^2-2xy+3y^2)\,dx + (4y^3+6xy-x^2)\,dy = 0$.

3) Solve $(1+e^{x/y})\,dx + e^{x/y}(1-x/y)\,dy = 0$

p-84 Ross

Bernoulli differential equation

An equation of the form

$$\frac{dy}{dx} + P(x)y = Q(x)y^n \quad \text{---}(1)$$

is called Bernoulli differential equation.

Suppose $n \neq 0$ or 1. Then transformation $v = y^{1-n}$ reduces the Bernoulli equation (1) to a linear equation in v.

~~Linear Second order Differential Equations with Constant Coefficient~~

We first multiply eqn (1) by y^{-n}. we get

$$y^{-n}\frac{dy}{dx} + P(x)\,y^{-n+1} = Q(x) \quad \cdots (2)$$

Now $v = y^{1-n}$.

$$\therefore \frac{dv}{dx} = (1-n)\,y^{1-n-1}\frac{dy}{dx} = (1-n)\,y^{-n}\frac{dy}{dx}.$$

so eqn (2) reduces to

$$\frac{1}{(1-n)}\frac{dv}{dx} + P(x)\,v = Q(x)$$

ie
$$\frac{dv}{dx} + (1-n)P(x)\,v = (1-n)Q(x) \quad \cdots (3)$$

Let $P_1(x) = (1-n)P(x)$ and $Q_1(x) = (1-n)Q(x)$

so (3) becomes

$$\frac{dv}{dx} + P_1(x)\,v = Q_1(x)$$

which is linear in v.

Example
$$\frac{dy}{dx} + y = xy^3.$$

Here $n = 3$.

so $v = y^{1-3} = y^{-2}$, $\frac{dv}{dx} = (-2)y^{-3}\frac{dy}{dx}$

$$y^{-3}\frac{dy}{dx} + y^{-2} = x$$

$$-\frac{1}{2}\frac{dv}{dx} + v = x$$

ie $$\frac{dv}{dx} - 2v = -2x$$

$$\text{I.F} = e^{-\int 2\,dx} = e^{-2x}$$

$$\therefore e^{-2x}\frac{dv}{dx} - 2ve^{-2x} = -2xe^{-2x}$$

$$\frac{d}{dx}\left(e^{-2x}v\right) = -2xe^{-2x}$$

$$\therefore e^{-2x}v = -2\int \underset{\mathrm{I}}{x}\,\underset{\mathrm{II}}{e^{-2x}}\,dx + c$$

$$= -2\left[x\frac{e^{-2x}}{-2} - \int \frac{dx}{dx}\int e^{-2x}dx\right] + c$$

$$= -2\left[-x\frac{e^{-2x}}{2} - \int \frac{e^{-2x}}{-2}dx\right] + c$$

$$= -2\left[-x\frac{e^{-2x}}{2} + \frac{1}{2}\cdot\frac{e^{-2x}}{(-2)}\right] + c$$

$$= (2x+1)\frac{e^{-2x}}{2} + c$$

$$\therefore v = \frac{1}{2}(2x+1) + ce^{2x}$$

Here $v = \frac{1}{y^2}$

$$\therefore \frac{1}{y^2} = \frac{1}{2}(2x+1) + ce^{2x}$$

Ans

Step 2 Find out the terms in N which is free from x, here it is $y+1$

So on integrating we get

$$\int (y+1)\,dy = \frac{y^2}{2} + y.$$

The general solution is

$$y^4 x + x^4 y + \frac{3}{2}x^2 + \frac{1}{2}y^2 + y = c$$

p-57 Sharma.

Assignment

Ex 1) Solve $x(x^2+y^2-a^2)\,dx + y(x^2-y^2-b^2)\,dy = 0$

2) Q^2 Solve $(x^2-2xy+3y^2)\,dx + (4y^3+6xy-x^2)\,dy = 0$

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p-84 Ross

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Suppose $n \neq 0$ or 1. Then transformation $v = y^{1-n}$ reduces the Bernoulli equation (1) to a linear equation in v.

Assignment.

1) Solve $\frac{dy}{dx} = x^3y^3 - xy$

Ans. $\frac{1}{y^2} = 1 - x^2 + ce^{x^2}$

2) Solve. $\frac{dy}{dx} + xy = xy^2$

Ans: $\frac{1}{y} = ce^{x^2/2} + 1.$

3) $\frac{dy}{dx}(x^2y^3 + xy) = 1.$

Ans: $\frac{1}{x} = (2 - y^2) - ce^{-y^2/2}$

p-58
pairing? Problem sum

Q.1. Number of bacteria in a yeast culture grows at a rate which is proportional to the number present. If the population of a colony at ~~years~~ yeast bacteria triples in 1 hour. Find the number of bacteria which will be present at the end of ~~8~~ 5 hours.

Ans: 3^5 times at the end of 5 years.

[Hint $\frac{dx}{dt} \propto x$ $\frac{dx}{dt} = Kx$.]