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1) Solve: $x(x^2+y^2-a^2)\,dx + y(x^2-y^2-b^2)\,dy = 0$

Here $M = (x^2+y^2-a^2)x \qquad N = y(x^2-y^2-b^2)$

$M = x^3 + xy^2 - xa^2 \qquad N = yx^2 - y^3 - yb^2$

so $\dfrac{\partial M}{\partial y} = 2yx \qquad \dfrac{\partial N}{\partial x} = 2yx$

$\because \dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$

① $P = \int M\,dx = \int (x^3 - xy^2 - a^2x)\,dx = \dfrac{x^4}{4} + \dfrac{x^2y^2}{2} - \dfrac{a^2x^2}{2}$

② $N - \dfrac{\partial P}{\partial y} = (x^2y - y^3 - b^2y) - \left(\dfrac{2x^2y}{2}\right) = -y^3 - b^2y$

③ $P + \int\left(N - \dfrac{\partial P}{\partial y}\right)dy = C$

$\Rightarrow \left(\dfrac{x^4}{4} + \dfrac{x^2y^2}{2} - \dfrac{a^2x^2}{2}\right) - \int y^3\,dy - \int b^2y\,dy = C$

$\Rightarrow \dfrac{x^4}{4} + \dfrac{x^2y^2}{2} - \dfrac{a^2x^2}{2} - \dfrac{y^4}{4} - \dfrac{b^2y^2}{2} = C$

$\Rightarrow \dfrac{x^4}{4} - \dfrac{y^4}{4} + \dfrac{x^2y^2}{2} - \dfrac{a^2x^2}{2} - \dfrac{b^2y^2}{2} = C$

② Solve: $(x^2 - 2xy + 3y^2)\,dx + (4y^3 + 6xy - x^2)\,dy = 0$

$M = x^2 - 2xy + 3y^2 \qquad N = 4y^2 + 6xy - x^2$

$\dfrac{\partial M}{\partial y} = -2x + 6y \qquad \dfrac{\partial N}{\partial x} = 6y - 2x$

so $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$

Now, $P = \int M\,dx = \int (x^2 - 2xy + 3y^2)\,dx$

$= \dfrac{x^3}{3} - x^2y + 3y^2x$

$N - \dfrac{\partial P}{\partial y} = 4y^3 + \cancel{6xy} - \cancel{x^2} + \cancel{x^2} - \cancel{6yx} = 4y^3$

$\therefore P + \int\left(N - \dfrac{\partial P}{\partial y}\right)dy = C \Rightarrow \dfrac{x^3}{3} - x^2y + 3y^2x + \int 4y^3\,dy = C$

$\Rightarrow \dfrac{x^3}{3} + y^4 + 3xy^2 - x^2y = C$

2) $(x^2 - 2xy + 3y$

$M = x^2 - 2xy + 3y^2 \qquad N = 4y^2 + 6xy - x^2$

$$\frac{\delta M}{\delta y} = -2x + 6y \qquad \frac{\delta N}{\delta x} = 6y - 2x$$

$$\therefore \frac{\delta M}{\delta y} = \frac{\delta N}{\delta x}$$

$$P = \int M\,dx = \int (x^2 - 2xy + 3y^2)\,dx$$

$$= \frac{x^3}{3} - x^2 y + 3y^2 x$$

$$N - \frac{\delta P}{\delta y} = 4y^3 + 6\cancel{xy} - \cancel{x^2} + \cancel{x^2} - 6\cancel{xy} = 4y^3$$

$$\therefore P + \int \left(N - \frac{\delta P}{\delta y}\right) dy = c \Rightarrow \frac{x^3}{3} - x^2 y + 3y^2 x + \int 4y^3\,dy$$

$$\Rightarrow \frac{x^3}{3} + y^4 + 3y^2 x - x^2 y = c$$

5) $\left(1+e^{\frac{x}{y}}\right)dx + e^{\frac{x}{y}}\left(1-\frac{x}{y}\right)dy = 0$

$M = 1+e^{\frac{x}{y}}$ $\quad\quad N = e^{\frac{x}{y}}\left(1-\frac{x}{y}\right)$

$\therefore \frac{\delta M}{\delta y} = e^{\frac{x}{y}}\left(-\frac{x}{y^2}\right)$ $\quad\quad \therefore \frac{\delta N}{\delta x} = e^{\frac{x}{y}}\left(-\frac{x}{y^2}\right)$

So, $\frac{\delta M}{\delta y} = \frac{\delta N}{\delta x}$.

$\Rightarrow \int\left(1+e^{\frac{x}{y}}\right)dx + \int 0\, dy = C$

$\Rightarrow x + \frac{e^{\frac{x}{y}}}{1/y} = C$

$\Rightarrow x + y e^{\frac{x}{y}} = C$

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i) $\frac{dy}{dx} = x^3y^3 - xy$ (Solve)

$\Rightarrow \frac{1}{y^3}\left(\frac{dy}{dx}\right) = x^3 - \frac{x}{y^2}$

$\Rightarrow \frac{1}{y^3}\frac{dy}{dx} + \frac{x}{y^2} = x^3$ (Let) $\frac{1}{y^2} = t$ $\Rightarrow -\frac{2}{y^3}\frac{dy}{dx} = \frac{dt}{dx}$

$\Rightarrow -\frac{1}{2}\frac{dt}{dx} + (tx) = x^3$

$\Rightarrow \frac{dt}{dx} - 2tx = -2x^3 \rightarrow Q$

$\quad\quad\quad\quad \hookrightarrow P$

If $e^{\int P dx} = e^{-\frac{2x^2}{2}}$

$\Rightarrow e^{-x^2}$

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$$t \cdot I_f = \int (Q \, I_f) \, dx$$

$$\Rightarrow t \cdot e^{-x^2} = \int (-2x^3) e^{-x^2} dx$$

$$\Rightarrow t \cdot e^{-x^2} = -\int u e^{-u} du$$

$$\Rightarrow t \cdot e^{-x^2} = -\left[-u e^{-u} + \int e^{-u} du \right]$$

$$\Rightarrow t \cdot e^{-x^2} = u e^{-u} + e^{-u} + c \rightarrow \text{constant}$$

$$\Rightarrow \frac{1}{y^2} = x^2 + 1 + c e^{-x^2}$$

ii) Solve : $\frac{dy}{dx} + xy = xy^2$

$\frac{dy}{dx} + xy = xy^2$ [dividing by y^2]

$\Rightarrow \frac{1}{y^2}\frac{dy}{dx} + \frac{x}{y} = x$

$\frac{1}{y} = t$

$\Rightarrow \frac{dt}{dx} - xt = -x$

$\Rightarrow -\frac{1}{y^2}\frac{dy}{dx} = \frac{dt}{dx}$

$\Rightarrow$

$P(x) = -x$ $Q(x) = -x$

If $e^{\int -x dx} = e^{-x^2/2}$

$t \cdot e^{-x^2/2} = \int -x e^{-x^2/2} dx + c$

$\Rightarrow t \cdot e^{-x^2/2} = \int e^{u} du + c$ (Put) $-\frac{x^2}{2} = u$

$\Rightarrow t \cdot e^{-x^2/2} = e^{u} + c$ $\Rightarrow -x dx = du$

$\Rightarrow \frac{1}{y} e^{-x^2/2} = e^{-x^2/2} + c$ [dividing by $e^{-x^2/2}$]

$\Rightarrow \frac{1}{y} = 1 + ce^{-x^2/2}$

iii) Solve : $\frac{dy}{dx}(x^2y^3 + xy) = 1$ $\Rightarrow \frac{dy}{dx} = \frac{1}{x^2y^3 + xy}$ $\Rightarrow \frac{dx}{dy} = x^2y^3 + xy$

$\Rightarrow \frac{dx}{dy} - xy = x^2y^3$ $\Rightarrow \frac{1}{x^2}\frac{dx}{dy} - \frac{y}{x} = y^3$ [Dividing by x^2]

Substitute $\frac{1}{x} = u$ $\Rightarrow \frac{du}{dx} = -\frac{1}{x^2}\frac{dx}{dy}$

$\Rightarrow -\frac{du}{dy} - uy = y^3$

$\Rightarrow \frac{du}{dy} + uy = -y^3$

$I.f \Rightarrow e^{\int y dy} = e^{y^2/2}$

$u \times e^{y^2/2} = -\int y^3 e^{y^2/2} dy$

$\Rightarrow -2\left(\frac{y^2}{2} - 1\right) e^{y^2/2} + c$

$= (2 - y^2) e^{y^2/2} + c$

$= x(2 - y^2) + cxe^{-y^2/2} = 1$

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Q. The number of bacteria in a yeast culture grows at a rate which is proportional to the number of population of colony triples in 1 hour. Find the number of bacteria at the end of 5 hour

→ (let) P = population t = time (hours)

$\frac{dP}{dt} \propto P \Rightarrow \frac{dP}{dt} = kP$ → constant

$\Rightarrow \int \frac{dP}{P} = \int k\,dt$ → arbitrary constant

$\Rightarrow \ln|P| = kt + \ln|c|$

$\Rightarrow \ln|P| - \ln|c| = kt$

$\Rightarrow \ln\left|\frac{P}{c}\right| = kt$

$\Rightarrow \boxed{P = ce^{kt}}$

∴ From the above problem

(let) $C = x$ $P = 3x$ $t = 1$

$\therefore 3\cancel{x} = \cancel{x} e^{k} \Rightarrow e^{k} = 3$

∴ for $t = 5$

$\therefore yx = x e^{k \cdot 5}$

$y\cancel{x} = \cancel{x}\, 3^5$

$\boxed{y = 3^5}$ → Answer