Hybrid Complementary Filters on $\mathbb{SO}(3)$

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The purpose of this work is to design a complementary filter for a system evolving on $\mathbb{SO}(3)$ with atleast globally asymptotically stable error dynamics. We first consider the nominal case when there is no measurement noise and develop globally asymptotic stability results. Then we introduce noise in the measurements of the angular velocity Ω and the rotation matrix R and establish robustness of the filter to said noise.

1 Preliminaries

The set of all rotation matrices is defined as $\mathbb{SO}(3) \coloneqq \{R \in \mathbb{R}^{3\times3} \mid R^\top R = RR^\top = I,\ \det R = 1\}$. The Lie algebra of $\mathbb{SO}(3)$ is defined as $\mathfrak{so}(3) \coloneqq \{X \in \mathbb{R}^{3\times3} \mid X + X^\top = 0\}$. The n-sphere, denoted by $\mathbb{S}^n$, is defined as $\mathbb{S}^n \coloneqq \{v \in \mathbb{R}^{n+1} \mid v^\top v = 1\}$. For any vector $v = (v_1, v_2, v_3)^\top \in \mathbb{R}^3$, we define the 'cross' map as

$$\cdot_\times : \mathbb{R}^3 \to \mathfrak{so}(3), \quad v \mapsto v_\times \coloneqq \begin{bmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{bmatrix} \tag{1}$$

The inverse map is defined by $\mathrm{vex} : \mathfrak{so}(3) \to \mathbb{R}^3$, $v_\times \mapsto \mathrm{vex}\left(v_\times\right) \coloneqq v$. For any matrix $A \in \mathbb{R}^{3\times3}$, its skew symmetric projection is given by $\mathbb{P}_a(A) \coloneqq (A - A^\top)/2$, and its symmetric projection is given by $\mathbb{P}_s(A) \coloneqq (A + A^\top)/2$. Any rotation matrix $R \in \mathbb{SO}(3)$ can be expressed as a rotation about an axis $v \in \mathbb{S}^2$ by an angle $\theta \in \mathbb{R}$. Consider the function $\mathcal{R} : \mathbb{R} \times \mathbb{S}^2 \to \mathbb{SO}(3)$, $(\theta, v) \mapsto \mathcal{R}(\theta, v) \coloneqq I + v_\times \sin\theta + v_\times^2(1 - \cos\theta)$ that represents a rotation about the axis v by an angle of θ in the counterclockwise direction. Next, we define the function $\mathrm{axis} : \mathbb{SO}(3) \to \mathbb{S}^2$, $\mathcal{R}(\theta, v) \mapsto \mathrm{axis}(\mathcal{R}(\theta, v)) \coloneqq v$ where $\theta \in [0, \pi]$.

The tangent space of $\mathbb{SO}(3)$ at $R \in \mathbb{SO}(3)$ is given by $\mathrm{T}_R\mathbb{SO}(3)$ and its dual is given by $\mathrm{T}^*_R\mathbb{SO}(3)$. The action of a dual element $w \in \mathrm{T}^*_R\mathbb{SO}(3)$ on $v \in \mathrm{T}_R\mathbb{SO}(3)$ is given by the duality pairing $\mathrm{T}^*_R\mathbb{SO}(3) \times \mathrm{T}_R\mathbb{SO}(3) \ni (w, v) \mapsto \langle\langle w, v\rangle\rangle \coloneqq \mathrm{tr}\left(A^\top B\right) \in \mathbb{R}^1$. The differential of a scalar valued function $f : \mathbb{SO}(3) \to \mathbb{R}$ is denoted by $\nabla_R f \in \mathrm{T}^*_R\mathbb{SO}(3)$. For any $A, B \in \mathbb{R}^{m\times n}$, the Frobenius inner product on the space of real-valued matrices is given by $\langle\langle A, B\rangle\rangle_F \coloneqq \mathrm{tr}\left(A^\top B\right)$. The closure of a set $\mathcal{X}$ is denoted by $\overline{\mathcal{X}}$.

Definition 1 (Distance metric on $\mathbb{SO}(3)$). *The distance between two points $x, y \in \mathbb{SO}(3)$ is given by a function $d : \mathbb{SO}(3) \times \mathbb{SO}(3) \to \mathbb{R}$, $(x, y) \mapsto d(x, y)$, where the following hold:*

- *$d(x, y) > 0$ if $x \neq y$;*
- *$d(x, y) = 0 \Leftrightarrow x = y$;*
- *$d(x, y) = d(y, x) \quad \forall x, y \in \mathbb{SO}(3)$;*
- *$d(x, y) \leq d(x, z) + d(y, z) \quad \forall x, y, z \in \mathbb{SO}(3)$.*

[1] Elements of $\mathrm{T}_R\mathbb{SO}(3)$ and $\mathrm{T}^*_R\mathbb{SO}(3)$ are matrices. Double angles are used for the duality pairing to avoid confusion with the inner product notation.

Definition 2 (Distance of a point from a set). *The distance of a point $x \in \mathbb{SO}(3)$ from the closed set $\mathcal{A} \subset \mathbb{SO}(3)$, denoted as $|x|_{\mathcal{A}}$, is defined as*

$$|x|_{\mathcal{A}} := \inf_{y \in \mathcal{A}} d(x, y)$$

Definition 3 (Neighbourhood of a point). *The open ϵ-neighbourhood of a point $R \in \mathbb{SO}(3)$ for $\epsilon > 0$, denoted by $\mathcal{N}_\epsilon(x)$, is defined as*

$$\mathcal{N}_\epsilon(R) := \{X \in \mathbb{SO}(3) \mid d(X, R) < \epsilon\}$$

Definition 4 (Neighbourhood of a set). *For $0 < \epsilon \leq 1$, the open ϵ-neighbourhood of a set $U \subset \mathbb{SO}(3)$ is given by*

$$\mathcal{N}_\epsilon(U) := \{X \in \mathbb{SO}(3) \mid d(X, R) < \epsilon, \;\; R \in U\}$$

Proposition 1 (Baker-Campbell-Hausdorff formula). *For $X, Y \in \mathfrak{so}(3)$, there exists $Z \in \mathfrak{so}(3)$ such that $\exp Z = \exp X \exp Y$ and*

$$Z = X + Y + \frac{1}{2}[X, Y] + \frac{1}{12}\left([X, [X, Y]] + [Y, [X, Y]]\right) - \frac{1}{24}[Y, [X, [X, Y]]] + \ldots$$

where $[X, Y] := XY - YX$.

Remark 1. *When $XY = YX$, $\exp(X) \exp(Y) = \exp(X + Y)$.*

Remark 2. *For $R_1, R_2 \in \mathbb{SO}(3)$, if $R_2 = \mathcal{R}(\theta, v)$ for some $\theta \in \mathbb{R}$ and $v \in \mathbb{S}^2$, then $R_1 R_2 R_1^\top = \mathcal{R}(\theta, R_1 v)$.*

Definition 5 (Distance between two points). *The distance between two points $R_1, R_2 \in \mathbb{SO}(3)$ is given by $|R_1|_{R_2}$, where*

$$|R_1|_{R_2}{}^2 := \frac{1}{4}\left(I - R_2^\top R_1\right) \quad \forall R_1, R_2 \in \mathbb{SO}(3).$$

Definition 6 (Configuration error function [2]). *The configuration error function for $\mathbb{SO}(3)$ is defined by $|\cdot|_I : \mathbb{SO}(3) \to \mathbb{R}$, where*

$$|R|_I{}^2 := \frac{1}{4} tr\left(I - R\right) \quad \forall R \in \mathbb{SO}(3).$$

Definition 7 (Nondegenerate function). *A function $f : \mathbb{R}^n \times \mathbb{R}^n \supset A \times B \to \mathbb{R}$ is said to be nondegenerate if the Hessian of f is full ranked.*

Definition 8 (Asymptotically Independent Signals [6]). *Two signals $x : \mathbb{R} \to M_x$ and $y : \mathbb{R} \to M_y$ are* asymptotically dependent *if there exists[3] a nondegenerate function $f : M_x \times M_y \to \mathbb{R}$ and a time $T > 0$ such that*

$$f(x(t), y(t)) = 0 \quad \forall t > T$$

The two signals are asymptotically independent if they are not asymptotically dependent.

Lemma 1. *For any rotation matrix $R \in \mathbb{SO}(3)$, the following holds.*

$$|R^2|_I{}^2 = 4|R|_I{}^2\left(1 - |R|_I{}^2\right) \tag{2a}$$

$$\|vex\left(\mathbb{P}_a(R)\right)\|^2 = |R^2|_I{}^2 \tag{2b}$$

[2] I have changed the definition of $|R|_I$ as it is now a distance metric. See [4].

[3] Existence of such a function is not trivial as it has to be nondegenerate while satisfying $f(x(t), y(t)) = 0$.

Proof. For any $R \in \mathbb{SO}(3)$, there exists $\theta \in [0, \pi]$ and $v \in \mathbb{S}^2$ such that $R = \mathcal{R}(\theta, v) = \exp(\theta v_\times)$. Now, $R^2 = RR = \exp(\theta v_\times)\exp(\theta v_\times)$. Using Remark 1, we have $R^2 = \exp(2\theta v_\times)$. Next, using Definition 6 and the fact that $\mathrm{tr}\,(R) = 1 + 2\cos\theta$, we get

$$|R|_I{}^2 = (1 - \cos\theta)/2 \implies \cos\theta = 1 - 2|R|_I{}^2. \tag{3}$$

Since $\theta \in [0, \pi]$, this leads to $\sin\theta = \sqrt{1 - (\cos\theta)^2} = 2\sqrt{|R|_I{}^2\left(1 - |R|_I{}^2\right)}$. Now using (3) results in

$$\begin{aligned} |R^2|_I{}^2 &= \frac{1}{2}(1 - \cos 2\theta) \\ &= \sin^2\theta \\ &= 4|R|_I{}^2\left(1 - |R|_I{}^2\right) \end{aligned}$$

This proves (2a). To prove (2b), we expand the LHS as follows.

$$\begin{aligned} \|\mathrm{vex}\,(\mathbb{P}_a(R))\,\|^2 &= \langle \mathrm{vex}\,(\mathbb{P}_a(R))\,, \mathrm{vex}\,(\mathbb{P}_a(R))\rangle \\ &= \frac{1}{2}\langle\langle \mathbb{P}_a(R), \mathbb{P}_a(R)\rangle\rangle_F \\ &= \frac{1}{2}\mathrm{tr}\left(\left(\frac{R - R^\top}{2}\right)^\top\left(\frac{R - R^\top}{2}\right)\right) \\ &= \frac{1}{4}\mathrm{tr}\left(I - R^2\right) \\ &= |R^2|_I{}^2 \end{aligned}$$

This proves (2b). $\square$

Lemma 2. *For any $X, Y \in \mathbb{SO}(3)$ with $X = \mathcal{R}(x, v_x)$ and $Y = \mathcal{R}(y, v_y)$ where $x, y \in [0, \pi]$ and $v_x, v_y \in \mathbb{S}^2$, the following holds:*

$$|XY|_I{}^2 = |X|_I{}^2 + |Y|_I{}^2 - |X|_I{}^2|Y|_I{}^2 + (v_x^\top v_y)|X|_I|Y|_I\left[2\sqrt{\left(1 - |X|_I{}^2\right)\left(1 - |Y|_I{}^2\right)} - v_x^\top v_y|X|_I|Y|_I\right] \tag{4}$$

Proof. Expanding $XY = \mathcal{R}(x, v_x)\mathcal{R}(y, v_y)$ we get the following.

$$XY = \left(I + v_{x\times}\sin x + v_{x\times}^2(1 - \cos x)\right)\left(I + v_{y\times}\sin y + v_{y\times}^2(1 - \cos y)\right)$$

Taking the trace on both sides results in

$$\mathrm{tr}\,(XY) = \mathrm{tr}\,(X) + (\sin x \sin y)\mathrm{tr}\left(v_{x\times}v_{y\times}\right) + (1 - \cos y)\mathrm{tr}\left(v_{y\times}^2\right) + (1 - \cos x)(1 - \cos y)\mathrm{tr}\left(v_{x\times}^2 v_{y\times}^2\right)$$

It is easy to verify that for any $v, w \in \mathbb{S}^2$, $\mathrm{tr}\,(v_\times w_\times) = -2v^\top w$ and $\mathrm{tr}\left(v_\times^2 w_\times^2\right) = 1 + (v^\top w)^2$. Using this, we obtain the following:

$$\mathrm{tr}\,(XY) = \mathrm{tr}\,(X) - 2(v^\top w)\sin x \sin y - 2(1 - \cos y) + (1 - \cos x)(1 - \cos y)(1 + (v^\top w)^2)$$

Now using (3), we have $\cos x = 1 - 2|X|_I{}^2$ and $\sin x = 4\sqrt{|X|_I{}^2(1 - |X|_I{}^2)}$ for all $X \in \mathbb{SO}(3)$. This leads to the following equation:

$$\mathrm{tr}\,(XY) = \mathrm{tr}\,(X) - 8(v^\top w)\sqrt{|X|_I{}^2(1 - |X|_I{}^2)}\sqrt{|Y|_I{}^2(1 - |Y|_I{}^2)} - 4|Y|_I{}^2 + 4|X|_I{}^2|Y|_I{}^2(1 + (v^\top w)^2)$$

Using Definition 6 results in

$$\implies |XY|_I{}^2 = |X|_I{}^2 + |Y|_I{}^2 - |X|_I{}^2|Y|_I{}^2 + (v_x^\top v_y)|X|_I|Y|_I\left[2\sqrt{\left(1 - |X|_I{}^2\right)\left(1 - |Y|_I{}^2\right)} - v_x^\top v_y|X|_I|Y|_I\right]$$

This completes the proof. $\square$

1.1 Attitude kinematics

The rotation kinematics of a rigid body on $\mathbb{SO}(3)$ are given by

$$\dot{R} = R\Omega_{\times} \tag{5}$$

where $\Omega \in \mathbb{R}^3$ represents the angular velocity in the body fixed frame.

Next, we define local exponential stability and unstable set for a closed-loop system on $\mathbb{SO}(3)$ given as

$$\dot{R} = f(R, \Omega) \tag{6}$$

for some $f : \mathbb{SO}(3) \times \mathbb{R}^3 \to \mathrm{T}_R\mathbb{SO}(3)$. It is important to note here that for the purpose of observer design, $t \mapsto \Omega(t)$ is not a control input and it cannot be altered, and hence it can be considered as a time-varying parameter.

Definition 9 (Local exponential stability)**.** *For an absolutely continuous $t \mapsto \Omega(t)$, the point $R^* \in \mathbb{SO}(3)$ is locally exponentially stable for* (6) *if there exist positive constants λ, k, δ such that every solution $t \mapsto R(t)$ to* (6) *satisfies*

$$|R(0)|_{R^*} \leq \delta \implies |R(t)|_{R^*} \leq ke^{-\lambda t}|R(0)|_{R^*} \quad \forall t \in dom\ R.$$

Definition 10 (Unstable set of a dynamical system)**.** *For an absolutely continuous $t \mapsto \Omega(t)$, a set $U \subset \mathbb{SO}(3)$ is unstable for* (6) *if for any ϵ such that $0 < \epsilon < 1$, every solution $t \mapsto R(t)$ to* (6) *with $R(0) \in \mathcal{N}_\epsilon(U) \backslash U$ satisfies $R(t) \notin U$ for all $t \in dom\ R$.*

2 Passive Complementary Fitler

One of the standard tools for estimation on the special orthogonal groups is the nonlinear complementary filter [6]. The measurements of the angular velocity Ω and the rotation matrix R are given as Ω^y and R^y respectively. As in [6], the following assumption is made.

Assumption 1. *The measurements are assumed noise-free and angular velocity measurements are assumed bias-free, i.e., $\Omega^y = \Omega$ and $R^y = R$.*

Now, we re-state below [6, Theorem 4.2] with the simplification of considering angular velocity measurements bias-free as in Assumption 1.

Theorem 1 (Passive Complementary Filter [6, Theorem 4.2])**.** *Consider the state $R \in \mathbb{SO}(3)$ that follows the kinematics in* (5)*. The measurements satisfy Assumption 1. Let the estimate of R be denoted by $\hat{R} \in \mathbb{SO}(3)$ and the estimation error be defined as $\tilde{R} := \hat{R}^\top R$. Let $k_p > 0$ be a positive gain value. With the initial estimate being $\hat{R}(0) = \hat{R}_0 \in \mathbb{SO}(3)$, let the estimates satisfy the following kinematics:*

$$\dot{\hat{R}} = \hat{R}\left(\Omega^y + k_p\omega(\tilde{R}^y)\right)_\times, \quad \omega(\tilde{R}^y) = vex\left(\mathbb{P}_a(\tilde{R}^y)\right) \quad where\ \tilde{R}^y := \hat{R}^\top R^y. \tag{7}$$

Assume that $t \mapsto \Omega(t) \in \mathbb{R}^3$ is a bounded, absolutely continuous signal and that the pair of signals $t \mapsto (\Omega(t), \tilde{R}(t))$ are asymptotically independent. Define $\mathbb{U}_0 \subseteq \mathbb{SO}(3)$ as

$$\mathbb{U}_0 = \left\{\tilde{R} \in \mathbb{SO}(3) \mid tr(\tilde{R}) = -1\right\}.$$

Then:

(a) The set $\mathbb{U}_0$ is forward invariant and unstable with respect to the system given in (7)*.*

(b) The identity element I is locally exponentially stable for the dynamics of the error $\tilde{R}$.

(c) *For all initial conditions such that $\hat{R}_0^\top R^y(0) \notin \mathbb{U}_0$, $\hat{R}(t)$ converges to $R(t)$ and, for each positive k such that $0 < k < 1$ and $|\tilde{R}^y(0)|_I \leq k$, every solution $t \mapsto \tilde{R}(t)$ to the error dynamics satisfies*

$$|\tilde{R}(t)|_I \leq e^{-2k_p(1-k^2)t}|\tilde{R}(0)|_I \quad \forall t \geq 0. \tag{8}$$

Proof. See [6, Theorem 4.2] for the proof of (a), (b). Next, $\tilde{R}^y(t) = \tilde{R}(t)$ for all $t \geq 0$ due to Assumption 1. For the proof of (c), consider a Lyapunov function candidate candidate $V(\tilde{R}) = |\tilde{R}|_I{}^2$ which is positive definite as shown in Lemma 3. Now, computing its derivative along the trajectories of $\tilde{R}$, we get

$$\langle\langle \nabla_{\tilde{R}} V, \dot{\tilde{R}} \rangle\rangle = 2|\tilde{R}|_I \frac{d|\tilde{R}(t)|_I}{dt} = -\frac{1}{4}\mathrm{tr}\left(\dot{\tilde{R}}(t)\right).$$

Now, since $\dot{\tilde{R}} = [R, \Omega_\times] - k_p \omega(\tilde{R})_\times \tilde{R}$ and using the fact that the trace of the Lie bracket is 0, we get the following.

$$\begin{aligned}
\langle\langle \nabla_{\tilde{R}} V, \dot{\tilde{R}} \rangle\rangle &= \frac{k_p}{4}\mathrm{tr}\left(\omega(\tilde{R})_\times \tilde{R}\right) \\
&= \frac{k_p}{4}\mathrm{tr}\left(\omega(\tilde{R})_\times \mathbb{P}_a(\tilde{R})\right) \\
&= \frac{k_p}{4}\mathrm{tr}\left(\omega(\tilde{R})_\times \omega(\tilde{R})_\times\right) \\
&= -\frac{k_p}{4}\langle\langle \omega(\tilde{R})_\times, \omega(\tilde{R})_\times \rangle\rangle_F \\
&= -\frac{k_p}{2}\|\omega(\tilde{R})\|^2.
\end{aligned}$$

Now using Lemma 1, we have the following:

$$\begin{aligned}
\langle\langle \nabla_{\tilde{R}} V, \dot{\tilde{R}} \rangle\rangle &= -2k_p|\tilde{R}|_I{}^2(1 - |\tilde{R}|_I{}^2) \\
2|\tilde{R}|_I \frac{d|\tilde{R}(t)|_I}{dt} &\leq -2k_p(1-k^2)|\tilde{R}|_I \\
\implies |\tilde{R}(t)|_I &\leq e^{-k_p(1-k^2)t}|\tilde{R}(0)|_I \quad \forall t \geq 0.
\end{aligned}$$

$\square$

Lemma 3 ([2, Lemma 5.6])**.** *The configuration error function $|\tilde{R}|_I$ defined in Definition 6 is positive definite, and has 4 critical points given by $\Theta = \{I, \exp(\pi e_{1\times}), \exp(\pi e_{2\times}), \exp(\pi e_{3\times})\}$ where $e_1, e_2, e_3 \in \mathbb{S}^2$ are the eigenvectors of I. Note that $\Theta \backslash \{I\} \subset \mathbb{U}_0$.*

3 Hybrid passive complementary filter

Since $\mathbb{SO}(3)$ is a compact Lie group, it follows that the Lyapunov function V considered in Theorem 1 has compact sublevel sets. Thus, using Theorem 1, we have that the error dynamics corresponding to the passive complementary filter are almost globally asymptotically and locally exponentially stable, with the unstable set being the measure zero set of $\mathbb{SO}(3)$ that represents a 180° rotation, i.e., the set $\mathbb{U}_0$. We leverage this property by using hybrid control theory to obtain a globally asymptotically stable observer on $\mathbb{SO}(3)$.

Definition 11 (Hybrid Observer)**.** *A hybrid observer $\hat{\mathcal{H}}$ with data $(\hat{C}, F, \hat{D}, G, \zeta)$ is given by*

$$\hat{\mathcal{H}} : \begin{cases} \dot{z} &= F(z, v), \quad (z, v) \in \hat{C} \\ z^+ &= G(z, v), \quad (z, v) \in \hat{D} \\ \zeta &= f(z, v) \end{cases} \tag{9}$$

where z is the state of the hybrid observer, v is the input and ζ is the output, $\hat{C}$ and $\hat{D}$ represent the flow set and the jump set for the hybrid observer respectively.

Due to Assumption 1, we have $R^y = R$. Our design approach for the globally asymptotically convergent observer is the following: when $\tilde{R}^y = \tilde{R}$ is *close enough* to I, the observer is defined by (7). When $\tilde{R}^y$ is *far enough* from I, a different passive complementary filter is defined such that $\tilde{R}$ tracks a constant $R^* \in \mathbb{SO}(3)$ (which is *close* to identity) which brings $\tilde{R}$ in a *small enough* neighbourhood of I so that the observer (7) can be used. Thus, what we desire is the following. For a fixed $R^* \in \mathbb{SO}(3)$:

$$R_1 := R^* R, \qquad R_1^y := R^* R^y \tag{10a}$$

$$\tilde{R}_1 := \hat{R}^\top R_1^y = \hat{R}^\top R^* R^y \tag{10b}$$

With $\overline{k_p} > 0$, we define the following observer for the case when $\tilde{R}^y$ is *far enough* from I.

$$\dot{\hat{R}} = \hat{R}\left(\Omega^y + \overline{k_p}\omega(\tilde{R}_1^y)\right)_\times, \quad \hat{R}(0) = \hat{R}_0 \tag{11a}$$

$$\omega(\tilde{R}_1^y) := \mathrm{vex}\left(\mathbb{P}_a(\tilde{R}_1^y)\right), \qquad \tilde{R}_1^y := \hat{R}^\top R_1^y \tag{11b}$$

Using Theorem 1, it is straightforward to see that $I \in \mathbb{SO}(3)$ is almost globally and locally exponentially stable for the dynamics of $\tilde{R}_1$. We employ a hysteresis based hybrid observer for reasons mentioned in Remark 4. Consider constants $0 < c_1 < c_0 < 1$, $0 < \delta < 1$ and $\theta^* \in \mathbb{R}$ with $|R^*|_I := (1 - \cos\theta^*)/2$. Consider the following sets

$$C_0 := \left\{\tilde{R}^y \in \mathbb{SO}(3) : |\tilde{R}^y|_I \le c_0\right\} \tag{12a}$$

$$C_1 := \left\{\tilde{R}^y \in \mathbb{SO}(3) : |\tilde{R}^y|_I \ge c_1\right\} \tag{12b}$$

$$D_0 := \left\{\tilde{R}^y \in \mathbb{SO}(3) : |\tilde{R}^y|_I \ge c_0\right\} \tag{12c}$$

$$D_1 := \left\{\tilde{R}^y \in \mathbb{SO}(3) : |\tilde{R}^y|_I \le c_1\right\} \tag{12d}$$

$$\mathcal{B} := \left\{(r, \tilde{R}^y, \hat{R}) \in \mathbb{S}^2 \times (\mathbb{SO}(3))^2 : \left| |\tilde{R}^y|_I{}^2 - \frac{1 - |R^*|_I{}^2}{1 - |R^*|_I{}^2 + \left(r^\top \hat{R}\mathrm{axis}(\tilde{R}^y)\right)^2 |R^*|_I{}^2} \right| > \delta \right\} \cap \mathbb{S}^2. \tag{12e}$$

The intersection $\Gamma := C_0 \cap C_1$ represents the hysteresis region on $\mathbb{SO}(3)$. The hybrid observer, denoted by $\hat{\mathcal{H}}_{PCF}$, has the state $x := (\hat{R}, q, r) \in \mathbb{SO}(3) \times Q \times \mathbb{S}^2$ where $Q := \{0, 1\}$, the input $v := (R^y, \Omega^y) \in \mathcal{V} := \mathbb{SO}(3) \times \mathbb{R}^3$ and the output $\zeta := \hat{R}$. Note that $r \in \mathbb{S}^2$ denotes the axis of rotation of R^*. Define $\gamma_i(x, v) := (\Omega^y + k_i\omega_i)_\times$, $i = 0, 1$ where $\omega_0 := \omega(\tilde{R}^y)$, $\omega_1 := \omega(\tilde{R}_1^y)$ and $k_0 := k_p$, $k_1 := \overline{k_p}$. The data of the hybrid observer $\mathcal{H}_{PCF} = (\hat{C}, \hat{F}, \hat{D}, \hat{G}, \zeta)$ is now given in (13).

$$\hat{C} = \bigcup_{q \in Q}\left(C_q \times q \times \hat{A}_q\right), \quad \begin{cases} \hat{A}_0 := \mathbb{S}^2 \\ \hat{A}_1 := \overline{\mathcal{B}} \end{cases} \tag{13a}$$

$$\hat{F}(x, v) = \left(\hat{R}h(x, v),\ 0,\ 0\right) \tag{13b}$$

$$\hat{D} = (C_1 \times \{1\} \times (\mathbb{S}^2 \backslash \mathcal{B})) \cup \left(\bigcup_{q \in Q}(D_q \times q \times \mathbb{S}^2)\right) \tag{13c}$$

$$\hat{G}(x, v) = \begin{cases} \left(\hat{R},\ q,\ \underset{u \in \mathbb{S}^2}{\mathrm{argmax}}\left((u^\top y)^2 - (r^\top y)^2\right)^2\right), & (\tilde{R}^y, q, r) \in C_1 \times \{1\} \times (\mathbb{S}^2 \backslash \mathcal{B}) \\ (\hat{R},\ 1 - q,\ r), & (\tilde{R}^y, q, r) \in \bigcup_{q \in Q}\left(D_q \times q \times \mathbb{S}^2\right) \end{cases} \tag{13d}$$

$$\zeta = f(x, v) := \hat{R} \tag{13e}$$

where $h(x,v) := (q\gamma_1(x,v) + (1-q)\gamma_0(x,v))$. Note that by definition of the flow (resp. jump) set $\hat{C}$ (resp. $\hat{D}$) are with respect to the measurement error $\tilde{R}^y$.

Consider the observer error dynamics, represented by the hybrid system $\tilde{\mathcal{H}}_{PCF} = (\tilde{C}, \tilde{F}, \tilde{G}, \tilde{D})$, with the states $z := (\tilde{R}, q, r) \in \mathbb{SO}(3) \times Q \times \mathbb{S}^2$ and inputs[4] $\rho := (\hat{R}, \Omega) \in \mathbb{SO}(3) \times \mathbb{R}^3$ has the following data

$$\tilde{C} = \hat{C} \tag{14a}$$

$$\tilde{F}(z,\rho) = \left(\tilde{R}\left(-\tilde{R}^\top h(x,v)\tilde{R} + \Omega_\times\right),\ 0,\ 0\right) \tag{14b}$$

$$\tilde{D} = \hat{D} \tag{14c}$$

$$\tilde{G}(z,\rho) = \begin{cases} \left(\tilde{R},\ q,\ \underset{u\in\mathbb{S}^2}{\operatorname{argmax}}\left((u^\top y)^2 - (r^\top y)^2\right)^2\right), & (\tilde{R}^y, q, r) \in C_1 \times \{1\} \times (\mathbb{S}^2 \backslash \mathcal{B}) \\ (\tilde{R},\ 1-q,\ r), & (\tilde{R}^y, q, r) \in \bigcup_{q\in Q}\left(D_q \times q \times \mathbb{S}^2\right) \end{cases} \tag{14d}$$

where (14b) is obtained by substituting $\dot{\hat{R}}$ from (13b) in $\dot{\tilde{R}} = \dot{\hat{R}}^\top R + \hat{R}^\top \dot{R}$. It is important to note that $\tilde{R}$ does not change after jumps.

Theorem 2 (Hybrid passive complementary filter)**.** *Let $\mathcal{A} := \{I\} \times \{0\} \times \mathbb{S}^2 \subset \mathbb{SO}(3) \times Q \times \mathbb{S}^2$. Assume that Assumption 1 holds and $\Omega(t)$ is a bounded and absolutely continuous signal. If there exist positive constants $\underline{k_p}$, $\overline{k_p}$, c_0, c_1, δ and $\theta^* \in \mathbb{R}$ such that*

(a) $0 < \frac{1-\cos\theta^*}{2} < c_1 < c_0 < 1$,

(b) $1 - \sqrt{\frac{1}{c_1} - 1} > \frac{1-\cos\theta^*}{2}$,

then

(1) The hybrid observer error system $\tilde{\mathcal{H}}_{PCF} = (\tilde{C}, \tilde{F}, \tilde{D}, \tilde{G})$ with data (14) satisfies the hybrid basic conditions.

(2) Every maximal solution to $\tilde{\mathcal{H}}_{PCF}$ from $\tilde{C} \cup \tilde{D}$ is complete.

(3) Every maximal solution to $\tilde{\mathcal{H}}_{PCF}$ exhibits no more than 3 jumps.

(4) The set $\mathcal{A}$ is globally exponentially stable for $\tilde{\mathcal{H}}_{PCF}$.

Proof. Since $\tilde{C}, \tilde{D}$ are closed subsets of $\mathbb{SO}(3) \times Q \times \mathbb{S}^2$ and the maps $\tilde{F}$ and $\tilde{G}$ are single valued and continuous on $\tilde{C}$ and $\tilde{D}$ respectively, (14) satisfies the hybrid basic conditions. This proves (1). Next, [3, Proposition 6.10] completes the proof of (2).

We prove the globally exponentially stability of $\mathcal{A}$ to (14) by considering all five cases where the state z of $\tilde{\mathcal{H}}_{PCF}$ can lie.

Case 1: $(\tilde{R}^y(0,0), q(0,0), r(0,0)) \in C_0 \times \{0\} \times \mathbb{S}^2$. The local exponential stability for this case is evident using (8) as follows:

$$|\tilde{R}(t,0)|_I \leq e^{-k_p(1-c_0^2)t}|\tilde{R}(0,0)|_I \quad \forall (t,j) \in \operatorname{dom} \tilde{R}. \tag{15}$$

Case 2: $(\tilde{R}^y(0,0), q(0,0), r(0,0)) \in D_1 \times \{1\} \times \mathbb{S}^2$. Since the initialization is in the jump set, the state jumps once which results in $(\tilde{R}(0,1), q(0,1), r(0,1)) \in C_0 \times \{0\} \times \mathbb{S}^2$. Now we refer to Case 1 for further analysis.

Case 3: $(\tilde{R}^y(0,0), q(0,0), r(0,0)) \in D_0 \times \{0\} \times \mathbb{S}^2$. Similar to Case 2, the initialization is in the jump set resulting in one jump and $(\tilde{R}(0,1), q(0,1), r(0,1)) \in C_1 \times \{1\} \times \mathbb{S}^2$. We refer to Case 4 and Case 5 for further analysis.

The only remaining cases to be considered are when $(\tilde{R}^y(0,0), q(0,0), r(0,0)) \in C_1 \times \{1\} \times (\overline{\mathcal{B}} \cup (\mathbb{S}^2 \backslash \mathcal{B}))$. When flow occurs, the rotated observer in (11) is used with $R^* := \mathcal{R}(\theta^*, r(0,0))$. Using

[4] $\hat{R}$ is also an input because the term 'y' in the jump map depends on $\hat{R}$.

Theorem 1, the rotated observer fails when the observer error $\tilde{R}_1$ is such that $|\tilde{R}_1|_I = 1$. Using Lemma 2, we see that

$$|\tilde{R}_1|_I = 1 \iff |\tilde{R}|_I^{\,2} = \frac{1 - |R^*|_I^{\,2}}{1 - |R^*|_I^{\,2} + \left(\mathrm{axis}(R^*)^\top \hat{R}\mathrm{axis}(\tilde{R})\right)^2 |R^*|_I^{\,2}} \tag{16}$$

It is now clear that when $\mathrm{axis}(R^*)$ is *far enough* from its the value where (16) holds, $|\tilde{R}_1|_I \neq 1$. This motivates the definition of the set $\mathcal{B}$ and results in the following two cases.

Case 4: $(\tilde{R}^y(0,0), q(0,0), r(0,0)) \in C_1 \times \{1\} \times (\mathbb{S}^2 \backslash \mathcal{B})$. This is the case when $r(0,0)$ is close to its value that satisfies (16). The jump map results in $(\tilde{R}(0,1), q(0,1), r(0,1)) \in C_1 \times \{1\} \times \overline{\mathcal{B}}$ which leads to Case 5.

Case 5: $(\tilde{R}^y(0,0), q(0,0), r(0,0)) \in C_1 \times \{1\} \times \overline{\mathcal{B}}$. This case employs the rotated observer defined in (11). $r(0,0) \in \overline{\mathcal{B}}$ ensures that the rotated observer never fails in this domain of operation. Since $c_1 > |R^*|_I$, there exists a finite time $T > 0$ such that $|\tilde{R}(T,0)|_I = c_1$. Now, following (8), there exists d_0 with $1 - |R^*|_I < d_0 < 1$ such that when $|\tilde{R}_1(0,0)|_I \leq d_0$, every solution to $(t,0) \mapsto \tilde{R}_1(t,0)$ to the rotated observer error dynamics $\tilde{R}_1$ satisfies

$$|\tilde{R}_1(t,0)|_I \leq e^{-\overline{k_p}(1-d_0^2)t} |\tilde{R}_1(0,0)|_I \quad \forall t \in \{s \in \mathbb{R}_{\geq 0} \mid s \leq T,\ (s,j) \in \mathrm{dom}\ \tilde{R}_1\} \tag{17}$$

where $T > 0$ is such that $(\tilde{R}(T,0), q(T,0), r(T,0)) \in D_1 \times \{1\} \times \overline{\mathcal{B}}$ which leads to Case 2. We now wish to find upper and lower bounds on $|\tilde{R}_1(t,0)|_I$ in terms of $|\tilde{R}(t,0)|_I$ to prove that the estimation error $|\tilde{R}|_I$ exponentially decreases. We note that $|\tilde{R}_1|_I = |\hat{R}^\top R^* R|_I = |R^* R \hat{R}^\top|_I$ and $|R\hat{R}^\top|_I = |\tilde{R}|_I$. Let $R^* = \mathcal{R}(\theta^*, r)$ and $R\hat{R}^\top = \mathcal{R}(\tilde{\theta}, w)$. Now, using Lemma 2 and $0 < |R^*|_I < c_1 < |\tilde{R}(t,0)|_I < 1$ with $t \leq T$, we get

$$\begin{aligned}
|\tilde{R}_1|_I^{\,2} &= |\tilde{R}|_I^{\,2} + |R^*|_I^{\,2} - |\tilde{R}|_I^{\,2}|R^*|_I^{\,2} + |\tilde{R}|_I^{\,2}|R^*|_I^{\,2}(r^\top w)\left[2\sqrt{\left(\frac{1}{|\tilde{R}|_I^{\,2}} - 1\right)\left(\frac{1}{|R^*|_I^{\,2}} - 1\right)} - r^\top w\right] \\
&\leq |\tilde{R}|_I^{\,2} + |\tilde{R}|_I^{\,2} - |\tilde{R}|_I^{\,2}|R^*|_I^{\,2} + 2|\tilde{R}|_I^{\,2}\sqrt{(1 - |R^*|_I^{\,2})(1 - c_1)} \\
&\leq |\tilde{R}|_I^{\,2}\left(2 - |R^*|_I^{\,2} + 2\sqrt{(1 - |R^*|_I^{\,2})(1 - c_1)}\right)
\end{aligned} \tag{18}$$

where $\tilde{R} \equiv \tilde{R}(t,0)$ for brevity. To calculate the lower bound of $|\tilde{R}_1|_I$, it is straightforward to see that $|\tilde{R}_1|_I$ will be the smallest when $r^\top w = -1$. Thus, we have

$$\begin{aligned}
|\tilde{R}_1|_I^{\,2} &\geq |\tilde{R}|_I^{\,2} + |R^*|_I^{\,2}|\tilde{R}|_I^{\,2} - 2|\tilde{R}|_I^{\,2}|R^*|_I^{\,2} - 2|\tilde{R}|_I^{\,2}|R^*|_I^{\,2}\sqrt{\left(\frac{1}{|\tilde{R}|_I^{\,2}} - 1\right)\left(\frac{1}{|R^*|_I^{\,2}} - 1\right)} \\
&\geq |\tilde{R}|_I^{\,2} - |R^*|_I^{\,2}|\tilde{R}|_I^{\,2} - 2|\tilde{R}|_I^{\,2}\sqrt{\left(\frac{1}{c_1} - 1\right)|R^*|_I^{\,2}(1 - |R^*|_I^{\,2})} \\
&\geq |\tilde{R}|_I^{\,2} - |\tilde{R}|_I^{\,2}|R^*|_I^{\,2} - |\tilde{R}|_I^{\,2}\sqrt{\frac{1}{c_1} - 1} \\
|\tilde{R}_1|_I^{\,2} &\geq |\tilde{R}|_I^{\,2}\left(1 - |R^*|_I^{\,2} - \sqrt{\frac{1}{c_1} - 1}\right)
\end{aligned} \tag{19}$$

Using (17), (18) and (19) results in

$$|\tilde{R}(t,0)|_I \leq \lambda e^{-\overline{k_p}(1-d_0^2)t} |\tilde{R}(0,0)|_I \quad \forall t \in \{s \in \mathbb{R}_{\geq 0} \mid s > T, (s,j) \in \mathrm{dom}\ \tilde{R}\} \tag{20}$$

where

$$\lambda := \sqrt{\frac{2 - |R^*|_I + 2\sqrt{(1 - |R^*|_I)(1 - c_1)}}{1 - |R^*|_I - \sqrt{\frac{1}{c_1} - 1}}}. \tag{21}$$

Note that (b) results in $\lambda > 0$. Next, since $(\tilde{R}(T,0), q(T,0), r(T,0)) \in D_1 \times \{1\} \times \overline{\mathcal{B}}$ results in $(\tilde{R}(T,1), q(T,1), r(T,1)) \in C_0 \times \{0\} \times \overline{\mathcal{B}}$, we have

$$|\tilde{R}(t,1)|_I \leq e^{-k_p(1-c_0^2)(t-T)}|\tilde{R}(T,1)|_I \quad \forall t \in \{s \in \mathbb{R}_{\geq 0} \mid s > T, (s,j) \in \text{dom } \tilde{R}\} \tag{22}$$

Using (20), (22) and the fact that $|\tilde{R}(t,j)|_I = |\tilde{R}(t,k)|_I$ for all $(t,j), (t,k) \in \text{dom } \tilde{R}$ results[5] in

$$|\tilde{R}(t,j)|_I \leq \min\{1, \alpha e^{-\min\{\overline{k_p}(1-d_0^2), k_p(1-c_0^2)\}t}|\tilde{R}(0,0)|_I\} \quad \forall (t,j) \in \text{dom } \tilde{R}. \tag{23}$$

where $\alpha := \max\{\lambda, \lambda \exp\left(-T[\overline{k_p}(1-d_0^2) - k_p(1-c_0^2)]\right)\}$. Thus, if $(\tilde{R}(0,0), q(0,0), r(0,0)) \in C_0 \times \{0\} \times \mathbb{S}^2$ or $(\tilde{R}(0,0), q(0,0), r(0,0)) \in D_1 \times \{1\} \times \mathbb{S}^2$, then (15) holds. For all other cases, (23) holds. This proves (4). Further, the system exhibits no jumps in case 1, one jump in case 2 and case 5, two jumps in case 3 and three jumps in case 4. This proves (3) and concludes the proof. □

Remark 3. *If condition (b) in Theorem 2 does not hold, the set $\mathcal{A}$ becomes only globally asymptotically stable for $\tilde{\mathcal{H}}_{PCF}$.*

Remark 4. *If we set $c_0 = c_1$, we get rid of the hysteresis region. Such an observer would work in ideal conditions where there is no noise. Introduction of the slightest noise will cause chattering (see [7]), which is the motivation to introduce the hysteresis region in the observer.*

Remark 4 motivates the analysis of the hybrid observer under measurement noise.

3.1 Robustness analysis of the hybrid passive complementary filter

We consider the filter in Theorem 1 and analyze the effect of measurement noise, i.e. noise in gyro measurements and attitude measurements. We first define the notions of input-to-state stability (see [1, 5]). Consider the hybrid system $\mathcal{H}_u$ with state $x \in \mathcal{X}$ and disturbance $u \in \mathcal{U}$ as follows:

$$\mathcal{H}_u := \begin{cases} \dot{x} &= f(x,u), \quad (x,u) \in C, \\ x^+ &= g(x,u), \quad (x,u) \in D. \end{cases} \tag{24}$$

Let $\mathcal{A} \subset \mathcal{X}$ be a nonempty and compact set.

Definition 12 (Local input-to-state stability)**.** *The system (24) is said to be locally input-to-state stable with respect to a nonempty, compact set $\mathcal{A}$ if there exist a class $\mathcal{KL}$ function β, a class $\mathcal{K}$ function γ and a scalar κ such that for each solution pair (x,u) to (24) with $|x(0,0)|_{\mathcal{A}} \leq \kappa$ and $\|u_\#\| := \sup_{(t,j)\in dom\ u}\|u(t,j)\|$, every solution $(t,j) \mapsto x(t,j)$ to (24) satisfies*

$$|x(t,j)|_{\mathcal{A}} \leq \beta(|x(0,0)|_{\mathcal{A}}, t+j) + \gamma(\|u_\#\|) \quad \forall (t,j) \in dom\ x.$$

If this holds for $\kappa \to +\infty$[6], then the system (24) is said to be input-to-state stable (ISS).

Definition 13 (Eventual input-to-state stability)**.** *The system (24) is said to be eventually input-to-state stable with respect to $\mathcal{A}$ if there exist a class $\mathcal{KL}$ function β, a class $\mathcal{K}$ function γ and strictly positive scalars κ and T such that for each solution pair (x,u) to (24) with $|x(0,0)|_{\mathcal{A}} \leq \kappa$ and $u_\# := \sup_{(t,j)\in dom\ u}\|u(t,j)\|$, every solution $(t,j) \mapsto x(t,j)$ to (24) satisfies*

$$|x(t,j)|_{\mathcal{A}} \leq \beta(|x(0,0)|_{\mathcal{A}}, t+j) + \gamma(\|u_\#\|) \quad \forall (t,j) \in \{(\bar{t},\bar{j}) \in dom\ x \mid \bar{t}+\bar{j} \geq T\}.$$

[5] I'm wondering if the result in (23) is any different than local exponential stability (given that we already have GAS). This is because after enough time has passed (alternatively, once $|\tilde{R}|_I$ enters some neighbourhood of I), the RHS in (23) takes the form in Defintion 8.

[6] $\kappa \to +\infty$ is required for ISS when considering proper indicator functions. On $\mathbb{SO}(3)$, $\kappa = +1$ ensures ISS. How should I write it for $\mathbb{SO}(3)$ as I cannot use proper indicator functions?

References

[1] Chaohong Cai and Andrew R. Teel. Characterizations of input-to-state stability for hybrid systems. *Systems & Control Letters*, 58(1):47–53, 2009.

[2] D. R. J. Chillingworth, J. E. Marsden, and Y. H. Wan. Symmetry and bifurcation in three-dimensional elasticity, part i. *Archive for Rational Mechanics and Analysis*, 80(4):295–331, Dec 1982.

[3] R. Goebel, R. G. Sanfelice, and A. R. Teel. *Hybrid Dynamical Systems: Modeling, Stability, and Robustness*. Princeton University Press, New Jersey, 2012.

[4] Du Huynh. Metrics for 3d rotations: Comparison and analysis. *Journal of Mathematical Imaging and Vision*, 35:155–164, 10 2009.

[5] H. K. Khalil. *Nonlinear Systems*. Prentice Hall, 3rd edition, 2002.

[6] Robert Mahony, Tarek Hamel, and Jean-Michel Pflimlin. Nonlinear complementary filters on the special orthogonal group. *IEEE Transactions on Automatic Control*, 53(5):1203–1218, 2008.

[7] Christopher G. Mayhew, Ricardo G. Sanfelice, and Andrew R. Teel. Quaternion-based hybrid control for robust global attitude tracking. *IEEE Transactions on Automatic Control*, 56(11):2555–2566, 2011.