CHAPTER VII

Duality

7.1 CONCEPT OF DUALITY.

Associated with every linear programming problem there is a linear programming problem which is called its *dual problem* ; the original problem is called the *primal problem*.

In order to have a clear concept of duality, let us consider the following problem of a cattle breeding firm :

Let us consider the problem of a dealer who sells three nutrients *A, B, C* to be purchased by shopkeepers who make two fodders I and II for the consumption of the animals. The dealer knows that the fodders contain the nutrients sold by him and they have their market values because of the presence of the nutrients.

The table in the next page shows the amount of three nutrients available in the two fodders along with the minimum requirement of each nutrient for one animal and the market price of unit quantity of each fodder.

Fodder	Nutrient			Market price per unit
	A	B	C	
I	2	2	1	3
II	1	3	1	2
Min. requirement	14	22	1	

Now the problem of the dealer is to fix maximum per unit selling prices of the three nutrients *A, B, C* such that the resulting prices of the fodders do not exceed the existing market price, at the same time he gets maximum sale price z.

To formulate the problem mathematically, let x_1, x_2, x_3 be the prices per unit of the three nutrients *A, B, C* that are to be fixed by the dealer. The problem then is to find x_1, x_2, x_3 which

$$\text{Maximize } z = 14x_1 + 22x_2 + x_3$$

subject to the constraints

$$2x_1 + 2x_2 + x_3 \leq 3 \text{ (market price of fodder I)},$$
$$x_1 + 3x_2 + x_3 \leq 2 \text{ (market price of fodder II)},$$
$$x_1,\ x_2,\ x_3 \geq 0 .$$

Let us consider the same problem from a different angle. Consider the case of the shopkeeper who is to determine the minimum quantities of the two fodders to be made so that the minimum requirements of the three nutrients for each animal is met and at the same time the cost (w) of purchasing the fodders is minimum.

To formulate this problem, let v_1 and v_2 be the units of fodder I and fodder II which are made for each animal. The mathematical formulation of the problem is to find v_1, v_2 which will

$$\text{Minimize } w = 3v_1 + 2v_2$$

subject to
$$2v_1 + v_2 \geq 14 \text{ (for nutrient } A),$$
$$2v_1 + 3v_2 \geq 22 \text{ (for nutrient } B),$$
$$v_1 + v_2 \geq 1 \text{ (for nutrient } C)$$

and
$$v_1,\ v_2 \geq 0 .$$

Either of the above problems can be considered as the primal, with the remaining problem as the dual. The two problems possess very closely related properties. These two have different mathematical formulations, one being a maximization problem and the other a minimization problem, although they are expressed in terms of the same basic data with different arrangements. The optimal values of the objective functions of the two problems, as will be seen later, are the same and the final tableau of the optimal solution for one will contain necessary indications for the optimal solution of the other. We give the mathematical formulations of primal and dual problems in the next article.

7.2 MATHEMATICAL FORMULATION OF DUALS.

Let us consider a linear programming problem in the following form which will be called the primal problem :

$$\text{Maximize } z = \sum_{j=1}^{n} c_j x_j$$

$$\text{subject to } \sum_{j=1}^{n} a_{ij} x_j \leq b_i, \quad i = 1, 2, \ldots\ldots, m$$

$$x_j \geq 0, \quad j = 1, 2, \ldots\ldots, n$$

in which $x_1, x_2, \ldots\ldots, x_n$ are the primal variables and z is the primal objective function.

The associated dual problem will be given by

$$\text{Minimize } w = \sum_{i=1}^{m} b_i v_i$$

$$\text{subject to } \sum_{i=1}^{m} a_{ji} v_i \geq c_j, \quad j = 1, 2, \ldots\ldots, n$$

$$v_i \geq 0, \quad i = 1, 2, \ldots\ldots, m$$

in which $v_1, v_2, \ldots\ldots, v_m$ are the dual variables and w is the dual objective function.

To be more explicit, if the primal problem be

$$\text{Maximize } z = c_1 x_1 + c_2 x_2 + \cdots\cdots + c_n x_n$$

$$\begin{aligned} \text{subject to} \quad & a_{11} x_1 + a_{12} x_2 + \cdots\cdots + a_{1n} x_n \leq b_1, \\ & a_{21} x_1 + a_{22} x_2 + \cdots\cdots + a_{2n} x_n \leq b_2, \\ & \ldots \quad \ldots\ldots \quad \ldots\ldots\ldots \quad \ldots \\ & a_{m1} x_1 + a_{m2} x_2 + \cdots\cdots + a_{mn} x_n \leq b_m, \\ & x_1, x_2, \ldots\ldots, x_n \geq 0. \end{aligned}$$

then its dual is

$$\begin{aligned}
\text{Minimize } w &= b_1v_1 + b_2v_2 + \cdots\cdots + b_mv_m\\
\text{subject to } & a_{11}v_1 + a_{21}v_2 + \cdots\cdots + a_{m1}v_m \geq c_1,\\
& a_{12}v_1 + a_{22}v_2 + \cdots\cdots + a_{m2}v_m \geq c_2,\\
& \cdots\cdots\cdots\cdots\cdots\cdots \quad \cdots\\
& a_{1n}y_1 + a_{2n}v_2 + \cdots\cdots + a_{mn}v_m \geq c_n,\\
& v_1,\ v_2,\ \ldots\ldots,\ v_m \geq 0.
\end{aligned}$$

From the above formulation of the primal and dual problems, we observe the following :

(a) Number of variables in the dual is equal to the number of constraints in the primal and vice-versa.

(b) The elements of the requirement vector (not necessarily all positive) in one problem are the respective prices in the objective function of the other problem.

(c) The row coefficients of the primal constraints become the column coefficients of the dual constraints.

(d) One of the problems seeks maximization while the other seeks minimization.

(e) In general, the primal maximization problem has less than ($\leq$) type of constraints and the dual minimization problem has greater than ($\geq$) type of constraints.

(f) The variables in both the problems are non-negative.

7.3 CONSTRUCTION OF DUALS.

Consider the linear programming problem in the matrix form

$$\begin{aligned}
&\textit{Maximize } z = \mathbf{cx},\ \mathbf{x} \geq \mathbf{0}\\
&\textit{subject to} \qquad \mathbf{Ax} \leq \mathbf{b}.
\end{aligned} \qquad \ldots \quad (1)$$

Here $\mathbf{A}$ is an $m \times n$ matrix, $\mathbf{x}$ and $\mathbf{c}$ are n-component column and row vectors respectively and $\mathbf{b}$ is an m component column vector.

Considering this problem as the primal problem, we define its dual problem as

$$\begin{aligned}
&\textit{Minimize } w = \mathbf{b'v},\ \mathbf{v} \geq \mathbf{0}\\
&\textit{subject to } \mathbf{A'v} \geq \mathbf{c'},
\end{aligned} \qquad \ldots \quad (2)$$

in which $\mathbf{v}$ is an m-component column vector. The primes denote as usual the transposes. Notice that the primal has n variables and m constraints whereas the dual has m variables and n constraints.

Thus the dual of the following minimization problem

$$\text{Minimize} \quad z = \mathbf{cx}$$

$$\text{subject to} \quad \mathbf{Ax} \geq \mathbf{b}, \quad \mathbf{x} \geq \mathbf{0} \qquad \dots \text{(}$$

$$\text{is Maximize} \quad w = \mathbf{b}'\,\mathbf{v}$$

$$\text{subject to} \quad \mathbf{A}'\mathbf{v} \leq \mathbf{c}', \quad \mathbf{v} \geq \mathbf{0}. \qquad \dots \text{(}$$

Primal and dual problems consisting of constraints with inequations only are *symmetric* while if the constraints be equations in the primal then the problem is *unsymmetric*. The third type, called *mixed* type, is such that the constraints of the primal are both equations and inequations and the variables may be unrestricted in sign.

The following examples will illustrate how a primal can be converted to its dual. Before that we discuss two theorems.

92 **Theorem 1.** *If any of the constraints in the primal problem be a perfect equality, then the corrresponding dual variable is unrestricted in sign.*

Without any loss of generality, we assume that the last of the constraints is a perfect equality.

Let the linear programming problem be

$$\text{Maximize} \quad z = \sum_{j=1}^{n} c_j x_j$$

$$\text{subject to} \quad \sum_{j=1}^{n} a_{ij} x_j \leq b_i, \quad i = 1, 2, \dots\dots, (m-1)$$

$$\text{and} \quad \sum_{j=1}^{n} a_{ij} x_j = b_i, \quad i = m \text{ (equality constraint)}$$

$$x_j \geq 0, \text{ for } j = 1, 2, \dots\dots, n.$$

Before writing the dual of this primal problem, we replace the m-th constraint with '$\leq$' sign. Thus we write the problem as

$$\text{Maximize} \quad z = \sum_{j=1}^{n} c_j x_j$$

$$\text{subject to} \quad \sum_{j=1}^{n} a_{ij} x_j \leq b_i, \quad i = 1, 2, \dots\dots, (m-1)$$

$$\text{and} \quad \sum_{j=1}^{n} a_{mj} x_j \leq b_m,$$

$$-\sum_{j=1}^{n} a_{mj} x_j \leq -b_m.$$

Dual of this problem is

$$\text{Minimize } w = \sum_{i=1}^{m-1} b_i v_i + b_m (v'_m - v''_m)$$

$$\text{subject to } \sum_{i=1}^{m-1} a_{ji} v_i + a_{mj} (v'_m - v''_m) \geq c_j, \; j = 1, 2, \ldots\ldots, n$$

$$v_i \geq 0, \text{ for } i = 1, 2, \ldots\ldots, (m-1),$$

$$v'_m, v''_m \geq 0.$$

If we now write $v_m = v'_m - v''_m$, then the dual becomes

$$\text{Minimize } w = \sum_{i=1}^{m} b_i v_i$$

$$\text{subject to } \sum_{i=1}^{m} a_{ji} v_i \geq c_j, \text{ for all } j$$

$$v_i \geq 0, \; i = 1, 2, \ldots\ldots, (m-1).$$

Obviously v_m is unrestricted in sign, since the difference $(v'_m - v''_m)$ can be positive, negative or zero.

Note. There is nothing to prevent the occurrence of both equality and unrestricted variable in the same problem. There will be more unrestricted variables in the dual if there be more equality constraints in the primal.

Theorem 2. *If any variable of the primal problem be unrestricted in sign, then the corresponding constraint of the dual will be an equality.*

Without any loss of generality, we assume that the p-th primal variable is unrestricted in sign.

Let the linear programming problem be

$$\text{Maximize } z = \sum_{j=1}^{n} c_j x_j$$

$$\text{subject to } \sum_{j=1}^{n} a_{ij} x_j \leq b_i, \; i = 1, 2, \ldots\ldots, m$$

and $x_1, x_2, \ldots\ldots, x_{p-1}, x_{p+1}, \ldots\ldots, x_n \geq 0$, but x_p is unrestricted in sign.

Since the variable x_p can always be expressed as the difference of two non-negative variables, say, x_p' and x_p'', $(x_p', x_p'' \geq 0)$, therefore x_p can be replaced by $(x_p' - x_p'')$ in the above primal problem.

14. Minimize $z = x_3 + x_4 + x_5$

subject to
$$x_1 - x_3 + x_4 - x_5 = -2,$$
$$x_2 - x_3 - x_4 + x_5 = 1,$$
$$x_j \geq 0, \; j = 1, 2, \ldots, 5.$$
[*Guahati, 1975*]

[**Ans.** Maximize $w = -2v_1 + v_2$

subject to $v_1 + v_2 \geq -1$, $v_1 - v_2 \leq 1$, $-v_1 + v_2 \leq 1$,

v_1, v_2 are unrestricted.]

15. Maximize $z = 6x_1 + 4x_2 + 6x_3 + x_4$

subject to
$$4x_1 + 4x_2 + 4x_3 + 8x_4 = 21,$$
$$3x_1 + 17x_2 + 80x_3 + 2x_4 \leq 48,$$
$x_1, x_2 \geq 0$; x_3, x_4 are unrestricted.

[**Ans.** Minimize $w = 21v_1 + 48v_2$

subject to $4v_1 + 3v_2 \geq 6$, $4v_1 + 17v_2 \geq 4$, $4v_1 + 80v_2 = 6$,

$8v_1 + 2v_2 = 1$, $v_2 \geq 0$, v_1 is unrestricted.]

16. Maximize $z = x_1 - x_2 + 3x_3 + 2x_4$

subject to
$$x_1 + x_2 \geq -1,$$
$$x_1 - 3x_2 - x_3 \leq 7,$$
$$x_1 + x_3 - 3x_4 = -2,$$
$x_1, x_4 \geq 0$ and x_2, x_3 are unrestricted in sign. [*Calcutta Hons.,1982*]

[**Ans.** Minimize $w = v_1 + 7v_2 - 2v_3$, s. t. $-v_1 + v_2 + v_3 \geq 1$,

$v_1 + 3v_2 = 1$, $-v_2 + v_3 = 3$, $3v_3 \leq -2$,

$v_1, v_2 \geq 0$, v_3 is unrestricted in sign.]

7.5 DUALITY THEOREMS.

The relationship as well as its formulation make us believe that every linear programming problem has a dual problem. The properties of duality evolve new algorithms for solving linear programming problems. With this in view, we establish a few theorems on duality relations.

For that, let us take the primal problem as

$$\begin{aligned} &\textit{Maximize} \quad z = \mathbf{cx} \\ &\textit{subject to} \; \mathbf{Ax} \leq \mathbf{b}, \\ &\qquad\qquad \mathbf{x} \geq \mathbf{0} \end{aligned} \qquad \ldots \quad (1)$$

so that its dual is

$$\begin{aligned} &\textit{Minimize} \quad w = \mathbf{b'v} \\ &\textit{subject to} \; \mathbf{A'v} \geq \mathbf{c'}, \\ &\qquad\qquad \mathbf{v} \geq \mathbf{0}. \end{aligned} \qquad \ldots \quad (2)$$

Theorem 1. *The dual of the dual is the primal.*

Consider the problem (2) which is the dual of (1) as given in previous page. We can rewrite the objective function as

$$\text{Min. } w = \text{Max. } (-w) = \text{Max. } w_1 = -\,\mathbf{b'v},$$

where $w_1 = -w$.

Multiplying the constraint inequation by (-1), we get

$$-\mathbf{A'v} \leq -\mathbf{c'}.$$

Hence (2) becomes

$$\begin{aligned} &\text{Maximize } w_1 = -\,\mathbf{b'v} \\ &\text{subject to } -\mathbf{A'v} \leq -\mathbf{c'}, \\ &\qquad\qquad \mathbf{v} \geq \mathbf{0}. \end{aligned} \qquad \ldots (3)$$

Now this problem has the same form as (1). Hence we can write the dual of this problem as

$$\begin{aligned} &\text{Minimize} \quad z_1 = -\,\mathbf{cx} \\ &\text{subject to } -\mathbf{Ax} \geq -\mathbf{b}, \\ &\qquad\qquad \mathbf{x} \geq \mathbf{0}. \end{aligned} \qquad \ldots (4)$$

Notice that $\text{Min. } z_1 = \text{Max. } (-z_1) = -\,\text{Max. } z_1$

$$= \mathbf{cx} = \text{Max. } z$$

and $\qquad -\mathbf{Ax} \geq -\mathbf{b}$

is equivalent to $\quad \mathbf{Ax} \leq \mathbf{b}$.

Hence the dual problem can be written as

$$\begin{aligned} &\text{Maximize} \quad z = \mathbf{cx} \\ &\text{subject to} \quad \mathbf{Ax} \leq \mathbf{b}, \\ &\qquad\qquad \mathbf{x} \geq \mathbf{0} \end{aligned}$$

which is the same primal problem (1). Hence the theorem follows.

Theorem 2. *If* $\mathbf{x}$ *be any feasible solution to the primal problem and* $\mathbf{v}$ *be any feasible solution to the dual problem, then*

$$\mathbf{cx} \leq \mathbf{b'v}.$$

Since $\mathbf{x}$ is any feasible solution to the primal problem (1), we have

$$\mathbf{Ax} \leq \mathbf{b}, \ \mathbf{x} \geq \mathbf{0}. \qquad \ldots (3)$$

Similarly, since $\mathbf{v}$ is any feasible solution to the dual problem (2), we have

$$\mathbf{A'v} \geq \mathbf{c'}, \ \mathbf{v} \geq \mathbf{0}. \qquad \ldots (4)$$