Differential Equations

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1. Simple defn:

A Differential equation is an equation with a function and one or more of its derivatives:

$$y + \frac{dy}{dx} = 5x$$

→ differential (derivative)

↓ Equation

Thus, it is an equation with the function y and its derivative $\frac{dy}{dx}$.

2. Order and Degree of a differential equation:

$$\left(\frac{d^2y}{dx^2}\right)^3 + \frac{dy}{dx} + y = 4x^5$$

order 2

degree 3

- **Order of a differential equation:**

Order of a differential equation is defined as the order of the highest order derivative of the dependent variable with respect to the independent variable involved in the given differential equation.

Example :

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(i) $\frac{dy}{dx} = e^x$

(ii) $\frac{d^2y}{dx^2} + y = 0$

(iii) $\left(\frac{d^3y}{dx^3}\right) + x^2\left(\frac{d^2y}{dx^2}\right)^3 = 0$

The equations (i), (ii) and (iii) involve the highest derivative of first, second and third order respectively.
Therefore, the order of these equations are respectively 1, 2 and 3.

- Degree of a differential equation

The degree of a differential equation is the degree (exponent) of the derivative of the highest order in the equation, after the equation is free from negative and fractional powers of the derivatives.

Example: (iv) $\frac{d^3y}{dx^3} + 2\left(\frac{d^2y}{dx^2}\right)^2 - \frac{dy}{dx} + y = 0$

(V) $\left(\frac{dy}{dx}\right)^2 + \frac{dy}{dx} - \sin^2 y = 0$

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(VI) $\frac{dy}{dx} + \sin\left(\frac{dy}{dx}\right) = 0$

By the degree of a differential equation, when it is a polynomial equation in derivatives, we mean the highest power (positive integral index) of the highest order derivative involved in the given differential equation.

Therefore, differential equations (i), (ii) and (iii) each are of degree one, also equation (IV) is of degree 1. Equation (V) is of degree 2. But equation (VI) is not a polynomial equation in $\left(\frac{dy}{dx}\right)$. So degree of such differential equation cannot be defined.

Note :- Order and degree (if defined) of a differential equation are always positive integers.

Exercise →

Determine order and degree of each of the following differential equations (if possible) :-

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(i) $\dfrac{dy}{dx} - \cos x = 0$

(ii) $xy\dfrac{d^2y}{dx^2} + x\left(\dfrac{dy}{dx}\right)^2 - y\dfrac{dy}{dx} = 0$

(iii) $y''' + y^2 + e^{y'} = 0$ $\qquad \left[y' \equiv \dfrac{dy}{dx}\right]$

(iv) $y' + 5y = 0$

(v) $y''' + 2y'' + y' = 0$

(vi) $\left(\dfrac{d^2y}{dx^2}\right)^2 + \cos\left(\dfrac{dy}{dx}\right) = 0$

(vii) $\left(\dfrac{ds}{dt}\right)^4 + 3s\dfrac{d^2s}{dt^2} = 0$

(viii) $y'' + (y')^2 + 2y = 0$

(ix) $\left(\dfrac{d^2y}{dx^2}\right)^3 + \left(\dfrac{dy}{dx}\right)^2 + \sin\left(\dfrac{dy}{dx}\right) + 1 = 0$

(x) $2x^2\dfrac{d^2y}{dx^2} - 3\dfrac{dy}{dx} + y = 0$

(xi) $\left[\dfrac{d^2y}{dx^2} + \left(\dfrac{dy}{dx}\right)^3\right]^{6/5} = 6y$

Solutions of Differential equation of the First order and First degree :-

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1. Inspection Method -

If the differential equation can be written as $f[f_1(x,y)\, d\{f_1(x,y)\}]$

$$+ \phi[f_2(x,y)\, d\{f_2(x,y)\}] + \cdots = 0$$

then each term can be integrated seperately.

For this, remember the following results:

i) $x\,dy + y\,dx = d(xy)$

ii) $d(x+y) = dx + dy$

iii) $d\left(\frac{y}{x}\right) = \frac{x\,dy - y\,dx}{x^2}$

iv) $d\left(\frac{x}{y}\right) = \frac{y\,dx - x\,dy}{y^2}$

v) $\frac{x\,dy + y\,dx}{xy} = d(\log xy)$

vi) $\frac{y\,dx - x\,dy}{xy} = d\left(\log \frac{x}{y}\right)$

vii) $\frac{x\,dy - y\,dx}{xy} = d\left(\log \frac{y}{x}\right)$

Continued→

Example :-

$$y(2xy+1)\,dx - x\,dy = 0$$

$$\Rightarrow (2xy^2+y)\,dx - x\,dy = 0$$

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$$\Rightarrow 2xy^2\,dx + (y\,dx - x\,dy) = 0$$

$$\Rightarrow 2x\,dx + \frac{y\,dx - x\,dy}{y^2} = 0$$ [Dividing both sides by y^2]

Integrating, we have,

$$\Rightarrow 2\int x\,dx + \int d\left(\frac{x}{y}\right) = 0$$ [Since $\frac{y\,dx - x\,dy}{y^2} = d\left(\frac{x}{y}\right)$]

$$\Rightarrow 2\frac{x^2}{2} + \frac{x}{y} = c$$

where c is a constant of integration.

$$\Rightarrow x^2 + \frac{x}{y} = c$$

$$\Rightarrow x(xy+1) = cy$$

<u>Exercise</u> :- i) $y(y^3-x)\,dx + x(y^3+x)\,dy = 0$

ii) $y(y^2+1)\,dx + x(y^2-1)\,dy = 0$

Linear Differential Equations

- A First order Linear Differential Equation is a first order differential equation which can be put in the form

$$\boxed{\frac{dy}{dx} + P(x)\,y = Q(x)}$$

(Brim)

where $P(x), Q(x)$ are functions of x.

Example:- Solve the differential equation

$$x\frac{dy}{dx} = x^2 + 3y. \quad —(1)$$

put the equation in the standard form

$$\frac{dy}{dx} = \frac{x^2+3y}{x}$$

$$\Rightarrow \frac{dy}{dx} = x + 3\frac{y}{x}$$

$$\Rightarrow \frac{dy}{dx} \underbrace{- \frac{3}{x}}_{P(x)} y = \underbrace{x}_{Q(x)}. \quad —(11)$$

The IF (Integrating Factor) is given by,

$$(\text{say}) \quad I(x) = e^{-\int \frac{3}{x} dx} = e^{-3\log x} = e^{\log x^{-3}} = x^{-3} = \frac{1}{x^3}$$

$$e^{\int P(x)dx}$$

Multiply the standard eqn (ii) with the integrating factor $I(x)$, we have

$$\frac{1}{x^3} \cdot \frac{dy}{dx} - \frac{3}{x^4} y = \frac{x}{x^3}$$

$$\Rightarrow x^{-3} \frac{dy}{dx} - \frac{3}{x^4} y = x^{-2}$$

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$$\Rightarrow \frac{d(x^{-3} y)}{dx} = x^{-2}$$

Now, Integrating both sides with respect to 'x', we get

$$\int d(x^{-3} y) = \int x^{-2} dx$$

$$\Rightarrow x^{-3} y = \frac{x^{-2+1}}{-2+1} + c$$

$$\Rightarrow x^{-3} y = -x^{-1} + c$$

$$\Rightarrow x^{-3} y = -\frac{1}{x} + c$$

$$\Rightarrow y = -\frac{x^3}{x} + cx^3$$

$$\Rightarrow y = -x^2 + cx^3 .$$

This is the solution of the given differential equation.

$\boxed{\text{Exercise:-}}$

i) $y' + xy = x$, $\quad y' \equiv \frac{dy}{dx}$

ii) $\frac{dy}{dx} + \frac{1}{x} y = x^2$,

iii) $\frac{dy}{dx} + \frac{2}{x} y = x - 1$.