**Solution of algebraic and transcendental equation:**

Determination of roots of algebraic and transcendental is a very important problem in science and engineering.

A function is called algebraic if, to get the values of the function starting from the given values of , we have to perform arithmetic operations between some real numbers and rational power of On the other hand, transcendental functions include all non-algebraic functions, i.e. And others.

An equation is called algebraic or transcendental as is algebraic or transcendental.

The equations etc. are the examples of algebraic equations and on the other hand etc. are the examples of transcendental equation. Though we know some methods like cardan’s method, Euler’s method , Ferrari’s method, Descart’s method in algebra to solve algebraic equation up to fourth order.In general there is no closed form formula to evaluate the algebraic equation of degree greater than two.

The definition of roots of an equation can be given in two different ways:

Algebraically, a number is called a root of an equation iff and geometrically, the real roots of the equation are the values of where the graph of meets the -axis.

Throughout our discussion, we assume that

1. The function is continuous and continuously differentiable up to a sufficient number of times.
2. has no multiple root i.e., if is a real root of , in a sufficiently small interval , then and either or in

Most of the numerical methods, used to solve an equation are based on iterative technics. Different numerical methods are available to solve the equation But each method has some advantage and disadvantage over another method. Generally, the following aspects are considered to compare the methods:

Convergence or divergence, rate of convergence , applicability of the method, amount of pre-calculation needed before application of the method. Etc.

The process of finding the approximate values of the roots of an equation can be divided into two stages:

1. Location of the roots.
2. Computation of the values of the roots with the specified degree of accuracy.

1.1.The interval is said to be the location of a real root if for . There are two methods used to locate the real roots of an equation

1. Graphical method
2. Method of tabulation which is an analytic method.
   * 1. Graphical method

* In this method the graph of is drawn in rectangular co-ordinate system. Then the points at which graph meets the –axis are the location of the roots of the equation

As an example, we consider the equation .We draw the graph of with respect to as rectangular axes, which meets the -axis at Thus the equation has two real roots , one is positive and other is negative. From the graph it is clear that the co-ordinate of is lies between and that of is between -1.6 to -1.7. Thus is an approximate value of the positive root (say) and is an approximate value of the negative root (say).

* If is not simple, rather complicated in form, we rewrite the equation as , where and are simple functions such that, we can draw conveniently the graphs of and with respect to rectangular axes. Then the -co-ordinate of the point of intersection of the graphs give the location of the real roots of the equation .

As an example, we consider an equation , we rewrite the equation as The graphs are drawn with respect to the rectangular axes. From the graph it is seen that the roots are in .

DISADVANTAGE:

The graphical method to locate the roots is not very useful. Because the drawing of the location of the function is itself complicated. But it makes possible to roughly determine the interval of the roots. Then an analytic method is used to locate the root.

* + 1. METHOD OF TABULATION

This method depends on the continuity of the function . Before applying the tabulation method, the following nature should be noted.

Theorem: if is continuous in the interval and if and have the opposite signs, then at least one real root of the equation lies within the interval .

Geometrically we can explain the theorem as:

Let, and . Then from the graph we can say that there must be a point in such that

If the curve touches the axis at some point, say at then is a root of , though and may have same sign where For example , touches the axis at Although but is a root of the equation

A trial method for tabulation is as follows:

From the table of signs of , setting If the signs of changes it’s signs for two consecutive values of then at least one root lies between these two values.

Example 1: Find the location of the roots of the equation

Solution: we form a table:

|  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- |
|  | 0 | -1 | 1 | 0.5 | -0.5 | -1.6 | -1.7 |
|  | - | - | + | - | - | - | + |

Since the has two roots. Since and , then the location of one root is . Also and . Then the location of the other root is .

Example 1: Find the number of real roots of the equation and locate them.

Solution:. The domain of definition of the function is .

we form a table:

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
|  |  | 0 | 1 |  |
| Sign of | + | - | - | + |

has two real roots, since the function has twice changes sign, among them one is negative root and other is greater than one.

A new table with small intervals of the location of the root is constructed in the following:

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
|  |  | -1 | 1 |  |
| Sign of | - | + | - | + |

Then the roots are in .

ORDER OF CONVERGENCE:

Assume that the sequence of numbers to and let If there exists two positive constants & such that Then the sequence is said to converge to wth the order of convergence . The number is called the asymptotic error constant.

If , the error of convergence of is called linear and if , the error of convergence of is called quadratic etc.

BISECTION METHOD:

It is an iterative method and is based on a well-known theorem which states that if be a continuous function in a closed interval and , then at least one real root of the equation between and . If further exists and maintains same sign in , i.e. is strictly monotonic, then there is only one real root of in This method is nothing but a repeated application of the above theorem.

First we consider a sufficiently small interval by graphical or tabulation method , in which and maintains same sign in then there is only one real root of , in . Now divide the interval into two equal intervals where . If then is an exact root of the equation. If then the root lies either in or in If then we take the interval as the new interval, otherwise we take Let the new interval be and use the same process to select the next new interval. In the next step, let the new interval be . The process of bisection is continued until either the midpoint of the interval is a root, or the length of the interval is sufficiently small. The number and are approximate roots of the equation Finally is taken as the approximate value of the root

Now the length of the interval is and the length of the interval is and at the n-th step the length of the interval is . In the final step is chosen as root , then the length of the interval being and hence the error does not exceed .

Thus, if be the error at the n-th step then the lower bound of n is obtained from the following relation

**.**

**CONVERGENCY:** letbe the error in approximating by , then

. Thus the iterative method must be convergent. To get a root of correct up to p-significant figures, we are to go up to q-th iteration so that and have same p-significant figures.

**DISADVANTAGE :**This method is very slow, but it is very simple and will converge surely to the exact root. So the method for any function only if the function is continuous within the interval , where the root lies.

Another popular method is the regula falsi method. This method was developed because the bisection method converges at fairly slow speed. In general regula falsi method is faster than bisection method.

**REGULA FALSI METHOD:** This method is also known as “method of false position”, ”method of chords”, “method of linear interpolation”.

Let a root of the equation be lies in the interval , i.e. . The idea of this method is that on a sufficiently small , the arc of the is replaced by the chord joining the points and . The abscissa of the point of intersection of the chord and the -axis is taken as the approximate value of the root.

Let, and . The equation of the chord joining the points and is

To find the point of intersection, set and let be the point. Then,

Therefore,

This is the second approximation of the root. Now if and are opposite signs then the root lies between and and replace by in . Then the next approximation is obtained as:

If and are opposite signs then the root lies between and and the new approximation is obtained as:

The procedure is repeated till the root is obtained to the desired accuracy. If the n-th approximate root lies between and , then the approximate root is thus obtained as:

**GEOMETRICAL INTERPRETATION:**

In this figure we assume that one real root of lies in the small interval and and . Let PRQ be the graph of in intersecting the x-axis at R.

Thus gives the exact value of the root If we consider the graph PRQ as the chord PQ, in the small interval , which intersect the x-axis at C, then (say). Now, and . Therefore, lies in in . Then in the similar process we get a point D such that OD=At the -th approximation must converge to .

**CONVERGENCE OF REGULA FALSI METHOD:**

As considering the proper sign of and we can write the equation as follows:

or

Since, , we have for both relation of (4) as

Or,

Or, when

Or,

, where Min

Or,

The approximation lies in and is continuous, then there exist two numbers such that

.

Then from (5) we get,

Now putting for n successively and multiplying relations we get:

If we choose the interval such that

Then

Therefore the method is convergent. Thus for the convergence of the Regula Falsi Method, the interval must be very small.

**ADVANTAGE:**

The advantage of this method is that it is very simple and the sequence is sure to converge. The another advantage of this method is that it does not require the evaluation of derivatives and pre-calculation.

**DISADVANTAGE:**

The method is very slow and not suitable for hand calculation.

**ITERATION METHOD OR FIXED POINT ITERATION:**

Let be a continuous function on the interval and the equation has at least one root on The equation can be written in the form

Thus a root of the given equation satisfies . Therefore the point remains fixed under the mapping and so a root of the equation is a fixed point of

is called the iteration function. Here we also assume that is continuously differentiable in

Using graphical or tabulation method, we first find a location or crude approximation of a real root (say) of and let be the initial approximation of Thus Satisfies the equation

Putting , we get first approximation of as , and then the successive approximations are calculated as:

The above iteration is generated by the formula and is called the iteration formula, where is the n-th approximation of the root of

These successive iterations are repeated till the approximate numbers ’s converges to the root with desired accuracy, i.e., where is a sufficiently small number.

The sequence of iterations or the successive better approximations may or may not be converge to a limit. If converges, then it converges to and the number of iterations required depends upon the desired degree of accuracy of the root

**CONVERGENCE OF METHOD OF ITERATION:**

The presentation of as is not unique, therefore the convergence of depends upon the nature of Now we investigate about the nature of which yields a convergent sequence

By Lagrange’s mean value theorem we get,

where

where

……………………………………………………………………………………………………………

where

Thus, =

Assuming , we have

Thus,

Therefore the method is convergent for in .

**ESTIMATION OF ERROR:**

Let, be an exact root of the equation  and .

Therefore, , , where

,[where, ]

After rearrangement, this relation becomes

Let the maximum number of iteration needed to achieve the accuracy be . Then

, i.e.

For the estimation of the error is given by the following simple form:

**ORDER OF CONVERGENCE:**

The convergence of an iteration method depends on the suitable choice of the iteration function and the initial guess

Let, converges to the exact root , so that .

Thus . Let, . Note that Then the above relation becomes

i.e.

hence the order of convergence of the iteration method is linear.

**GEOMETRICAL INTERPRETATION:**

**ADVANTAGE AND DISADVANTAGE:**

The disadvantage of this method is that a pre-calculation is required to re-write to in such a way that

The advantage of this method is that the operation carried out at each stage are of same kind, and this makes easier to develop computer program.

**NEWTON RAPHSON METHOD**

This is also an iterative method and is used to find isolated roots of an equation . The object of this method is to correct the approximate root (say) successively to the exact root Initially, a crude approximation of a small interval is found out in which only one root (say) of les.

Let, is an approximation of the root of the equation Let, be a small correction on , then is the correct root.

Using Taylor’s series expansion,

since of

Neglecting the second and the higher order derivatives, the above equation reduces to-

Or,

Therefore,

Further if be the correction on , then is the correct root of

Then using the previous process we get,

Therefore,

Processing in this way, we get th corrected root as

This expression generates a sequence of approximate values each successive term of which is closer to the exact value of the root . The method will terminate when becomes very small.

In this method the arc of the curve is replaced by the tangent to the curve, hence this method is sometimes called method of tangent.

Note: the newton Raphson method may also used to find a complex root of an equation when the initial guess is taken as a complex number.

**GEOMETRICAL INTERPRETATION:**

The geometrical interpretation of this method is shown in the figure 1. In this method, a tangent is drawn at to the curve The tangent cuts the x-axis at . Again the tangent is drawn at , which cuts the x-axis at . This process is continued until .

The choice of initial guess of this method is very important. If the initial guess is near the root then the method converges very fast. If it is not so near the root or if the starting point is wrong, then the method may lead to an endless cycle.

This illustrated in figure2. In this figure the initial guess gives the fast convergence to the root, the initial guess leads to an endless cycle and the initial guess gives a divergent solution.

Even if the initial guess is not close to the exact root, the method may diverge. To chose the initial guess the following rule may be followed. If the initial guess be and if then be the initial guess.

**CONVERGENCE OF NEWTON RAPHSON METHOD:**

Comparing with the iteration method, we may assume the iteration function as:

Thus the above sequence will be convergent, if and only if

i.e.

**RATE OF CONVERGENCE OF NR METHOD:**

Let, be a root of the equation . Then, . The iteration scheme for NR-method is

Let, Then from the above relation we get-

Or,

Or,

Or,

Or ,

Neglecting the terms of order and higher power the expression becomes

This relation shows that NR method has quadratic convergence or second order convergence.