6.19. FOURTH ORDER RUNGE-KUTTA METHOD

It is one of the most widely used methods and is particularly suitable in cases when the computation of higher derivatives is complicated.

Consider the differential equation $y' = f(x, y)$ with the initial condition $y(x_0) = y_0$. Let h be the interval between equidistant values of x then the first increment in y is computed from the formulae

$$\left.\begin{aligned} k_1 &= hf(x_0, y_0) \\ k_2 &= hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) \\ k_3 &= hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) \\ k_4 &= hf(x_0 + h, y_0 + k_3) \\ \Delta y &= \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \end{aligned}\right\} \quad \ldots(1)$$

taken in the given order.

Then, $x_1 = x_0 + h$ and $y_1 = y_0 + \Delta y$

In a similar manner, the increment in y for the II interval is computed by means of the formulae

$$\begin{aligned} k_1 &= hf(x_1, y_1) \\ k_2 &= hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right) \\ k_3 &= hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right) \\ k_4 &= hf(x_1 + h, y_1 + k_3) \\ \Delta y &= \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \end{aligned}$$

and similarly for the next intervals.

This method is also termed as *Runge-Kutta's* method simply.

It is to be noted that the calculations for the first increment are exactly the same as for any other increment. The change in the formula for the different intervals is only in the values of x and y to be substituted. Hence to obtain Δy for the n^{th} interval, we substitute x_{n-1}, y_{n-1}, in the expressions for k_1, k_2, etc.

The inherent error in the fourth order Runge-Kutta method is of order h^5.

6.19.1. Algorithm of Runge Kutta Method

[*U.P.T.U. MCA(C.O.) 2005*]

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1. Function F(x)=(x-y)/(x+y)
2. Input x0,y0,h,xn
3. n=(xn-x0)/h
4. x=x0
5. y=y0
6. For i=0, n
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Example 1. *Apply Runge-Kutta Method to solve.*

$$\frac{dy}{dx} = xy^{1/3}, y(1) = 1 \text{ to obtain } y(1.1).$$

Sol. Here, $x_0 = 1, y_0 = 1$ and $h = 0.1$. Then, we can find

$$k_1 = hf(x_0, y_0)$$

$$= 0.1(1)(1)^{1/3} = 0.1$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$

$$= 0.1\left(1 + \frac{0.1}{2}\right)\left(1 + \frac{0.1}{2}\right)^{1/3} = 0.10672$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$$

$$= 0.1\left(1 + \frac{0.1}{2}\right)\left(1 + \frac{0.10672}{2}\right)^{1/3} = 0.10684$$

$$k_4 = hf(x_0 + h, y_0 + k_3)$$

$$= 0.1(1 + 0.1)(1 + 0.10684)^{1/3} = 0.11378 \quad = 0.11378$$

$$\therefore \quad y(1.1) = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= 1 + \frac{1}{6}(0.1 + 2 \times 0.10672 + 2 \times 1.0684 + 0.11378)$$

$$= 1 + 0.10682 = 1.10682. \quad \textbf{Ans.}$$

Example 2. *The unique solution of the problem*

$$y' = -xy \text{ with } y_0 = 1 \text{ is } y = e^{-x^2/2}.$$

Find approximately the value of $y_{(0.2)}$ using one application of Runge-Kutta method of order four.

Sol. Let $h = 0.2,$ we have $y_0 = 1$ when $x_0 = 0.$

$$k_1 = hf(x_0, y_0)$$

$$= 0.2[(0)1] = 0$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$

$$= 0.2\left[-\left(0 + \frac{0.2}{2}\right)(1 + 0)\right] = -0.02$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$$

$$= 0.2\left[-\left(0 + \frac{0.2}{2}\right)\left(1 - \frac{0.02}{2}\right)\right] = -0.0198$$

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$$\therefore \quad y(1.1) = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= 1 + \frac{1}{6}(0.1 + 2 \times 0.10672 + 2 \times 1.0684 + 0.11378)$$

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$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$$

$$= 0.2\left[-\left(0 + \frac{0.2}{2}\right)\left(1 - \frac{0.02}{2}\right)\right] = -0.0198$$

$$k_4 = hf(x_0 + h, y_0 + k_3)$$

$$= 0.2(0 + 0.2)(1 - 0.0198) = 0.039208$$

$$\therefore \quad y(0.2) = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= 1 + \frac{1}{6}[0 + 2(-0.02) + 2(-0.0198) + (-0.039208)]$$

$$= 1.0000 - 0.198013$$

$$= 0.9801986 \cong 0.9802$$

The exact value of $y(0.2)$ is 0.9802.

Example 3. *Solve the equation $y' = (x + y)$ with $y_0 = 1$ by Runge-Kutta rule from $x = 0$ to $x = 0.4$ with $h = 0.1$.*

Sol. Here $f(x, y) = x + y, h = 0.1$, given $y_0 = 1$ when $x_0 = 0$.

We have,

$$k_1 = hf(x_0, y_0)$$

$$= 0.1(0 + 1) = 0.1$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$

$$= 0.1(0.05 + 1.05) = 0.11$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$$

$$= 0.1(0.05 + 1.055) = 0.1105$$

$$k_4 = hf(x_0 + h, y_0 + k_3)$$

$$= 0.1(0.1 + 1.1105) = 0.12105$$

$$y_1 = y_{(x=0.1)} = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= 1 + \frac{1}{6}(0.1 + 0.22 + 0.2210 + 0.12105) = 1.11034$$

Similarly for finding $y_2 = y(x = 0.2)$, we get

$$k_1 = hf(x_1 y_1 = 0.1)[(0.1) + 1.11034] = 0.121034$$

$$k_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right)$$

$$= 0.1[0.15 + 1.11034 + 0.660517] = 0.13208$$

$$k_3 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right)$$

$$k_4 = hf(x_1 + h, y_1 + k_3)$$
$$= 0.1[0.20 + 1.11034 + 0.13263] = 0.14263$$

$$\therefore \quad y_2 = y_{(x=0.2)} = y_1 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$
$$= 1.11034 + \frac{1}{6}[0.121034 + 2(0.13208 + 0.13263 + 0.14429)] = 1.2428$$

Similarly, for finding $y_3 = y(x = 0.3)$, we get

$$k_1 = hf(x_2 y_2) = 0.1[(0.2) + 1.2428] = 0.14428$$
$$k_2 = hf\left(x_2 + \frac{h}{2}, y_2 + \frac{k_1}{2}\right)$$
$$= 0.1[0.25 + 132428 + 0.07214] = 0.15649$$
$$k_3 = hf\left(x_2 + \frac{h}{2}, y_2 + \frac{k_2}{2}\right)$$
$$= 0.1[0.25 + 1.2428 + 0.07824] = 0.15710$$
$$k_4 = hf(x_2 + h, y_2 + k_3)$$
$$= 0.1[0.30 + 1.2428 + 0.15710] = 0.16999$$
$$\therefore \quad y_3 = y_{(x=0.3)} = y_2 + \frac{1}{6}(k_1 + 2K_2 + 2K_3 + k_4)$$
$$= 0.13997$$

Similarly, for finding $y_4 = y(x = 0.4)$, we get

$$k_1 = (0.1)[0.3 + 1.3997] = 0.16997 \qquad \Rightarrow k_1 = 0.16997$$
$$k_2 = (0.1)[0.35 + 1.3997 + 0.08949] = 0.18347 \qquad \Rightarrow k_2 = 0.18347$$
$$k_3 = (0.1)[0.35 + 1.3997 + 0.9170] = 0.18414 \qquad \Rightarrow k_3 = 0.18414$$
$$k_4 = (0.1)[0.4 + 1.3997 + 0.18414] = 0.19838 \qquad \Rightarrow k_1 = 0.19838$$
$$\therefore \quad y_4 = 1.3997 + \frac{1}{6}[0.16997 + 2(0.18347 + 0.18414 + 0.19838)]$$
$$y_4 = 1.5836.$$ **Ans.**

Example 4. *Given* $\frac{dy}{dx} = y - x$ *with y(0) = 2, find y(0.1) and y(0.2) correct to 4 decimal places.*

Sol. We have $x_0 = 0,\ y_0 = 2,\ h = 0.1$

Then, we get

$$k_1 = hf(x_0, y_0)$$
$$= 0.1(2 - 0) = 0.2$$