Method of substitution

1. Solve the differential equation

$$\frac{dy}{dx} = \frac{x^2+y^2}{xy}$$

D BISWAS

$$\Rightarrow \frac{dy}{dx} = \frac{x^2}{xy} + \frac{y^2}{xy}$$

$$= \frac{x}{y} + \frac{y}{x} \quad \text{——— (1)}$$

Put, $\boxed{\frac{y}{x} = v}$ $(x \neq 0)$

$\Rightarrow y = vx$

$\Rightarrow \frac{d}{dx}(y) = \frac{d}{dx}(vx)$

$\Rightarrow \boxed{\frac{dy}{dx} = v + x\frac{dv}{dx}}$

Put, the value in equation (1),

$$v + x\frac{dv}{dx} = \frac{1}{v} + v$$

$$\Rightarrow x\frac{dv}{dx} = \frac{1}{v}$$

$$\Rightarrow \frac{dx}{x} = v\,dv$$

Integrating, we have,

$$\log|x| = \frac{v^2}{2} + c$$

$$\Rightarrow \log|x| = \frac{y^2}{2x^2} + c \quad \left[v = \frac{y}{x}\right]$$

$$\Rightarrow 2\log|x| = \frac{y^2}{x^2} + 2c$$

$\Rightarrow \quad 2x^2 \log|x| = y^2 + 2cx^2$

$\Rightarrow \quad y^2 = 2x^2 \log|x| - 2cx^2$

$\Rightarrow \quad y = \pm x\sqrt{2\log|x| - 2c}$

D BISWAS

2. Solve the differential equation

$$\frac{dy}{dx} = \frac{x-y}{x+y}$$

$$\Rightarrow \frac{dy}{dx} = \frac{\frac{x-y}{x}}{\frac{x+y}{x}} \qquad (x \neq 0)$$

$$= \frac{1-\frac{y}{x}}{1+\frac{y}{x}} \qquad \text{——(1)}$$

Put, $\frac{y}{x} = v$

$\Rightarrow y = vx$

$\Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$

From (1)

$$v + x\frac{dv}{dx} = \frac{1-v}{1+v}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{1-v}{1+v} - v$$

$$\Rightarrow x\frac{dv}{dx} = \frac{1-v-v-v^2}{1+v}$$

D BISWAS

$$\Rightarrow x\frac{dv}{dx} = \frac{1-2v-v^2}{1+v}$$

$$\Rightarrow \frac{(1+v)\,dv}{1-2v-v^2} = \frac{dx}{x}$$

Integrating, we have,

$$\int\frac{(1+v)\,dv}{1-2v-v^2} = \int\frac{dx}{x}$$

Let, $1-2v-v^2=u$

$\Rightarrow -2-2v=\frac{du}{dv}$

$\Rightarrow -2(1+v)\,dv=du$

$$\Rightarrow -\frac{1}{2}\int\frac{du}{u} = \int\frac{dx}{x}$$

$$\Rightarrow -\frac{1}{2}\log|u| = \log|x| + c$$

$$\Rightarrow -\frac{1}{2}\log|u| - \log|x| - \log|c_1| = 0$$

[$c=\log|c_1|$ say]

$$\Rightarrow \log u^{-1/2}x^{-1}c_1^{-1} = 0$$

$$\Rightarrow \log(1-2v-v^2)^{-1/2}x^{-1}c_1^{-1} = 0$$

$$\Rightarrow \log\left(1-2\frac{y}{x}-\frac{y^2}{x^2}\right)^{-1/2}x^{-1}c_1^{-1} = 0$$

$$\Rightarrow \left\{1-2\left(\frac{y}{x}\right)-\frac{y^2}{x^2}\right\}^{-1/2}x^{-1}c_1^{-1} = 1 \quad [\because e^0=1]$$

$$\Rightarrow \left(1 - \frac{2y}{x} - \frac{y^2}{x^2}\right)^{-1} x^{-2} c_1^{-2} = 1$$

$$\Rightarrow \left(\frac{x^2 - 2xy - y^2}{x^2}\right)^{-1} \cdot x^{-2} c_1^{-2} = 1$$

$$\Rightarrow \left(\frac{x^2 - 2xy - y^2}{x^2}\right) x^2 c_1^2 = 1$$

$$\Rightarrow x^2 - 2xy - y^2 = \frac{1}{c_1^2}.$$

3. Solve the differential equation

$$\frac{dy}{dx} = \frac{x^2 + 3y^2}{2xy}$$

$$= \frac{(x^2+3y^2)/x^2}{\frac{2xy}{x^2}} \quad (x \neq 0)$$

$$\Rightarrow \frac{dy}{dx} = \frac{1 + 3\left(\frac{y}{x}\right)^2}{2\left(y/x\right)}.$$

Put, $\frac{y}{x} = v$

$$\Rightarrow y = vx$$

$$\Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$\Rightarrow v + x\frac{dv}{dx} = \frac{1+3v^2}{2v}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{1+3v^2}{2v} - v$$

D BISWAS

$$\Rightarrow x\frac{dv}{dx} = \frac{1+3v^2-2v^2}{2v}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{1-v^2}{2v}$$

$$\Rightarrow \int \frac{2v\,dv}{1-v^2} = \int \frac{dx}{x}$$

$$\Rightarrow -\int \frac{du}{u} = \int \frac{dx}{x}$$

$1-v^2 = u$

$-2v\,dv = du$

$$\Rightarrow -\log|u| = \log|x| + \log|c|$$

$$\Rightarrow -\log|u| - \log|x| - \log|c| = 0$$

$$\Rightarrow uxc = e^0 = 1$$

$$\Rightarrow (1-v^2)xc = 1$$

$$\Rightarrow \left\{1-\left(\frac{y}{x}\right)^2\right\}xc = 1$$

$$\Rightarrow \frac{(x^2-y^2)}{x^2}xc = 1$$

$$\Rightarrow \frac{x^2-y^2}{x}.c = 1$$

$$\Rightarrow \frac{x^2-y^2}{x} = \frac{1}{c}$$

Exercise :-

i) Solve $\dfrac{dy}{dx} = \dfrac{y-4x}{x-y}$

ii) Solve $\dfrac{dy}{dx} = \dfrac{y}{x} + 1$

iii) Find a general solution of the differential equation :-

$$y y' + x = \sqrt{x^2 + y^2}.$$

$\left[y' = \dfrac{dy}{dx} \right]$

iv) solve $\dfrac{dy}{dx} = \dfrac{y^2 - x^2}{xy}$

v) solve $\dfrac{dy}{dx} = (y+x)^2$ put $y+x = v$

vi) solve $\dfrac{dy}{dx} = \dfrac{-3y}{3x-7y}$.