3. Employ Picard's method to obtain the solution of

$$\frac{dy}{dx} = x^2 + y^2 \quad \text{for} \quad x = 0.1$$

correct to four places of decimal given that $y = 0$ when $x = 0$.

4. Find an approximate value of y when $x = 0.1$ if $\frac{dy}{dx} = x - y^2$ and $y = 1$ at $x = 0$ using Pic[illegible] method.

5. Solve numerically $\frac{dy}{dx} = 2x - y, y(0) = 0.9$ at $x = 0.4$ by Picard's method with three iterations [illegible] compare the result with the exact value.

6. Employ Picard's method to find y (0.2) and y (0.4) given that $\frac{dy}{dx} = 1 + y^2$ and y (0) = 0.

7. Explain Picard's method of successive approximation for numerical solution of ordinary diff[illegible] ential equations.

8. Approximate y and z by using Picard's method for the solution of simultaneous differential equ[illegible] tions

$$\frac{dy}{dx} = 2x + z, \quad \frac{dz}{dx} = 3xy + x^2z$$

with $y = 2$, $z = 0$ at $x = 0$ upto third approximation.

9. Using Picard's method, obtain the solution of

$$\frac{dy}{dx} = x(1 + x^3y), \; y\,(0) = 3$$

Tabulate the values of $y(0.1)$, $y(0.2)$.

Answers

1. .019984, .0200 **2.** 1.0082

3. 0.0214 **4.** 0.9138

5. 0.7432, 0.7439 **6.** 0.2027, 0.4227

8. $y^{(3)} = 2 + x^2 + x^3 + \frac{3}{20}x^5 + \frac{1}{10}x^6$

$z^{(3)} = 3x^2 + \frac{3}{4}x^3 + \frac{6}{5}x^5 + \frac{3}{20}x^7 + \frac{3}{40}x$

9. $y(0.1) = 3.005, y(0.2) = 3.020.$

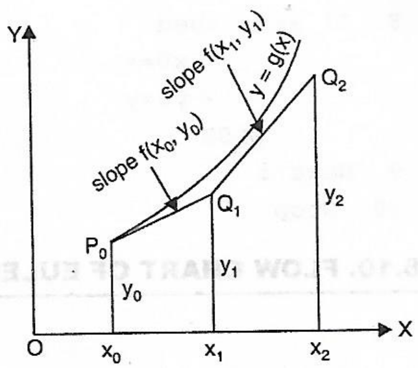
6.8. EULER'S METHOD

It is the simplest one-step method and has a limited application because of its low accuracy. This method yields solution of an ordinary diff. eqn. in the form of a set of tabulated values.

In this method, we determine the change Δy is y corresponding to small increase in the argument x. Consider the differential equation

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0 \qquad ...(1)$$

Let $y = g(x)$ be the solution of (1). Let x_0, x_1, x_2 ...... be equidistant values of x.



In this method, we use the property that in a small interval, a curve is nearly a straight line. Thus at the point (x_0, y_0), we approximate the curve by the tangent at the point (x_0, y_0).

The eqn. of the tangent at $P_0(x_0, y_0)$ is

$$y - y_0 = \left(\frac{dy}{dx}\right)_{P_0} (x - x_0)$$

$$= f(x_0, y_0)\,(x - x_0)$$

$$\Rightarrow \qquad y = y_0 + (x - x_0)\, f(x_0, y_0) \qquad \ldots(2)$$

This gives the y-coordinate of any point on the tangent. Since the curve is approximated by the tangent in the interval (x_0, x_1), the value of y on the curve corresponding to $x = x_1$ is given by the above value of y in eqn. (2) approximately.

Putting $x = x_1 (= x_0 + h)$ in eqn. (2), we get

$$y_1 = y_0 + hf(x_0, y_0)$$

Thus Q_1 is (x_1, y_1)

Similarly, approximating the curve in the next interval (x_1, x_2) by a line through $Q_1(x_1, y_1)$ with slope $f(x_1, y_1)$, we get

$$y_2 = y_1 + hf(x_1, y_1)$$

In general, it can be shown that,

$$\boxed{y_{n+1} = y_n + hf(x_n, y_n)}$$

This is called Euler's formule.

A great disadvantage of this method lies in the fact that if $\frac{dy}{dx}$ changes rapidly over an interval, its value at the beginning of the interval may give a poor approximation as compared to its average value over the interval and thus the value of y calculated from Euler's method may be in much error from its true value. These errors accumulate in the succeeding intervals and the value of y becomes much erroroneous ultimately.

Note. In Euler's method, the curve of actual solution $y = g(x)$ is approximated by a sequence of short lines. The process is very slow. If h is not properly chosen, the curve $P_0Q_1Q_2$ ...... of short lines representing numerical solution deviates significantly from the curve of actual solution.

To avoid this error, **Euler's modified method** is preferred because in this, we consider the curvature of the actual curve inplace of approximating the curve by sequence of short lines.

6.9. ALGORITHM OF EULER'S METHOD

```
1. Function F(x,y)=(x-y)/(x+y)
2. Input x0,y0,h,xn
3. n=((xn-x0)/h)+1
4. For i=1,n
5. y=y0+h*F(x0,y0)
6. x=x+h
7. Print x0,y0
```

```
      X=0.020000        Y=0.980400
N=1   Y=0.961584
N=2   Y=0.961598
      X=0.040000        Y=0.961584
N=1   Y=0.943572
N=2   Y=0.943593
      X=0.060000        Y=0.943572
```

6.15.2. Notations used in the Program

(*i*) **x(1)** is an array of the initial value of x.

(*ii*) **y(1)** is an array of the initial value of y.

(*iii*) **h** is the spacing value of x.

(*iv*) $\mathbf{x_n}$ is the last value of x at which value of y is required.

EXAMPLES

Example 1. *Given* $\frac{dy}{dx} = \frac{y-x}{y+x}$ *with* $y = 1$ *for* $x = 0$. *Find* y *approximately for* $x = 0.1$ *by Euler's method.*

Sol. We have

$$\frac{dy}{dx} = f(x, y) = \frac{y-x}{y+x}\ ; x_0 = 0, y_0 = 1, h = 0.1$$

Hence the approximate value of y at $x = 0.1$ is given by

$$y_1 = y_0 + hf(x_0, y_0) \qquad | \text{ using } y_{n+1} = y_n + hf(x_n, y_n)$$

$$= 1 + (.1) + \left(\frac{1-0}{1+0}\right) = 1.1$$

Much better accuracy is obtained by breaking up the interval 0 to 0.1 into five steps. The approximate value of y at $x_A = .02$ is given by,

$$y_1 = y_0 + hf(x_0, y_0)$$

$$= 1 + (.02)\left(\frac{1-0}{1+0}\right) = 1.02$$

At $x_B = 0.04$, $\qquad y_2 = y_1 + hf(x_1, y_1)$

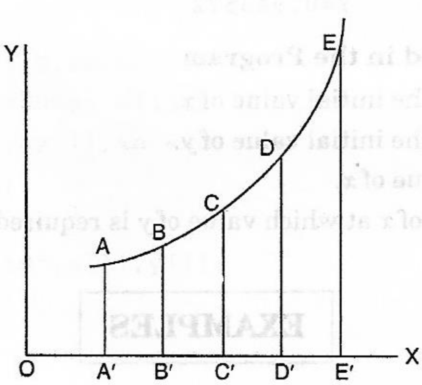
$$= 1.02 + (.02)\left(\frac{1.02 - .02}{1.02 + .02}\right) = 1.0392$$

At $x_C = .06$, $\qquad y_3 = 1.0392 + (.02)\left(\frac{1.0392 - .04}{1.0392 + .04}\right) = 1.0577$

At $x_D = .08$, $\quad y_4 = 1.0577 + (.02)\left(\dfrac{1.0577 - .06}{1.0577 + .06}\right) = 1.0756$

At $x_E = .1$, $\quad y_5 = 1.0756 + (.02)\left(\dfrac{1.0756 - .08}{1.0756 + .08}\right) = 1.0928$

Hence $y = 1.0928$ when $x = 0.1$



Example 2. *Solve the equation* $\dfrac{dy}{dx} = 1 - y$ *with the initial condition* $x = 0$, $y = 0$ *using Euler's algorithm and tabulate the solutions at* $x = 0.1, 0.2, 0.3$.

Sol. Here, $\quad f(x, y) = 1 - y$

Taking $h = 0.1$, $x_0 = 0$, $y_0 = 0$, we obtain

$$y_1 = y_0 + hf(x_0, y_0)$$
$$= 0 + (.1)(1 - 0) = .1$$

$\therefore \quad y(0.1) = 0.1$

Again,
$$y_2 = y_1 + hf(x_1, y_1)$$
$$= 0.1 + (0.1)(1 - .1)$$
$$= 0.1 + .09 = .19$$

$\therefore \quad y(0.2) = 0.19$

Again,
$$y_3 = y_2 + hf(x_2, y_2)$$
$$= .19 + (.1)(1 - .19)$$
$$= .19 + (.1)(.81) = .271$$

$\therefore \quad y(0.3) = .271$

Tabulated values are

x	$y(x)$
0	0
0.1	0.1
0.2	0.19
0.3	0.271