Hybrid Globally Convergent Complementary Filter for $\mathbb{SO}(3)$

Piyush Jirwankar

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The purpose of this work is to design a globally asymptotically convergent observer for a system evolving on $\mathbb{SO}(3)$. We first consider the nominal case when there is no measurement noise and develop globally asymptotic stability results. Then we introduce noise in the measurements of the angular velocity Ω and the rotation matrix R and establish robustness of the filter to said noise.

1 Preliminaries

The set of all rotation matrices is defined as $\mathbb{SO}(3) \coloneqq \{R \in \mathbb{R}^{3\times 3} \mid R^\top R = RR^\top = I,\ \det R = 1\}$. The Lie algebra of $\mathbb{SO}(3)$ is defined as $\mathfrak{so}(3) \coloneqq \{X \in \mathbb{R}^{3\times 3} \mid X + X^\top = 0\}$. The n-sphere, denoted by $\mathbb{S}^n$, is defined as $\mathbb{S}^n \coloneqq \{v \in \mathbb{R}^{n+1} \mid v^\top v = 1\}$. For any vector $v = (v_1, v_2, v_3)^\top \in \mathbb{R}^3$, we define the 'cross' map as

$$\cdot_\times : \mathbb{R}^3 \to \mathfrak{so}(3), \quad v \mapsto v_\times \coloneqq \begin{bmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{bmatrix} \tag{1}$$

The inverse map is defined by $\mathrm{vex} : \mathfrak{so}(3) \to \mathbb{R}^3$, $v_\times \mapsto \mathrm{vex}\,(v_\times) \coloneqq v$. For any matrix $A \in \mathbb{R}^{3\times 3}$, its skew symmetric projection is given by $\mathbb{P}_a(A) \coloneqq (A - A^\top)/2$, and its symmetric projection is given by $\mathbb{P}_s(A) \coloneqq (A + A^\top)/2$. Any rotation matrix $R \in \mathbb{SO}(3)$ can be expressed as a rotation about an axis $v \in \mathbb{S}^2$ by an angle $\theta \in \mathbb{R}$. Consider the function $\mathcal{R} : \mathbb{R} \times \mathbb{S}^2 \to \mathbb{SO}(3)$, $(\theta, v) \mapsto \mathcal{R}(\theta, v) \coloneqq I + v_\times \sin\theta + v_\times^2(1 - \cos\theta)$. Now, $\mathcal{R}(\theta, v)$ represent a rotation about the axis v by an angle of θ in the counterclockwise direction. We see that $\mathcal{R}(\theta, v) = \exp\,(\theta v_\times)$, where exp is the exponential map that maps values from the Lie algebra $\mathfrak{so}(3)$ to $\mathbb{SO}(3)$. The tangent space of $\mathbb{SO}(3)$ at $R \in \mathbb{SO}(3)$ is given by $\mathrm{T}_R\mathbb{SO}(3)$ and its dual is given by $\mathrm{T}_R^*\mathbb{SO}(3)$. The action of a dual element $w \in \mathrm{T}_R^*\mathbb{SO}(3)$ on $v \in \mathrm{T}_R\mathbb{SO}(3)$ is given by the duality pairing $\mathrm{T}_R^*\mathbb{SO}(3) \times \mathrm{T}_R\mathbb{SO}(3) \ni (w, v) \mapsto \langle\langle w, v\rangle\rangle \in \mathbb{R}$[1]. For matrix Lie groups, we have $\langle\langle A, B\rangle\rangle = \mathrm{tr}\left(A^\top B\right)$. The differential of a scalar valued function $f : \mathbb{SO}(3) \to \mathbb{R}$ is denoted by $\nabla_R f \in \mathrm{T}_R^*\mathbb{SO}(3)$.

The inner product on the space of real-valued matrices is given by the Frobenius inner product $\langle\langle A, B\rangle\rangle_F \coloneqq \mathrm{tr}\left(A^\top B\right)$ for any $A, B \in \mathbb{R}^{m\times n}$.

Definition 1 (Distance metric on $\mathbb{SO}(3)$)**.** *The distance between two points $x, y \in \mathbb{SO}(3)$ is given by a function $d : \mathbb{SO}(3) \times \mathbb{SO}(3) \to \mathbb{R}$, $(x, y) \mapsto d(x, y)$, where the following hold:*

- $d(x, y) > 0$ *if* $x \neq y$;
- $d(x, y) = 0 \Leftrightarrow x = y$;
- $d(x, y) = d(y, x) \quad \forall x, y \in \mathbb{SO}(3)$;
- $d(x, y) \leq d(x, z) + d(y, z) \quad \forall x, y, z \in \mathbb{SO}(3)$.

[1] Elements of $\mathrm{T}_R\mathbb{SO}(3)$ and $\mathrm{T}_R^*\mathbb{SO}(3)$ are matrices. Double angles are used for the duality pairing to avoid confusion with the inner product notation.

Definition 2 (Distance of a point from a set). *The distance of a point $x \in \mathbb{SO}(3)$ from the closed set $\mathcal{A} \subset \mathbb{SO}(3)$, denoted as $|x|_{\mathcal{A}}$, is defined as*

$$|x|_{\mathcal{A}} := \inf_{y \in \mathcal{A}} d(x, y)$$

Definition 3 (Neighbourhood of a point). *The open ϵ-neighbourhood of a point $R \in \mathbb{SO}(3)$ for $\epsilon > 0$, denoted by $\mathcal{N}_\epsilon(x)$, is defined as*

$$\mathcal{N}_\epsilon(R) := \{X \in \mathbb{SO}(3) \mid d(X, R) < \epsilon\}$$

Definition 4 (Neighbourhood of a set). *For $0 \leq \epsilon \leq 1$, the open ϵ-neighbourhood of a set $U \subset \mathbb{SO}(3)$ is given by*

$$\mathcal{N}_\epsilon(U) := \{X \in \mathbb{SO}(3) \mid d(X, R) < \epsilon, \;\; R \in U\}$$

Proposition 1 (Baker-Campbell-Hausdorff formula). *For $X, Y \in \mathfrak{so}(3)$, there exists $Z \in \mathfrak{so}(3)$ such that $\exp Z = \exp X \exp Y$ and*

$$Z = X + Y + \frac{1}{2}[X, Y] + \frac{1}{12}\left([X, [X, Y]] + [Y, [X, Y]]\right) - \frac{1}{24}[Y, [X, [X, Y]]] + \ldots$$

where $[X, Y] := XY - YX$.

Remark 1. *When $XY = YX$, $\exp(X)\exp(Y) = \exp(X + Y)$.*

Remark 2. *For $R_1, R_2 \in \mathbb{SO}(3)$, if $R_2 = \mathcal{R}(\theta, v)$ for some $\theta \in \mathbb{R}$ and $v \in \mathbb{S}^2$, then $R_1 R_2 R_1^\top = \mathcal{R}(\theta, R_1 v)$.*

Definition 5 (Distance between two points). *The distance between two points $R_1, R_2 \in \mathbb{SO}(3)$ is given by $|R_1|_{R_2}$, where*

$$|R_1|_{R_2}{}^2 := \frac{1}{4}\left(I - R_2^\top R_1\right) \quad \forall R_1, R_2 \in \mathbb{SO}(3).$$

Definition 6 (Configuration error function). *The configuration error function[2] for $\mathbb{SO}(3)$ is defined by $|\cdot|_I : \mathbb{SO}(3) \to \mathbb{R}$, where*

$$|R|_I{}^2 := \frac{1}{4} tr\left(I - R\right) \quad \forall R \in \mathbb{SO}(3).$$

Definition 7 (Nondegenerate function). *A function $f : \mathbb{R}^n \times \mathbb{R}^n \supset A \times B \to \mathbb{R}$ is said to be a nondegenerate function if the Hessian of f is full ranked.*

Definition 8 (Asymptotically Independent Signals [6]). *Two signals $x : \mathbb{R} \to M_x$ and $y : \mathbb{R} \to M_y$ are* asymptotically dependent *if there exists[3] a nondegenerate function $f : M_x \times M_y \to \mathbb{R}$ and a time $T > 0$ such that*

$$f(x(t), y(t)) = 0 \quad \forall t > T$$

The two signals are asymptotically independent if they are not asymptotically dependent.

Lemma 1. *For any rotation matrix $R \in \mathbb{SO}(3)$, the following holds.*

$$|R^2|_I{}^2 = 4|R|_I{}^2\left(1 - |R|_I{}^2\right) \tag{2a}$$

$$\|vex\left(\mathbb{P}_a(R)\right)\|^2 = |R^2|_I{}^2 \tag{2b}$$

[2]Using Definition 5, the configuration error function is the distance of $R \in \mathbb{SO}(3)$ to the identity element I.
[3]Existence of such a function is not trivial as it has to be nondegenerate while satisfying $f(x(t), y(t)) = 0$.

Proof. For any $R \in \mathbb{SO}(3)$, there exists $\theta \in [0, \pi]$ and $v \in \mathbb{S}^2$ such that $R = \mathcal{R}(\theta, v) = \exp(\theta v_\times)$. Now, $R^2 = RR = \exp(\theta v_\times) \exp(\theta v_\times)$. Using Remark 1, we have $R^2 = \exp(2\theta v_\times)$. Next, using Definition 6 and the fact that $\operatorname{tr}(R) = 1 + 2\cos\theta$, we get

$$|R|_I^2 = (1 - \cos\theta)/2 \implies \cos\theta = 1 - 2|R|_I^2. \tag{3}$$

Since $\theta \in [0, \pi]$, this leads to $\sin\theta = \sqrt{1 - (\cos\theta)^2} = 2\sqrt{|R|_I^2 \left(1 - |R|_I^2\right)}$. Now using (3) results in

$$\begin{aligned} |R^2|_I^2 &= \frac{1}{2}(1 - \cos 2\theta) \\ &= \sin^2\theta \\ &= 4|R|_I^2 \left(1 - |R|_I^2\right) \end{aligned}$$

This proves (2a). To prove (2b), we expand the LHS as follows.

$$\begin{aligned} \|\operatorname{vex}(\mathbb{P}_a(R))\|^2 &= \langle \operatorname{vex}(\mathbb{P}_a(R)), \operatorname{vex}(\mathbb{P}_a(R)) \rangle \\ &= \frac{1}{2} \langle\langle \mathbb{P}_a(R), \mathbb{P}_a(R) \rangle\rangle_F \\ &= \frac{1}{2} \operatorname{tr}\left(\left(\frac{R - R^\top}{2} \right)^\top \left(\frac{R - R^\top}{2} \right) \right) \\ &= \frac{1}{4} \operatorname{tr}\left(I - R^2\right) \\ &= |R^2|_I^2 \end{aligned}$$

This proves (2b). □

Lemma 2. *For any $X, Y \in \mathbb{SO}(3)$ with $X = \mathcal{R}(x, v_x)$ and $Y = \mathcal{R}(y, v_y)$ where $x, y \in [0, \pi]$ and $v_x, v_y \in \mathbb{S}^2$, the following holds:*

$$|XY|_I^2 = |X|_I^2 + |Y|_I^2 - |X|_I^2 |Y|_I^2 + (v_x^\top v_y)|X|_I |Y|_I \left[2\sqrt{\left(1 - |X|_I^2\right)\left(1 - |Y|_I^2\right)} - v_x^\top v_y |X|_I |Y|_I \right] \tag{4}$$

Proof. Expanding $XY = \mathcal{R}(x, v_x)\mathcal{R}(y, v_y)$ we get the following.

$$XY = \left(I + v_{x\times} \sin x + v_{x\times}^2 (1 - \cos x)\right) \left(I + v_{y\times} \sin y + v_{y\times}^2 (1 - \cos y)\right)$$

Taking the trace on both sides results in

$$\operatorname{tr}(XY) = \operatorname{tr}(X) + (\sin x \sin y) \operatorname{tr}\left(v_{x\times} v_{y\times}\right) + (1 - \cos y) \operatorname{tr}\left(v_{y\times}^2\right) + (1 - \cos x)(1 - \cos y) \operatorname{tr}\left(v_{x\times}^2 v_{y\times}^2\right)$$

It is easy to verify that for any $v, w \in \mathbb{S}^2$, $\operatorname{tr}(v_\times w_\times) = -2v^\top w$ and $\operatorname{tr}\left(v_\times^2 w_\times^2\right) = 1 + (v^\top w)^2$. Using this, we obtain the following:

$$\operatorname{tr}(XY) = \operatorname{tr}(X) - 2(v^\top w) \sin x \sin y - 2(1 - \cos y) + (1 - \cos x)(1 - \cos y)(1 + (v^\top w)^2)$$

Now using (3), we have $\cos x = 1 - 2|X|_I^2$ and $\sin x = 4\sqrt{|X|_I^2 (1 - |X|_I^2)}$ for all $X \in \mathbb{SO}(3)$. This leads to the following equation:

$$\operatorname{tr}(XY) = \operatorname{tr}(X) - 8(v^\top w)\sqrt{|X|_I^2 (1 - |X|_I^2)}\sqrt{|Y|_I^2 (1 - |Y|_I^2)} - 4|Y|_I^2 + 4|X|_I^2 |Y|_I^2 (1 + (v^\top w)^2)$$

Using Definition 6 results in

$$\implies |XY|_I^2 = |X|_I^2 + |Y|_I^2 - |X|_I^2 |Y|_I^2 + (v_x^\top v_y)|X|_I |Y|_I \left[2\sqrt{\left(1 - |X|_I^2\right)\left(1 - |Y|_I^2\right)} - v_x^\top v_y |X|_I |Y|_I \right]$$

This completes the proof. □

1.1 Attitude kinematics

The rotation kinematics on $\mathbb{SO}(3)$ are given by

$$\dot{R} = R\Omega_\times \tag{5}$$

where $\Omega \in \mathbb{R}^3$ represents the angular velocity in the body fixed frame.

Next, we define local exponential stability and unstable set for a closed-loop system on $\mathbb{SO}(3)$ given as

$$\dot{R} = f(R, \Omega) \tag{6}$$

for some $f : \mathbb{SO}(3) \times \mathbb{R}^3 \to \mathrm{T}_R\mathbb{SO}(3)$. It is important to note here that for the purpose of observer design, $t \mapsto \Omega(t)$ is not a control input and it cannot be altered, and hence it can be considered as a time-varying parameter.

Definition 9 (Local exponential stability)**.** *For an absolutely continuous $t \mapsto \Omega(t)$, the point $R^* \in \mathbb{SO}(3)$ is locally exponentially stable for (6) if there exist positive constants λ, k, δ such that every solution $t \mapsto R(t)$ to (6) satisfies*

$$|R(0)|_{R^*} \leq \delta \implies |R(t)|_{R^*} \leq k e^{-\lambda t}|R(0)|_{R^*} \quad \forall t \in dom\ R.$$

Definition 10 (Unstable set of a dynamical system)**.** *For an absolutely continuous $t \mapsto \Omega(t)$, a set $U \subset \mathbb{SO}(3)$ is unstable for (6) if for any ϵ such that $0 < \epsilon < 1$, every solution $t \mapsto R(t)$ to (6) with $R(0) \in \mathcal{N}_\epsilon(U)\backslash U$ satisfies $R(t) \notin U$ for all $t \in dom\ R$.*

2 Passive Complementary Fitler

One of the standard tools for estimation on the special orthogonal groups is the nonlinear complementary filter [6]. The measurements of the angular velocity Ω and the rotation matrix R are given as Ω^y and R^y respectively. As in [6], the following assumption is made.

Assumption 1. *The measurements are assumed noise-free and angular velocity measurements are assumed bias-free, i.e., $\Omega^y = \Omega$ and $R^y = R$.*

Now, we re-state below [6, Theorem 4.2] with the simplification of considering angular velocity measurements bias-free as in Assumption 1.

Theorem 1 (Passive Complementary Filter [6, Theorem 4.2])**.** *Consider the state $R \in \mathbb{SO}(3)$ that follows the kinematics in (5). The measurements satisfy Assumption 1. Let the estimate of R be denoted by $\hat{R} \in \mathbb{SO}(3)$ and the estimation error be defined as $\tilde{R} := \hat{R}^\top R$. Let $k_p > 0$ be a positive gain value. With the initial estimate being $\hat{R}(0) = \hat{R}_0 \in \mathbb{SO}(3)$, let the estimates satisfy the following kinematics:*

$$\dot{\hat{R}} = \hat{R}\left(\Omega^y + k_p\omega(\tilde{R}^y)\right)_\times, \quad \omega(\tilde{R}^y) = vex\left(\mathbb{P}_a(\tilde{R}^y)\right) \quad where\ \tilde{R}^y := \hat{R}^\top R^y. \tag{7}$$

Assume that $t \mapsto \Omega(t) \in \mathbb{R}^3$ is a bounded, absolutely continuous signal and that the pair of signals $t \mapsto (\Omega(t), \tilde{R}(t))$ are asymptotically independent. Define $\mathbb{U}_0 \subseteq \mathbb{SO}(3)$ as

$$\mathbb{U}_0 = \left\{\tilde{R} \in \mathbb{SO}(3) \mid tr(\tilde{R}) = -1\right\}$$

Then:

(a) The set $\mathbb{U}_0$ is forward invariant and unstable with respect to the system given in (7).

(b) The identity element I is locally exponentially stable for the dynamics of the error $\tilde{R}$.

(c) *For all initial conditions such that $\hat{R}_0^\top R^y(0) \notin \mathbb{U}_0$, $\hat{R}(t)$ converges to $R(t)$ and, for each positive k such that $0 < k < 1$ and $|\tilde{R}^y(0)|_I \leq k$, every solution $t \mapsto \tilde{R}(t)$ to the error dynamics satisfies*

$$|\tilde{R}(t)|_I \leq e^{-2k_p(1-k)t}|\tilde{R}(0)|_I \quad \forall t \geq 0. \tag{8}$$

Proof. See [6, Theorem 4.2] for the proof of (a), (b). Next, $\tilde{R}^y(t) = \tilde{R}(t)$ for all $t \geq 0$ due to Assumption 1. For the proof of (c), consider a Lyapunov function candidate candidate $V(\tilde{R}) = |\tilde{R}|_I{}^2$ which is positive definite as shown in Lemma 3. Now, computing its derivative along the trajectories of $\tilde{R}$, we get

$$\langle\langle \nabla_{\tilde{R}} V, \dot{\tilde{R}} \rangle\rangle = 2|\tilde{R}|_I \frac{d|\tilde{R}(t)|_I}{dt} = -\frac{1}{4}\mathrm{tr}\left(\dot{\tilde{R}}(t)\right).$$

Now, since $\dot{\tilde{R}} = [R, \Omega_\times] - k_p \omega(\tilde{R})_\times \tilde{R}$ and using the fact that the trace of the Lie bracket is 0, we get the following.

$$\begin{aligned}
\langle\langle \nabla_{\tilde{R}} V, \dot{\tilde{R}} \rangle\rangle &= \frac{k_p}{4}\mathrm{tr}\left(\omega(\tilde{R})_\times \tilde{R}\right) \\
&= \frac{k_p}{4}\mathrm{tr}\left(\omega(\tilde{R})_\times \mathbb{P}_a(\tilde{R})\right) \\
&= \frac{k_p}{4}\mathrm{tr}\left(\omega(\tilde{R})_\times \omega(\tilde{R})_\times\right) \\
&= -\frac{k_p}{4}\langle\langle \omega(\tilde{R})_\times, \omega(\tilde{R})_\times \rangle\rangle_F \\
&= -\frac{k_p}{2}\|\omega(\tilde{R})\|^2.
\end{aligned}$$

Now using Lemma 1, we have the following:

$$\begin{aligned}
\langle\langle \nabla_{\tilde{R}} V, \dot{\tilde{R}} \rangle\rangle &= -2k_p|\tilde{R}|_I{}^2(1 - |\tilde{R}|_I{}^2) \\
2|\tilde{R}|_I \frac{d|\tilde{R}(t)|_I}{dt} &\leq -2k_p(1-k^2)|\tilde{R}|_I \\
\implies |\tilde{R}(t)|_I &\leq e^{-k_p(1-k^2)t}|\tilde{R}(0)|_I \quad \forall t \geq 0.
\end{aligned}$$

□

Lemma 3 ([3, Lemma 5.6]). *The configuration error function $|\tilde{R}|_I$ defined in Definition 6 is positive definite, and has 4 critical points given by $\Theta = \{I, \exp(\pi {e_1}_\times), \exp(\pi {e_2}_\times), \exp(\pi {e_3}_\times)\}$ where $e_1, e_2, e_3 \in \mathbb{S}^2$ are the eigenvectors of I. Note that $\Theta \backslash \{I\} \subset \mathbb{U}_0$.*

3 Hybrid passive complementary filter

Since $\mathbb{SO}(3)$ is a compact Lie group, it follows that the Lyapunov function V considered in Theorem 1 has compact sublevel sets. Thus, using Theorem 1, we have that the passive complementary filter is almost globally asymptotically convergent and has exponentially stable error dynamics, with the unstable set being the measure zero set of $\mathbb{SO}(3)$ that represents a 180° rotation, i.e., the set $\mathbb{U}_0$. We leverage this property by using hybrid control theory to obtain a globally asymptotically stable observer on $\mathbb{SO}(3)$.

Definition 11 (Hybrid Observer). *A hybrid observer $\hat{\mathcal{H}}$ with data $(\hat{C}, F, \hat{D}, G, \zeta)$ is given by*

$$\hat{\mathcal{H}} : \begin{cases} \dot{z} &= F(z, v), \quad (z, v) \in \hat{C} \\ z^+ &= G(z, v), \quad (z, v) \in \hat{D} \\ \zeta &= f(z, v) \end{cases} \tag{9}$$

where z is the state of the hybrid observer, v is the input and ζ is the output, $\hat{C}$ and $\hat{D}$ represent the flow set and the jump set for the hybrid observer respectively.

Due to Assumption 1, we have $R^y = R$. Our design approach for the globally asymptotically convergent observer is the following: when $\tilde{R}^y = \tilde{R}$ is *close enough* to I, the observer is defined by (7). When $\tilde{R}^y$ is *far enough* from I, a different passive complementary filter is defined such that $\tilde{R}$ tracks a constant $R^* \in \mathbb{SO}(3)$ (which is *close* to identity) which brings $\tilde{R}$ in a *small enough* neighbourhood of I so that the observer (7) can be used. Thus, what we desire is the following. For a fixed $R^* \in \mathbb{SO}(3)$:

$$R_1 \coloneqq R^* R, \qquad R_1^y \coloneqq R^* R^y \tag{10a}$$

$$\tilde{R}_1 \coloneqq \hat{R}^\top R_1^y = \hat{R}^\top R^* R^y \tag{10b}$$

With $\overline{k_p} > 0$, we define the following observer for the case when $\tilde{R}^y$ is *far enough* from I.

$$\dot{\hat{R}} = \hat{R}\left(\Omega^y + \overline{k_p}\omega(\tilde{R}_1^y)\right)_\times, \quad \hat{R}(0) = \hat{R}_0 \tag{11a}$$

$$\omega(\tilde{R}_1^y) \coloneqq \mathrm{vex}\left(\mathbb{P}_a(\tilde{R}_1^y)\right), \qquad \tilde{R}_1^y \coloneqq \hat{R}^\top R_1^y \tag{11b}$$

Using Theorem 1, it is straightforward to see that $I \in \mathbb{SO}(3)$ is almost globally and locally exponentially stable for the dynamics of $\tilde{R}_1$. We employ a hysteresis based hybrid observer for reasons mentioned in Remark 3. Consider two constants $0 < c_1 < c_0 < 1$. Consider the following sets

$$C_0 \coloneqq \left\{\tilde{R}^y \in \mathbb{SO}(3) \mid |\tilde{R}^y|_I \le c_0\right\} \tag{12a}$$

$$C_1 \coloneqq \left\{\tilde{R}^y \in \mathbb{SO}(3) \mid |\tilde{R}^y|_I \ge c_1\right\} \tag{12b}$$

$$D_0 \coloneqq \left\{\tilde{R}^y \in \mathbb{SO}(3) \mid |\tilde{R}^y|_I \ge c_0\right\} \tag{12c}$$

$$D_1 \coloneqq \left\{\tilde{R}^y \in \mathbb{SO}(3) \mid |\tilde{R}^y|_I \le c_1\right\} \tag{12d}$$

The intersection $\Gamma \coloneqq C_0 \cap C_1$ represents the hysteresis region. The state of the hybrid observer, denoted by $\hat{\mathcal{H}}_1$, is $q \in Q \coloneqq \{0, 1\}$, the input is $v \coloneqq (\hat{R}, R^y, \Omega^y) \in \mathbb{SO}(3) \times \mathbb{SO}(3) \times \mathbb{R}^3$ and the output $\zeta = f(q, v) \coloneqq (f_1(q, v), f_2(q, v)) \in \mathfrak{so}(3) \times Q$ is defined in (13e). Define $\gamma_i(v) \coloneqq (\Omega^y + k_i\omega_i)_\times$, $i = 0, 1$ where $\omega_0 \coloneqq \omega(\tilde{R})$, $\omega_1 \coloneqq \omega(\tilde{R}_1)$ and $k_0 \coloneqq k_p$, $k_1 \coloneqq \overline{k_p}$. The data for the hybrid observer $\hat{\mathcal{H}}_1 = (\hat{C}, \hat{F}, \hat{D}, \hat{G}, \zeta)$ is given as

$$\hat{C} = \bigcup_{q \in Q}\left(\hat{C}_q \times q\right), \quad \begin{cases} \hat{C}_0 \coloneqq C_0 \\ \hat{C}_1 \coloneqq C_1 \end{cases} \tag{13a}$$

$$\hat{F}(q, v) = 0 \tag{13b}$$

$$\hat{D} = \bigcup_{q \in Q}\left(\hat{D}_q \times q\right), \quad \begin{cases} \hat{D}_0 \coloneqq D_0 \\ \hat{D}_1 \coloneqq D_1 \end{cases} \tag{13c}$$

$$\hat{G}(q, v) = 1 - q \tag{13d}$$

$$\zeta = f(q, v) \coloneqq (\underbrace{q\gamma_1(v) + (1 - q)\gamma_0(v)}_{f_1(q, v)}, \underbrace{q}_{f_2(q, v)}) \tag{13e}$$

Using the above hybrid observer, the estimates are updated as

$$\underbrace{\dot{\hat{R}} = \hat{R}f_1(q, v), \quad \dot{q} = 0,}_{\text{during flows}} \quad \text{and} \quad \underbrace{\hat{R}^+ = \hat{R}, \quad q^+ = 1 - q}_{\text{during jumps}} \tag{14}$$

Working of the hybrid observer: Let $(t_0, j_0) \in \operatorname{dom} \tilde{R}$. If $(\tilde{R}(t_0, j_0), q(t_0, j_0)) \in C_0 \times \{0\}$, the system flows. The observer in Theorem 1 is used and $\tilde{R}(t, j_0)$ converges to $R(t)$, for $t \geq t_0$ without any jumps in q . Since $D_1 \times \{0\} \subset C_0 \times \{0\}$, this analysis also applies to the other case when $(\tilde{R}(t_0, j_0), q(t_0, j_0)) \in D_1 \times \{0\}$. Next, if $(\tilde{R}(t_0, j_0), q(t_0, j_0)) \in D_1 \times \{1\}$, a jump is triggered causing $(\tilde{R}(t_0, j_0 + 1), q(t_0, j_0 + 1)) \in C_0 \times \{0\}$ which has been analysed already. On the other hand, when $(\tilde{R}(t_0, j_0), q(t_0, j_0)) \in C_1 \times \{1\}$, we resort to the observer in (11) such that the observer error $\tilde{R}$ tracks a constant rotation R^* that is defined such that $|R^*|_I < c_1$. This ensures that while $|\tilde{R}(t, j_0)|_I < c_1$ with $t \geq t_0$ and is tracking R^*, there exists a finite $T > t_0$ such that $|\tilde{R}(t_0 + T, j_0)|_I = c_1$ which causes the switch to $(\tilde{R}(t_0 + T, j_0 + 1), q(t_0 + T, j_0 + 1)) \in C_0 \times \{0\}$. If $(\tilde{R}(t_0, j_0), q(t_0, j_0)) \in D_0 \times \{0\}$, this triggers a jump such that $(\tilde{R}(t_0, j_0 + 1), q(t_0, , j_0 + 1)) \in C_1 \times \{1\}$ which we have analyzed already.

Consider the hybrid system $\mathcal{H}_{PCF} = (C, F, G, D)$ that represents the observer error, with the states $\eta := (\tilde{R}, q) \in \mathbb{SO}(3) \times Q$ and inputs $\rho := (\hat{R}, R^y, \Omega^y, \Omega) \in \mathbb{SO}(3) \times \mathbb{SO}(3) \times \mathbb{R}^3 \times \mathbb{R}^3$ has the following data

$$C := \hat{C} \tag{15a}$$

$$F(\eta, \rho) := \begin{bmatrix} \tilde{R}\left(\tilde{R}^\top f_1(q, v)^\top \tilde{R} + \Omega_\times\right) \\ 0 \end{bmatrix} \tag{15b}$$

$$D := \hat{D} \tag{15c}$$

$$G(\eta, \rho) := \begin{bmatrix} \tilde{R} \\ 1 - q \end{bmatrix} \tag{15d}$$

where $v = (\hat{R}, R^y, \Omega^y)$ and (15b) is obtained by substituting $\dot{\hat{R}}$ from (14) in $\dot{\tilde{R}} = \dot{\hat{R}}^\top R + \hat{R}^\top \dot{R}$.

Theorem 2 (Hybrid passive complementary filter)**.** *Given the system* (5)*, and consider the set* $\mathcal{A} := \{I\} \times \{0\} \subset \mathbb{SO}(3) \times Q$. *Assume that* $\Omega(t)$ *is a bounded and absolutely continuous signal. If there exist positive constants* k_p, $\overline{k_p}$, c_0, c_1 *and a fixed* $R^* \in \mathbb{SO}(3)$ *such that*

(a) $0 < |R^*|_I < c_1 < c_0 < 1$,

(b) $v^\top \hat{R}(0, 0)x \neq 0$ *where* x *and* v *are the axis of rotation of* $\tilde{R}^y(0, 0)$ *and* R^* *respectively,*

then

(1) The hybrid observer error system $\mathcal{H}_{PCF} = (C, F, D, G)$ *with data* (15) *satisfies the hybrid basic conditions.*

(2) Every maximal solution to $\mathcal{H}_{PCF}$ *from* $C \cup D$ *is complete.*

(3) Every maximal solution to $\mathcal{H}_{PCF}$ *exhibits no more than 2 jumps.*

(4) The set $\mathcal{A}$ *is globally asymptotically stable and locally exponentially stable for* $\mathcal{H}_{PCF}$.

Proof. Since C_0, C_1, D_0, D_1 are closed subsets of $\mathbb{SO}(3) \times Q$ and the maps F and G are single valued and continuous on C and D respectively , (15) satisfies the hybrid basic conditions. This proves (1). Next, using Proposition 6.10 from [4], the proof of (2) is complete.

We prove the globally exponentially stability of $\mathcal{A}$ to (15) by considering all four cases where $\tilde{R}^y(0, 0)$ can lie.

Case 1: $(\tilde{R}^y(0, 0), q(0, 0)) \in C_0 \times \{0\}$. The local exponential stability for this case is evident using (8) as follows:

$$|\tilde{R}(t, j)|_I \leq e^{-2k_p(1-c_0)t} |\tilde{R}(0, 0)|_I \quad \forall (t, j) \in \operatorname{dom} \tilde{R}. \tag{16}$$

Case 2: $(\tilde{R}^y(0, 0), q(0, 0)) \in D_1 \times \{1\}$. Since the initialization is in the jump set, the state jumps once which results in $(\tilde{R}(0, 1), q(0, 1)) \in C_0 \times \{0\}$. Now we refer to Case 1 for further analysis. Further, due to the jump map definition in (14), $|\tilde{R}(t, j)|_I$ does not change with jumps.

Case 3: $(\tilde{R}^y(0,0), q(0,0)) \in D_0 \times \{0\}$. Similar to Case 2, the initialization is in the jump set resulting in one jump and $(\tilde{R}(0,1), q(0,1)) \in C_1 \times \{0\}$. We refer to Case 4 for further analysis.

Case 4: $(\tilde{R}^y(0,0), q(0,0)) \in C_1 \times \{1\}$. This case employs the rotated observer defined in (11). From Lemma 2, we see that when $|\tilde{R}(0,0)|_I = 1 \implies |\tilde{R}_1(0,0)|_I = 1 - (v^\top w)^2 |R^*|_I$, where $v, w \in \mathbb{S}^2$ are the axis of rotation corresponding to R^* and $\hat{R}(0,0)R(0,0)^\top$ respectively. From Theorem 1, the unstable set for $\tilde{R}_1$ dynamics is the set $\mathbb{U}_0$. By definition of R^*, we have $|R^*|_I > 0$. Thus, $v^\top w \neq 0 \implies |\tilde{R}_1(0,0)|_I \neq 1$. Using Remark 2, we get $w = \hat{R}(0,0)x$ which is the condition (b). This ensures that there is no initial condition that renders $\mathbb{U}_0 \times Q$ forward invariant with respect to (15). Next, there exists a positive scalar d_0 such that $1 - |R^*|_I \leq d_0 < 1$. Considering a Lyapunov function $V(\tilde{R}_1) = |\tilde{R}_1|_I$ for the rotated observer in (11), following (8), we have that when $|\tilde{R}_1(0,0)|_I \leq d_0$, every solution $(t,0) \mapsto \tilde{R}_1(t,0)$ to the rotated error dynamics $\tilde{R}_1$ satisfies

$$|\tilde{R}_1(t,0)|_I \leq e^{-2\overline{k_p}(1-d_0)t} |\tilde{R}_1(0,0)|_I \quad \forall t < T \text{ and } t \in \operatorname{dom} \tilde{R}_1 \tag{17}$$

and $T > 0$ is such that $(\tilde{R}(T,0), q(T,0)) \in D_1 \times \{1\}$. We now wish to find upper and lower bounds on $|\tilde{R}_1(t,0)|_I$ in terms of $|\tilde{R}(t,0)|_I$ to prove that the estimation error $|\tilde{R}|_I$ exponentially decreases. Firstly, we see that $|\hat{R}^\top R^* R|_I = |R^* R \hat{R}^\top|_I$ and $|R\hat{R}^\top|_I = |\tilde{R}|_I$. Let $R^* = \mathcal{R}(\theta^*, v)$ and $R\hat{R}^\top = \mathcal{R}(\tilde{\theta}, w)$. Now, using Lemma 2, we get

$$\begin{aligned}
|\tilde{R}_1|_I &= |\tilde{R}|_I + |R^*|_I - |\tilde{R}|_I |R^*|_I + |\tilde{R}|_I |R^*|_I (v^\top w) \left[2\sqrt{\left(\frac{1}{|\tilde{R}|_I} - 1\right)\left(\frac{1}{|R^*|_I} - 1\right)} - v^\top w \right] \\
&\leq |\tilde{R}|_I + |R^*|_I - |\tilde{R}|_I |R^*|_I + 2|\tilde{R}|_I |R^*|_I (v^\top w) \sqrt{\left(\frac{1}{|\tilde{R}|_I} - 1\right)\left(\frac{1}{|R^*|_I} - 1\right)}
\end{aligned}$$

Since $(\tilde{R}(t,0), q(t,0)) \in C_1 \times \{1\}$ for all $t < T$, by construction, $|\tilde{R}|_I \geq c_1$. Furthermore, by definition of R^*, there exist positive scalars e_0, f_0 such that $e_0 < |R^*|_I < f_0 < c_1$. Since $e_0 < c_1$ by construction, this also implies that $|\tilde{R}(t,0)|_I < |R^*|_I$ for $t < T$. This leads to the following.

$$\begin{aligned}
|\tilde{R}_1|_I &\leq |\tilde{R}|_I + |R^*|_I - f_0 |\tilde{R}|_I + 2e_0 |\tilde{R}|_I \sqrt{\left(\frac{1}{c_1} - 1\right)\left(\frac{1}{f_0} - 1\right)} \\
|\tilde{R}_1|_I &\leq |\tilde{R}|_I \left(2 - f_0 + 2e_0 \sqrt{\left(\frac{1}{c_1} - 1\right)\left(\frac{1}{f_0} - 1\right)} \right)
\end{aligned} \tag{18}$$

It is straightforward to see that $|\tilde{R}_1|_I$ will be the smallest when $v^\top w = -1$. Thus, we have

$$\begin{aligned}
|\tilde{R}_1|_I &\geq |\tilde{R}|_I + |R^*|_I - 2|\tilde{R}|_I |R^*|_I - 2|\tilde{R}|_I |R^*|_I \sqrt{\left(\frac{1}{|\tilde{R}|_I} - 1\right)\left(\frac{1}{|R^*|_I} - 1\right)} \\
&\geq |\tilde{R}|_I + |R^*|_I - 2e_0 |\tilde{R}|_I - 2e_0 |\tilde{R}|_I \sqrt{\left(\frac{1}{c_1} - 1\right)\left(\frac{1}{f_0} - 1\right)} \\
&\geq |\tilde{R}|_I - 2e_0 |\tilde{R}|_I - 2e_0 |\tilde{R}|_I \sqrt{\left(\frac{1}{c_1} - 1\right)\left(\frac{1}{f_0} - 1\right)} \\
|\tilde{R}_1|_I &\geq |\tilde{R}|_I \left(1 - 2e_0 \left[1 + \sqrt{\left(\frac{1}{c_1} - 1\right)\left(\frac{1}{f_0} - 1\right)} \right] \right)
\end{aligned} \tag{19}$$

Using (17), (18) and (19) results in

$$|\tilde{R}(t,j)|_I \leq \frac{\left(2 - f_0 + 2e_0 \sqrt{\left(\frac{1}{c_1} - 1\right)\left(\frac{1}{f_0} - 1\right)} \right)}{\left(1 - 2e_0 \left[1 + \sqrt{\left(\frac{1}{c_1} - 1\right)\left(\frac{1}{f_0} - 1\right)} \right] \right)} e^{-2\overline{k_p}(1-d_0)t} |\tilde{R}(0,0)|_I \quad \forall (t,j) \in \operatorname{dom} \tilde{R} \tag{20}$$

where

$$\lambda \coloneqq \frac{\left(2 - f_0 + 2e_0\sqrt{\left(\frac{1}{c_1} - 1\right)\left(\frac{1}{f_0} - 1\right)}\right)}{\left(1 - 2e_0\left[1 + \sqrt{\left(\frac{1}{c_1} - 1\right)\left(\frac{1}{f_0} - 1\right)}\right]\right)}. \tag{21}$$

For every $(\tilde{R}(0,0), q(0,0)) \in C_1 \times \{1\}$, there exists $T > 0$ such that $(\tilde{R}(T,0), q(T,0)) \in D_1 \times \{1\}$ which results in one jump and $(\tilde{R}(T,1), q(T,1)) \in C_0 \times \{0\}$. This results in

$$\begin{aligned}
|\tilde{R}(T,0)|_I &\leq \lambda e^{-2\overline{k_p}(1-d_0)T}|\tilde{R}(0,0)|_I \\
|\tilde{R}(t,j)|_I &\leq \lambda e^{-2\overline{k_p}(1-d_0)T} e^{-2k_p(1-c_0)(t-T)}|\tilde{R}(0,0)|_I \quad \forall t \geq T \\
&= \lambda e^{-2T[\overline{k_p}(1-d_0)-k_p(1-c_0)]} e^{-2k_p(1-c_0)t}|\tilde{R}(0,0)|_I \quad \forall t \geq T
\end{aligned}$$

This leads to the following bound.

$$|\tilde{R}(t,j)|_I \leq \min\{1, \alpha e^{-2\min\{\overline{k_p}(1-d_0), k_p(1-c_0)\}t}|\tilde{R}(0,0)|_I\} \quad \forall (t,j) \in \text{dom } \tilde{R}. \tag{22}$$

where $\alpha \coloneqq \max\{\lambda, \lambda \exp\left(-2T[\overline{k_p}(1-d_0) - k_p(1-c_0)]\right)\}$. Thus, if $(\tilde{R}(0,0), q(0,0)) \in C_0 \times \{0\}$ or $(\tilde{R}(0,0), q(0,0)) \in D_1 \times \{1\}$, then (16) holds. If $(\tilde{R}(0,0), q(0,0)) \in C_1 \times \{1\}$ or $(\tilde{R}(0,0), q(0,0)) \in D_0 \times \{0\}$, then (22) holds. Further, the system exhibits no jumps when $(\tilde{R}(0,0), q(0,0)) \in C_0 \times \{0\}$; exhibits 1 jump if $(\tilde{R}(0,0), q(0,0)) \in D_1 \times \{1\}$ or $(\tilde{R}(0,0), q(0,0)) \in C_1 \times \{1\}$; and exhibits 2 jumps if $(\tilde{R}(0,0), q(0,0)) \in D_0 \times \{0\}$. The other cases when $(\tilde{R}(0,0), q(0,0)) \in C_i \times j$ or $(\tilde{R}(0,0), q(0,0)) \in D_i \times j$ where $i, j \in Q$ and $i \neq j$ are subsumed in the four cases already considered. This proves (3) and (4) and concludes the proof. □

Remark 3. *If we set $c_0 = c_1$, we get rid of the hysteresis region. Such an observer would work in ideal conditions where there is no noise. Introduction of the slightest noise will cause chattering (see [7]), which is the motivation to introduce the hysteresis region in the observer.*

Remark 3 motivates the analysis of the hybrid observer under measurement noise.

3.1 Robustness analysis of the hybrid passive complementary filter

We consider the filter in Theorem 1 and analyze the effect of measurement noise, i.e. noise in gyro measurements and attitude measurements. We first define the notions of input-to-state stability (see [2, 5]). Consider the hybrid system $\mathcal{H}_u$ with state $x \in \mathcal{X}$ and disturbance $u \in \mathcal{U}$ as follows:

$$\mathcal{H}_u \coloneqq \begin{cases} \dot{x} &= f(x,u), \quad (x,u) \in C, \\ x^+ &= g(x,u), \quad (x,u) \in D. \end{cases} \tag{23}$$

Let $\mathcal{A} \subset \mathcal{X}$ be a nonempty and compact set.

Definition 12 (Local input-to-state stability)**.** *The system (23) is said to be locally input-to-state stable with respect to a nonempty, compact set $\mathcal{A}$ if there exist a class $\mathcal{KL}$ function β, a class $\mathcal{K}$ function γ and a scalar κ such that for each solution pair (x,u) to (23) with $|x(0,0)|_{\mathcal{A}} \leq \kappa$ and $\|u_\#\| \coloneqq \sup_{(t,j)\in dom\ u}\|u(t,j)\|$, every solution $(t,j) \mapsto x(t,j)$ to (23) satisfies*

$$|x(t,j)|_{\mathcal{A}} \leq \beta(|x(0,0)|_{\mathcal{A}}, t+j) + \gamma(\|u_\#\|) \quad \forall (t,j) \in dom\ x.$$

If this holds for $\kappa \to +\infty$[4], then the system (23) is said to be input-to-state stable (ISS).

[4] $\kappa \to +\infty$ is required for ISS when considering proper indicator functions. On $\mathbb{SO}(3)$, $\kappa = +1$ ensures ISS. How should I write it for $\mathbb{SO}(3)$ as I cannot use proper indicator functions?

Definition 13 (Eventual input-to-state stability)**.** *The system* (23) *is said to be eventually input-to-state stable with respect to* $\mathcal{A}$ *if there exist a class* $\mathcal{KL}$ *function* β*, a class* $\mathcal{K}$ *function* γ *and strictly positive scalars* κ *and* T *such that for each solution pair* (x, u) *to* (23) *with* $|x(0,0)|_{\mathcal{A}} \leq \kappa$ *and* $u_{\#} := \sup_{(t,j)\in dom\ u} \|u(t,j)\|$*, every solution* $(t,j) \mapsto x(t,j)$ *to* (23) *satisfies*

$$|x(t,j)|_{\mathcal{A}} \leq \beta(|x(0,0)|_{\mathcal{A}}, t+j) + \gamma(\|u_{\#}\|) \quad \forall (t,j) \in \{(\bar{t},\bar{j}) \in dom\ x \mid \bar{t}+\bar{j} \geq T\}.$$

3.1.1 Noise in gyro measurements

The following measurement model is used

$$\Omega^y = \Omega + \eta_\omega \qquad\qquad R^y = R \tag{24}$$

where the bounds on the measurement noise are given by $\eta_\omega \in \mathbb{R}^3$ and $\overline{\eta_\omega} := \sup_{t\geq 0} \|\eta_\omega(t)\|$.

Theorem 3. *Assuming the conditions (a)-(b) in Theorem 2 hold and there exist positive scalars* d_0, e_0, f_0 *such that* $1 - |R^*|_I \leq d_0 < 1$*,* $0 < e_0 < |R^*|_I < f_0 < c_1$ *and*

$$\frac{\overline{\eta_\omega}}{8\gamma}\left\{\frac{1}{k_p(1-c_0)} + \frac{1}{\overline{k_p}(1-d_0)}\right\} < 1$$

where $\gamma := 1 - 2e_0\left[1 + \sqrt{\left(\frac{1}{c_1}-1\right)\left(\frac{1}{f_0}-1\right)}\right]$*. Then the hybrid observer defined in* (14) *renders the resulting hybrid error dynamics eventually input-to-state stable with respect to* $\mathcal{A}$ *to noisy gyroscope measurements of the form presented in* (24)*, where the set* $\mathcal{A}$ *is defined in Theorem 2.*

Proof. We analyze the observer given in (14) assuming $(\tilde{R}^y(0,0), q(0,0)) \in C_0 \times \{0\}$ with the measurement model in (24). The other case when $(\tilde{R}^y(0,0), q(0,0)) \in C_1 \times \{1\}$ can be studied similarly. Consider the Lyapunov function candidate $V(\tilde{R}) = |\tilde{R}|_I$. The derivative of the Lyapunov function along the trajectories of $\tilde{R}$ is given by

$$\begin{aligned}
\langle\langle \nabla_{\tilde{R}} V, \dot{\tilde{R}} \rangle\rangle &= -\frac{1}{4}\mathrm{tr}\left(\dot{\tilde{R}}\right) \\
&= -\frac{1}{4}\mathrm{tr}\left(\left[\Omega + \eta_\omega + k_p\omega(\tilde{R})\right]_\times \tilde{R} + \tilde{R}\Omega_\times\right) \\
&= \frac{1}{4}\mathrm{tr}\left({\eta_\omega}_\times \tilde{R}\right) + \frac{k_p}{4}\mathrm{tr}\left(\omega(\tilde{R})_\times \tilde{R}\right) \\
&= \frac{1}{4}\mathrm{tr}\left({\eta_\omega}_\times, \omega(\tilde{R})_\times\right) + \frac{k_p}{4}\mathrm{tr}\left(\omega(\tilde{R})_\times \omega(\tilde{R})_\times\right) \\
&= -\frac{k_p}{2}\langle \omega(\tilde{R}), \omega(\tilde{R})\rangle - \frac{1}{2}\langle \eta_\omega, \omega(\tilde{R})\rangle \\
&\leq -2k_p(1-c_0)|\tilde{R}|_I + \frac{1}{2}\overline{\eta_\omega}
\end{aligned}$$

$$|\tilde{R}(t,j)|_I \leq e^{-2k_p(1-c_0)t}|\tilde{R}(0,0)|_I + \frac{\overline{\eta_\omega}}{8k_p(1-c_0)} \quad \forall (t,j) \in \mathrm{dom}\ \tilde{R}. \tag{25}$$

Equation (25) shows that when $(\tilde{R}^y(0,0), q(0,0)) \in C_0 \times \{0\}$, the observer error dynamics are locally ISS to gyro noise. We obtain a similar ISS bound for the rotated observer (11) when $(\tilde{R}^y(0,0), q(0,0)) \in C_1 \times \{1\}$ as given below.

$$|\tilde{R}_1(t,j)|_I \leq e^{-2\overline{k_p}(1-d_0)t}|\tilde{R}_1(0,0)|_I + \frac{\overline{\eta_\omega}}{8\overline{k_p}(1-d_0)} \quad \forall (t,j) \in \{(t,j) \in \mathrm{dom}\ \tilde{R} \mid t \leq T\} \tag{26}$$

where T is defined in Case 4 of the proof of Theorem 2. Following (22), we obtain the following ISS bound for all $t \geq 0$:

$$|\tilde{R}(t,j)|_I \leq \min\left\{1, \alpha e^{-2\min\{\overline{k_p}(1-d_0), k_p(1-c_0)\}t}|\tilde{R}(0,0)|_I + \frac{\overline{\eta_\omega}}{8\gamma}\left\{\frac{1}{k_p(1-c_0)} + \frac{1}{\overline{k_p}(1-d_0)}\right\}\right\} \tag{27}$$

where $\gamma := 1 - 2e_0\left[1 + \sqrt{\left(\frac{1}{c_1} - 1\right)\left(\frac{1}{f_0} - 1\right)}\right]$ and α is defined in (22). After sufficient time has passed, the second term in the parentheses in (27) will become smaller than 1, which will remove the min and complete the proof. □

3.1.2 Noise in attitude measurements

The following measurement model is used

$$\Omega^y = \Omega \qquad\qquad R^y = R\mathcal{N}_R \tag{28}$$

where $\mathcal{N}_R \in \mathbb{SO}(3)$ is the noise in attitude measurements and $|\overline{\mathcal{N}_R}|_I := \sup_{t\geq 0}|\mathcal{N}_R(t)|_I$ is the bound on the noise. Now,

(A) set $c_0 \in (0,1)$ such that $\overline{c_0} := c_0 + 2|\overline{\mathcal{N}_R}|_I + \sqrt{|\overline{\mathcal{N}_R}|_I(1 - |\overline{\mathcal{N}_R}|_I)} < 1$.

(B) set $c_1 \in (0, c_0)$ such that $\overline{c_1} := c_1 - |\overline{\mathcal{N}_R}|_I - \sqrt{|\overline{\mathcal{N}_R}|_I(1 - |\overline{\mathcal{N}_R}|_I)} > 0$.

(C) Set $R^* \in \mathbb{SO}(3)$ such that $0 < |R^*|_I < \overline{c_1}$.

(D) Add the noise bound obtained from ?? of Theorem 2.

(E) Check if this list of items is exhaustive.

Lemma 4. *For any $\tilde{R} \in \mathbb{SO}(3)$ and $\mathcal{N}_R \in \mathbb{SO}(3)$, the following holds for $|\overline{\mathcal{N}_R}|_I \leq 1/2$.*

$$|\tilde{R}|_I - 2|\overline{\mathcal{N}_R}|_I - \sqrt{|\overline{\mathcal{N}_R}|_I(1 - |\overline{\mathcal{N}_R}|_I)} \leq |\tilde{R}\mathcal{N}_R|_I \leq |\tilde{R}|_I + |\overline{\mathcal{N}_R}|_I + \sqrt{|\overline{\mathcal{N}_R}|_I(1 - |\overline{\mathcal{N}_R}|_I)}$$

Proof. Let $v, w \in \mathbb{S}^2$ be the axis of rotation for $\tilde{R}$ and $\mathcal{N}_R$ respectively. Using Lemma 2, $|\tilde{R}\mathcal{N}_R|_I$ is written below.

$$\begin{aligned}
|\tilde{R}\mathcal{N}_R|_I &= |\tilde{R}|_I + |\mathcal{N}_R|_I - |\tilde{R}|_I|\mathcal{N}_R|_I - (v^\top w)^2|\tilde{R}|_I + 2(v^\top w)\sqrt{|\tilde{R}|_I(1 - |\tilde{R}|_I)}\sqrt{|\mathcal{N}_R|_I(1 - |\mathcal{N}_R|_I)} \\
&\geq |\tilde{R}|_I + |\mathcal{N}_R|_I - 2|\tilde{R}|_I|\mathcal{N}_R|_I - 2\sqrt{|\tilde{R}|_I(1 - |\tilde{R}|_I)}\sqrt{|\mathcal{N}_R|_I(1 - |\mathcal{N}_R|_I)} \\
&\geq |\tilde{R}|_I - 2|\tilde{R}|_I|\mathcal{N}_R|_I - \sqrt{|\mathcal{N}_R|_I(1 - |\mathcal{N}_R|_I)} \\
&\geq |\tilde{R}|_I - 2|\mathcal{N}_R|_I - \sqrt{|\mathcal{N}_R|_I(1 - \mathcal{N}_R)}
\end{aligned}$$

Now, since $|\overline{\mathcal{N}_R}|_I < 1/2$, we get

$$\geq |\tilde{R}|_I - 2|\overline{\mathcal{N}_R}|_I - \sqrt{|\overline{\mathcal{N}_R}|_I(1 - |\overline{\mathcal{N}_R}|_I)}$$

This proves the first inequality.

$$\begin{aligned}
|\tilde{R}\mathcal{N}_R|_I &= |\tilde{R}|_I + |\mathcal{N}_R|_I - |\tilde{R}|_I|\mathcal{N}_R|_I - (v^\top w)^2|\tilde{R}|_I + 2(v^\top w)\sqrt{|\tilde{R}|_I(1 - |\tilde{R}|_I)}\sqrt{|\mathcal{N}_R|_I(1 - |\mathcal{N}_R|_I)} \\
&\leq |\tilde{R}|_I + |\mathcal{N}_R|_I - |\tilde{R}|_I|\mathcal{N}_R|_I + 2\sqrt{|\tilde{R}|_I(1 - |\tilde{R}|_I)}\sqrt{|\mathcal{N}_R|_I(1 - |\mathcal{N}_R|_I)} \\
&\leq |\tilde{R}|_I + |\mathcal{N}_R|_I - |\tilde{R}|_I|\mathcal{N}_R|_I + \sqrt{|\mathcal{N}_R|_I(1 - |\mathcal{N}_R|_I)} \\
&\leq |\tilde{R}|_I + |\overline{\mathcal{N}_R}|_I + \sqrt{|\overline{\mathcal{N}_R}|_I(1 - |\overline{\mathcal{N}_R}|_I)}
\end{aligned}$$

This proves the second inequality, thus concluding the proof. □

Theorem 4. *Given the hybrid observer in* (14) *with the measurement model in* (28)*, and consider the set $\mathcal{A}$ as defined in Theorem 2. If (A)-(C) are satisfied and there exist positive constants d_0, e_0, f_0 such that $1 - |R^*|_I \leq d_0 < 1$, $0 < e_0 < |R^*|_I < f_0 < c_1$ and*

$$\frac{\sqrt{|\overline{\mathcal{N}_R}|_I}}{2\gamma}\left\{\frac{1}{1-\overline{c_0}}+\frac{1}{1-d_0}\right\}<1, \qquad |\overline{\mathcal{N}_R}|_I<\frac{1}{13}\left(5-2\sqrt{3}\right).$$

where $\gamma := 1 - 2e_0\left[1+\sqrt{\left(\frac{1}{c_1}-1\right)\left(\frac{1}{f_0}-1\right)}\right]$, then the hybrid observer defined in (14) *renders the resulting error dynamics eventually input-to-state stable with respect to $\mathcal{A}$.*

Proof. Firstly, we see that for (A) and (B) to hold simultaneously, $|\overline{\mathcal{N}_R}|_I + \sqrt{|\overline{\mathcal{N}_R}|_I(1-|\overline{\mathcal{N}_R}|_I)} < c_1 < c_0 < 1 - 2|\overline{\mathcal{N}_R}|_I - \sqrt{|\overline{\mathcal{N}_R}|_I(1-|\overline{\mathcal{N}_R}|_I)} \implies 3|\overline{\mathcal{N}_R}|_I + 2\sqrt{|\overline{\mathcal{N}_R}|_I(1-|\overline{\mathcal{N}_R}|_I)} < 1$. This implies that $|\overline{\mathcal{N}_R}|_I < \frac{1}{13}\left(5-2\sqrt{3}\right)$.

We analyze the observer given in (14) assuming $(\tilde{R}^y(0,0), q(0,0)) \in C_0 \times \{0\}$ with the measurement model in (24). The other case when $(\tilde{R}^y(0,0), q(0,0)) \in C_1 \times \{1\}$ can be analyzed similarly. Consider the Lyapunov function candidate $V(\tilde{R}) = |\tilde{R}|_I$.

$$\begin{aligned}
\langle\langle \nabla_{\tilde{R}} V, \dot{\tilde{R}}\rangle\rangle &= -\frac{1}{4}\mathrm{tr}\left(\dot{\tilde{R}}\right) \\
&= -\frac{1}{4}\mathrm{tr}\left(-\left[\Omega + k_p\omega(\tilde{R}\mathcal{N}_R)\right]_\times \tilde{R} + \tilde{R}\Omega_\times\right) \\
&= \frac{k_p}{4}\mathrm{tr}\left(\omega(\tilde{R}\mathcal{N}_R)_\times \tilde{R}\right)
\end{aligned} \tag{29}$$

Now, the above term is simplified below.

$$\begin{aligned}
\mathrm{tr}\left(\omega(\tilde{R}\mathcal{N}_R)_\times \tilde{R}\right) &= \frac{1}{2}\left(\mathrm{tr}\left(\tilde{R}^2\mathcal{N}_R\right) - \mathrm{tr}\left(\mathcal{N}_R\right)\right) \\
&= 2\left(|\mathcal{N}_R|_I - |\tilde{R}^2\mathcal{N}_R|_I\right)
\end{aligned} \tag{30}$$

Let the axis angle representation of $\tilde{R}^2$ and $\mathcal{N}_R$ be given as $\tilde{R}^2 = \mathcal{R}(2\tilde{\theta}, u)$ and $\mathcal{N}_R = \mathcal{R}(\xi, v)$, where $2\tilde{\theta}, \xi \in [0, \pi]$ and $u, v \in \mathbb{S}^2$. Lemma 2 results in the following.

$$\begin{aligned}
\langle\langle \nabla_{\tilde{R}} V, \dot{\tilde{R}}\rangle\rangle &= -\frac{kp}{2}\left(|\tilde{R}^2\mathcal{N}_R|_I - |\mathcal{N}_R|_I\right) \\
&= -\frac{k_p}{2}\left(|\tilde{R}^2|_I - |\tilde{R}^2|_I(u^\top v)\left[2\sqrt{\frac{1}{|\tilde{R}^2|_I}-1}\sqrt{|\mathcal{N}_R|_I(1-|\mathcal{N}_R|_I)} - (u^\top v)|\mathcal{N}_R|_I\right]\right) \\
&= -\frac{k_p}{2}|\tilde{R}^2|_I - \frac{k_p}{2}(u^\top v)^2|\tilde{R}^2|_I|\mathcal{N}_R|_I + k_p(u^\top v)|\tilde{R}^2|_I\sqrt{\frac{1}{|\tilde{R}^2|_I}-1}\sqrt{|\mathcal{N}_R^2|_I(1-|\mathcal{N}_R^2|_I)}
\end{aligned}$$

$$\langle\langle \nabla_{\tilde{R}} V, \dot{\tilde{R}}\rangle\rangle \leq -\frac{k_p}{2}|\tilde{R}^2|_I + k_p(u^\top v)|\tilde{R}^2|_I\sqrt{\frac{1}{|\tilde{R}^2|_I}-1}\sqrt{|\mathcal{N}_R|_I(1-|\mathcal{N}_R|_I)}$$

Since $\sqrt{|\mathcal{N}_R|_I(1-|\mathcal{N}_R|_I)} \leq \sqrt{|\mathcal{N}_R|_I}$, we have the following:

$$\begin{aligned}
\langle\langle \nabla_{\tilde{R}} V, \dot{\tilde{R}}\rangle\rangle &\leq -\frac{k_p}{2}|\tilde{R}^2|_I + k_p(u^\top v)\sqrt{|\mathcal{N}_R|_I}|\tilde{R}^2|_I\sqrt{\frac{1}{|\tilde{R}^2|_I}-1} \\
&= -\frac{k_p}{2}|\tilde{R}^2|_I + k_p(u^\top v)\sqrt{|\mathcal{N}_R|_I}\sqrt{|\tilde{R}^2|_I(1-|\tilde{R}^2|_I)}
\end{aligned}$$

Now, since $|\tilde{R}^2|_I(1-|\tilde{R}^2|_I) \leq 1/4$, we have the following:

$$\langle\langle \nabla_{\tilde{R}} V, \dot{\tilde{R}} \rangle\rangle \leq -\frac{k_p}{2}|\tilde{R}^2|_I + \frac{k_p(u^\top v)}{2}\sqrt{|\mathcal{N}_R|_I}$$

Now, the following holds for any $u, v \in \mathbb{S}^2$:

$$\begin{aligned}\langle\langle \nabla_{\tilde{R}} V, \dot{\tilde{R}} \rangle\rangle &\leq -\frac{k_p}{2}|\tilde{R}^2|_I + \frac{k_p}{2}\sqrt{|\overline{\mathcal{N}_R}|_I} \\ &\leq -2k_p|\tilde{R}|_I\left(1-|\tilde{R}|_I\right) + \frac{k_p}{2}\sqrt{|\overline{\mathcal{N}_R}|_I}\end{aligned} \tag{31}$$

We have $|\tilde{R}^y|_I \leq c_0$. Using (A) and Lemma 4, we have that $|\tilde{R}^y|_I \leq c_0 \implies |\tilde{R}|_I \leq \overline{c_0}$.

$$\begin{aligned}&\langle\langle \nabla_{\tilde{R}} V, \dot{\tilde{R}} \rangle\rangle \leq -2k_p(1-\overline{c_0})|\tilde{R}|_I + \frac{k_p}{2}\sqrt{|\overline{\mathcal{N}_R}|_I} \\ \implies &|\tilde{R}(t,j)|_I \leq e^{-2k_p(1-\overline{c_0})t}|\tilde{R}(0,0)|_I + \frac{1}{4(1-\overline{c_0})}\sqrt{|\overline{\mathcal{N}_R}|_I}\end{aligned} \tag{32}$$

Equation (32) shows that the observer when $(\tilde{R}^y(0,0), q(0,0)) \in C_0 \times \{0\}$ is locally ISS to attitude measurement noise. When $(\tilde{R}^y(0,0), q(0,0)) \in C_1 \times \{1\}$, a similar bound can be obtained as written below. There exists $d_0 < 1 - |R^*|_I$ such that the following holds:

$$|\tilde{R}_1(t,j)|_I \leq e^{-2\overline{k_p}(1-d_0)t}|\tilde{R}_1(0,j)|_I + \frac{1}{4(1-\overline{d_0})}\sqrt{|\overline{\mathcal{N}_R}|_I} \tag{33}$$

When $|\tilde{R}^y(0,j)|_I \geq c_1$, $\exists T \geq 0$ such that $|\tilde{R}^y(T,j)|_I = c_1 \implies (\tilde{R}^y(T,j+1), q(T,j+1)) \in C_0 \times \{0\}$. Using (32), the following holds.

$$|\tilde{R}(t,j)|_I \leq e^{-2k_p(1-\overline{c_0})(t-T)}|\tilde{R}(T,j)|_I + \frac{1}{4(1-\overline{c_0})}\sqrt{|\overline{\mathcal{N}_R}|_I} \quad \forall t \geq T \tag{34}$$

Combining equations (32), (33) and (34) results in the following.

$$|\tilde{R}(t,j)|_I \leq \min\left\{1, \alpha e^{-\min\{2k_p(1-\overline{c_0}), 2\overline{k_p}(1-d_0)\}t}|\tilde{R}(0,0)|_I + \frac{\sqrt{|\overline{\mathcal{N}_R}|_I}}{2\gamma}\left\{\frac{1}{1-\overline{c_0}} + \frac{1}{1-\overline{d_0}}\right\}\right\} \tag{35}$$

where α and γ are defined in (22) and (27) respectively. This completes the proof. □

Remark 4. *It is important to note that [1] has proved that the complementary filter is locally ISS to noise in gyro measurements and attitude measurements. However, to show local ISS to noise in attitude measurements, [1] assumes that the attitude measurement error $\mathcal{N}_R = \mathcal{R}(\xi, v)$ has its axis v aligned with the axis of rotation of $\tilde{R}$ while also assuming a small angle approximation for ξ. The local ISS shown above in* (32) *makes no such assumptions.*

3.2 Simulations for the hybrid passive complementary filter

We simulate the hybrid passice complementary filter with the following initial conditions:

$$R(0,0) = I, \qquad \hat{R}(0,0) = \mathcal{R}(\pi, [0,1,0]^\top), \qquad q(0,0) = 0.$$

The true angular velocity Ω is set as $\Omega(t,j) = [0,0,\sin t]$. We set $R^y(t,j) = R(t,j)$. The gains and the switching parameters are set as follows.

$$k_p = 1, \qquad c_0 = 0.75, \qquad c_1 = 0.25.$$

Next, R^* is set such that $|R^*|_I = 0.117$ and the axis of rotation of R^* is given by $\hat{R}(0,0)x$ where $x \in \mathbb{S}^2$ is the axis of rotation of $\tilde{R}^y(0,0)$. The trajectory of the error function $V : \mathbb{SO}(3) \times Q \to \mathbb{R}$ and $q(t,j)$ with time is shown in Figures 1 and 2 respectively. Note that the error function V is such that $V(\tilde{R}, q) = |\tilde{R}|_I$ if $\tilde{R} \times q \in C_0 \times \{0\}$, and $V(\tilde{R}, q) := |\tilde{R}_1|_I$ if $\tilde{R} \times q \in C_1 \times \{1\}$. The states jump twice and there are no more jumps required as shown in Theorem 2. The error function goes to 0 which shows that the estimates converge to the true values of the rotation matrix. Also, $q(t) \to 0$ which shows that $(\tilde{R}(t), q(t))$ converge to $\mathcal{A}$.

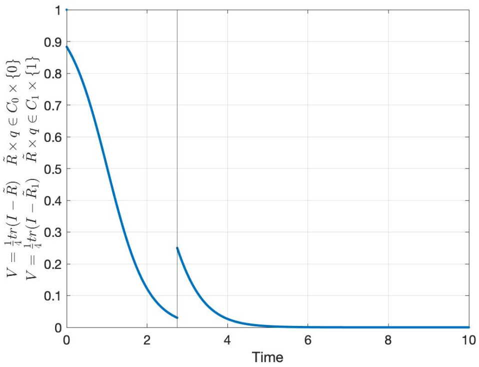

Figure 1: Plot of $V(|\tilde{R}|_I, q)$ vs time. Jumps are seen at $t_{j1} = 0$ sec and $t_{j2} = 2.747$ sec.

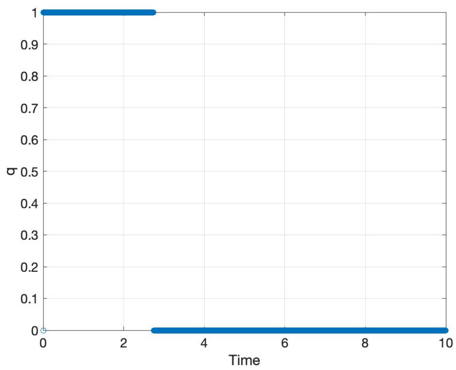

Figure 2: Plot of $q(t)$ vs time.

4 Hybrid Explicit Complementary Filter

This section develops a hybrid globally exponentially stable observer derived from the explicit complementary filter presented in [6]. Unlike the passive complementary filter, vector measurements are considered instead of measurements on $\mathbb{SO}(3)$ which allows for an implementable version of the filter. The IMU sensor onboard is assumed to measure some fixed inertial directions r_i, where $i = 1, 2, ..n$. This measurement of r_i in the body frame is denoted by b_i. The true vectors and the measured vectors are related as $b_i = R^\top r_i \quad \forall i = 1, 2, ..., n$. To be able to extract the attitude estimate $\hat{R}$ from the measurements, we assume that $n \geq 2$ and at least two of the fixed inertial directions are noncollinear.

Further, each of the vector measurements is assigned a weight $k_i > 0$ where $i = 1, 2, ...n$. This allows to increase the weight of the measurement in which we have more confidence.

Assumption 2. *The measurements are assumed to be obtained from gyroscope and IMU sensors onboard the robot. We assume the following:*

1. *The gyroscope and IMU measurements are noise free, i.e.*

$$\Omega^y := \Omega \qquad\qquad b_i = R^\top r_i \quad \forall i = 1, 2, ..n \tag{36}$$

where $n \geq 2$, and atleast two fixed inertial vectors r_i are noncollinear.

2. *The gains k_i are normalized, i.e. $\sum_{i=1}^n k_i = 1$.*

Assumption 3. *Consider $M_0 := \sum_{i=1}^n k_i r_i r_i^\top$. The gains $k_i > 0$ are chosen such that Assumption 2 is satisfied and $M_0 := \sum_{i=1}^n k_i r_i r_i^\top$ has three distinct eigenvalues.*

Lemma 5. *Given inertial directions $r_i, i = 1, ..., n$ and the corresponding measurements $b_i, i = 1, ..., n$, and the estimate $\hat{R}$ of the true attitude R, the left invariant estimation error is $\tilde{R} := \hat{R}^\top R$ and* *the right invariant estimation error is* $\bar{R} \coloneqq \hat{R}R^\top$*. Define* $M_0 \coloneqq \sum_{i=1}^n k_i r_i r_i^\top$ *and* $M \coloneqq R^\top M_0 R$*. Then*

$$tr\left(\bar{R}M_0\right) = 1 + (1-\cos\tilde{\theta})tr\left(w_\times^2 M_0\right) \tag{37}$$

$$tr\left(w_\times^2 M_0\right) = -1 + \sum_{i=1}^n k_i (w^\top r_i)^2 \tag{38}$$

where $\tilde{\theta} \in \mathbb{R}$ *and* $w \in \mathbb{S}^2$ *are such that* $\bar{R} = \mathcal{R}(\tilde{\theta}, w)$*.*

Proof. We can write $\bar{R}$ using the Rodrigues rotation formula as $\bar{R} = I + w_\times \sin\tilde{\theta} + w_\times^2(1-\cos\tilde{\theta})$. Now, $\mathrm{tr}\left(\bar{R}M_0\right) = \mathrm{tr}\left(M_0 + (1-\cos\tilde{\theta})w_\times^2 M_0\right)$. Since k_i are normalized weights, $\mathrm{tr}\left(M_0\right) = 1$, this proves (37). Next, $\mathrm{tr}(w_\times^2 M_0) = \sum_{i=1}^n k_i \mathrm{tr}\left(w_\times^2 r_i r_i^\top\right)$. Now, with $w = [w_1, w_2, w_3]^\top \in \mathbb{S}^1$ and $r_i = [r_{i,1}, r_{i,2}, r_{i,3}]^\top \in \mathbb{S}^1$, we get

$$\begin{aligned} w_\times^2 r_i r_i^\top &= \begin{bmatrix} 0 & -w_3 & w_2 \\ w_3 & 0 & -w_1 \\ -w_2 & w_1 & 0 \end{bmatrix} \begin{bmatrix} r_{i,1}^2 & r_{i,1}r_{i,2} & r_{i,1}r_{i,3} \\ r_{i,1}r_{i,2} & r_{i,2}^2 & r_{i,2}r_{i,3} \\ r_{i,1}r_{i,3} & r_{i,2}r_{i,3} & r_{i,3}^2 \end{bmatrix} \\ \implies \mathrm{tr}\left(w_\times^2 r_i r_i^\top\right) &= -1 + (w^\top r_i) \end{aligned}$$

Now, $\mathrm{tr}\left(w_\times^2 M_0\right) = -\sum_{i=1}^n k_i + \sum_{i=1}^n k_i (w^\top r_i)^2 = -1 + \sum_{i=1}^n (w^\top r_i)^2$. This completes the proof. □

Next, similar to Definition 6, we define a configuration error function in terms of vectorial measurements.

Definition 14 (Estimation error function)**.** *Given* n *fixed inertial directions* r_i*, the corresponding measurements* b_i*, the corresponding relative weights as* $k_i, i = 1, ..., n$ *and the attitude estimate as* $\hat{R}$*, the estimation error function* $\Psi : \mathbb{SO}(3) \times \mathbb{R}^{3\times n} \times \mathbb{R}^{3\times n} \to \mathbb{R}$ *is given below:*

$$\Psi(\hat{R}, \{b_i\}_{i=1}^n, \{r_i\}_{i=1}^n) \coloneqq \frac{1}{4}\sum_{i=1}^n k_i \|b_i - \hat{R}^\top r_i\|^2 \tag{39}$$

Lemma 6. *The estimation error function in* (39) *can be simplified as follows.*

$$\Psi(\hat{R}, \mathbf{b}, \mathbf{r}) = |\tilde{R}|_I \left[1 - \sum_{i=1}^n k_i (w^\top r_i)^2\right] \tag{40}$$

where $\mathbf{b} \coloneqq \{b_i\}_{i=1}^n$*,* $\mathbf{r} \coloneqq \{r_i\}_{i=1}^n$*, and* $\tilde{\theta} \in [0, \pi]$*,* $v \in \mathbb{R}^3$ *is such that* $\bar{R} \coloneqq \hat{R}R^\top = \mathcal{R}(\tilde{\theta}, w)$*.*

Proof. For brevity, $\Psi \equiv \Psi(\hat{R}, \mathbf{b}, \mathbf{r})$.

$$\begin{aligned} \Psi &= \frac{1}{4}\sum_{i=1}^n k_i \|b_i - \hat{R}^\top r_i\|^2 \\ &= \frac{1}{4}\sum_{i=1}^n k_i \|R^\top r_i - \hat{R}^\top r_i\|^2 \\ &= \frac{1}{2}\sum_{i=1}^n k_i \left[1 - r_i^\top \hat{R}R^\top r_i\right] \\ &= \frac{1}{2}\sum_{i=1}^n k_i \left[1 - \mathrm{tr}\left(\hat{R}R^\top r_i r_i^\top\right)\right] \\ &= \frac{1}{2}\sum_{i=1}^n \left[1 - \mathrm{tr}\left(\bar{R}M_0\right)\right] \end{aligned}$$

Using Lemma 5 results in the following.

$$\Psi = \frac{1-\cos\tilde{\theta}}{2}\left[1-\sum_{i=1}^{n} k_i(w^\top r_i)^2\right] = |\tilde{R}|_I\left[1-\sum_{i=1}^{n} k_i(w^\top r_i)^2\right]$$

□

The explicit complementary filter from [6] is now stated below.

Theorem 5 (Explicit Complementary Filter [6, Theorem 5.1])**.** *Consider the system in* (5) *with the measurement model in* (36) *with the observer gain* $k_p > 0$*, the measurement gains* $k_i > 0$ *for* $i = 1, ..., n$ *following Assumption 2 and Assumption 3. Consider the following filter dynamics:*

$$\dot{\hat{R}} = \hat{R}\left(\Omega^y + k_p\omega_{mes}(\hat{R}, \mathbf{b}.\mathbf{r})\right)_\times, \quad \hat{R}(0) = \hat{R}_0 \tag{41}$$

$$\omega_{mes}(\hat{R}, \mathbf{b}, \mathbf{r}) \coloneqq \sum_{i=1}^{n} k_i\left[b_i \times (\hat{R}^\top r_i)\right], \quad k_i > 0 \text{ for } i = 1, ..., n \tag{42}$$

where $\mathbf{b} \coloneqq \{b_i\}_{i=1}^n$, $\mathbf{r} \coloneqq \{r_i\}_{i=1}^n$ *and let* $t \mapsto \hat{R}(t)$ *denote the solution to* (41)*. Consider the left invariant estimation error* $\tilde{R} \coloneqq \hat{R}^T R$*. Assume that* $\Omega(t)$ *is a bounded, absolutely continuous function and the pair* $(\Omega(t), \tilde{R}(t))$ *are asymptotically independent. Then:*

1. *There are three unstable equilibria of the filter characterized by*

$$\hat{R}_{*i} = U_0 D_i U_0^\top R, \quad i = 1, 2, 3$$

where $D_i = \mathbf{diag}(1, -1, -1)$*,* $D_2 = \mathbf{diag}(-1, 1, -1)$ *and* $D_3 = \mathbf{diag}(-1, -1, 1)$*, and* $U_0 \in \mathbb{SO}(3)$ *such that* $M_0 = U_0 \Lambda U_0^\top$ *(where* M_0 *is defined in Assumption 3) and* $\Lambda = \mathbf{diag}(\lambda_1, \lambda_2, \lambda_3)$ *is the diagonal matrix of distinct eigenvalues of* M_0*.*

2. *The set* $\{I\}$ *is locally exponentially stable for* $\tilde{R}$ *dynamics.*

3. *For almost all initial conditions* $\tilde{R}_0 \neq \hat{R}_{*i}^\top R(0), i = 1, 2, 3$*,* $\hat{R}(t)$ *converges to* $R(t)$*, and for each positive* k *such that* $0 < k < 1 - \xi$ *where* $\xi \coloneqq \max_{u\in\mathbb{S}^2}\sum_{i=1}^n k_i(u^T r_i)^2$ *and* $|\tilde{R}^y(0)|_I \leq k$*, every solution* $t \mapsto \tilde{R}(t)$ *to the observer error dynamics satisfies*

$$|\tilde{R}(t)|_I \leq e^{-k_p(1-\xi)(1-k)t}|\tilde{R}(0)|_I \quad \forall t \in dom\ \tilde{R}. \tag{43}$$

Proof. See [6, Theorem 5.1] for the proof of 1, 2. For the proof of 3, consider the Lyapunov function candidate $V(\tilde{R}) = |\tilde{R}|_I$. Note that this Lyapunov function candidate is different from the one used in [6, Theorem 5.1]. The derivative of V along the trajectory of $\tilde{R}$ gives

$$\begin{aligned}
\langle\langle \nabla_{\tilde{R}} V, \dot{\tilde{R}}\rangle\rangle &= -\frac{1}{4}\mathrm{tr}\left(\dot{\tilde{R}}\right) \\
&= \frac{k_p}{4}\mathrm{tr}\left(\omega_{mes}(\hat{R}, \mathbf{b}, \mathbf{r})_\times \mathbb{P}_a(\tilde{R})\right) \\
&= \frac{k_p}{4}\mathrm{tr}\left(\mathbb{P}_a(\tilde{R}M)\mathbb{P}_a(\tilde{R})\right)
\end{aligned}$$

where $M = R^\top M_0 R$.

$$\langle\langle \nabla_{\tilde{R}} V, \dot{\tilde{R}}\rangle\rangle = \frac{k_p}{8}\mathrm{tr}\left(\tilde{R}^2 M - M\right)$$

Now, with $\tilde{R} = \mathcal{R}(\tilde{\theta}, x)$ for some $\tilde{\theta} \in \mathbb{R}$ and $x \in \mathbb{S}^2$, using $\tilde{R}^2 = I + x_\times \sin\tilde{\theta} + x_\times^2(1-\cos\tilde{\theta})$ results in

$$\begin{aligned}
\langle\langle \nabla_{\tilde{R}} V, \dot{\tilde{R}}\rangle\rangle &= \frac{k_p}{4}|\tilde{R}^2|_I \mathrm{tr}\left(x_\times^2 R^\top M_0 R\right) \\
&= \frac{k_p}{4}\mathrm{tr}\left(w_\times^2 M_0\right)
\end{aligned}$$

where $w \coloneqq Rx$. Now using Lemma 5 results in the following.

$$\begin{aligned}\langle\langle \nabla_{\tilde{R}} V, \dot{\tilde{R}} \rangle\rangle &= -\frac{k_p}{4} |\tilde{R}^2|_I \left[1 - \sum_{i=1}^{n} k_i (w^\top r_i)^2\right] \\ &\leq -k_p |\tilde{R}|_I (1-k)(1-\xi) \\ \implies |\tilde{R}(t)|_I &\leq e^{-k_p(1-\xi)(1-k)t} |\tilde{R}(0)|_I \quad \forall t \in \text{dom } \tilde{R}.\end{aligned}$$

□

Remark 5. *Note that in Theorem 5, $U_0 D_i U_0^\top$ represents a rotation of 180° about the axis given by $U_0 e_i$ where e_i is the standard basis vector. Thus, the unstable equilibria of the explicit complementary filter are when the right invariant estimation error ($\bar{R} \coloneqq \hat{R} R^\top$) represents a 180° rotation about the axis of rotation given by either $U_0 e_1$, $U_0 e_2$ or $U_0 e_3$. This is unlike the passive complementary filter in Theorem 1 which is unstable for 180° observer error about any axis of rotation.*

5 Hybrid explicit complementary filter

In this section, we extend the explicit complementary filter in Theorem 5 to obtain a globally convergent observer. Next, similar to (11), we define a rotated observer below.

$$\dot{\hat{R}} = \hat{R}\left(\Omega^y + \overline{k_p}\omega_{mes}(\hat{R}, \mathbf{b}, R^{*\top}\mathbf{r})\right)_\times, \quad \hat{R}(0) = \hat{R}_0, \quad \overline{k_p} > 0 \tag{44a}$$

$$\omega_{mes}(\hat{R}, \mathbf{b}, R^{*\top}\mathbf{r}) \coloneqq \sum_{i=1}^{n} \overline{k_i}\left[b_i \times (\hat{R}^\top R^{*\top} r_i)\right], \quad \overline{k_i} > 0, \quad i = 1, ..., n \tag{44b}$$

where $\mathbf{b} \coloneqq \{b_i\}_{i=1}^n$ and $\mathbf{r} \coloneqq \{r_i\}_{i=1}^n$. It is straightforward from Theorem 5 to see that (44) ensures that R^* is a locally exponentially stable equilibrium point of the right invariant observer error ($\bar{R}$) dynamics. We now move towards deriving the hybrid explicit complementary filter.

Let $\xi = \max_{u \in \mathbb{S}^2} \sum_{i=1}^{n} k_i (u^\top r_i)^2$. Define positive constants c_0, c_1 such that $0 < c_1 < c_0 < 1 - \xi$. For the hybrid observer, we define the flow and jump sets below.

$$C_0 \coloneqq \{(\hat{R}, \mathbf{b}, \mathbf{r}) \in \mathbb{SO}(3) \times \mathbb{R}^{3\times n} \times \mathbb{R}^{3\times n} \mid \Psi(\hat{R}, \mathbf{b}, \mathbf{r}) \leq c_0\} \tag{45a}$$

$$D_0 \coloneqq \{(\hat{R}, \mathbf{b}, \mathbf{r}) \in \mathbb{SO}(3) \times \mathbb{R}^{3\times n} \times \mathbb{R}^{3\times n} \mid \Psi(\hat{R}, \mathbf{b}, \mathbf{r}) \geq c_0\} \tag{45b}$$

$$C_1 \coloneqq \{(\hat{R}, \mathbf{b}, \mathbf{r}) \in \mathbb{SO}(3) \times \mathbb{R}^{3\times n} \times \mathbb{R}^{3\times n} \mid \Psi(\hat{R}, \mathbf{b}, \mathbf{r}) \geq c_1\} \tag{45c}$$

$$D_1 \coloneqq \{(\hat{R}, \mathbf{b}, \mathbf{r}) \in \mathbb{SO}(3) \times \mathbb{R}^{3\times n} \times \mathbb{R}^{3\times n} \mid \Psi(\hat{R}, \mathbf{b}, \mathbf{r}) \leq c_1\} \tag{45d}$$

The state of the hybrid observer, denoted by $\hat{\mathcal{H}}_2$, is $q \in Q \coloneqq \{0, 1\}$, the input is $v \coloneqq (\hat{R}, \mathbf{b}, \mathbf{r}, \Omega^y) \in \mathbb{SO}(3) \times \mathbb{R}^{3\times n} \times \mathbb{R}^{3\times n} \times \mathbb{R}$ and the output $\zeta = f(q, v) \coloneqq (f_1(q, v), f_2(q, v)) \in \mathfrak{so}(3) \times Q$ is defined in (46e). Define $\gamma_i(v) \coloneqq (\Omega^y + k_i \omega_i)_\times$, $i = 1, 2$ where $\omega_0 \coloneqq \omega_{mes}(\hat{R}, \mathbf{b}, \mathbf{r})$, $\omega_1 \coloneqq \omega_{mes}(\hat{R}, \mathbf{b}, R^{*\top}\mathbf{r})$ and $k_0 \coloneqq k_p$, $k_1 \coloneqq \overline{k_p}$. The data for the hybrid observer $\hat{\mathcal{H}}_2 = (\hat{C}, \hat{F}, \hat{D}, \hat{G}, \zeta)$ is given as

$$\hat{C} = \bigcup_{q \in Q}\left(\hat{C}_q \times q\right), \quad \begin{cases}\hat{C}_0 \coloneqq C_0 \\ \hat{C}_1 \coloneqq C_1\end{cases} \tag{46a}$$

$$\hat{F}(q, v) = 0 \tag{46b}$$

$$\hat{D} = \bigcup_{q \in Q}\left(\hat{D}_q \times q\right), \quad \begin{cases}\hat{D}_0 \coloneqq D_0 \\ \hat{D}_1 \coloneqq D_1\end{cases} \tag{46c}$$

$$\hat{G}(q, v) = 1 - q \tag{46d}$$

$$\zeta = f(q, v) \coloneqq (\underbrace{q\gamma_1(v) + (1-q)\gamma_0(v)}_{f_1(q,v)}, \underbrace{\quad q \quad}_{f_2(q,v)}) \tag{46e}$$

Using the above hybrid observer, the estimates are updated as

$$\underbrace{\dot{\hat{R}} = \hat{R}f_1(q,v), \quad \dot{q}=0,}_{\text{during flows}} \quad \text{and} \quad \underbrace{\hat{R}^+ = \hat{R}, \quad q^+ = 1-q}_{\text{during jumps}} \tag{47}$$

Consider the hybrid system $\mathcal{H}_2 = (C, F, G, D)$, that represents the observer error, with the states $\eta := (\tilde{R}, q) \in \mathbb{SO}(3) \times Q$ and inputs $\rho := (\hat{R}, \mathbf{b}, \mathbf{r}, \Omega^y, \Omega) \in \mathbb{SO}(3) \times \mathbb{R}^{3\times n} \times \mathbb{R}^{3\times n} \times \mathbb{R}^3 \times \mathbb{R}^3$ has the following data.

$$C := \hat{C} \tag{48a}$$

$$F(\eta,\rho) := \begin{bmatrix} \tilde{R}\left(\tilde{R}^\top f_1(q,v)^\top \tilde{R} + \Omega_\times\right) \\ 0 \end{bmatrix} \tag{48b}$$

$$D := \hat{D} \tag{48c}$$

$$G(\eta,\rho) := \begin{bmatrix} \tilde{R} \\ 1-q \end{bmatrix} \tag{48d}$$

where $v = (\hat{R}, \mathbf{b}, \mathbf{r}, \Omega^y)$ and (48b) is obtained by substituting $\dot{\hat{R}}$ from (47) in $\dot{\tilde{R}} = \dot{\hat{R}}^\top R + \hat{R}^\top \dot{R}$.

Theorem 6 (Hybrid explicit complementary filter)**.** *Given the system* (5) *and consider the set* $\mathcal{A} := \{I\} \times \{0\} \in \mathbb{SO}(3) \times Q$. *Define* $\xi := \max_{u\in\mathbb{S}^2} \sum_{i=1}^n k_i(u^\top r_i)^2$. *Assume that* $\Omega(t)$ *is a bounded and absolutely continuous. If there exist constants* $k_p, \overline{k_p}, c_0, c_1 > 0$ *and* $R^* \in \mathbb{SO}(3)$ *such that*

(a) $0 < |R^*|_I < c_1 < c_0 < 1-\xi$,

(b) $v^\top \hat{R}(0,0)x \neq 0$ *where* x *and* v *are the axis of rotation of* $\tilde{R}^y(0,0)$ *and* R^* *respectively,*

then

(1) The hybrid observer error system $\mathcal{H}_2 = (C, F, D, G)$ *with data given in* (48) *satisfies the hybrid basic conditions.*

(2) Every maximal solution to $\mathcal{H}_2$ *from* $C \cup D$ *is complete.*

(3) Every maximal solution to $\mathcal{H}_2$ *exhibits no more than 2 jumps.*

(4) The set $\mathcal{A}$ *is globally exponentially stable for* $\mathcal{H}_2$.

Proof. The proof follows in the same manner as the proof of Theorem 2. The only difference is in the exponential bounds that we obtain. We now calculate just the bounds since the rest of it follows directly from Theorem 2. Due to the construction of the set C_0, $(\hat{R}(0,0), \mathbf{b}, \mathbf{r}) \in C_0 \implies \Psi(\hat{R}(0,0), \mathbf{b}, \mathbf{r}) \leq c_0$. Using 6, we have that $\Psi(\hat{R}, \mathbf{b}, \mathbf{r}) \leq c_0 \implies |\tilde{R}|_I \leq \frac{c_0}{1-\xi}$. Now, using (43), we obtain the exponential bound when $(\hat{R}(0,0), \mathbf{b}, \mathbf{r}, q(0,0)) \in C_0 \times \{0\}$ as below.

$$|\tilde{R}(t,0)|_I \leq e^{-k_p(1-\xi-c_0)t}|\tilde{R}(0,0)|_I \quad \forall t \in \text{dom } \tilde{R}. \tag{49}$$

Similarly, when $(\hat{R}(0,0), \mathbf{b}, \mathbf{r}, q(0,0)) \in C_1 \times \{1\}$ and using the fact that there exists d_0 such that $1 - |R^*|_I < d_0 < 1$, we have

$$|\tilde{R}_1(t,0)|_I \leq e^{-\overline{k_p}(1-\xi-d_0)t}|\tilde{R}_1(0,0)|_I \quad \forall t \in \text{dom } \tilde{R}_1 \tag{50}$$

Now, due to (a), there exist positive constants e_0, f_0 such that $e_0 < |R^*|_I < f_0 < c_1$. Using (18) and (19), we obtain the following.

$$\implies |\tilde{R}(t,0)|_I \leq \alpha e^{-\overline{k_p}(1-\xi-d_0)t}|\tilde{R}(0,0)|_I \quad \forall t \in \text{dom } \tilde{R} \tag{51}$$

where α is defined in (22). Since $V(\tilde{R}) = |\tilde{R}|_I$ is always decreasing, there exists $T > 0$ such that $(\hat{R}(0,0), \mathbf{b}, \mathbf{r}, q(0,0)) \in C_1 \times \{1\}$ which implies that $(\hat{R}(T,0), \mathbf{b}, \mathbf{r}, q(T,0)) \in D_1 \times \{1\}$, resulting in $(\hat{R}(T,1), \mathbf{b}, \mathbf{r}, q(T,1)) \in C_0 \times \{0\}$. Combining (50) and (51) results in

$$|\tilde{R}(t,0)|_I \leq \min\{1, \alpha e^{-\min\{\overline{k_p}(1-d_0), k_p(1-c_0)\}t} |\tilde{R}(0,0)|_I\} \quad \forall t \in \operatorname{dom} \tilde{R} \tag{52}$$

Following similar arguments as the proof of Theorem 2 completes the proof. $\square$

5.1 Robustness analysis of the hybrid explicit complementary filter

We consider measurement noise in gyro measurements Ω^y as well as the vectorial direction measurements $\{b_i\}_{i=1}^n$. Consider the following measurement model

$$\Omega^y = \Omega + \eta_\omega \qquad\qquad b_i = R^\top r_i + \eta_i, \quad i = 1, ..., n \tag{53}$$

with the bounds on the noise as given by $\overline{\eta_\omega} := \sup_{t\geq 0} \|\eta_\omega(t)\|$ and $\overline{\eta} := \max_i \{\sup_{t\geq 0} \|\eta_i(t)\|\}_{i=1}^n$.

Theorem 7. *Assuming the conditions (a)-(b) in Theorem 6 hold and* $\frac{\overline{\eta}\kappa}{4} + \frac{\overline{\eta_\omega}\kappa_\omega}{4} < 1$, *where* $\kappa := \frac{1}{1-\xi-d_0} + \frac{1}{1-\xi-c_0}$ *and* $\kappa_\omega := \frac{1}{\overline{k_p}(1-\xi-d_0)} + \frac{1}{k_p(1-\xi-c_0)}$. *Consider the set* $\mathcal{A}$ *defined in Theorem 6. Then, the hybrid observer defined in* (47) *renders the resulting observer error dynamics eventually input-to-state stable with respect to* $\mathcal{A}$ *to noisy gyroscope and direction measurements of the form presented in* (53).

Proof. Consider the case when $(\hat{R}(0,0), \mathbf{b}, \mathbf{r}, q(0,0)) \in C_0 \times \{0\}$. This implies that $\Psi(\hat{R}(0,0), \mathbf{b}, \mathbf{r}) \leq c_0$. It follows from Lemma 6 that $|\tilde{R}(0,0)|_I \leq \frac{c_0}{1-\xi}$. This case employs the observer in (41). Consider the Lyapunov function candidate $V(\tilde{R}) = |\tilde{R}|_I$. The derivative of the Lyapunov function candidate along the trajectories of $\tilde{R}$ is given below.

$$\begin{aligned}
\langle\langle \nabla_{\tilde{R}} V, \dot{\tilde{R}} \rangle\rangle &= -\frac{1}{4}\operatorname{tr}\left(\dot{\tilde{R}}\right) \qquad (54)\\
&= \frac{k_p}{4}\operatorname{tr}\left(\omega_{mes}(\hat{R}, \mathbf{b}, \mathbf{r})_\times \tilde{R}\right) + \frac{k_p}{4}\operatorname{tr}\left(\left[\sum_{i=1}^n k_i(\eta_i \times \hat{R}^\top r_i)\right]_\times \tilde{R}\right) + \frac{1}{4}\operatorname{tr}\left(\eta_{\omega\times} \tilde{R}\right)\\
&\leq \frac{k_p}{4}|\tilde{R}^2|_I \operatorname{tr}\left(v_\times^2 R^\top M_0 R\right) + \frac{k_p}{4}\overline{\eta} + \frac{1}{4}\overline{\eta_\omega} \qquad (55)
\end{aligned}$$

Following the proof of 43, we obtain the following.

$$\begin{aligned}
&\langle\langle \nabla_{\tilde{R}} V, \dot{\tilde{R}} \rangle\rangle \leq -k_p(1-\xi-c_0)|\tilde{R}|_I + \frac{k_p}{4}\overline{\eta} + \frac{1}{4}\overline{\eta_\omega}\\
&|\tilde{R}(t,0)|_I \leq e^{-k_p(1-\xi-c_0)t}|\tilde{R}(0,0)|_I + \frac{\overline{\eta}}{4(1-\xi-c_0)} + \frac{\overline{\eta_\omega}}{4k_p(1-\xi-c_0)} \quad \forall t \in \operatorname{dom}\tilde{R} \qquad (56)
\end{aligned}$$

Considering the other case when $(\hat{R}(0,0), \mathbf{b}, \mathbf{r}, q(0,0)) \in C_1 \times \{1\}$ results in the following bound.

$$|\tilde{R}_1(t,0)|_I \leq e^{-k_p(1-\xi-d_0)t}|\tilde{R}_1(0,0)|_I + \frac{\overline{\eta}}{4(1-\xi-d_0)} + \frac{\overline{\eta_\omega}}{4k_p(1-\xi-d_0)} \quad \forall t \in \{\bar{t} \in \operatorname{dom}\tilde{R}_1 \mid \bar{t} \leq T\} \tag{57}$$

where $T \geq 0$ is such that $(\hat{R}(0,0), \mathbf{b}, \mathbf{r}, q(0,0)) \in C_1 \times \{1\} \implies (\hat{R}(T,0), \mathbf{b}, \mathbf{r}, q(T,0)) \in D_1 \times \{1\}$. Performing a similar analysis as in Theorem 3, we obtain

$$|\tilde{R}(t,j)|_I \leq \min\left\{1, \alpha e^{-\min\{k_p(1-\xi-c_0), \overline{k_p}(1-\xi-d_0)\}t}|\tilde{R}(0,0)|_I + \frac{\overline{\eta}\kappa}{4} + \frac{\overline{\eta_\omega}\kappa_\omega}{4}\right\} \quad \forall t \in \operatorname{dom}\tilde{R} \tag{58}$$

where $\kappa := \frac{1}{1-\xi-d_0} + \frac{1}{1-\xi-c_0}$, $\kappa_\omega := \frac{1}{\overline{k_p}(1-\xi-d_0)} + \frac{1}{k_p(1-\xi-c_0)}$ and α is defined in (22). Thus, it is evident that the observer error dynamics are eventually ISS with respect to noise in gyroscope and direction measurements. This concludes the proof. $\square$

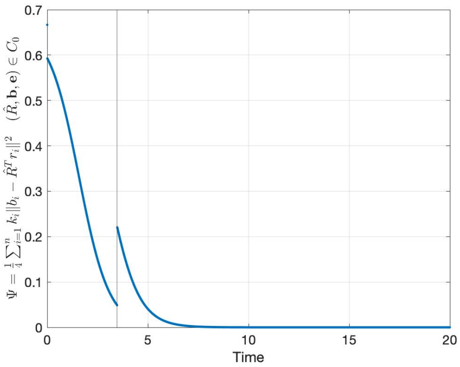

Figure 3: Plot of Ψ vs time. Jumps are seen at $t_{j1} = 0$ sec and $t_{j2} = 3.472$ sec. For t such that $t_{j1} \leq t \leq j_{j_2}$, $\Psi = \Psi(\hat{R}, \mathbf{b}, {R^*}^\top \mathbf{r})$, otherwise $\Psi = \Psi(\hat{R}, \mathbf{b}, \mathbf{r})$. This hybrid estimation error function is strictly decreasing during flows.

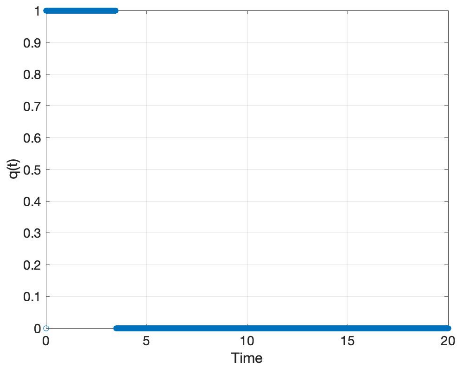

Figure 4: Plot of $q(t)$ vs time. Jumps are seen at $t_{j1} = 0$ sec and at $t_{j2} = 3.472$ sec. $q(0,0) = 0$, after which a jump occurs since $\Psi(\hat{R}, \mathbf{b}, \mathbf{r}) = 1 - \xi$.

5.2 Simulations for hybrid explicit complementary filter

We simulate the hybrid ECF with the following initial conditions:

$$R(0) = I, \qquad \hat{R}(0) = \mathcal{R}(\pi, [0,0,1]^\top), \qquad q(0) = 0,$$

with $n = 3$ and the three fixed inertial directions are taken to be $r_1 = [1,0,0]^\top, r_2 = [0,1,0]^\top, r_3 = [0,0,1]^\top$. The true angular velocity Ω is set as $\Omega(t) = [0,0,\sin t]$. Next, the gains and the switching parameters are set as follows.

$$k_p = 1, \qquad \overline{k_p} = 1, \qquad k_i = 1/3 \quad i = 1,2,3,$$
$$c_0 = \frac{1-\xi}{1.5}, \qquad c_1 = \frac{c_0}{2}.$$

Next, R^* is set such that $|R^*|_I = \frac{c_1}{2}$, with the axis of rotation of R^* given by $\hat{R}(0,0)x$ where $x \in \mathbb{S}^2$ is the axis of rotation of $\hat{R}(0,0)^\top \tilde{R}^y(0,0)$. The trajectory of $|\tilde{R}(t,j)|_I$ and $q(t,j)$ with time is shown in Figures 3 and 4 respectively. The states jump twice and there are no more jumps required as shown in Theorem 6. The error function goes to 0 which shows that the estimates converge to the true values of the rotation matrix.

6 ECF with biased angular velocity measurements

The measurement model used in (53) did not consider bias in the angular velocity measurements that is inadvertently introduced by the gyroscope. Here, we consider the bias $\mathfrak{b} \in \mathbb{R}^3$ in the angular velocity measurements, resulting in the following measurement model

$$\Omega^y = \Omega + \mathfrak{b} + \eta_\omega \qquad b_i = R^\top r_i + \eta_i, \quad i = 1, ..., n \tag{59}$$

where $\eta_\omega \in \mathbb{R}^3$ represents the noise in gyro measurements and $\eta_i, i = 1, ..., n$ represents the measurement noise in the vectorial direction measurements. Now, the bias is also estimated, and the estimate is denoted by $\hat{\mathfrak{b}}$. Following [6], the hybrid explicit complementary filter with bias estimation is given below. Define $\omega_0 := \omega_{mes}(\hat{R}, \mathbf{b}, \mathbf{r})$ and $\omega_1 := \omega_{mes}(\hat{R}, \mathbf{b}, {R^*}^\top \mathbf{r})$.

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