Module – I (Calculus – Integration)

1. * Find the evolute of the Ellipse-- $\frac{x2}{a2} + \frac{y2}{b2} = 1$
2. Find the surface area of the solid generated by revolving the cycloid $x = a(\theta + \sin\theta), y = a(1 + \cos\theta)$ about its base.
3. Find the radius of curvature of the curve $\sqrt{x} + \sqrt{y} = \sqrt{a}$ at the point where the straight line y = x cuts it.
4. Find the evolute of the curve given by $x = a(\cos\theta + \theta\sin\theta), y = a(\sin\theta - \theta\cos\theta)$

7. * Find the evolute of the asteroid $x^{2/3} + y^{2/3} = a^{2/3}$

8. Find the area of the region bounded by the curve $2y = x^2(x+2)(7-2x)$
9. Find the equation of the evolute of the parabola$y^2 = 12x$.
10. Find the circle of curvature of the following curves $y = x^2 - 6x + 10$ at the point (3, 1).
11. Show that (a) $\int_0^\infty e^{-x^4}dx \times \int_0^\infty x^2e^{-x^4} = \frac{\pi}{8\sqrt{2}}$ (b) $\int_0^\infty e^{-4x}x^{\frac{3}{2}}\,dx = \frac{3}{128}\sqrt{\pi}$
    (c) $\int_0^{\frac{\pi}{2}} \sin^4 x\cos^5 x\,dx = \frac{8}{315}$ (d) $\int_0^{\frac{\pi}{2}} \sqrt{\tan x}\,dx = \frac{\pi}{\sqrt{2}}$
12. Evaluate (a) $\int_{\frac{-1}{2}}^{\frac{1}{2}} \cos x\log(\frac{1-x}{1+x})dx$ (b) $\int_0^{\pi} |\sin x + \cos x|dx$
13. State the relation between Beta Function and Gamma Function and use it to show that $\int_0^1 x^{\frac{3}{2}}(1-x)^{\frac{3}{2}}\,dx = \frac{3\pi}{128}$
14. Test for convergence the improper integral $\int_0^{\frac{1}{e}} \frac{dx}{x(\log x)^2}$ and evaluate if possible .
15. Prove that $\int_{-1}^{1} \frac{dx}{x^3}$ exists in the Cauchy's principal value sense but not in the general sense.
16. Prove that $\int_0^1 \frac{x\,dx}{\sqrt{1-x^5}} = \frac{1}{5}\beta\left(\frac{2}{5}, \frac{1}{2}\right)$

Module – II (Calculus – Differentiation)

1. Use Mean- value theorem to prove the following inequality
$$\frac{x}{1+X} < \log(1+X) < X$$
2. If $y = \sin(m\sin^{-1}x)$, prove that $(1-x^2)y_{n+2} = (2n+1)xy_{n+1} + (n^2 - m^2)y_n$
3. In Cauchy's mean value theorem if $f(x) = e^x$ and $g(x) = e^{-x}$ then show that θ is independent of both x and h.
4. Show that the maximum value of $\left(\frac{1}{x}\right)^x$ is $e^{\frac{1}{e}}$.

5. Evaluate (a) $\lim_{x\to 0}\left(\frac{\tan x}{x}\right)^{\frac{1}{x}}$ (b) $\lim_{x\to 0+}\frac{\log x-\cot\left(\frac{\pi x}{2}\right)}{\cot \pi x}$

6. Prove that $\frac{x}{1+x^2} < \tan^{-1} < x$ if $x>0$, by using suitable Mean Value Theorem.

7. If $\lim \frac{\sin 2x + a\sin x}{x^3}$ be finite , find the value of 'a' and the limit.

8. * Use mean value theorem to prove that $\sin 46^o \approx \frac{1}{2}\sqrt{2}(1+\frac{\pi}{180})$

9. Use mean value theorem to prove that $\sin x > x - \frac{x^3}{6}$ $if\ 0 < x < \frac{\pi}{2}$

10. State Roll's theorem. Verify Rolle's theorem for the function f(x)= |x| in the interval [-1,1].

11. Use MVT to prove $\frac{x}{\sqrt{1-x^2}} \geq \sin^{-1} x \geq x$ if $0 \leq x < 1$.

12. Prove that the maximum rectangle inscribable a circle is a square.

13. Show that, $\frac{d^n}{dx^n}\left(\frac{\log x}{x}\right) = (-1)^n \frac{n!}{x^{n+1}}(\log x - 1 - \frac{1}{2} - \frac{1}{3} - \cdots - \frac{1}{n})$.

14. If $y = a\cos(\log x) + b\sin(\log x)$, prove that,

$$x^2 y_{n+2} + (2n+1)xy_{n+1} + (n^2+1)y_n = 0.$$

Module – III (Matrices)

1. Express the following matrix as the sum of a symmetric and a skew-symmetric Matrix A= $\begin{pmatrix} 4 & 2 & -3 \\ 1 & 3 & -6 \\ -5 & 0 & -7 \end{pmatrix}$

2. Show that $\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix}$ = abc $\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$

3. Prove that,

$$\begin{vmatrix} 2bc-a^2 & c^2 & b^2 \\ c^2 & 2ca-b^2 & a^2 \\ b^2 & a^2 & 2ab-c^2 \end{vmatrix} = (a^3+b^3+c^3-3abc)^2$$

4. By Gauss Jordan elimination method find the inverse of the matrix

(a) A= $\begin{pmatrix} 1 & 5 & 0 \\ 0 & 2 & -1 \\ 1 & 0 & 3 \end{pmatrix}$ (b) $\begin{pmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{pmatrix}$ (c) $\begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & -1 \\ -1 & 1 & -7 \end{pmatrix}$

5. ** Determine the conditions for which the systems admit of (i) only one solution, (ii) no solution, (iii) many solutions:

$$x + y + z = 1, x + 2y - z = b, 5x + 7y + az = b^2$$

6. Solve the equation by matrix inversion method:

$$x + 2y + 3z = 14$$
$$2x - y + 5z = 15$$
$$2y + 4z - 3x = 13$$

7. Prove that $\begin{vmatrix} a & b & c & d \\ -b & a & d & -c \\ -c & -d & a & b \\ -d & c & -b & a \end{vmatrix} = (a^2 + b^2 + c^2 + d^2)^2$ by Laplace expansion method.

8. If A and B are orthogonal matrix such that $\det A + \det B = 0$, then show that $(A + B)$ is singular matrix.

9. Find the rank and nullity of the matrix: $\begin{pmatrix} 1 & -1 & 2 & 0 & 4 \\ 2 & 3 & 1 & 5 & 2 \\ 1 & 3 & -1 & 0 & 3 \\ 1 & 7 & -4 & 1 & 1 \end{pmatrix}$

10. Determine the conditions for which the systems admit of i) only one solution, ii) no solution, iii) many solutions:

$$x + y + z = 6, x + 2y + 3z = 10, x + 2y + \lambda z = \mu$$

Module – IV (Vector Spaces)

1. Find whether the following transformation is linear or not :--

   $T: V_2 \rightarrow V_2$ defined by $T(X_1, X_2) = (X_1 + X_2, X_1)$

2. Prove that Ker (T) is a subspace of V where $T : V \rightarrow W$ is a linear transformation.
3. Define Vector Space.

   Prove that $R^2 = \{(x, y) : x, y \in R\}$ is a Vector Space over the field of Real numbers R Where composition +, °,are defined by $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$,
   a. $(x_1,y_1)=(ax_1,ay_1)$
4. Show that the set $S = \{(x, y, z) \in R^3 : x + y - z = 0, 2x + y - z = 0\}$ is a subspace of R^3. Hence find a basis of S and dim(S).
5. Extend the set $\{(2,1,1), (1,1,1)\}$ to a basis of R^3.
6. Determine the linear mapping $T: R^3 \rightarrow R^3$ which maps the basis vectors $(0,1,1), (1,0,1), (1,1,0)$ of $R^3$to$(1,1,1,), (1,1,1), (1,1,1)$ respectively.

   Verify Sylvester's law.
7. Let S and T be two subspaces of a vector space V over R, prove that $S \cap T$ is also a subspace of V.
8. If { α,β,γ} is a basis of a real vector space V, show that { α+β, β+γ, γ+α } is also a basis of V.
9. *Determine the linear mapping $T:R^3 \rightarrow R^3$ that maps the basis vectors (1,0,0), (0,1,0),(0,0,1) to the vectors (-1,2,1), (1,1.2),(2,1,1) respectively. Find Ker T and verify that dim Ker T + dim Im T =3.

10. Let U = $\left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} : a+b=0\right\}$ and W = $\left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} : c+d=0\right\}$ be a subspace of R_{2x2}. Find dim W, dim U, dim($U \cap W$)

11. Determine the linear transformation T : $R^3 \to R^2$ which maps the basis vectors (1,0,0),(0,1,0),(0,0,1) of R^3 to the vectors (1,1),(2,3),(3,2) respectively.Find Ker T and ImT.

12. S and T are subspaces of R^4 given by S={(x,y,z)ϵR^4 : 2x+y+3z+w=0} , T=={(x,y,z)ϵR^4 : x+2y+z+3w=0}. Find dim($S \cap T$)

13. A linear mapping T: $R^3 \to R^3$ is defined by T(x,y,z)= (2x+y-z,y+4z,x-y+3z). Find the matrix of T relative to the ordered basis (1,0,0),(0,1,0),(0,0,1).
Prove that { $1+x+x^2$, 1+x,1} is a basis of P_3.

14. Let $S = \{(x,y,z): x+y+z=0,\ 2x-3y+z=0\}$. Show that S is a subspace of R^3.
Find a basis of S and dim(S).

15. Prove that the set $\left\{\begin{pmatrix} x & y \\ y & z \end{pmatrix} : x,y,z \in R\right\}$ is a vector space under usual matrix addition and usual multiplication between a real number a matrix.

Module – V (Vector Spaces (continued.)

1. Find Eigen values of $\begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix}$
2. Prove that every Orthogonal matrix A is non-singular and $|A| = \pm 1$.
3. Find the eigen values and the corresponding eigen vectors of
$$\text{(a)} \begin{pmatrix} 1 & -1 & 2 \\ 2 & -2 & 4 \\ 3 & -3 & 6 \end{pmatrix} \text{(b) } A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$
4. Find non-singular matrix P such that $P^{-1}AP$ is a diagonal matrix, where
$$\text{(a) } A = \begin{pmatrix} 3 & 2 & 1 \\ 2 & 3 & 1 \\ 0 & 0 & 1 \end{pmatrix} \text{(b) } A = \begin{bmatrix} 1 & 1 & -2 \\ -1 & 2 & 1 \\ 0 & 1 & -1 \end{bmatrix} \text{(c) } A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$$
5. Prove that for any two vectors α, β in a Euclidean space V, $|\alpha, \beta| \le ||\alpha||\ ||\beta||$, the equality holds when α, β are linearly dependent.
6. Verify Cayley –Hamilton Theorem for the matrix (a) A = $\begin{bmatrix} 2 & 1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$ (b) $A = \begin{bmatrix} 2 & 1 \\ 3 & 5 \end{bmatrix}$ and hence find A^{-1}.
7. Prove that the set of vectors {(2,-1,1),(2,3,-1),(1,-2,-4)} is an orthogonal basis of R^3 with standard inner product.
8. If λ be an eigenvalue of a non-singular matrix$A$, then λ^{-1} is an eigen value of A^{-1}

9. Find whether the matrix $A = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$ is diagonalisable.

10. Use Gram-Schmidt process to obtain an orthogonal basis from the basis set $\{(1,1,0), (0,1,1), (1,0,1)\}$ of the Euclidean Space R^3 with Standard inner product.

11. Prove that the set of vectors $\{(1,2,2), (2,-2,1), (2,1,-2)\}$ is an orthogonal in the Euclidean space R^3 with standard inner product. Convert this orthogonal set to an orthonormal set. Express $(4,3,2)$ as a linear combination of this basis.

12. Apply Gram-Schmidt process to find an orthonormal basis for the Euclidean space

    (a) R^3 with standard inner product that contains the vectors $\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right), \left(\frac{2}{\sqrt{6}}, \frac{-1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)$

    (b) R^4 with standard inner product spanned by the vectors $(1,1,0,1), (1,-2,0,0), (1,0,-1,2)$.

13. State and prove (a) Triangle inequality (b) Pythagorus theorem and (c) Parallelogram law in an Euclidean space V.
