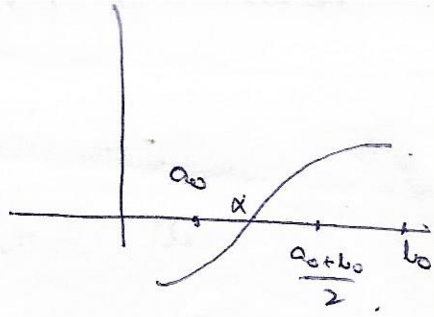
Method of bisection



Write $a_0 = a$, $b_0 = b$.

$f(a_0) f(b_0) < 0$, $a_0 < \alpha < b_0$.

write $c_0 = \frac{a_0 + b_0}{2}$

If $f(a_0) f(c_0) < 0$ then write $a_1 = a_0$, $b_1 = c_0$, s.t. $a_1 < \alpha < b_1$.

else if $f(b_0) f(c_0) < 0$ then write $a_1 = c_0$, $b_1 = b_0$ s.t. $a_1 < \alpha < b_1$.

$c_1 = \frac{a_1 + b_1}{2}$.

either $f(a_1) f(c_1) < 0$, $a_2 = a_1$, $b_2 = c_1$

$f(c_1) f(b_1) < 0$, $a_2 = c_1$, $b_2 = b_1$.

Thus we get.

$$[a_0, b_0] \supset [a_1, b_1] \cdots \supset [a_n, b_n]$$

$$b_1 - a_1 = \frac{(b_0 - a_0)}{2}$$

$$b_1 - a_1 = \frac{a_0 + b_0}{2} - a_0 = \frac{b_0 - a_0}{2}$$

$$b_2 - a_2 = \frac{b_1 - a_1}{2} = \frac{b_0 - a_0}{2^2}.$$

$\vdots$

$b_2 - a_2$

$$b_n - a_n = \frac{b_0 - a_0}{2^n}.$$

Suppose α is the root.

Initial approximation : $x_0 \equiv c_0$,

next : $x_1 \equiv c_1$,

$\vdots$

$x_n \equiv c_n$.

$$|\alpha - x_0| = |\alpha - c_0| \le \frac{b_0 - a_0}{2} = \frac{b - a}{2}$$

$$|\alpha - x_1| \le \frac{b - a}{2^2} = \frac{1}{2}|\alpha - x_0|$$

$$|\alpha - x_1| \le \frac{b_1 - a_1}{2} = \frac{b - a}{2^2}$$

$\vdots$

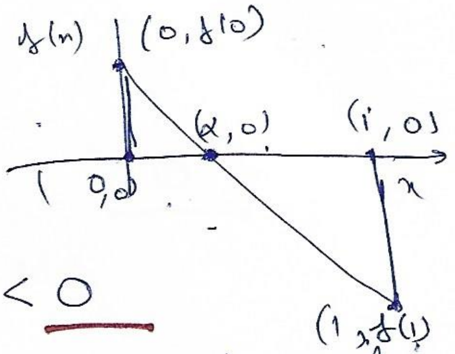
$$|\alpha - x_n| \le \frac{b_n - a_n}{2} = \frac{b - a}{2^{n+1}} = \frac{1}{2^n}|\alpha - x_0|.$$

Convergence is linear ~~(and rate of convergence is $\frac{1}{2}$~~

Bisection method

Example Perform five iteration of the bisection method to obtain the smallest positive root of the equation.

$$f(x) = x^3 - 5x + 1 = 0$$



Here

$f(0) = 1 > 0$

$f(1) = 1 - 5 + 1 = -3 < 0$

Therefore the roots lies between 0 and 1

Since it cut the x-axis. Hence there is a root $x = \alpha$.

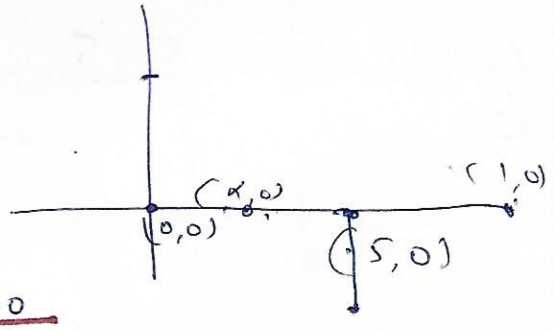
Now in bisection method. take

$a_0 = 0$ and $b_0 = 1$

we get $c_0 = \dfrac{a_0 + b_0}{2} = \dfrac{0+1}{2} = .5$.

$f(c_0) = (.5)^3 - 5 \times (.5) + 1 = -1.375 < 0$.

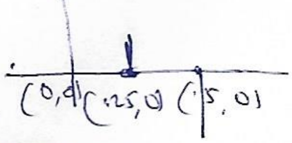
and $f(a_0) f(c_0) < 0$.



Thus the root lies ~~to~~ between $(0, 0.5)$. Again we write $a_1 = a_0$, $b_1 = c_0$

such that $a_1 < \alpha < b_1$

$c_1 = \dfrac{a_1 + b_1}{2} = \dfrac{0 + .5}{2} = .25$



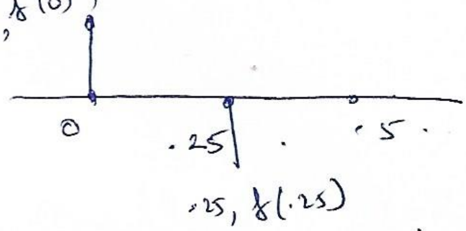
$f(c_1) = (.25)^3 - 5 \times (.25) + 1 = -.234375 < 0$.

$a_2 = .25$ & $b_2 = .5$.

~~$a_2 = c_1$, $b_2 = c_1$~~

~~Now $c_2 = \dfrac{a_2 + b_2}{2} = \dfrac{0 + .25}{2}$~~ $\dfrac{.25 + .5}{2}$

So $f(a_0)f(c_1) < 0$.



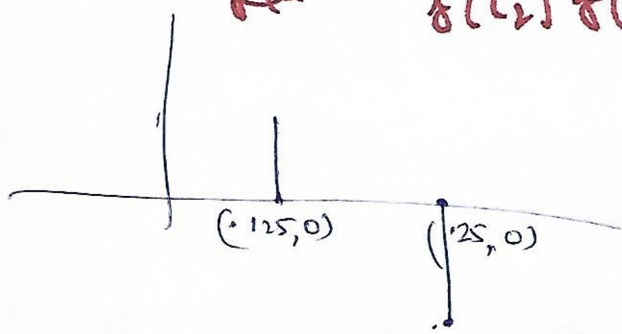
roots lies between $(0, .25)$ ∴ $a_2 = 0,\ b_2 = .25$

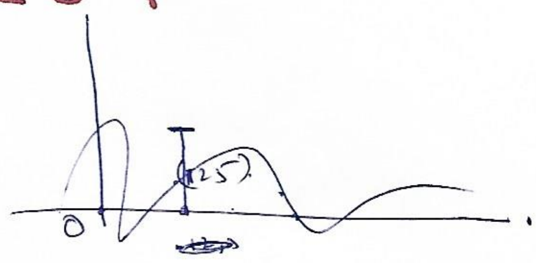
$$c_2 = \frac{a_2 + b_2}{2} = \frac{0 + .25}{2}$$

$$= .125$$

$f(c_2) = f(.125) = .37695 > 0$

$f(c_2)\,f(b_2) < 0$.





So roots lies between $(.125, .5)$

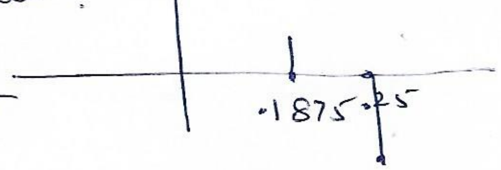
∴ $a_3 = .125,\ b_3 = .25$.

$$∴ c_3 = \frac{a_3 + b_3}{2} = \frac{.125 + .25}{2} = .1875.$$

$f(c_3) = ~~.2187~~\ .06909 > 0$ $\quad f(c_3)f(b_3) < 0$

∴ The root lies ~~bw~~ between

$(.1875, .25)$ $\quad a_4 = .1875$, $b_4 = .25$

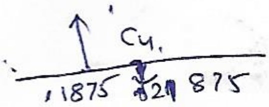


$$∴ c_4 = \frac{.1875 + .25}{2}$$

$$= .21875.$$

$f(c_4) = -0.08328 < 0$

∴ $f(a_4)\,f(c_4) < 0$,



$$c_5 = \frac{.1875 + .21875}{2} = .203125$$

$f(c_5) = -0.007366 < 0$ $\quad$ correct upto 2 Decimal places

$f(a_4)\,f(c_5) < 0$ $\quad$ ∴ $c_6 = \frac{.1875 + .203125}{2} = .195312$

Ans is .203125 Correct upto 2 decimal places

Newton Rapson's method (N-R method)

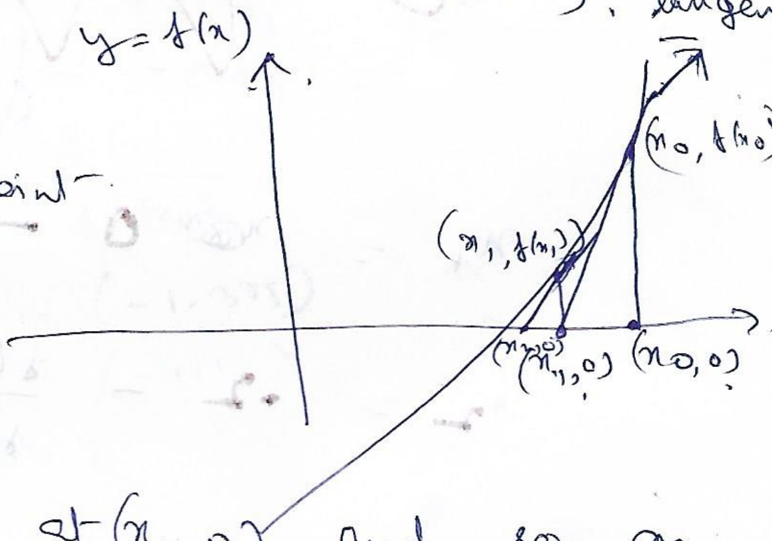
$f(x_0 + h) = 0 \qquad x_1 = x_0 + h$

Take Taylor's expansion about x_0 and h is small so neglect the terms containing $h^2, h^3, \ldots$

$$f(x_0) + h f'(x_0) = 0$$

$$\therefore \; h = -\frac{f(x_0)}{f'(x_0)}$$

Hence $x_1 = x_0 + h = x_0 - \dfrac{f(x_0)}{f'(x_0)}$



Next, from the point $(x_1, f(x_1))$ we draw a tangent to the curve $y = f(x)$ it cut the x axis at $(x_2, 0)$, and so on

$$\therefore \; x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$\vdots$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Example. Find the smallest +ve root of the equation $x^3 - 5x + 1 = 0$, by using N. R. method

$f(x) = x^3 - 5x + 1$

$f'(x) = 3x^2 - 5$.

$f(0) = 1 > 0$

$f(1) < 0$

$\therefore$ The root lies between $(0,1)$.

Take $x_0 = 0$.

$f(x_0) = 1$.

$f'(x_0) = -5$

$\therefore x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$

$= \cancel{.5} - \frac{\cancel{(-1.375)}}{\cancel{(1.25)}}$

$x_1 = 0 - \frac{1}{(-5)} = \frac{1}{5} = .2$

$x_2 = .2 - \frac{f(x_1)}{f'(x_1)}$

$f(x_1) = .008$.

$f'(x_1) = -4.88$,

$x_2 = .2 - \frac{.008}{(-)4.88} = .2 + \frac{.008}{4.88}$,

$= .201639$

$f(x_2) = .00019$

$f'(x_2) = -4.8780$

$\therefore x_3 = .201639 - \frac{.00019}{(-)4.8780} = .20163$

The root $\simeq .20163$

So in bisection we have seen that it converges slowly.
But in N-R. method it is quadratically convergent and rate of convergence is fast.