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ORIGINAL RESEARCH

Stress-strength reliability estimation of xgamma distribution under generalized progressive hybrid censoring scheme

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Abstract

This article aims to estimate the parameters and stress-strength reliability $R = P(Y < X)$ based on the generalized progressive hybrid censored data when X and Y follow independent xgamma distributions with different scale parameters. The maximum likelihood estimators and asymptotic confidence intervals for parameters and stress-strength reliability R have been obtained based on both classical and Bayesian setups. The Bayes estimators for the model parameters and stress-strength reliability R are derived under the assumption of independent gamma prior using symmetric and asymmetric loss functions. The Markov chain Monte Carlo technique is employed for Bayesian computations due to the complexity of the posterior, which lacks a closed-form expression for Bayesian estimators. We have also computed the highest probability density credible intervals for the Bayes estimators. Additionally, a simulation study is conducted to study the effectiveness of Bayes and maximum likelihood estimators using mean squared errors. In the end, a real data set has been utilized for illustrative purposes.

Keywords Maximum likelihood estimates · Generalized progressive hybrid censoring · Stress-strength reliability · Markov chain Monte Carlo · Bayes estimates

1 Introduction

In the discipline of engineering science, stress-strength reliability models are frequently used to design and analyse the reliability of systems. According to reliability theory, stress-strength reliability is a gauge of how well a system performs under stress (load). Stress can include electrical voltage, temperatures, pressure, and other atmospheric variables that impact how a system functions and may even lead to a system failure (see Church and Harris 1970). In a nutshell, if the system's accumulated stress is greater than its capacity, the system will fail and vice versa. The groundbreaking work of Birnbaum and McCarty (1958) introduced the topic of stress-strength reliability estimation in reliability theory. In the monograph by Kotz et al. (2003) various stress-strength models are thoroughly discussed. Stress-strength reliability (SSR) of a system may be comprehended in a probabilistic approach since stress and strength variables are random and cannot be anticipated, as a result, it is calculated using parametric frameworks for strength and stress variables. The probability $R = P(Y < X)$ is used to calculate the SSR in parametric inference, where Y and X stand for the stress and strength random variables, respectively. Owen et al. (1964) were the first to examine $P(Y < X)$ using a parametric setup under the consideration that X and Y were normally distributed random variables. Following that, numerous writers investigated the estimation of $P(Y < X)$ while assuming different lifetime models in a variety of contexts, including complete and censored samples, bivariate samples, record values, etc. Using entire sample observation estimation of $P(Y < X)$ has been done by several authors by considering different distributions. For example, in exponential distribution by Chung (1982), the estimate and confidence interval of R in the case of generalized exponential distribution by Kundu and Gupta (2005) and Hajebi et al. (2012), respectively, inference on stress-strength reliability of two-parameter exponential distribution by Krishnamoorthy et al. (2007), estimation of $P(Y < X)$ for the three-parameter generalized exponential distribution by Raqab et al. (2008), Weibull by Kundu and Gupta (2006), inference on stress-strength reliability for weighted Weibull distribution by Salem (2013), three-parameter

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Weibull by Kundu and Raqab (2009), Comparison of different estimators of $P(Y < X)$ for a scaled burr type X distribution by Raqab and Kundu (2005), Lindley by Al-Mutairi et al. (2013) and Ali (2013), power Lindley by Ghitany et al. (2015), generalized Lindley by Singh et al. (2014), weighted Lindley by Al-Mutairi et al. (2015), stress-strength reliability with general form distributions by Mokhlis et al. (2017), interval estimation of $P(Y < X)$ for generalized Pareto distribution by Wong (2012), generalized logistic by Asgharzadeh et al. (2013), exponentiated Gumbel by Kakade et al. (2008), gamma by Huang et al. (2012), Topp-Leone by Genç (2013), inverse Chen distribution by Agiwal (2023) Power-Modified Lindley distribution Al-Babtain et al. (2022), stress-strength reliability for two-parameter bathtub-shaped lifetime distribution based on maximum ranked set sampling with unequal samples Basikhasteh et al. (2020), stress-strength reliability using discrete phase type distribution Jose (2022) etc. In addition to these studies, several work on the estimation of $P(Y < X)$ have been conducted with bivariate, censored, and record data, for detailed information the reader may see (Pak et al. 2014; Hor and Seal 2017; Hanagal 1997; Asgharzadeh et al. 2017; Nadar and Kızılaslan 2014; Baklizi 2008) etc.

In real-life scenarios, we are unable to obtain complete information on the event of interest due to several reasons. As a result, we must conduct a study using censored data. Therefore, several authors have studied stress-strength reliability under censored cases. For further detailed information, the reader may follow, Saraçoğlu et al. (2012), who have addressed how to estimate SSR using a progressive Type-II censoring scheme where stress and strength follow an independent exponential distribution, Asgharzadeh et al. (2011) focused on the estimation of SSR based on progressively censored samples. They considered a scenario where X and Y are two independent Weibull distributions with different scale parameters but the same shape parameter. The SSR estimate for the Weibull distribution, generalized Pareto distribution, proportional hazard models, and inverse Weibull distribution were each examined by Valiollahi et al. (2013), Rezaei et al. (2015), Basirat et al. (2015), and Yadav et al. (2018) using progressive Type-II censored samples, respectively. Mirjalili et al. (2016) discussed SSR of exponential distribution based on Type-I progressive hybrid censored samples. Alslman and Helu (2022) and Asadi and Panahi (2022) have considered inverse Weibull distribution and inverted exponentiated Rayleigh distribution under adaptive Type-II progressive hybrid censoring for the estimation of stress-strength reliability, respectively. Saini et al. (2021) and Krishna et al. (2019) have studied the estimation of stress-strength reliability for generalized Maxwell failure distribution and inverse Weibull distribution under progressive first failure censoring. Recently, Çetinkaya (2023) have studied inference of P $(Y < X)$ for the Burr-XII model under generalized progressive hybrid censored data with binomial removals. To the best of our knowledge, neither the classical paradigm nor the Bayesian paradigm has been taken into account to estimate the SSR for the xgamma distribution. Hence, the considered work using xgamma distribution under the generalized progressive hybrid censoring scheme has been proposed to fill up the gap.

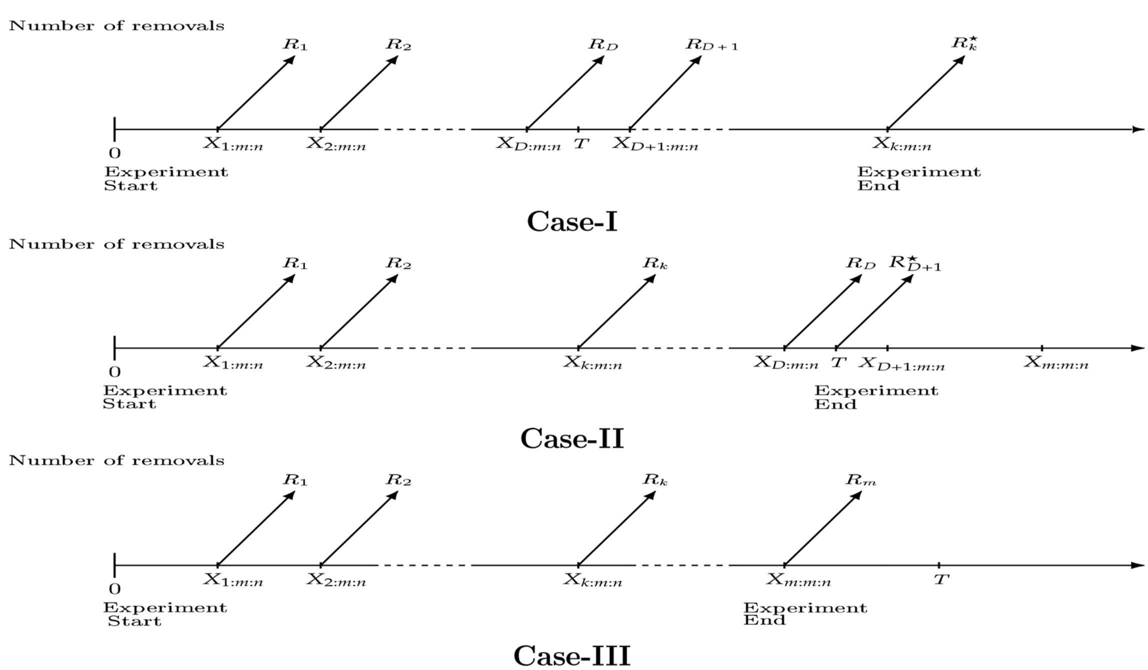

Fig. 1 Graphical representation of generalized progressive hybrid censoring Scheme

In the literature, xgamma distribution was first introduced by Sen et al. (2016) using a finite mixture of exponential and gamma distributions. The following equations provide the cumulative density function, probability density function, and hazard function of the considered model;

$$F(x) = 1 - \frac{\left(1 + \theta + \theta x + \frac{\theta^2 x^2}{2}\right)}{(\theta + 1)} e^{-\theta x}; \qquad x > 0, \theta > 0 \tag{1}$$

$$f(x) = \frac{\theta^2}{(\theta + 1)} \left(1 + \frac{\theta}{2} x^2\right) e^{-\theta x}; \qquad x > 0, \theta > 0 \tag{2}$$

and,

$$h(x) = \frac{\theta^2 \left(1 + \frac{\theta}{2} x^2\right)}{\left(1 + \theta + \theta x + \frac{\theta^2 x^2}{2}\right)}; \qquad x > 0, \theta > 0 \tag{3}$$

respectively, where θ is a scale parameter.

The applicability of the xgamma distribution in reliability theory has been extensively addressed by several authors, viz. Elshahhat and Elemary (2021) have discussed the problem of analysis for xgamma parameters under Type-II adaptive progressively hybrid censored data. Pathak et al. (2024) examined E-Bayesian inference for the xgamma distribution under progressive Type-II censoring with binomial removals. Saha and Yadav (2021) have considered the problem of estimation of the reliability characteristics by using classical and Bayesian methods of estimation for xgamma distribution. Using Type-II hybrid censored data, Yadav et al. (2019) discuss Bayesian estimation of the parameter and the reliability properties of the xgamma distribution. Recently, Yadav (2022) has considered an asymmetric loss function for Bayesian estimation of the xgamma distribution under Type-I hybrid censored data, etc.

Censoring is a typical element of life-testing experiments that might occur spontaneously or as a result of confinement. Experimenters prefer censored observations to make accurate deductions about the traits of any lifetime model in clinical trial investigations. Type-I and Type-II censoring schemes are the most often utilized censoring scheme out of all censoring schemes. In Type-I censoring strategy, the experiment stops at a specified time T, but when using a Type-II censoring scheme, the experiment is ended when a certain amount of failures m realized. However, neither of these two schemes permits the removal of experimental units while the experiment is still running. Therefore, Progressive censoring is suggested in the literature to overcome the demerits of Type-I and Type-II censoring scheme by Cohen (1963). In the problem of life testing and reliability assessment, using the mentioned censoring scheme, there may not be adequate failure data, or even worse still, there may be no failure at all, or the items may take a long time to fail. Hence, these circumstances could significantly impact the efficacy of inferential analysis. To overcome this problem Epstein (1954) introduced a hybrid censoring approach based on the combination of Type-I and Type-II censoring schemes. In the similar scenario of life testing experiments, the Type-I progressive hybrid censoring scheme (Type-I PHCS) has been investigated by Kundu and Joarder (2006) and Childs et al. (2008). In which, n identical units are tested with the progressive censoring scheme $(R_1, R_2, \cdots \cdots R_m)$, where R_i is the number of removals at i^{th} failure and the experiment ends at time $T^\star = min\{X_{m:m:n}, T\}$, where $T \in (0, \infty)$ and $1 \le m \le n$ are predefined and the resulting ordered failure are obtained as $X_{1:m:n} \le X_{2:m:n} \le \cdots \cdots \le X_{m:m:n}$. The test can never exceed T, which is a benefit of this censoring strategy. Thus, this plan has control over the experiment's duration and expense. There can be fewer than m failures, though, if the unknown average lifetime is significantly higher than the termination time T. This progressively diminishes the effectiveness of the conclusions drawn from such censored data, keeping this in mind, Childs et al. (2008) suggested the Type-II progressive hybrid censoring technique, which ends the experiment at a time $T^\star = max\{X_{m:m:n}, T\}$ and guarantees at least m failures to achieve a desired level of efficiency with prefixed removal $(R_1, R_2, \cdots \cdots R_m)$ at each failures. When $X_{m:m:n} > T$, signifying the experiment's termination at the m^{th} failure, it is important to note that the duration of the termination period may considerably extend. However, failures are observed up to time T when $X_{m:m:n} < T$. If D is the number of failures that happen before time T, then the progressive removal scheme in this situation is $(R_1, R_2, \cdots R_m, R_{m+1}, \cdots R_D)$, where $R_m = R_{m+1} = \cdots = R_D = 0$. As a result, we can observe that the experimenter is unaware of the experiment's stopping time under such a technique. The Type-I hybrid censoring technique preserves the experiment's termination time below a certain value by sacrificing efficiency, while the Type-II hybrid censoring scheme assures efficiency beyond the predetermined level while forfeiting the experiment's termination time. As a result, there is a necessity for a censoring scheme that can control termination time and efficiency at the same time. For instance, it must ensure the lowest possible amount of failures, and set constraints on time as well. Therefore, the generalized progressive hybrid (GPH) censoring approach has been proposed by Cho et al. (2015), which overcomes the disadvantages of both Type-I PHC and Type-II PHC. The purpose of this work is to propose a point and interval estimate of the SSR parameter of the xgamma distribution using generalized progressive hybrid censored samples under both classical and Bayesian setups.

The following sections serve as the structure for the remaining portions of the paper. Section 2 provides an extensive overview of the considered censoring mechanism. The formulation of R and its maximum likelihood estimator (MLE) are obtained in Section 3. We have calculated 95% asymptotic confidence intervals (ACIs) based on ML estimates using asymptotic normality criteria of MLE and bootstrap confidence intervals have been also obtained using bootstrap samples, respectively. In Section 4, we obtained Bayes estimators under squared error and LINEX loss functions. These estimations utilize the Markov chain Monte Carlo (MCMC) technique, considering independent gamma distributions as prior information with hyper parameters a_1, b_1 and a_2, b_2. A simulation study is conducted to assess the performance of ML and Bayes estimators in section 5. In Section 6, the demonstration of the practical applicability of the study has been given by analyzing a real-world data set. The study concludes with a summary of the findings presented in Section 7.

2 The GPHC scheme

Let us consider a life-testing study where n identical components are placed to the test. Let $X_1, X_2, \cdots, X_n$ be the independent and identical units from the distribution with probability density (pdf) $f(x)$ and cumulative distribution function (cdf) $F(x)$, which are provided in equations (2) and (1), respectively. To establish the generalized progressive hybrid censoring scheme, we consider fixed number of failures k and m with $k < m$, belonging to the positive integers. The units are eliminated randomly by a pre-decided removal scheme as the R_1 units are removed randomly at the time of the first failure. At the moment of the second failure from the surviving units, R_2 units of the remaining items are removed, and so on R_i units are eliminated randomly at i^{th} failure, meeting the constraints $\sum_{i=1}^{m} R_i + m = n$. This process is continued until the termination time $T^\star = max\{X_{k:m:n}, min\{X_{m:m:n}, T\}\}$ is reached. At this point, all of the remaining units are withdrawn from the experiment with pre-fixed time, where $T \in (0, \infty)$. The fact that this technique guarantees at least k failures rather than witnessing m failures should be emphasized. D should represent all failures up to time T. As a result, three situations occur under this scheme, resulting in the following set of observations

$$\begin{aligned}
&\textbf{Case-I:} X_{1:m:n}, \cdots, X_{2:m:n}, \cdots, X_{k:m:n}, && \text{if } T < X_{k:m:n} < X_{m:m:n},\\
&\textbf{Case-II:} X_{1:m:n}, \cdots, X_{k:m:n}, \cdots, X_{D:m:n}, && \text{if } X_{k:m:n} < T < X_{m:m:n},\\
&\textbf{Case-III:} X_{1:m:n}, \cdots, X_{k:m:n}, \cdots, X_{m:m:n}, && \text{if } X_{k:m:n} < X_{m:m:n} < T.
\end{aligned} \tag{4}$$

where,

$$T^\star = \begin{cases} X_{k:m:n}, & \text{if } T < X_{k:m:n} < X_{m:m:n}, \\ T, & \text{if } X_{k:m:n} < T < X_{m:m:n}, \\ X_{m:m:n}, & \text{if } X_{k:m:n} < X_{m:m:n} < T. \end{cases}$$

In Figure 1, the GPH censoring scheme is graphically depicted. The joint density function is defined under GPH censoring scheme as follows

$$L(data) = \begin{cases} C_1 \prod_{j=1}^{k-1} f(x_{j:m:n})[1 - F(x_{j:m:n})] \\ {}^{R_j} f(x_{k:m:n})[1 - F(x_{k:m:n})]^{R_k^\star}, \ \textbf{Case-I} \\ C_2 \prod_{j=1}^{D} f(x_{j:m:n})[1 - F(x_{j:m:n})] \\ {}^{R_j}[1 - F(T)]^{R_{D+1}^\star}, \qquad \textbf{Case-II} \\ C_3 \prod_{j=1}^{m} f(x_{j:m:n})[1 - F(x_{j:m:n})] \\ {}^{R_j}. \qquad \textbf{Case-III} \end{cases} \tag{5}$$

where, $C_1 = \prod_{j=1}^{k} \sum_{k=j}^{m} (R_k + 1)$, $C_2 = \prod_{j=1}^{D} \sum_{k=j}^{m} (R_k + 1)$, $C_3 = \prod_{j=1}^{m} \sum_{k=j}^{m} (R_k + 1)$, $R_k^\star = n - k - \sum_{i=1}^{k-1} R_i$ and $R_{D+1}^\star = n - D - \sum_{i=1}^{D} R_i$.

Therefore, the combined likelihood for the considered censoring scheme can be written as

$$L(\Theta, data) = C^\star \prod_{j=1}^{D^\star} f(x_{j:m:n}) \left[1 - F(x_{j:m:n})\right]^{R_j} W(\Theta) \tag{6}$$

where

$$W(\Theta) = \begin{cases} 1, & \textbf{Case-I, and III} \\ \left[1 - F(T)\right]^{R_{D+1}^\star}, & \textbf{Case-II} \end{cases}$$

$$D^\star = \begin{cases} k, & \textbf{Case-I} \\ D, & \textbf{Case-II} \\ m, & \textbf{Case-III} \end{cases}$$

and

$$C^\star = \sum_{j=1}^{D^\star} \sum_{k=j}^{m} (R_k + 1).$$

3 Stress-strength reliability and maximum likelihood estimation

Let X and Y be the independent strength and stress random variables follows xgamma distribution with parameters θ_1 and θ_2 respectively. Then, the expression of stress-strength reliability is obtained as

$$\begin{aligned}
R = P[Y < X] &= \int_{x=0}^{\infty} f(x|\theta_1) P(Y \leq x) dx \\
&= \int_{x=0}^{\infty} f(x|\theta_1) F(x|\theta_2) dx \\
&= \int_{x=0}^{\infty} \frac{\theta_1^2}{(1+\theta_1)} \left(1 + \frac{\theta_1}{2} x^2\right) \\
&\quad e^{-\theta_1 x} \left(1 - \frac{\left(1 + \theta_2 + \theta_2 x + \frac{\theta_2^2 x^2}{2}\right)}{(1+\theta_2)} e^{-\theta_2 x}\right) dx
\end{aligned}$$

after performing some algebraic calculations, the expression for the SSR parameter R is given by

$$R = \frac{\theta_2^2((\theta_1+1)\theta_2^4+(4\theta_1^2+6\theta_1+1)\theta_2^3+(6\theta_1^3+12\theta_1^2+5\theta_1)\theta_2^2+(4\theta_1^4+10\theta_1^3+10\theta_1^2)\theta_2+\theta_1^5+3\theta_1^4)}{(\theta_1+1)(\theta_2+1)(\theta_1+\theta_2)^5} \tag{7}$$

From the above equation (7), it is clear that that SSR (R) is function of parameters θ_1 and θ_2, respectively. Therefore, to compute the MLE of R under the GPH censoring scheme, first, we have to derive the MLE of the parameters θ_1 and θ_2 and then by using the invariance property, MLE of R is computed. Therefore, by using equations (1), (2) and (6) the combined likelihood for the considered model under the GPH censoring scheme can be written as

$$\begin{aligned} L(\theta_1,\theta_2,data) =& C_1^\star C_2^\star \frac{\theta_1^{2D_1^\star}\theta_2^{2D_2^\star}}{(\theta_1+1)^{D_1^\star}(\theta_2+1)^{D_2^\star}}\prod_{j=1}^{D_1^\star}\left(1+\frac{\theta_1x_j^2}{2}\right)\\ &\left\{\frac{\left(1+\theta_1+\theta_1x_j+\frac{\theta_1^2x_j^2}{2}\right)}{(\theta_1+1)}e^{-\theta_1x_j}\right\}^{R_{j1}}\\ &\times\left\{\frac{\left(1+\theta_1+\theta_1T_1+\frac{\theta_1^2T_1^2}{2}\right)}{(\theta_1+1)}e^{-\theta_1T_1}\right\}^{R^\star_{D_1+1}}\\ &e^{-\theta_1x_j}\prod_{j=1}^{D_2^\star}\left(1+\frac{\theta_2y_j^2}{2}\right)e^{-\theta_2y_j}\\ &\times\left\{\frac{\left(1+\theta_2+\theta_2y_j+\frac{\theta_2^2y_j^2}{2}\right)}{(\theta_2+1)}e^{-\theta_2y_j}\right\}^{R_{j2}}\\ &\left\{\frac{\left(1+\theta_2+\theta_2T_2+\frac{\theta_2^2T_2^2}{2}\right)}{(\theta_1+1)}e^{-\theta_1T_1}\right\}^{R^\star_{D_2+1}} \end{aligned} \tag{8}$$

Hence, the corresponding log-likelihood function $l(\theta_1,\theta_2,data)$ for the considered model is

$$\begin{aligned} &l(\theta_1,\theta_2,data)\\ &=\log C_1^\star+\log C_2^\star\\ &+D_1^\star\left(2\log\theta_1-\log(\theta_1+1)\right)+D_2^\star\left(2\log\theta_2-\log(\theta_2+1)\right)\\ &+\sum_{j=1}^{D_1^\star}\log\left(1+\frac{\theta_1x_j^2}{2}\right)+\sum_{j=1}^{D_1^\star}R_{j1}\\ &\left\{\log\frac{\left(1+\theta_1+\theta_1x_j+\frac{\theta_1^2x_j^2}{2}\right)}{(\theta_1+1)}-\theta_1x_j\right\}\\ &-\sum_{j=1}^{D_1^\star}\theta_1x_j+R^\star_{D_1+1}\left\{\log\frac{\left(1+\theta_1+\theta_1T_1+\frac{\theta_1^2T_1^2}{2}\right)}{(\theta_1+1)}-\theta_1T_1\right\}\\ &-\sum_{j=1}^{D_2^\star}\theta_2y_j+\sum_{j=1}^{D_2^\star}\log\left(1+\frac{\theta_2y_j^2}{2}\right)+\sum_{j=1}^{D_2^\star}R_{j2}\\ &\left\{\log\frac{\left(1+\theta_2+\theta_2y_j+\frac{\theta_2^2y_j^2}{2}\right)}{(\theta_2+1)}-\theta_2y_j\right\}\\ &+R^\star_{D_2+1}\left\{\log\frac{\left(1+\theta_2+\theta_2T_2+\frac{\theta_2^2T_2^2}{2}\right)}{(\theta_2+1)}-\theta_2T_1\right\} \end{aligned} \tag{9}$$

The MLE of the parameters θ_1 and θ_2 has been obtained by solving the following two non-linear equations simultaneously.

$$\begin{aligned} \frac{\partial l}{\partial\theta_1} =& D_1^\star\left(\frac{2}{\theta_1}-\frac{1}{(\theta_1+1)}\right)\\ &-\sum_{j=1}^{D_1^\star}x_j+\sum_{j=1}^{D_1^\star}\frac{x_j^2}{2\left(1+\frac{\theta_1x_j^2}{2}\right)}\\ &+\sum_{j=1}^{D_1^\star}R_{j1}\psi_1(x_1,\theta_1)+R^\star_{D_1+1}\psi_1(T_1,\theta_1) \end{aligned} \tag{10}$$

and

$$\begin{aligned}\frac{\partial l}{\partial \theta_2} =& D_2^\star\left(\frac{2}{\theta_2}-\frac{1}{(\theta_2+1)}\right)\\ &-\sum_{j=1}^{D_2^\star} y_j+\sum_{j=1}^{D_2^\star}\frac{y_j^2}{2\left(1+\frac{\theta_2 y_j^2}{2}\right)}\\ &+\sum_{j=1}^{D_2^\star} R_{j2}\psi_1(y,\theta_2)+R^\star_{D_2+1}\psi_1(T_2,\theta_2)\end{aligned} \tag{11}$$

$$\begin{aligned}\psi_1(x,\theta_1) =& \frac{\left(1+x_j+\theta_1 x_j^2\right)}{\left(1+\theta_1+\theta_1 x_j+\frac{\theta_1^2 x_j^2}{2}\right)}\\ &-\frac{1}{(\theta_1+1)}-x_j\\ \psi_1(T_1,\theta_1) =& \frac{\left(1+T_1+\theta_1 T_1^2\right)}{\left(1+\theta_1+\theta_1 T_1+\frac{\theta_1^2 T_1^2}{2}\right)}\\ &-\frac{1}{(\theta_1+1)}-T_1\\ \psi_1(y,\theta_2) =& \frac{\left(1+y_j+\theta_2 y_j^2\right)}{\left(1+\theta_2+\theta_2 y_j+\frac{\theta_2^2 y_j^2}{2}\right)}\\ &-\frac{1}{(\theta_2+1)}-y_j\\ \psi_1(T_2,\theta_2) =& \frac{\left(1+T_2+\theta_2 T_2^2\right)}{\left(1+\theta_2+\theta_2 T_2+\frac{\theta_2^2 T_2^2}{2}\right)}\\ &-\frac{1}{(\theta_2+1)}-T_2\end{aligned}$$

Since equations (10) and (11) do not have analytical solutions for θ_1 and θ_2, thus non-linear maximization procedure such as *nlm*() is used to obtain the MLE $(\hat{\theta}_1,\hat{\theta}_2)$ for (θ_1,θ_2). Once the MLE of the parameters are obtained then by using the invariance property of MLE, the MLE of SSR parameter R is obtained by

$$\hat{R}=\frac{\hat{\theta}_2^2((\hat{\theta}_1+1)\hat{\theta}_2^4+(4\hat{\theta}_1^2+6\hat{\theta}_1+1)\hat{\theta}_2^3+(6\hat{\theta}_1^3+12\hat{\theta}_1^2+5\hat{\theta}_1)\hat{\theta}_2^2+(4\hat{\theta}_1^4+10\hat{\theta}_1^3+10\hat{\theta}_1^2)\hat{\theta}_2+\hat{\theta}_1^5+3\hat{\theta}_1^4)}{(\hat{\theta}_1+1)(\hat{\theta}_2+1)(\hat{\theta}_1+\hat{\theta}_2)^5} \tag{12}$$

3.1 Asymptotic confidence intervals

Confidence intervals (CIs) provide a measure of uncertainty associated with estimates derived from sample data, indicating the range of values within the population parameter is likely to reside with a specified probability. We derive confidence intervals for the parameters by leveraging the asymptotic normality property of MLE. For sufficiently large sample sizes, the MLEs demonstrate consistency and asymptotic normality. Then from the asymptotic theory of normality

$$\begin{pmatrix}\hat{\theta}_1-\theta_1\\ \hat{\theta}_2-\theta_2\end{pmatrix}\sim N_2(0,\Sigma^{-1})$$

where, Σ represents the observed symmetric Fisher information matrix, which is defined as follows:

$$\Sigma=\begin{bmatrix}-\frac{\partial^2 l}{\partial\theta_1^2} & -\frac{\partial^2 l}{\partial\theta_1\partial\theta_2}\\ -\frac{\partial^2 l}{\partial\theta_2\partial\theta_1} & -\frac{\partial^2 l}{\partial\theta_2^2}\end{bmatrix}_{(\theta_1,\theta_2)=(\hat{\theta}_1,\hat{\theta}_2)}$$

where

$$\begin{aligned}\frac{\partial^2 l}{\partial\theta_1^2} =& D_1^\star\left(\frac{1}{(\theta_1+1)^2}-\frac{2}{\theta_1^2}\right)\\ &-\sum_{j=1}^{D_1^\star}\frac{x_j^4}{(\theta_1 x_j^2+2)^2}\\ &+\sum_{j=1}^{D_1^\star} R_{j1}\psi_2(x,\theta_1)+R^\star_{D_1+1}\psi_2(T_1,\theta_1)\end{aligned} \tag{13}$$

and

$$\begin{aligned}\frac{\partial^2 l}{\partial\theta_2^2} =& D_2^\star\left(\frac{1}{(\theta_2+1)^2}-\frac{2}{\theta_2^2}\right)\\ &-\sum_{j=1}^{D_2^\star}\frac{y_j^4}{(\theta_2 y_j^2+2)^2}\\ &+\sum_{j=1}^{D_2^\star} R_{j2}\psi_2(y,\theta_2)+R^\star_{D_2+1}\psi_2(T_2,\theta_2)\end{aligned} \tag{14}$$

$$\frac{\partial^2 l}{\partial\theta_1\partial\theta_2}=\frac{\partial^2 l}{\partial\theta_2\partial\theta_1}=0 \tag{15}$$

and the term $\psi_2(x,\theta_1)$, $\psi_2(y,\theta_2)$, $\psi_2(T_1,\theta_1)$, $\psi_2(T_2,\theta_2)$ are given by

$$\psi_2(x,\theta_1) = \frac{x_j^2}{\left(\frac{\theta_1^2 x_j^2}{2} + \theta_1 x_j + \theta_1 + 1\right)} - \frac{(\theta_1 x_j^2 + x_j + 1)}{\left(\frac{\theta_1^2 x_j^2}{2} + \theta_1 x_j + \theta_1 + 1\right)} + \frac{1}{(\theta_1+1)^2}$$

$$\psi_2(y,\theta_2) = \frac{y_j^2}{\left(\frac{\theta_2^2 y_j^2}{2} + \theta_2 y_j + \theta_2 + 1\right)} - \frac{(\theta_2 y_j^2 + y_j + 1)}{\left(\frac{\theta_2^2 y_j^2}{2} + \theta_2 y_j + \theta_2 + 1\right)} + \frac{1}{(\theta_2+1)^2}$$

$$\psi_2(T_1,\theta_1) = \frac{T_1^2}{\left(\frac{\theta_1^2 T_1^2}{2} + \theta_1 T_1 + \theta_1 + 1\right)} - \frac{(\theta_1 T_1^2 + T_1 + 1)}{\left(\frac{\theta_1^2 T_1^2}{2} + \theta_1 T_1 + \theta_1 + 1\right)} + \frac{1}{(\theta_1+1)^2}$$

$$\psi_2(T_2,\theta_2) = \frac{T_2^2}{\left(\frac{\theta_2^2 T_2^2}{2} + \theta_2 T_2 + \theta_2 + 1\right)} - \frac{(\theta_2 T_2^2 + T_2 + 1)}{\left(\frac{\theta_2^2 T_2^2}{2} + \theta_2 T_2 + \theta_2 + 1\right)} + \frac{1}{(\theta_2+1)^2}$$

respectively. The two-tailed $100(1-\alpha)\%$ asymptotic confidence intervals of the parameters θ_1 and θ_2 are easily derived using the using large sample theory as

$$\left(\hat{\theta}_1 - z_{\frac{\alpha}{2}}\sqrt{(\widehat{Var}(\hat{\theta}_1))}, \hat{\theta}_1 + z_{\frac{\alpha}{2}}\sqrt{(\widehat{Var}(\hat{\theta}_1))}\right)$$

and

$$\left(\hat{\theta}_2 - z_{\frac{\alpha}{2}}\sqrt{(\widehat{Var}(\hat{\theta}_2))}, \hat{\theta}_2 + z_{\frac{\alpha}{2}}\sqrt{(\widehat{Var}(\hat{\theta}_2))}\right)$$

respectively. Further, from equation (12), it is clear that the exact distribution of R is mathematically intractable. Hence, the concept of the delta method is applied to obtain the asymptotic variance of R. This method allows us to approximate the variance of a function of random variables using the first-order Taylor expansion. This approach yields the following asymptotic variance of R, say $\mathcal{V}$ is

$$\mathcal{V} = b^T \Sigma^{-1} b$$

where

$$b = \begin{pmatrix} \frac{\partial R}{\partial \theta_1} \\ \frac{\partial R}{\partial \theta_2} \end{pmatrix}$$

and the partial derivative of R with respect to θ_1 and θ_2 are given as follow

$$\begin{aligned}\frac{\partial R}{\partial \theta_1} =& -\frac{1}{(\theta_2+1)(\theta_1+1)^2(\theta_1+\theta_2)^6} \\ &\{\theta_2^2\theta_1(\theta_1^5 + (4\theta_2+6)\theta_1^4) + (6\theta_2^2 + 20\theta_2 + 3)\theta_1^3 \\ &+ (4\theta_2^3 + 24\theta_2^2 + 48\theta_2)\theta_1^2 + (\theta_2^4 \\ &+ 12\theta_2^3 + 21\theta_2^2 + 30\theta_2)\theta_1 + 2\theta_2^4 + 6\theta_2^3\}\end{aligned}$$

$$\begin{aligned}\frac{\partial R}{\partial \theta_2} =& \frac{1}{(\theta_1+1)(\theta_2+1)^2(\theta_1+\theta_2)^6} \\ &\{\theta_1^2\theta_2(\theta_2^5 + (4\theta_1+6)\theta_2^4) + (6\theta_1^2 + 20\theta_1 + 3)\theta_2^3 \\ &+ (4\theta_1^3 + 24\theta_1^2 + 48\theta_1)\theta_2^2 + (\theta_1^4 \\ &+ 12\theta_1^3 + 21\theta_1^2 + 30\theta_1)\theta_2 + 2\theta_1^4 + 6\theta_1^3\}\end{aligned}$$

estimating the variance, $\mathcal{V}$ is crucial for determining the confidence interval of R. Therefore, an $100(1-\alpha)\%$ asymptotic confidence interval of R is

$$\left(\hat{R} - z_{\frac{\alpha}{2}}\sqrt{\hat{\mathcal{V}}}, \hat{R} + z_{\frac{\alpha}{2}}\sqrt{\hat{\mathcal{V}}}\right)$$

where, $z_{\frac{\alpha}{2}}$ stands for the upper $\left(\frac{\alpha}{2}\right)^{th}$ percentile point of a standard normal distribution.

3.2 Bootstrap confidence intervals

In this section, an alternative approach for constructing confidence intervals, as proposed by Efron (1982) has been considered. Efron introduced a computer-based technique that offers a practical and accessible method for computing confidence intervals without requiring extensive theoretical deliberation. The Computational algorithm for implementing the bootstrap method is provided below:

Algorithm:

1. Generate random GPH samples $\underline{X} = \{x_1, x_2, ..., x_{n_1}\}$ and $\underline{Y} = \{y_1, y_2, ..., y_{n_2}\}$ of size n_1 and n_2, respectively from xgamma(θ_1) and xgamma(θ_2) and compute the MLEs of θ_1 and θ_2.
2. Again, generate GPH bootstrap samples $\underline{X}^\star = \{x_1^\star, x_2^\star, ..., x_{n_1}^\star\}$ and $\underline{Y}^\star = \{y_1^\star, y_2^\star, ..., y_{n_2}^\star\}$ of size n_1 and n_2, respectively from xgamma($\hat{\theta}_1$) and xgamma($\hat{\theta}_2$), and compute the bootstrap MLEs of $\theta_1^\star, \theta_2^\star$ and $\hat{R}^\star$ of θ_1, θ_2 and R, respectively, using generated bootstrap GPH samples.
3. Execute step 2, **B** times to obtain the bootstrap estimates $\theta_{1:i}^\star$, $\theta_{2:i}^\star$ and $\hat{R}_i^\star$, $i = 1, 2, ..., B$. The different bootstrap

CIs based on the generated bootstraps samples from the above step are constructed below

(i) **Boot-p confidence interval:** Let the sequence of ordered values of the bootstrap estimates $\{\theta_1^\star, \theta_2^\star, ..., \theta_B^\star\}$ is given as $\{\theta_{(1)}^\star, \theta_{(2)}^\star, ..., \theta_{(B)}^\star\}$ for the parameter of interest θ. Let $\hat{\theta}^{\star(\epsilon)}$ be the ϵ^{th} percentile of $\{\hat{\theta}_{(i)}^\star; i = 1, 2, ..., B\}$ such that

$$\frac{1}{B}\sum_{i=1}^{B} I\left(\hat{\theta}_i^\star \leq \hat{\theta}^{\star(\epsilon)}\right), \qquad 0 < \epsilon < 1$$

here, $I(.)$ is a indicator function. The $100(1-\alpha)\%$ boot-p confidence interval for θ is given by

$$\left(\hat{\theta}^{\star(\alpha/2)}, \hat{\theta}^{\star(1-\alpha/2)}\right)$$

(ii) **Boot-t confidence interval:** Define $\bar{\hat{\theta}}^\star$ as the sample mean and $se(\hat{\theta}^\star)$ as the sample standard deviation calculated from $\{\theta_1^\star, \theta_2^\star, ..., \theta_B^\star\}$. Consider a statistic $\hat{t}_i^\star = \frac{\hat{\theta}_i^\star - \bar{\hat{\theta}}^\star}{se(\hat{\theta}^\star)}; i = 1, 2, ..., B$, and let $\hat{t}^{\star(\epsilon)}$ be the ϵ^{th} percentile of $\{\hat{t}_{(i)}^\star; i = 1, 2, ..., B\}$ such that,

$$\frac{1}{B}\sum_{i=1}^{B} I\left(\hat{t}_i^\star \leq \hat{t}^{\star(\epsilon)}\right), \qquad 0 < \epsilon < 1$$

then, the $100(1-\alpha)\%$ boot-t confidence interval of θ is given by

$$(\hat{\theta} - \hat{t}^{\star(\alpha/2)} se(\hat{\theta}^\star), \hat{\theta} + \hat{t}^{\star(\alpha/2)} se(\hat{\theta}^\star))$$

by following the procedure outlined above we can compute the boot-p and boot-t confidence interval for θ_1, θ_2, and R.

4 Bayesian estimation

The Bayesian estimators for the unknown parameters of the xgamma distribution and the stress-strength reliability parameter R are presented in this section. Bayesian estimation requires the specification of a prior distribution for the unknown parameter. The choice of prior distribution depends on the available information about the parameter. In cases of limited or no information available, it is advisable to use a non-informative prior. On the other hand, an informative prior can be used if sufficient information is available. The gamma distribution is commonly chosen as an informative prior for the parameter in Bayesian estimation due to its flexibility, richness, and computational simplicity (see Yadav 2022; Singh et al. 2015). Therefore, in this study, we assume independent gamma priors for the unknown scale parameters θ_1 and θ_2 with hyper-parameters (a_1, b_1) and (a_2, b_2), respectively.

$$\pi_1(\theta_1; a_1, b_1) \propto \theta_1^{a_1-1} e^{-\frac{\theta_1}{b_1}}, \qquad \theta_1 > 0,\ a_1 > 0,\ b_1 > 0$$

and

$$\pi_2(\theta_2; a_2, b_2) \propto \theta_2^{a_2-1} e^{-\frac{\theta_2}{b_2}}, \qquad \theta_2 > 0,\ a_2 > 0,\ b_2 > 0$$

The hyperparameters associated with prior distributions can be easily computed when the prior mean and variance are specified. For instance, if an experimenter possesses prior knowledge regarding the expected value M_{θ_1} of the parameter θ_1, along with an associated variance V_{θ_1} reflecting the degree of confidence in this prior belief, the parameters of the prior distribution can be determined accordingly. Specifically, assuming a Gamma prior, we can express $M_{\theta_1} = a_1 b_1$ and $V_{\theta_1} = a_1 b_1^2$, which leads to the expressions $b_1 = \frac{V_{\theta_1}}{M_{\theta_1}}$ and $a_1 = \frac{M_{\theta_1}^2}{V_{\theta_1}}$. An analogous approach can be employed for determining the prior parameters of θ_2. It is important to emphasize that, in scenarios where prior knowledge about the parameters is limited or unavailable, it is advisable to set the prior variance to a sufficiently large value (approaching infinity), thereby yielding a non-informative or diffuse prior distribution that minimizes the influence of prior assumptions on the posterior inference.

The joint prior for the parameters (α, β) is written as

$$\pi(\theta_1, \theta_2) \propto \theta_1^{a_1-1} e^{-\frac{\theta_1}{b_1}} \theta_2^{a_2-1} e^{-\frac{\theta_2}{b_2}} \tag{16}$$

by combining the likelihood function (8) with the joint prior distribution (16), the posterior probability density function of (θ_1, θ_2) for the given GPH data is given by

$$\pi(\theta_1, \theta_2|data) = K^{-1} \frac{\theta_1^{2D_1^\star + a_1 - 1} \theta_2^{2D_2^\star + a_2 - 1}}{(\theta_1 + 1)^{D_1^\star} (\theta_2 + 1)^{D_2^\star}} \times e^{-\sum_{j=1}^{D_1^\star} \left[\theta_1 x_j - \log\left(1 + \frac{\theta_1 x_j^2}{2}\right) - R_{j1} \left\{ \log\left(\psi_3(x, \theta_1)\right) - \theta_1 x_j \right\} + \frac{\theta_1}{b_1}\right] + R^\star_{D_1+1} \left\{ \log\left(\psi_3(T_1, \theta_1)\right) - \theta_1 T_1 \right\}} \times e^{-\sum_{j=1}^{D_2^\star} \left[\theta_2 y_j - \log\left(1 + \frac{\theta_2 y_j^2}{2}\right) - R_{j2} \left\{ \log\left(\psi_3(y, \theta_2)\right) - \theta_2 y_j \right\} + \frac{\theta_2}{b_1}\right] + R^\star_{D_2+1} \left\{ \log\left(\psi_3(T_2, \theta_2)\right) - \theta_2 T_2 \right\}} \tag{17}$$

where

$$\psi_3(x, \theta_1) = \frac{1 + \theta_1 x_j + \theta_1^2 x_j^2}{\theta_1 + 1} \qquad \psi_3(T_1, \theta_1) = \frac{1 + \theta_1 T_1 + \theta_1^2 T_1^2}{\theta_1 + 1}$$
$$\psi_3(y, \theta_2) = \frac{1 + \theta_2 y_j + \theta_2^2 y_j^2}{\theta_2 + 1} \qquad \psi_3(T_2, \theta_2) = \frac{1 + \theta_2 T_2 + \theta_2^2 T_2^2}{\theta_2 + 1}$$

and K is a normalizing constant given as

$$K = \int_0^\infty \int_0^\infty \frac{\theta_1^{2D_1^\star + a_1 - 1} \theta_2^{2D_2^\star + a_2 - 1}}{(\theta_1 + 1)^{D_1^\star} (\theta_2 + 1)^{D_2^\star}} \times e^{-\sum_{j=1}^{D_1^\star} \left[\theta_1 x_j - \log\left(1 + \frac{\theta_1 x_j^2}{2}\right) - R_{j1} \left\{ \log\left(\psi_3(x, \theta_1)\right) - \theta_1 x_j \right\} + \frac{\theta_1}{b_1}\right] + R^\star_{D_1+1} \left\{ \log\left(\psi_3(T_1, \theta_1)\right) - \theta_1 T_1 \right\}} \times e^{-\sum_{j=1}^{D_2^\star} \left[\theta_2 y_j - \log\left(1 + \frac{\theta_2 y_j^2}{2}\right) - R_{j2} \left\{ \log\left(\psi_3(y, \theta_2)\right) - \theta_2 y_j \right\} + \frac{\theta_2}{b_1}\right] + R^\star_{D_2+1} \left\{ \log\left(\psi_3(T_2, \theta_2)\right) - \theta_2 T_2 \right\}} d\theta_1 d\theta_2$$

In Bayesian inference, the selection of the best estimator from a posterior distribution involves the use of a loss function. Here, both symmetric and asymmetric loss functions have been considered. The symmetric loss function, such as the squared error loss function (SELF), is commonly employed in Bayesian statistics. This loss function assigns equal importance to both overestimation and underestimation. On the other hand, the asymmetric loss function, such as the LINEX loss function (LLF), is helpful when overestimation or underestimation is more detrimental than the other. If $\hat{\zeta}$ be the estimator of ζ, then the SELF and LLF are given as

$$L_{SELF}(\zeta, \hat{\zeta}) = (\zeta - \hat{\zeta})^2, \tag{18}$$

$$L_{LLF}(\zeta, \hat{\zeta}) = e^{c(\hat{\zeta} - \zeta)} - c(\hat{\zeta} - \zeta) - 1; \qquad c \neq 0, \tag{19}$$

respectively. Where c is the loss parameter and if value of c in (19) is negative ($c < 0$), underestimation is considered more serious than overestimation. Conversely, when c is positive ($c > 0$), overestimation is considered more serious than underestimation. It is important to mention that when c in equation (19) approaches zero, the LLF behaves similarly to the SELF.

Under the squared error loss function and LINEX loss function, the Bayes estimators of any general parametric function $\Phi(\theta_1, \theta_2)$ are defined as follows:

$$\widehat{\Phi}_{SELF} \propto K^{-1} \int_0^\infty \int_0^\infty \Phi(\theta_1, \theta_2) \frac{\theta_1^{2D_1^\star + a_1 - 1} \theta_2^{2D_2^\star + a_2 - 1}}{(\theta_1 + 1)^{D_1^\star} (\theta_2 + 1)^{D_2^\star}} \times e^{-\sum_{j=1}^{D_1^\star} \left[\theta_1 x_j - \log\left(1 + \frac{\theta_1 x_j^2}{2}\right) - R_{j1} \left\{ \log\left(\psi_3(x, \theta_1)\right) - \theta_1 x_j \right\} + \frac{\theta_1}{b_1}\right] + R^\star_{D_1+1} \left\{ \log\left(\psi_3(T_1, \theta_1)\right) - \theta_1 T_1 \right\}} \times e^{-\sum_{j=1}^{D_2^\star} \left[\theta_2 y_j - \log\left(1 + \frac{\theta_2 y_j^2}{2}\right) - R_{j2} \left\{ \log\left(\psi_3(y, \theta_2)\right) - \theta_2 y_j \right\} + \frac{\theta_2}{b_1}\right] + R^\star_{D_2+1} \left\{ \log\left(\psi_3(T_2, \theta_2)\right) - \theta_2 T_2 \right\}} d\theta_1 d\theta_2 \tag{20}$$

and

$$\hat{\Phi}_{LLF} \propto -\left(\frac{1}{c}\right)\log\left[K^{-1}\int_0^\infty\int_0^\infty e^{-c\Phi(\theta_1,\theta_2)}\frac{\theta_1^{2D_1^\star+a_1-1}\theta_2^{2D_2^\star+a_2-1}}{(\theta_1+1)^{D_1^\star}(\theta_2+1)^{D_2^\star}} \times e^{-\sum_{j=1}^{D_1^\star}\left[\theta_1 x_j-\log\left(1+\frac{\theta_1 x_j^2}{2}\right)-R_{j1}\left\{\log\left(\psi_3(x,\theta_1)\right)-\theta_1 x_j\right\}+\frac{\theta_1}{b_1}\right]+R^\star_{D_1+1}\left\{\log\left(\psi_3(T_1,\theta_1)\right)-\theta_1 T_1\right\}} \times e^{-\sum_{j=1}^{D_2^\star}\left[\theta_2 y_j-\log\left(1+\frac{\theta_2 y_j^2}{2}\right)-R_{j2}\left\{\log\left(\psi_3(y,\theta_2)\right)-\theta_2 y_j\right\}+\frac{\theta_2}{b_1}\right]+R^\star_{D_2+1}\left\{\log\left(\psi_3(T_2,\theta_2)\right)-\theta_2 T_2\right\}}\right]d\theta_1 d\theta_2 \tag{21}$$

respectively. In equations (20) and (21), it is important to replace the general function $\Phi(\theta_1,\theta_2)$ with θ_1, θ_2 and R to obtain the desired Bayes estimates of parameters. However, evaluating the integrals in (20) and (21) explicitly is not feasible. Therefore, any Bayes approximation technique might be used to obtain the Bayes estimates. One commonly used approach for computing approximate Bayes estimates is the Markov chain Monte Carlo (MCMC) method. Several authors have employed MCMC techniques, for the same viz. Oramulu et al. (2023), Kumar et al. (2023), Hassan et al. (2024). Here, the same technique has been implemented to obtain the Bayes solution, the detail of the MCMC technique is given in the following subsection.

4.1 Markov Chain Monte Carlo methods

This section emphasises the use of Markov chain Monte Carlo (MCMC) methods, which aim to generate samples from complex distributions or posterior distributions of interest. In actuality, the name "Markov chain Monte Carlo" reflects the nature of the process, as it involves iteratively sampling new random values from the posterior distribution based on the previous value. A Markov chain, which is a series of values taken from the posterior distribution, is produced by this iterative process. The Metropolis-Hastings (MH) algorithm and the Gibbs sampler are the two most widely used MCMC techniques. See Hastings (1970) for further information on the Metropolis-Hastings (MH) method, which generates samples from any given probability distribution function. This approach has several advantages over other sampling like inversion and accept-reject sampling, including the ability to work successfully with multivariate distributions and the lack of the necessity for an envelop function and a normalising constant in rejection sampling.

By employing the Metropolis-Hastings (M-H) algorithm, we can compute the Bayes estimates and determine the corresponding highest posterior density (HPD) credible intervals for the unknown parameters θ_1 and θ_2 of the xgamma distribution, using generalised progressive hybrid censored (GPHC) sample. The conditional posterior densities of θ_1 and θ_2 can be formulated in the following manner.

$$\pi^\star(\theta_1|\theta_2, data) \propto \frac{\theta_1^{2D_1^\star+a_1-1}}{(\theta_1+1)^{D_1^\star}} \times e^{-\sum_{j=1}^{D_1^\star}\left[\theta_1\left(x_j+R_{j1}x_j+\frac{1}{b_1}\right)-\log\left(1+\frac{\theta_1 x_j^2}{2}\right)-R_{j1}\log\psi_3(x,\theta_1)\right]+R^\star_{D_1+1}\left(\log\psi_3(T_1,\theta_1)-\theta_1 T_1\right)} \tag{22}$$

and

$$\pi^\star(\theta_2|\theta_1, data) \propto \frac{\theta_2^{2D_2^\star+a_2-1}}{(\theta_2+1)^{D_2^\star}} \times e^{-\sum_{j=1}^{D_2^\star}\left[\theta_2\left(y_j+R_{j2}y_j+\frac{1}{b_2}\right)-\log\left(1+\frac{\theta_2 y_j^2}{2}\right)-R_{j2}\log\psi_3(y,\theta_2)\right]+R^\star_{D_2+1}\left(\log\psi_3(T_2,\theta_2)-\theta_2 T_2\right)} \tag{23}$$

In order to extract the posterior sample from the above full conditional distribution the following steps are used.

- Initialize (θ_1,θ_2) as $\left(\theta_1^{(0)}, \theta_2^{(0)}\right)$ and set $j = 1$.
- Draw $\theta_1^{(j)}$ and $\theta_2^{(j)}$ by sampling from the normal proposal distributions $N\left(\theta_1^{(j-1)}, Var(\theta_1)\right)$ and $N\left(\theta_2^{(j-1)}, Var(\theta_2)\right)$, respectively, for $j = 1, 2, \dots, N$
- Compute the acceptance probabilities for the generated samples.

$$\xi_{\theta_1} = min\left\{1, \frac{\pi^\star\left(\theta_1^{(j)}|\theta_2^{(j-1)}, data\right)}{\pi^\star\left(\theta_1^{(j-1)}|\theta_2^{(j-1)}, data\right)}\right\},$$

$$\xi_{\theta_2} = min\left\{1, \frac{\pi^\star\left(\theta_2^{(j)}|\theta_1^{(j)}, data\right)}{\pi^\star\left(\theta_2^{(j-1)}|\theta_1^{(j)}, data\right)}\right\}.$$

- Generate sample u_1 and u_2 from the uniform distribution on the interval (0, 1).
- If u_1 is greater than the threshold $\xi_{\theta_1}(u_1 > \xi_{\theta_1})$, then set $\theta_1^{(j)} = \theta_1^{(j)}$; otherwise set $\theta_1^{(j)} = \theta_1^{(j-1)}$. Similarly, if u_2 is greater than the threshold $\xi_{\theta_2}(u_2 > \xi_{\theta_2})$, then set $\theta_2^{(j)} = \theta_2^{(j)}$; otherwise set $\theta_2^{(j)} = \theta_2^{(j-1)}$.
- Compute $R^{(j)}$ from equation (7).
- Set $j = j + 1$.
- Replicate **steps 2-7** for a total of N iterations and obtain $\left(\theta_1^{(j)}, \theta_2^{(j)}, R^{(j)}\right)$ for $j = 1, 2, \dots, N$.

Based on the generated posterior sample, the Bayes estimates of θ_1, θ_2 and R under SELF and LLF are obtained as

$$\hat{\theta}_{1,self} = \frac{1}{N - N_0} \sum_{j=N_0+1}^{N} \theta_1^{(j)},$$

$$\hat{\theta}_{1,llf} = -\left(\frac{1}{c}\right) \log\left[\frac{1}{N - N_0} \sum_{j=N_0+1}^{N} e^{-c\theta_1^{(j)}}\right],$$

$$\hat{\theta}_{2,self} = \frac{1}{N - N_0} \sum_{j=N_0+1}^{N} \theta_2^{(j)},$$

$$\hat{\theta}_{2,llf} = -\left(\frac{1}{c}\right) \log\left[\frac{1}{N - N_0} \sum_{j=N_0+1}^{N} e^{-c\theta_2^{(j)}}\right],$$

and

$$\hat{R}_{self} = \frac{1}{N - N_0} \sum_{j=N_0+1}^{N} R^{(j)},$$

$$\hat{R}_{llf} = -\left(\frac{1}{c}\right) \log\left[\frac{1}{N - N_0} \sum_{j=N_0+1}^{N} e^{-cR^{(j)}}\right].$$

respectively, where N_0 represents the burn-in period of the Markov chain and c is not equal to 0. The $100(1-\alpha)\%$ HPD credible intervals for θ_1, θ_2 and R can be obtained using the idea of Chen and Shao (1999). Hence, the obtained intervals are given as below

$$\left(\theta_1^{\lfloor (N-N_0)\alpha/2 \rfloor}, \theta_1^{\lfloor (N-N_0)(1-\alpha/2) \rfloor}\right),$$

$$\left(\theta_2^{\lfloor (N-N_0)\alpha/2 \rfloor}, \theta_2^{\lfloor (N-N_0)(1-\alpha/2) \rfloor}\right)$$

and

$$\left(R^{\lfloor (N-N_0)\alpha/2 \rfloor}, R^{\lfloor (N-N_0)(1-\alpha/2) \rfloor}\right).$$

where, $\lfloor . \rfloor$ is the greatest integer function.

5 Simulation study

In this section, the performance of the proposed estimators are studied using a Monte Carlo simulation study. To generate the GPH censored samples from xgamma distribution, we have considered different values of censoring parameters n_1, n_2, m_1, m_2, k_1 and k_2, with model parameters $\theta_1 = 0.5, 1$, $\theta_2 = 1$ and time thresholds $T_1 = 0.5, 0.75$ and $T_2 = 0.5, 0.75$. The simulation results are obtained through 10,000 replications. Based on the mean square error (MSE) and the average width of the confidence intervals, the performance of the point and interval estimates are examined. The Coverage probabilities are also determined to assess the confidence level of the intervals for the considered censoring scheme. Following are the three censoring schemes have been considered.

- **Scheme I:** $R_1 = n_i - m_i, R_2 = \cdots = R_{m_i} = 0$.
- **Scheme II:** $R_1 = \cdots = R_{m_i - 1} = 0, R_{m_i} = n_i - m_i$.
- **Scheme III:** $R_1 = \cdots = R_{n_i - m_i} = 1, R_{n_i - m_i + 1} = \cdots = R_{m_i} = 0 i = 1, 2$.

Under considered loss function, we use non-informative prior (NIP) and informative prior (IP) distributions to calculate the Bayes estimates. For the LLF, we have examined three different values of c, namely $-1.5, 0.05$, and 1.5. In the case of IP, the hyperparameters are determined using a specific formula, while in the case of NIP, they are assumed to be approximately zero.

The MLEs of the parameters along with the interval estimates, for the aforementioned considered variation of the censoring schemes have been computed. boot-p and boot-t confidence intervals are also calculated for the the same variation of censoring parameters. Furthermore, the Bayes estimates of the parameters using informative and

Table 1 Average estimates and their MSEs (in parenthesis) for the unknown model parameter θ_1, when $\theta_1 = 1$ and $\theta_2 = 1$ with different values of censoring parameters

(T_1, T_2)	CS	(n_1, n_2)	(m_1, m_2)	(k_1, k_2)	MLE	SELF		LLF					
								−1.5		0.05		1.5	
					$\hat{\theta}_1$	NIP	IP	NIP	IP	NIP	IP	NIP	IP
(0.5,0.5)	I	(20,25)	(15,18)	(12,16)	1.05788 (0.06051)	1.05780 (0.06019)	1.04246 (0.01797)	1.09478 (0.07699)	1.06172 (0.02141)	1.05663 (0.05973)	1.04184 (0.01787)	1.02432 (0.04893)	1.02413 (0.01548)
			(17,22)	(15,20)	1.04646 (0.04742)	1.04683 (0.04719)	1.03424 (0.01415)	1.07511 (0.05719)	1.04925 (0.01629)	1.04592 (0.04691)	1.03375 (0.01408)	1.02060 (0.04008)	1.01980 (0.01255)
		(30,40)	(25,30)	(20,24)	1.04098 (0.02936)	1.03624 (0.03159)	1.02959 (0.00878)	1.05622 (0.0365)	1.03976 (0.00986)	1.03559 (0.03144)	1.02926 (0.00875)	1.01728 (0.02791)	1.01669 (0.00795)
			(27,35)	(23,30)	1.03052 (0.02653)	1.03054 (0.02692)	1.01858 (0.00636)	1.04764 (0.03048)	1.02683 (0.00696)	1.02998 (0.02682)	1.01831 (0.00634)	1.01420 (0.02423)	1.01051 (0.00592)
		(50,60)	(40,50)	(30,40)	1.02390 (0.01716)	1.02439 (0.01763)	1.01702 (0.00452)	1.03700 (0.01946)	1.02327 (0.00489)	1.02398 (0.01758)	1.01681 (0.00451)	1.01219 (0.01622)	1.00986 (0.00423)
			(45,56)	(35,46)	1.02213 (0.01627)	1.02197 (0.01624)	1.01731 (0.00430)	1.03272 (0.01765)	1.02291 (0.00460)	1.02162 (0.01619)	1.01713 (0.00429)	1.01153 (0.01513)	1.00980 (0.00407)
	II	(20,25)	(15,18)	(12,16)	1.05501 (0.05583)	1.05459 (0.05557)	1.03951 (0.01623)	1.09125 (0.07099)	1.05802 (0.01930)	1.05343 (0.05515)	1.03891 (0.01614)	1.02134 (0.04527)	1.02006 (0.01402)
			(17,22)	(15,20)	1.04602 (0.04548)	1.04590 (0.04537)	1.03375 (0.01349)	1.07390 (0.05500)	1.04844 (0.01554)	1.04501 (0.04510)	1.03327 (0.01343)	1.01992 (0.03855)	1.01860 (0.01197)
		(30,40)	(25,30)	(20,24)	1.03940 (0.03182)	1.03924 (0.03185)	1.03013 (0.00931)	1.05939 (0.03691)	1.04070 (0.01047)	1.03959 (0.03170)	1.02999 (0.00927)	1.01913 (0.02805)	1.01785 (0.00841)
			(27,35)	(23,30)	1.03477 (0.02794)	1.03472 (0.02792)	1.02705 (0.00828)	1.05190 (0.03173)	1.03623 (0.00919)	1.03416 (0.02781)	1.02675 (0.00826)	1.01831 (0.02501)	1.01810 (0.00758)
		(50,60)	(40,50)	(30,40)	1.02953 (0.01963)	1.02918 (0.01955)	1.02367 (0.00575)	1.04199 (0.02162)	1.03037 (0.00628)	1.02886 (0.01949)	1.02344 (0.00574)	1.01707 (0.01791)	1.01678 (0.00533)
			(45,56)	(35,46)	1.02357 (0.01669)	1.02341 (0.01669)	1.01866 (0.00486)	1.03417 (0.01816)	1.02435 (0.00523)	1.02305 (0.01665)	1.01847 (0.00485)	1.01305 (0.01552)	1.01195 (0.00457)
	III	(20,25)	(15,18)	(12,16)	1.06351 (0.06269)	1.06319 (0.06223)	1.04778 (0.01946)	1.10062 (0.07971)	1.06748 (0.02327)	1.06201 (0.06176)	1.04714 (0.01935)	1.02927 (0.05039)	1.02804 (0.01666)
			(17,22)	(15,20)	1.05010 (0.04772)	1.05040 (0.04754)	1.03767 (0.01456)	1.07891 (0.05785)	1.05284 (0.01684)	1.04949 (0.04725)	1.03718 (0.01449)	1.02398 (0.04020)	1.02208 (0.01285)
		(30,40)	(25,30)	(20,24)	1.03433 (0.03172)	1.03426 (0.03170)	1.02536 (0.00911)	1.05420 (0.03655)	1.03587 (0.01017)	1.03361 (0.03166)	1.02502 (0.00908)	1.01535 (0.02808)	1.01413 (0.00831)
			(27,35)	(23,30)	1.03128 (0.02678)	1.03143 (0.02670)	1.02386 (0.00766)	1.04854 (0.03029)	1.03285 (0.00846)	1.03087 (0.02660)	1.02356 (0.00764)	1.01507 (0.02398)	1.01407 (0.00705)
		(50,60)	(40,50)	(30,40)	1.02775 (0.02015)	1.02773 (0.02014)	1.02191 (0.00589)	1.04049 (0.02224)	1.02868 (0.00640)	1.02732 (0.02008)	1.02169 (0.00587)	1.01441 (0.01849)	1.01325 (0.00548)
			(45,56)	(35,46)	1.02176 (0.01636)	1.02162 (0.01635)	1.01703 (0.00465)	1.03232 (0.01775)	1.02264 (0.00498)	1.02127 (0.01631)	1.01684 (0.00464)	1.01150 (0.01525)	1.01021 (0.00438)

Table 2 Average estimates and their MSEs (in parenthesis) for the unknown model parameter θ_2, when $\theta_1 = 1$ and $\theta_2 = 1$ with different values of censoring parameters

(T_1, T_2)	CS	(n_1, n_2)	(m_1, m_2)	(k_1, k_2)	MLE	SELF		LLF −1.5		LLF 0.05		LLF 1.5	
					$\hat{\theta}_2$	NIP	IP	NIP	IP	NIP	IP	NIP	IP
(0.5,0.5)	I	(20,25)	(15,18)	(12,16)	1.03104 (0.03618)	1.03146 (0.03619)	1.02108 (0.00940)	1.05666 (0.04287)	1.03369 (0.01060)	1.03065 (0.03600)	1.02066 (0.00936)	1.00888 (0.03151)	1.00788 (0.00856)
			(17,22)	(15,20)	1.04131 (0.03107)	1.04152 (0.03099)	1.03307 (0.00919)	1.06191 (0.03595)	1.04363 (0.01038)	1.04086 (0.03085)	1.03272 (0.00915)	1.02278 (0.02728)	1.02219 (0.00826)
		(30,40)	(25,30)	(20,24)	1.03078 (0.02152)	1.02785 (0.02125)	1.02305 (0.00562)	1.04390 (0.02398)	1.03093 (0.00623)	1.02732 (0.02117)	1.02279 (0.00560)	1.01534 (0.01922)	1.01246 (0.00515)
			(27,35)	(23,30)	1.02209 (0.01871)	1.02199 (0.01826)	1.01228 (0.00438)	1.03462 (0.02005)	1.01845 (0.00469)	1.02158 (0.01821)	1.01208 (0.00437)	1.00979 (0.01689)	1.00695 (0.00416)
		(50,60)	(40,50)	(30,40)	1.01611 (0.01265)	1.01570 (0.01139)	1.01151 (0.00300)	1.02487 (0.01223)	1.01612 (0.00319)	1.01540 (0.01137)	1.01135 (0.00300)	1.00675 (0.01074)	1.00642 (0.00286)
			(45,56)	(35,46)	1.01369 (0.01100)	1.01387 (0.01103)	1.01038 (0.00289)	1.02182 (0.01171)	1.01437 (0.00304)	1.01361 (0.01101)	1.01024 (0.00289)	1.00620 (0.01050)	1.00609 (0.00277)
	II	(20,25)	(15,18)	(12,16)	1.03245 (0.03499)	1.03227 (0.03498)	1.02197 (0.00899)	1.05735 (0.04154)	1.03434 (0.01018)	1.03146 (0.03479)	1.02157 (0.00896)	1.01143 (0.03038)	1.00880 (0.00815)
			(17,22)	(15,20)	1.02851 (0.02669)	1.02848 (0.02665)	1.02082 (0.00701)	1.04794 (0.03056)	1.03044 (0.00779)	1.02785 (0.02654)	1.02050 (0.00699)	1.01100 (0.02381)	1.00800 (0.00644)
		(30,40)	(25,30)	(20,24)	1.02574 (0.02106)	1.02560 (0.02103)	1.01923 (0.00535)	1.04159 (0.02369)	1.02701 (0.00589)	1.02508 (0.02095)	1.01897 (0.00534)	1.01060 (0.01905)	1.00726 (0.00494)
			(27,35)	(23,30)	1.01769 (0.01867)	1.01771 (0.01868)	1.01214 (0.00519)	1.03016 (0.02039)	1.01861 (0.00554)	1.01730 (0.01863)	1.01193 (0.00518)	1.00578 (0.01738)	1.00557 (0.00493)
		(50,60)	(40,50)	(30,40)	1.01284 (0.01271)	1.01269 (0.01271)	1.00871 (0.00325)	1.02180 (0.01357)	1.01331 (0.00342)	1.01239 (0.01268)	1.00856 (0.00324)	1.00418 (0.01204)	1.00378 (0.00312)
			(45,56)	(35,46)	1.00789 (0.01066)	1.00801 (0.01181)	1.00378 (0.00334)	1.01706 (0.01260)	1.00842 (0.00354)	1.00772 (0.01179)	1.00426 (0.00267)	0.99919 (0.01118)	0.99919 (0.00318)
	III	(20,25)	(15,18)	(12,16)	1.04426 (0.03747)	1.04444 (0.03744)	1.03576 (0.01036)	1.06499 (0.04454)	1.04658 (0.01179)	1.04377 (0.03724)	1.03541 (0.01032)	1.02524 (0.03242)	1.02398 (0.00933)
			(17,22)	(15,20)	1.03732 (0.03237)	1.03784 (0.03220)	1.02680 (0.00985)	1.06347 (0.03746)	1.03970 (0.01115)	1.03702 (0.03204)	1.02638 (0.00981)	1.01818 (0.02822)	1.01741 (0.00882)
		(30,40)	(25,30)	(20,24)	1.02962 (0.02178)	1.02982 (0.02174)	1.02456 (0.00572)	1.04266 (0.02437)	1.03104 (0.00625)	1.02940 (0.02166)	1.02434 (0.00570)	1.01432 (0.01980)	1.01290 (0.00532)
			(27,35)	(23,30)	1.02391 (0.01809)	1.02398 (0.01812)	1.01739 (0.00511)	1.03994 (0.02008)	1.02528 (0.00560)	1.02346 (0.01806)	1.01713 (0.00509)	1.01065 (0.01659)	1.01059 (0.00471)
		(50,60)	(40,50)	(30,40)	1.01832 (0.01313)	1.01847 (0.01310)	1.01476 (0.00348)	1.02653 (0.01389)	1.01892 (0.00361)	1.01821 (0.01307)	1.01462 (0.00348)	1.00966 (0.01250)	1.00868 (0.00339)
			(45,56)	(35,46)	1.00789 (0.01180)	1.00801 (0.01181)	1.00378 (0.00334)	1.01706 (0.01260)	1.00842 (0.00354)	1.00772 (0.01179)	1.00362 (0.00333)	0.99919 (0.01118)	0.99919 (0.00318)

Table 3 Average estimates and their MSEs (in parenthesis) for the stress-strength parameter R, when $\theta_1 = 1$ and $\theta_2 = 1$ with different values censoring parameters

(T_1, T_2)	CS	(n_1, n_2)	(m_1, m_2)	(k_1, k_2)	MLE	SELF		LLF					
								−1.5		0.05		1.5	
					$\hat{R}$	NIP	IP	NIP	IP	NIP	IP	NIP	IP
(0.5,0.5)	I	(20,25)	(15,18)	(12,16)	0.49412 (0.00934)	0.49627 (0.00869)	0.49928 (0.00298)	0.50238 (0.00867)	0.49894 (0.00296)	0.49607 (0.00869)	0.49916 (0.00298)	0.49020 (0.00875)	0.49164 (0.00303)
			(17,22)	(15,20)	0.50076 (0.00783)	0.50217 (0.00740)	0.50155 (0.00251)	0.50717 (0.00744)	0.50453 (0.00253)	0.50200 (0.00740)	0.50145 (0.00251)	0.49718 (0.00740)	0.49858 (0.00251)
		(30,40)	(25,30)	(20,24)	0.49775 (0.00555)	0.49946 (0.00540)	0.49977 (0.00160)	0.50343 (0.00541)	0.50095 (0.00159)	0.49932 (0.00540)	0.49970 (0.00160)	0.49549 (0.00542)	0.49659 (0.00161)
			(27,35)	(23,30)	0.49804 (0.00489)	0.49939 (0.00461)	0.49961 (0.00125)	0.50276 (0.00461)	0.50042 (0.00125)	0.49927 (0.00461)	0.49955 (0.00125)	0.49601 (0.00462)	0.49681 (0.00126)
		(50,60)	(40,50)	(30,40)	0.49876 (0.00364)	0.49876 (0.00322)	0.49991 (0.00089)	0.50135 (0.00322)	0.50009 (0.00089)	0.49867 (0.00322)	0.49996 (0.00089)	0.49617 (0.00324)	0.49733 (0.00089)
			(45,56)	(35,46)	0.49787 (0.00309)	0.49866 (0.00301)	0.49992 (0.00087)	0.50091 (0.00301)	0.49944 (0.00087)	0.49858 (0.00301)	0.49998 (0.00087)	0.49641 (0.00302)	0.49700 (0.00088)
	II	(20,25)	(15,18)	(12,16)	0.49519 (0.00909)	0.49730 (0.00845)	0.49835 (0.00276)	0.50340 (0.00844)	0.49992 (0.00275)	0.49709 (0.00845)	0.49824 (0.00276)	0.49123 (0.00852)	0.49280 (0.00280)
			(17,22)	(15,20)	0.49682 (0.00733)	0.49844 (0.00692)	0.49971 (0.00225)	0.50341 (0.00693)	0.50059 (0.00225)	0.49828 (0.00692)	0.49962 (0.00225)	0.49349 (0.00695)	0.49484 (0.00227)
		(30,40)	(25,30)	(20,24)	0.49689 (0.00571)	0.49742 (0.00545)	0.49845 (0.00167)	0.50171 (0.00545)	0.49967 (0.00167)	0.49761 (0.00545)	0.49838 (0.00167)	0.49377 (0.00548)	0.49523 (0.00169)
			(27,35)	(23,30)	0.49737 (0.00498)	0.49774 (0.00479)	0.49881 (0.00153)	0.49977 (0.00478)	0.49974 (0.00152)	0.49731 (0.00479)	0.49875 (0.00153)	0.49389 (0.00482)	0.49589 (0.00155)
		(50,60)	(40,50)	(30,40)	0.49819 (0.00355)	0.49811 (0.00344)	0.49956 (0.00103)	0.49870 (0.00343)	0.49898 (0.00102)	0.49802 (0.00344)	0.49951 (0.00103)	0.49452 (0.00346)	0.49614 (0.00105)
			(45,56)	(35,46)	0.49937 (0.00319)	0.49912 (0.00311)	0.49967 (0.00092)	0.49936 (0.00310)	0.49989 (0.00091)	0.49904 (0.00311)	0.49963 (0.00092)	0.49587 (0.00313)	0.49745 (0.00093)
	III	(20,25)	(15,18)	(12,16)	0.49435 (0.00931)	0.49663 (0.00864)	0.49753 (0.00306)	0.50274 (0.00862)	0.49921 (0.00304)	0.49642 (0.00864)	0.49540 (0.00306)	0.49054 (0.00871)	0.49186 (0.00310)
			(17,22)	(15,20)	0.50040 (0.00791)	0.50185 (0.00746)	0.50022 (0.00259)	0.50684 (0.00750)	0.50422 (0.00260)	0.50168 (0.00746)	0.50012 (0.00259)	0.49687 (0.00746)	0.49823 (0.00259)
		(30,40)	(25,30)	(20,24)	0.49784 (0.00595)	0.49770 (0.00568)	0.49838 (0.00176)	0.50267 (0.00569)	0.50062 (0.00175)	0.49757 (0.00568)	0.49830 (0.00176)	0.49474 (0.00570)	0.49613 (0.00177)
			(27,35)	(23,30)	0.50091 (0.00474)	0.50183 (0.00456)	0.50139 (0.00139)	0.50521 (0.00458)	0.50329 (0.00140)	0.50172 (0.00456)	0.50133 (0.00139)	0.49846 (0.00455)	0.49949 (0.00139)
		(50,60)	(40,50)	(30,40)	0.49802 (0.00384)	0.49893 (0.00372)	0.49938 (0.00114)	0.49751 (0.00371)	0.49582 (0.00113)	0.49885 (0.00372)	0.49933 (0.00115)	0.49836 (0.00375)	0.49994 (0.00116)
			(45,56)	(35,46)	0.49956 (0.00324)	0.50030 (0.00316)	0.49981 (0.00094)	0.50255 (0.00317)	0.50104 (0.00094)	0.50023 (0.00316)	0.49977 (0.00094)	0.49906 (0.00316)	0.49957 (0.00094)

Table 4 Average estimates and their MSEs (in parenthesis) for the unknown model parameter θ_1, when $\theta_1 = 0.5$ and $\theta_2 = 1$ with different values of censoring parameters

(T_1, T_2)	CS	(n_1, n_2)	(m_1, m_2)	(k_1, k_2)	MLE	SELF		LLF −1.5		LLF 0.05		LLF 1.5	
					$\hat{\theta}_1$	NIP	IP	NIP	IP	NIP	IP	NIP	IP
(0.75,0.75)	I	(20,25)	(15,18)	(12,16)	0.52073 (0.01073)	0.52245 (0.01168)	0.51580 (0.00314)	0.52966 (0.01282)	0.51944 (0.00336)	0.52222 (0.01165)	0.51568 (0.00313)	0.51551 (0.01072)	0.51222 (0.00295)
			(17,22)	(15,20)	0.51651 (0.00815)	0.51830 (0.00936)	0.51277 (0.00243)	0.52399 (0.01008)	0.51562 (0.00256)	0.51811 (0.00934)	0.51267 (0.00242)	0.51277 (0.00875)	0.50995 (0.00231)
		(30,40)	(25,30)	(20,24)	0.51402 (0.00572)	0.51441 (0.00631)	0.51081 (0.00165)	0.51844 (0.00667)	0.51284 (0.00172)	0.51427 (0.00630)	0.51074 (0.00164)	0.51046 (0.00600)	0.50879 (0.00158)
			(27,35)	(23,30)	0.51287 (0.00476)	0.51219 (0.00540)	0.50960 (0.00136)	0.51569 (0.00567)	0.51134 (0.00142)	0.51208 (0.00539)	0.50955 (0.00136)	0.50876 (0.00517)	0.50789 (0.00131)
		(50,60)	(40,50)	(30,40)	0.50676 (0.00356)	0.50978 (0.00357)	0.50635 (0.00091)	0.51234 (0.00371)	0.50764 (0.00094)	0.50969 (0.00357)	0.50630 (0.00091)	0.50724 (0.00345)	0.50507 (0.00089)
			(45,56)	(35,46)	0.50729 (0.00307)	0.50894 (0.00330)	0.50632 (0.00086)	0.51114 (0.00341)	0.50743 (0.00088)	0.50886 (0.00329)	0.50628 (0.00085)	0.50676 (0.0032)	0.50521 (0.00083)
	II	(20,25)	(15,18)	(12,16)	0.52209 (0.01031)	0.52058 (0.01063)	0.51564 (0.00297)	0.52748 (0.01163)	0.51915 (0.00318)	0.52035 (0.0106)	0.51552 (0.00297)	0.51391 (0.00979)	0.51219 (0.00280)
			(17,22)	(15,20)	0.52308 (0.00890)	0.51769 (0.00889)	0.51545 (0.00266)	0.52319 (0.00955)	0.51837 (0.00282)	0.51751 (0.00887)	0.51535 (0.00265)	0.51234 (0.00832)	0.51257 (0.00252)
		(30,40)	(25,30)	(20,24)	0.51031 (0.00545)	0.51533 (0.00621)	0.50960 (0.00154)	0.51928 (0.00656)	0.51156 (0.00161)	0.51519 (0.00620)	0.50953 (0.00154)	0.51144 (0.00589)	0.50765 (0.00148)
			(27,35)	(23,30)	0.51217 (0.00476)	0.51386 (0.00554)	0.51016 (0.00140)	0.51732 (0.00583)	0.51189 (0.00145)	0.51374 (0.00554)	0.51011 (0.00139)	0.51045 (0.00530)	0.50846 (0.00134)
		(50,60)	(40,50)	(30,40)	0.50774 (0.00353)	0.51161 (0.00386)	0.50757 (0.00100)	0.51414 (0.00401)	0.50885 (0.00103)	0.51112 (0.00402)	0.50716 (0.00098)	0.50911 (0.00372)	0.50631 (0.00097)
			(45,56)	(35,46)	0.50418 (0.00276)	0.50869 (0.00332)	0.50358 (0.00075)	0.51089 (0.00343)	0.50564 (0.00077)	0.50861 (0.00332)	0.50355 (0.00075)	0.50651 (0.00322)	0.50354 (0.00074)
	III	(20,25)	(15,18)	(12,16)	0.51878 (0.00982)	0.52519 (0.01212)	0.51584 (0.00299)	0.53247 (0.01333)	0.51935 (0.00320)	0.52495 (0.01209)	0.51573 (0.00298)	0.51818 (0.01111)	0.51240 (0.00281)
			(17,22)	(15,20)	0.51209 (0.00855)	0.52000 (0.00945)	0.50915 (0.00264)	0.52573 (0.01019)	0.51908 (0.00281)	0.51981 (0.00943)	0.50906 (0.00264)	0.51444 (0.00881)	0.51327 (0.0025)
		(30,40)	(25,30)	(20,24)	0.51132 (0.00536)	0.51350 (0.00629)	0.50903 (0.00152)	0.51753 (0.00665)	0.51100 (0.00159)	0.51337 (0.00628)	0.50896 (0.00152)	0.50956 (0.00598)	0.50708 (0.00147)
			(27,35)	(23,30)	0.51135 (0.00523)	0.51262 (0.00540)	0.50920 (0.00147)	0.51613 (0.00567)	0.51101 (0.00153)	0.51251 (0.00539)	0.50914 (0.00147)	0.50917 (0.00516)	0.50740 (0.00142)
		(50,60)	(40,50)	(30,40)	0.50765 (0.00343)	0.51120 (0.00403)	0.50720 (0.00098)	0.51379 (0.00419)	0.50847 (0.00101)	0.51112 (0.00402)	0.50716 (0.00098)	0.50865 (0.00389)	0.50594 (0.00095)
			(45,56)	(35,46)	0.50418 (0.00276)	0.50869 (0.00332)	0.50358 (0.00075)	0.51089 (0.00343)	0.50564 (0.00077)	0.50861 (0.00332)	0.50355 (0.00075)	0.50651 (0.00322)	0.50354 (0.00074)

Table 5 Average estimates and their MSEs (in parenthesis) for the unknown model parameter θ_2, when $\theta_1 = 0.5$ and $\theta_2 = 1$ with different values of censoring parameters

(T_1,T_2)	CS	(n_1,n_2)	(m_1,m_2)	(k_1,k_2)	MLE	SELF		LLF					
								−1.5		0.05		1.5	
					$\hat{\theta}_2$	NIP	IP	NIP	IP	NIP	IP	NIP	IP
(0.75,0.75)	I	(20,25)	(15,18)	(12,16)	1.03601 (0.03439)	1.03166 (0.03616)	1.02731 (0.00909)	1.06193 (0.04290)	1.03763 (0.0103)	1.03085 (0.03598)	1.02791 (0.00906)	1.01207 (0.03147)	1.01138 (0.00823)
			(17,22)	(15,20)	1.02858 (0.03015)	1.03037 (0.03098)	1.02329 (0.00858)	1.05675 (0.03593)	1.03567 (0.00962)	1.03071 (0.03084)	1.02294 (0.00855)	1.01106 (0.02729)	1.01018 (0.00779)
		(30,40)	(25,30)	(20,24)	1.02249 (0.02125)	1.02776 (0.02120)	1.01909 (0.00538)	1.04383 (0.02392)	1.02691 (0.00592)	1.02724 (0.02112)	1.01883 (0.00537)	1.01038 (0.01918)	1.01003 (0.00498)
			(27,35)	(23,30)	1.02131 (0.01753)	1.02190 (0.01826)	1.01737 (0.00476)	1.03453 (0.02005)	1.02369 (0.00515)	1.02148 (0.01821)	1.01716 (0.00475)	1.00967 (0.01688)	1.00916 (0.00446)
		(50,60)	(40,50)	(30,40)	1.01737 (0.0128)	1.01573 (0.01140)	1.01299 (0.00308)	1.02491 (0.01225)	1.01762 (0.00329)	1.01543 (0.01138)	1.01284 (0.00308)	1.00977 (0.01075)	1.00841 (0.00292)
			(45,56)	(35,46)	1.01172 (0.01179)	1.01183 (0.01104)	1.00940 (0.00305)	1.02178 (0.01173)	1.01353 (0.00320)	1.01157 (0.01102)	1.00927 (0.00305)	1.00605 (0.01051)	1.00532 (0.00294)
	II	(20,25)	(15,18)	(12,16)	1.04029 (0.03687)	1.03229 (0.03489)	1.02566 (0.00966)	1.05740 (0.04146)	1.03834 (0.01101)	1.03148 (0.03471)	1.02525 (0.00962)	1.01339 (0.03030)	1.00880 (0.00868)
			(17,22)	(15,20)	1.03458 (0.03136)	1.02855 (0.02664)	1.02312 (0.00813)	1.04809 (0.03058)	1.03251 (0.00906)	1.02792 (0.02653)	1.02378 (0.00810)	1.01201 (0.02380)	1.00800 (0.00744)
		(30,40)	(25,30)	(20,24)	1.03154 (0.02314)	1.02556 (0.02105)	1.02249 (0.00604)	1.04158 (0.02371)	1.03163 (0.00669)	1.02504 (0.02098)	1.02222 (0.00602)	1.01152 (0.01909)	1.00719 (0.00553)
			(27,35)	(23,30)	1.02025 (0.01544)	1.01763 (0.01867)	1.01323 (0.00439)	1.03007 (0.02036)	1.01909 (0.00469)	1.01722 (0.01862)	1.01304 (0.00438)	1.00746 (0.01738)	1.00558 (0.00416)
		(50,60)	(40,50)	(30,40)	1.01499 (0.01208)	1.01281 (0.01270)	1.00991 (0.00312)	1.02193 (0.01357)	1.01439 (0.00329)	1.01251 (0.01268)	1.00976 (0.00311)	1.00548 (0.01203)	1.00390 (0.00299)
			(45,56)	(35,46)	1.01126 (0.01022)	1.00790 (0.01067)	1.00613 (0.00259)	1.01572 (0.01125)	1.00996 (0.00269)	1.00764 (0.01065)	1.00600 (0.00258)	1.00233 (0.01023)	1.00023 (0.00251)
	III	(20,25)	(15,18)	(12,16)	1.03940 (0.03605)	1.03881 (0.03644)	1.02772 (0.01007)	1.06549 (0.04453)	1.04340 (0.01147)	1.03799 (0.03725)	1.02731 (0.01003)	1.02509 (0.03242)	1.02321 (0.00905)
			(17,22)	(15,20)	1.03862 (0.02809)	1.03856 (0.02827)	1.02603 (0.00856)	1.06313 (0.03751)	1.04010 (0.00967)	1.03889 (0.02912)	1.02669 (0.00853)	1.01546 (0.02831)	1.01382 (0.00769)
		(30,40)	(25,30)	(20,24)	1.02644 (0.02195)	1.02695 (0.02172)	1.01893 (0.00584)	1.04292 (0.02435)	1.02787 (0.00640)	1.02643 (0.02165)	1.01867 (0.00582)	1.01325 (0.01978)	1.01263 (0.00541)
			(27,35)	(23,30)	1.02209 (0.01958)	1.02267 (0.01908)	1.01712 (0.00527)	1.03951 (0.02004)	1.02683 (0.00574)	1.02225 (0.01803)	1.01789 (0.00525)	1.01115 (0.01655)	1.00851 (0.00489)
		(50,60)	(40,50)	(30,40)	1.01023 (0.01242)	1.01378 (0.01213)	1.01202 (0.00331)	1.02683 (0.01391)	1.01644 (0.00345)	1.01348 (0.01310)	1.01207 (0.00331)	1.01040 (0.01253)	1.00055 (0.00322)
			(45,56)	(35,46)	1.01349 (0.01125)	1.00926 (0.01177)	1.00537 (0.00312)	1.01629 (0.01255)	1.00954 (0.00329)	1.00900 (0.01174)	1.00524 (0.00312)	1.00835 (0.01114)	0.99895 (0.00298)

Table 6 Average estimates and their MSEs (in parenthesis) for the stress-strength parameter R, when $\theta_1 = 0.5$ and $\theta_2 = 1$ with different values censoring parameters

(T_1,T_2)	CS	(n_1,n_2)	(m_1,m_2)	(k_1,k_2)				LLF					
					MLE	SELF		−1.5		0.05		1.5	
					$\hat{R}$	NIP	IP	NIP	IP	NIP	IP	NIP	IP
(0.75,0.75)	I	(20,25)	(15,18)	(12,16)	0.73702 (0.00554)	0.72959 (0.00595)	0.73732 (0.00172)	0.73361 (0.00566)	0.73949 (0.00166)	0.72946 (0.00596)	0.73725 (0.00172)	0.72546 (0.00628)	0.73511 (0.00179)
			(17,22)	(15,20)	0.73749 (0.00492)	0.73626 (0.00482)	0.74067 (0.00140)	0.73945 (0.00465)	0.74243 (0.00136)	0.73615 (0.00483)	0.74062 (0.00140)	0.73298 (0.00502)	0.73890 (0.00144)
		(30,40)	(25,30)	(20,24)	0.73879 (0.00340)	0.73660 (0.00346)	0.74063 (0.00095)	0.73910 (0.00335)	0.74195 (0.00093)	0.73651 (0.00346)	0.74059 (0.00095)	0.73404 (0.00358)	0.73931 (0.00098)
			(27,35)	(23,30)	0.74040 (0.00287)	0.73765 (0.00295)	0.74141 (0.00078)	0.73978 (0.00286)	0.74251 (0.00077)	0.73758 (0.00295)	0.74137 (0.00078)	0.73549 (0.00304)	0.74030 (0.00080)
		(50,60)	(40,50)	(30,40)	0.74328 (0.00212)	0.73910 (0.00200)	0.74286 (0.00053)	0.74069 (0.00196)	0.74368 (0.00053)	0.73904 (0.00200)	0.74283 (0.00053)	0.73748 (0.00205)	0.74203 (0.00054)
			(45,56)	(35,46)	0.74160 (0.00183)	0.73940 (0.00188)	0.74197 (0.00052)	0.74079 (0.00184)	0.74270 (0.00051)	0.73936 (0.00188)	0.74195 (0.00052)	0.73800 (0.00192)	0.74124 (0.00053)
	II	(20,25)	(15,18)	(12,16)	0.73677 (0.00566)	0.73078 (0.00559)	0.73780 (0.00167)	0.73470 (0.00532)	0.73953 (0.00161)	0.73065 (0.00560)	0.73773 (0.00167)	0.72674 (0.00589)	0.73562 (0.00174)
			(17,22)	(15,20)	0.73758 (0.00503)	0.73380 (0.00466)	0.73874 (0.00148)	0.73697 (0.00448)	0.73994 (0.00143)	0.73369 (0.00466)	0.73868 (0.00148)	0.73054 (0.00486)	0.73592 (0.00153)
		(30,40)	(25,30)	(20,24)	0.74409 (0.00333)	0.73533 (0.00358)	0.74234 (0.00096)	0.73782 (0.00346)	0.74364 (0.00094)	0.73525 (0.00358)	0.74229 (0.00096)	0.73279 (0.00371)	0.74102 (0.00098)
			(27,35)	(23,30)	0.74329 (0.00283)	0.73530 (0.00316)	0.74110 (0.00080)	0.73742 (0.00307)	0.74117 (0.00079)	0.73523 (0.00316)	0.740126 (0.00080)	0.73315 (0.00326)	0.73902 (0.00082)
		(50,60)	(40,50)	(30,40)	0.74209 (0.00214)	0.73702 (0.00217)	0.74030 (0.00057)	0.73861 (0.00212)	0.74101 (0.00056)	0.73697 (0.00217)	0.74037 (0.00057)	0.73541 (0.00223)	0.74048 (0.00058)
			(45,56)	(35,46)	0.74178 (0.00177)	0.73745 (0.00197)	0.74004 (0.00051)	0.73883 (0.00193)	0.74005 (0.00050)	0.73740 (0.00197)	0.74002 (0.00051)	0.73605 (0.00202)	0.74043 (0.00052)
	III	(20,25)	(15,18)	(12,16)	0.73786 (0.00530)	0.72964 (0.00584)	0.73824 (0.00164)	0.73365 (0.00555)	0.74338 (0.00158)	0.72950 (0.00585)	0.73817 (0.00164)	0.72551 (0.00617)	0.73607 (0.0017)
			(17,22)	(15,20)	0.73883 (0.00450)	0.73597 (0.00489)	0.74048 (0.00141)	0.73917 (0.00472)	0.74221 (0.00137)	0.73586 (0.00490)	0.74042 (0.00141)	0.73270 (0.00509)	0.73872 (0.00145)
		(30,40)	(25,30)	(20,24)	0.74389 (0.00332)	0.73578 (0.00367)	0.74146 (0.00097)	0.73929 (0.00356)	0.74205 (0.00095)	0.73569 (0.00368)	0.74141 (0.00097)	0.73321 (0.00381)	0.74014 (0.00099)
			(27,35)	(23,30)	0.74139 (0.00321)	0.73960 (0.00288)	0.74100 (0.00084)	0.74171 (0.00281)	0.74158 (0.00083)	0.73953 (0.00288)	0.74098 (0.00084)	0.73746 (0.00296)	0.73919 (0.00086)
		(50,60)	(40,50)	(30,40)	0.74056 (0.00232)	0.73994 (0.00242)	0.74050 (0.00062)	0.74126 (0.00236)	0.74085 (0.00061)	0.73958 (0.00243)	0.74000 (0.00062)	0.73800 (0.00249)	0.74137 (0.00064)
			(45,56)	(35,46)	0.74051 (0.00182)	0.74059 (0.00195)	0.74002 (0.00049)	0.74097 (0.00192)	0.74052 (0.00049)	0.74055 (0.00195)	0.74000 (0.00049)	0.73920 (0.00199)	0.74013 (0.00050)

Table 7 Average length and coverage probability (in parenthesis) of 95% ACI, boot-p, boot-t and HPD credible interval for θ_1, θ_2 and R,when $\theta_1 = 1$ and $\theta_2 = 1$ with different values of censoring parameters

(T_1, T_2)	cs	(n_1, n_2)	(m_1, m_2)	(k_1, k_2)	ACI			Boot-p			Boot-t			NIP HPD			IP HPD		
					θ_1	θ_2	R	θ_1	θ_2	R	θ_1	θ_2	R	θ_1	θ_2	R	θ_1	θ_2	R
(0.5,0.5)	I	(20,25)	(15,18)	(12,16)	0.82212 (0.94700)	0.69296 (0.95000)	0.37842 (0.93300)	0.92683 (0.93500)	0.76003 (0.93700)	0.35312 (0.95500)	0.85925 (0.93900)	0.71095 (0.94300)	0.35334 (0.94800)	0.80072 (0.93600)	0.60214 (0.98600)	0.35015 (0.93700)	0.67703 (0.94400)	0.49215 (0.99300)	0.27233 (0.98900)
			(17,22)	(15,20)	0.72799 (0.94200)	0.62729 (0.94500)	0.34702 (0.92900)	0.80880 (0.91800)	0.66283 (0.94600)	0.31904 (0.94100)	0.77417 (0.94000)	0.63129 (0.95900)	0.31895 (0.94500)	0.71163 (0.93600)	0.53539 (0.98400)	0.31767 (0.93500)	0.61431 (0.94000)	0.45245 (0.99100)	0.24597 (0.98700)
		(30,40)	(25,30)	(20,24)	0.62390 (0.95400)	0.56272 (0.95800)	0.28898 (0.93200)	0.66334 (0.93900)	0.59149 (0.93500)	0.28409 (0.95400)	0.64166 (0.94900)	0.56672 (0.94600)	0.28458 (0.95200)	0.60781 (0.93800)	0.44353 (0.98800)	0.28377 (0.94400)	0.54996 (0.95500)	0.39265 (0.99100)	0.21101 (0.99500)
			(27,35)	(23,30)	0.57024 (0.95100)	0.49451 (0.95600)	0.26588 (0.92900)	0.60945 (0.92800)	0.51968 (0.93800)	0.26212 (0.94100)	0.58761 (0.94300)	0.50796 (0.94600)	0.26217 (0.94900)	0.56492 (0.93200)	0.40096 (0.99000)	0.26182 (0.93600)	0.49049 (0.94500)	0.34816 (0.99200)	0.19197 (0.99400)
		(50,60)	(40,50)	(30,40)	0.49661 (0.95000)	0.42757 (0.95200)	0.23217 (0.94100)	0.52153 (0.94300)	0.44310 (0.94000)	0.22935 (0.95200)	0.50314 (0.95100)	0.43082 (0.94900)	0.22951 (0.94800)	0.49000 (0.95800)	0.35033 (0.99000)	0.22986 (0.96400)	0.42085 (0.96000)	0.30134 (0.99700)	0.16779 (0.99800)
			(45,56)	(35,46)	0.46117 (0.94500)	0.39882 (0.94900)	0.21627 (0.93800)	0.47684 (0.94700)	0.40908 (0.93600)	0.21391 (0.96000)	0.46531 (0.95400)	0.40146 (0.94100)	0.21429 (0.95700)	0.45388 (0.94000)	0.33144 (0.98900)	0.21429 (0.94800)	0.39242 (0.94500)	0.28104 (0.99300)	0.15791 (0.99000)
	II	(20,25)	(15,18)	(12,16)	0.82024 (0.94900)	0.69098 (0.95200)	0.35904 (0.90900)	0.93435 (0.91700)	0.77413 (0.92800)	0.35242 (0.94900)	0.85040 (0.95800)	0.71945 (0.94700)	0.35196 (0.95000)	0.79895 (0.94700)	0.59121 (0.98800)	0.35011 (0.94100)	0.67526 (0.94700)	0.48824 (0.99400)	0.26871 (0.99300)
			(17,22)	(15,20)	0.72462 (0.93700)	0.61519 (0.94900)	0.32343 (0.91700)	0.80774 (0.90700)	0.66625 (0.93200)	0.31780 (0.94600)	0.76021 (0.94200)	0.63195 (0.92300)	0.31844 (0.94500)	0.70777 (0.93700)	0.52970 (0.98800)	0.31648 (0.93800)	0.60143 (0.94600)	0.43240 (0.99300)	0.24165 (0.99000)
		(30,40)	(25,30)	(20,24)	0.62236 (0.94900)	0.55975 (0.96300)	0.28854 (0.93000)	0.67025 (0.93800)	0.59729 (0.92600)	0.28444 (0.94100)	0.63415 (0.94500)	0.57440 (0.94500)	0.28492 (0.94800)	0.60984 (0.94200)	0.54217 (0.99000)	0.28365 (0.94600)	0.54901 (0.95900)	0.38978 (0.99400)	0.21276 (0.99000)
			(27,35)	(23,30)	0.57735 (0.94200)	0.49543 (0.93900)	0.26603 (0.93800)	0.61035 (0.93200)	0.52286 (0.94100)	0.26218 (0.95100)	0.58582 (0.93900)	0.50868 (0.94700)	0.26209 (0.95700)	0.56513 (0.93600)	0.41818 (0.99000)	0.26176 (0.95100)	0.49502 (0.94200)	0.35616 (0.99500)	0.19668 (0.99400)
		(50,60)	(40,50)	(30,40)	0.50121 (0.95000)	0.42466 (0.94700)	0.23203 (0.92800)	0.52411 (0.93900)	0.44372 (0.93500)	0.22942 (0.93900)	0.51126 (0.94700)	0.43381 (0.95600)	0.22930 (0.94200)	0.49186 (0.94100)	0.36401 (0.98700)	0.22893 (0.93700)	0.41706 (0.94100)	0.30237 (0.98700)	0.17176 (0.99000)
			(45,56)	(35,46)	0.46096 (0.95300)	0.40092 (0.94300)	0.21613 (0.93800)	0.47535 (0.94100)	0.41051 (0.94200)	0.21427 (0.95800)	0.46329 (0.94600)	0.40353 (0.94200)	0.21367 (0.94300)	0.45257 (0.94600)	0.33203 (0.99000)	0.21408 (0.94700)	0.39464 (0.93200)	0.28653 (0.99600)	0.15910 (0.99000)
	III	(20,25)	(15,18)	(12,16)	0.82770 (0.93100)	0.69779 (0.95100)	0.35961 (0.91000)	0.92275 (0.93200)	0.77318 (0.93200)	0.35278 (0.95500)	0.85035 (0.94400)	0.72101 (0.94200)	0.35185 (0.93900)	0.80485 (0.92400)	0.60867 (0.98400)	0.34994 (0.94100)	0.68244 (0.94500)	0.49773 (0.99300)	0.27330 (0.99300)
			(17,22)	(15,20)	0.73087 (0.93700)	0.62928 (0.95300)	0.32426 (0.91100)	0.79423 (0.93100)	0.66134 (0.94000)	0.31948 (0.94700)	0.75231 (0.93800)	0.62776 (0.94400)	0.31880 (0.94800)	0.71362 (0.93100)	0.53804 (0.98400)	0.31731 (0.93800)	0.61648 (0.94300)	0.45759 (0.98800)	0.24692 (0.98700)
		(30,40)	(25,30)	(20,24)	0.61959 (0.94300)	0.55879 (0.95300)	0.28862 (0.92400)	0.66902 (0.92700)	0.60069 (0.93000)	0.28434 (0.94100)	0.64137 (0.95000)	0.57846 (0.95700)	0.28448 (0.95400)	0.60680 (0.93100)	0.45068 (0.98700)	0.28353 (0.93900)	0.54801 (0.95100)	0.39217 (0.99200)	0.21386 (0.98900)
			(27,35)	(23,30)	0.57666 (0.94200)	0.50373 (0.95400)	0.26603 (0.93800)	0.61035 (0.93200)	0.52286 (0.94100)	0.26218 (0.95100)	0.58582 (0.93900)	0.50868 (0.94700)	0.26209 (0.95700)	0.56513 (0.93600)	0.41818 (0.99000)	0.26176 (0.95100)	0.49502 (0.94200)	0.35616 (0.99500)	0.19668 (0.99400)
		(50,60)	(40,50)	(30,40)	0.50121 (0.95000)	0.42466 (0.94700)	0.23203 (0.92800)	0.52411 (0.93900)	0.44372 (0.93500)	0.22942 (0.93900)	0.51126 (0.94700)	0.43381 (0.95600)	0.22930 (0.94200)	0.49186 (0.94100)	0.36401 (0.98700)	0.22893 (0.93700)	0.41706 (0.94100)	0.30237 (0.98700)	0.17176 (0.99000)
			(45,56)	(35,46)	0.46096 (0.95300)	0.40092 (0.94300)	0.21613 (0.93800)	0.47535 (0.94100)	0.41051 (0.94200)	0.21427 (0.95800)	0.46329 (0.94600)	0.40353 (0.94200)	0.21367 (0.94300)	0.45257 (0.94600)	0.33203 (0.99000)	0.21408 (0.94700)	0.39464 (0.93200)	0.28653 (0.99600)	0.15910 (0.99000)

Table 8 Average length and coverage probability (in parenthesis) of 95% ACI, boot-p, boot-t and HPD credible interval for θ_1, θ_2 and R, when $\theta_1 = 0.5$ and $\theta_2 = 1$ with different values of censoring parameters

(T_1, T_2)	CS	(n_1, n_2)	(m_1, m_2)	(k_1, k_2)	ACI			Boot-p			Boot-t			NIP HPD			IP HPD		
					θ_1	θ_2	R	θ_1	θ_2	R	θ_1	θ_2	R	θ_1	θ_2	R	θ_1	θ_2	R
(0.75,0.75)	I	(20,25)	(15,18)	(12,16)	0.36951 (0.95500)	0.69666 (0.96300)	0.28402 (0.93400)	0.40528 (0.95000)	0.76474 (0.93500)	0.27089 (0.94900)	0.38535 (0.95200)	0.72500 (0.94900)	0.27060 (0.96600)	0.36319 (0.94200)	0.67690 (0.94100)	0.28330 (0.94000)	0.26530 (0.99100)	0.48728 (0.99100)	0.21012 (0.99200)
			(17,22)	(15,20)	0.33072 (0.95400)	0.61852 (0.94600)	0.25628 (0.92500)	0.36036 (0.92500)	0.66384 (0.93200)	0.24023 (0.99200)	0.34449 (0.93900)	0.64581 (0.94700)	0.24059 (0.96800)	0.32541 (0.93400)	0.61419 (0.93900)	0.25307 (0.93500)	0.23568 (0.98500)	0.44820 (0.99300)	0.18871 (0.98700)
		(30,40)	(25,30)	(20,24)	0.28195 (0.95200)	0.55754 (0.95400)	0.22635 (0.93900)	0.29382 (0.93700)	0.59258 (0.92900)	0.21212 (0.94400)	0.28232 (0.93900)	0.57255 (0.93700)	0.21283 (0.96800)	0.27695 (0.93300)	0.55053 (0.95400)	0.22498 (0.94000)	0.19967 (0.98800)	0.39084 (0.99000)	0.16374 (0.99300)
			(27,35)	(23,30)	0.26384 (0.95800)	0.50023 (0.94500)	0.20799 (0.93900)	0.27609 (0.93600)	0.52421 (0.94300)	0.19577 (0.93800)	0.26921 (0.94600)	0.50932 (0.94400)	0.19532 (0.97600)	0.25891 (0.93400)	0.49067 (0.94700)	0.20730 (0.94000)	0.18441 (0.99100)	0.35217 (0.99000)	0.14987 (0.99400)
		(50,60)	(40,50)	(30,40)	0.22532 (0.94200)	0.42915 (0.95700)	0.17930 (0.93400)	0.23376 (0.94500)	0.44296 (0.94300)	0.16170 (0.94800)	0.22808 (0.94400)	0.43442 (0.93900)	0.16137 (0.98800)	0.22289 (0.95600)	0.42134 (0.96300)	0.17998 (0.96800)	0.15963 (0.99400)	0.30213 (0.99200)	0.12973 (0.99800)
			(45,56)	(35,46)	0.20945 (0.95000)	0.39792 (0.94400)	0.16784 (0.93800)	0.21505 (0.93900)	0.40805 (0.94300)	0.14762 (0.94700)	0.21032 (0.94700)	0.40232 (0.94300)	0.14816 (0.98400)	0.20710 (0.94300)	0.39232 (0.94400)	0.16760 (0.94800)	0.14831 (0.98900)	0.28525 (0.99000)	0.12188 (0.99200)
	II	(20,25)	(15,18)	(12,16)	0.36462 (0.95100)	0.69720 (0.94700)	0.28069 (0.92700)	0.41351 (0.92300)	0.75891 (0.93200)	0.27087 (0.94400)	0.37998 (0.94300)	0.71221 (0.93700)	0.27062 (0.97000)	0.35616 (0.93500)	0.67566 (0.94600)	0.28009 (0.94000)	0.26062 (0.99200)	0.49359 (0.99300)	0.20868 (0.99400)
			(17,22)	(15,20)	0.33047 (0.94400)	0.61767 (0.94300)	0.25518 (0.91900)	0.35302 (0.93700)	0.67247 (0.92500)	0.24074 (0.94700)	0.33362 (0.95600)	0.64130 (0.94800)	0.24073 (0.98500)	0.32031 (0.92900)	0.60270 (0.94400)	0.25266 (0.93700)	0.23825 (0.98800)	0.44851 (0.99200)	0.19104 (0.99000)
		(30,40)	(25,30)	(20,24)	0.27631 (0.94400)	0.56544 (0.96400)	0.22187 (0.93000)	0.29571 (0.92200)	0.59652 (0.93200)	0.21053 (0.94400)	0.28440 (0.93900)	0.57243 (0.94100)	0.21011 (0.97000)	0.27472 (0.94200)	0.54935 (0.95400)	0.22441 (0.94700)	0.19646 (0.99200)	0.39840 (0.99400)	0.16289 (0.99200)
			(27,35)	(23,30)	0.26094 (0.95500)	0.49672 (0.94600)	0.20661 (0.94500)	0.27388 (0.93100)	0.51857 (0.95300)	0.19354 (0.93000)	0.26484 (0.93600)	0.50025 (0.94300)	0.19639 (0.98600)	0.25735 (0.94100)	0.48630 (0.93100)	0.20703 (0.93700)	0.18418 (0.98600)	0.33929 (0.99100)	0.14787 (0.99200)
		(50,60)	(40,50)	(30,40)	0.22561 (0.95000)	0.42576 (0.95200)	0.18031 (0.92700)	0.23295 (0.95000)	0.43754 (0.94400)	0.17055 (0.95800)	0.22660 (0.94400)	0.42918 (0.94000)	0.17051 (0.98600)	0.22377 (0.94300)	0.41726 (0.93800)	0.18111 (0.94100)	0.15839 (0.99100)	0.29822 (0.99100)	0.12974 (0.99200)
			(45,56)	(35,46)	0.20801 (0.96500)	0.39876 (0.95100)	0.16689 (0.93400)	0.21487 (0.95000)	0.40669 (0.93900)	0.14803 (0.94700)	0.20979 (0.95700)	0.40042 (0.93400)	0.14832 (0.96900)	0.20699 (0.93900)	0.39422 (0.93900)	0.16697 (0.94800)	0.14436 (0.99200)	0.28338 (0.99900)	0.11928 (0.99200)
	III	(20,25)	(15,18)	(12,16)	0.36718 (0.95400)	0.69919 (0.95500)	0.28270 (0.93600)	0.41021 (0.93100)	0.76507 (0.92900)	0.26005 (0.93900)	0.37903 (0.94500)	0.72008 (0.93200)	0.26044 (0.97500)	0.36477 (0.92700)	0.68238 (0.94400)	0.28292 (0.94400)	0.26041 (0.99000)	0.49385 (0.99400)	0.20829 (0.99500)
			(17,22)	(15,20)	0.33427 (0.95800)	0.62538 (0.96100)	0.25638 (0.94600)	0.35612 (0.94700)	0.66582 (0.93100)	0.24016 (0.95200)	0.33891 (0.95600)	0.63466 (0.93300)	0.24051 (0.98300)	0.32625 (0.92600)	0.61675 (0.94600)	0.25314 (0.93500)	0.23867 (0.98800)	0.44216 (0.99100)	0.18771 (0.98800)
		(30,40)	(25,30)	(20,24)	0.28016 (0.95300)	0.56021 (0.95600)	0.22493 (0.94300)	0.29551 (0.94400)	0.59290 (0.94800)	0.21406 (0.95300)	0.28519 (0.94300)	0.56840 (0.94400)	0.21337 (0.97100)	0.27665 (0.93700)	0.54836 (0.94900)	0.22514 (0.93900)	0.19626 (0.99000)	0.39355 (0.99300)	0.16262 (0.99000)
			(27,35)	(23,30)	0.26302 (0.94700)	0.49962 (0.93600)	0.20757 (0.92600)	0.27383 (0.95200)	0.52511 (0.94700)	0.19675 (0.95500)	0.26616 (0.93800)	0.50825 (0.94000)	0.19639 (0.98600)	0.25918 (0.94200)	0.49433 (0.94500)	0.20613 (0.95100)	0.18876 (0.98600)	0.36280 (0.99300)	0.15297 (0.99500)
		(50,60)	(40,50)	(30,40)	0.22561 (0.95000)	0.42576 (0.95200)	0.18031 (0.92700)	0.23295 (0.95000)	0.43754 (0.94400)	0.17055 (0.95800)	0.22660 (0.94400)	0.42918 (0.94000)	0.17051 (0.98600)	0.22377 (0.94300)	0.41726 (0.93800)	0.18111 (0.94100)	0.15839 (0.99100)	0.29822 (0.99100)	0.12974 (0.99200)
			(45,56)	(35,46)	0.20801 (0.96500)	0.39876 (0.95100)	0.16689 (0.93400)	0.21487 (0.95000)	0.40669 (0.93900)	0.14803 (0.94700)	0.20979 (0.95700)	0.40042 (0.93400)	0.14832 (0.96900)	0.20699 (0.93900)	0.39422 (0.93900)	0.16697 (0.94800)	0.14436 (0.99200)	0.28338 (0.99900)	0.11928 (0.99200)

Table 9 MLE and K-S distance for considered real data set

Data Set	MLE	K-S Distance	P-Value
Bank-$A(X)$	0.2634	0.0624	0.8297
Bank-$B(Y)$	0.3822	0.1029	0.5482

non-informative priors under SELF and LLF are computed based on MCMC samples. The HPD credible interval is also computed using the considered variation of the censoring scheme, following the idea of Chen and Shao (1999) algorithm. The point and interval estimates of the parameters θ_1, θ_2, and R, as presented in Tables 1, 2, 3, 4, 5, 6, 7, 8, lead to the following conclusions.

C1. ML and Bayes estimates of parameters and SRR parameter (R) are quite good, as reflected by the average estimates converging towards the actual values when we increase the effective sample size (m_1, m_2, k_1, k_2) for fixed (n_1, n_2, T_1, T_2). Moreover, the estimates also converge to actual values as n_1 and n_2 increases.
C2. Indeed, with a rise in effective sample sizes (m_1, m_2, k_1, k_2) while maintaining (n_1, n_2, T_1, T_2) as fixed, it is as anticipated that the mean square errors (MSEs) decrease for both ML and Bayesian estimations. Additionally, as n_1 and n_2 increase, the MSEs also exhibit a declining trend.
C3. ML estimates are quite close to that of Bayes estimates, particularly when non-informative priors are used under SELF. However, when we incorporate informative priors under SELF, the Bayes estimates exhibit superiority over ML estimates in terms of the average estimate, MSE, and the length of Confidence Intervals (CI).
C4. Increasing the effective sample sizes (m_2, m_2, k_1, k_2) while keeping the values of (n_1, n_2, T_1, T_2) fixed led to a reduction in the average width of the 95% ACIs, bootstrap, and Bayes credible intervals. Notably, the average width of ACIs and HPD credible intervals, based on non-informative priors demonstrate a remarkable closeness. However, among all the methods considered, HPD credible intervals relying on informative priors exhibited the shortest width.
C5. Among boot-p, boot-t, and ACI, we observe that boot-t perform more effectively than boot-p for the parameters θ_1 and θ_2, and ACIs demonstrate the best performance in terms of average width. Additionally, when considering SSR, boot-p and boot-t show approximately identical average lengths and are smaller than the length of ACIs. When we consider informative priors, the HPD demonstrates superior performance, achieving a shorter average width among all considered methods.

6 Real data application

To provide an illustrative insight, this section deals with real lifetime data, which represent the duration of waiting times (measured in minutes) for customer service in two distinct banks reported by Ghitany et al. (2008). Here, we are interested in estimating the stress-strength parameter R, where X and Y denote the customer service time in Bank A and B, respectively.

To commence our investigation, we initially conducted the Kolmogorov-Smirnov (K-S) test to assess the fitting of the xgamma distribution to the datasets. The results obtained by the K-S test are outlined in Table 9. The K-S test statistic and p-values indicate that the xgamma distribution fits the data well. Hence, we can use the xgamma distribution to model these data sets. Proceeding to the subsequent phase, we assume that X and Y denote the durations of waiting times, measured in minutes, before customer service in two different banks identified as A and B, respectively. To analyze the provided real data, we have considered the following schemes under the GPH censoring schemes.

Table 10 ML and Bayes estimates for considered real data set

(T_1,T_2)	cs	(n_1,n_2)	(m_1,m_2)	(k_1,k_2)	ϕ	MLE	SELF	LLF		
								−1.5	0.05	1.5
(5,10)	I	(100,60)	(50,40)	(30,20)	θ_1	0.25834	0.25703	0.25754	0.25701	0.25652
					θ_2	0.37572	0.37514	0.37641	0.37515	0.37389
					R	0.65344	0.65246	0.65489	0.65238	0.65001
	II	(100,60)	(60,50)	(40,30)	θ_1	0.28096	0.28175	0.28222	0.28174	0.28129
					θ_2	0.38776	0.38763	0.38848	0.38760	0.38678
					R	0.63236	0.62999	0.63171	0.62991	0.62820
	III	(100,60)	(90,58)	(80,45)	θ_1	0.25794	0.25754	0.25776	0.25753	0.25732
					θ_2	0.37977	0.37901	0.37992	0.37898	0.37810
					R	0.65809	0.65593	0.65731	0.65588	0.65454

Table 11 The 95% ACI, and HPD credible interval with their corresponding interval length of θ_1, θ_2 and R for the considered real data set

(T_1, T_2)	cs	(n_1, n_2)	(m_1, m_2)	(k_1, k_2)	ACI			HPD		
					θ_1	θ_2	R	θ_1	θ_2	R
(5,10)	I	(100,60)	(50,40)	(30,20)	[0.20626,0.31043] [0.10417]	[0.29520,0.45624] [0.16103]	[0.54027,0.76662] [0.22634]	[0.20565,0.30649] [0.10083]	[0.29875,0.45474] [0.15599]	[0.53822,0.75699] [0.21877]
	II	(100,60)	(60,50)	(40,30)	[0.23222,0.32971] [0.09748]	[0.32022,0.45530] [0.13508]	[0.53587,0.72886] [0.19299]	[0.23544,0.33272] [0.09727]	[0.32622,0.45538] [0.12916]	[0.53844,0.72262] [0.18418]
	III	(100,60)	(90,58)	(80,45)	[0.22411,0.29177] [0.06766]	[0.31235,0.44718] [0.13483]	[0.57408,0.74209] [0.16801]	[0.22482,0.29090] [0.06607]	[0.31270,0.44572] [0.13302]	[0.57568,0.73636] [0.16068]

Table 12 The 95% Boot-p, Boot-t, confidence intervals with their corresponding interval length of θ_1, θ_2 and R for the considered real data set

(T_1, T_2)	cs	(n_1, n_2)	(m_1, m_2)	(k_1, k_2)	Boot-p			Boot-t		
					θ_1	θ_2	R	θ_1	θ_2	R
(5,10)	I	(100,60)	(50,40)	(30,20)	[0.21483,0.32677] [0.11194]	[0.27496,0.43709] [0.16213]	[0.48376,0.70604] [0.22228]	[0.21420,0.31995] [0.10574]	[0.27484,0.43576] [0.16092]	[0.49274,0.71898] [0.22624]
	II	(100,60)	(60,50)	(40,30)	[0.23953,0.33697] [0.09743]	[0.32777,0.47131] [0.14354]	[0.53095,0.72401] [0.19253]	[0.24104,0.33902] [0.09798]	[0.32694,0.46779] [0.14085]	[0.53043,0.72296] [0.19253]
	III	(100,60)	(90,58)	(80,45)	[0.23363,0.30920] [0.07557]	[0.33174,0.48484] [0.15310]	[0.56916,0.73526] [0.16609]	[0.23420,0.31012] [0.07592]	[0.33181,0.46839] [0.13657]	[0.56564,0.73311] [0.16747]

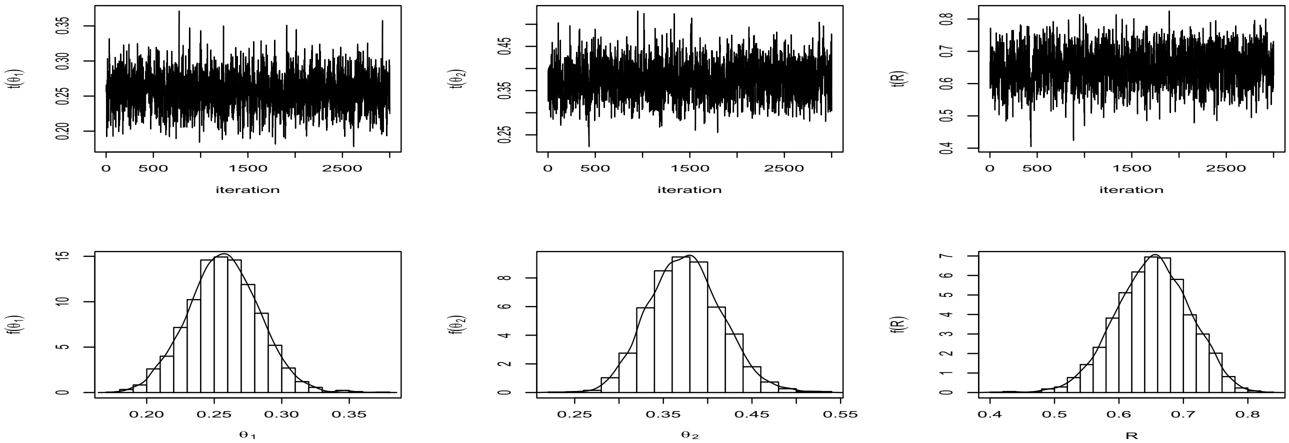

Fig. 2 Trace and posterior density plot of θ_1, θ_2 and R for real data set under CS-I

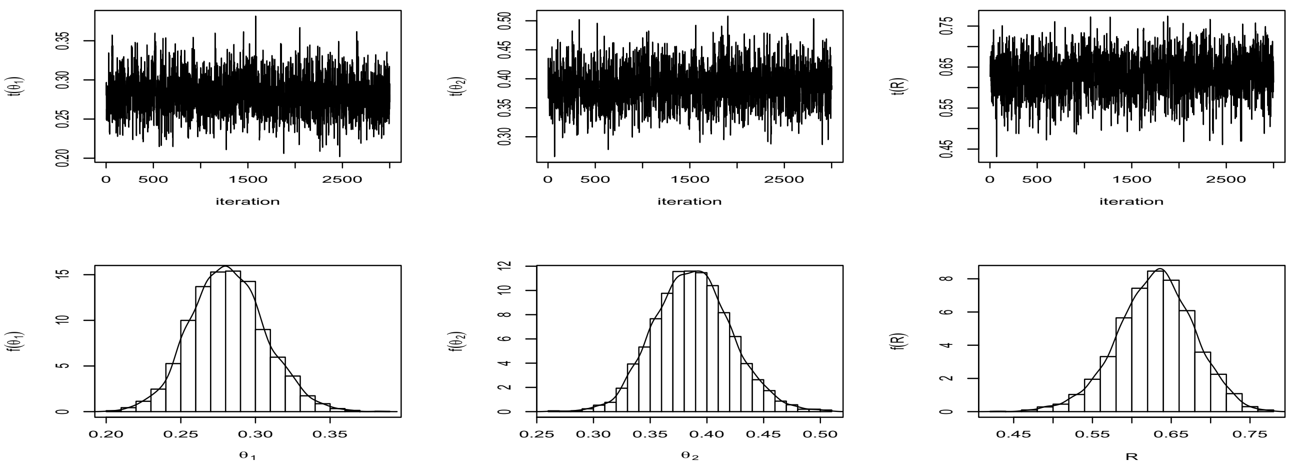

Fig. 3 Trace and posterior density plot of θ_1, θ_2 and R for real data set under CS-II

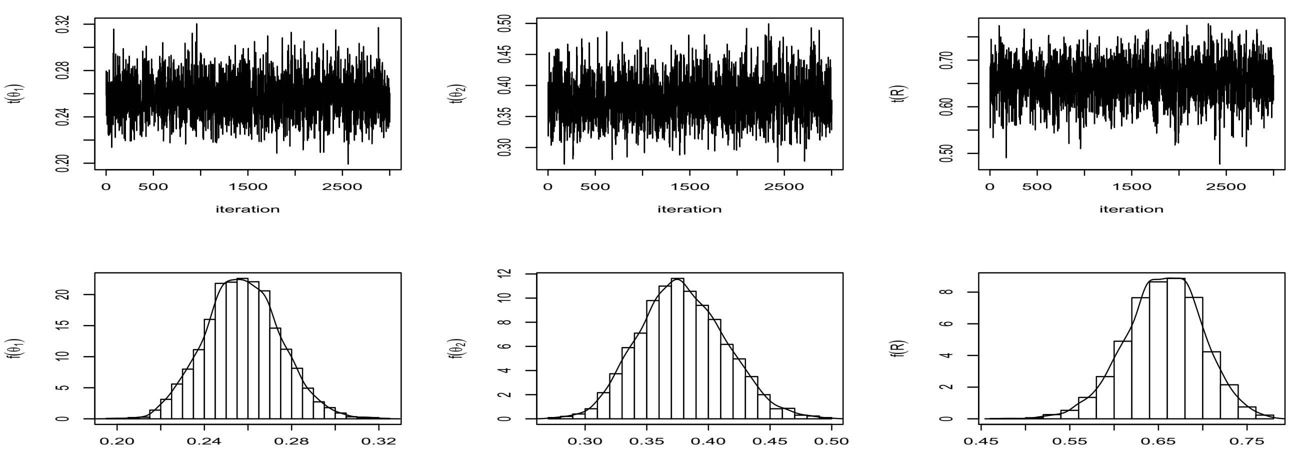

Fig. 4 Trace and posterior density plot of θ_1, θ_2 and R for real data set under CS-III

- **Scheme I:** $(n_1, n_2) = (100, 60)$, $(m_1, m_2) = (50, 40)$, $(k_1, k_2) = (30, 20)$, $(T_1, T_2) = (5, 10)$, $R_1 = (50, 0^{\star 49})$, and $R_2 = (20, 0^{\star 39})$.
- **Scheme II:** $(n_1, n_2) = (100, 60)$, $(m_1, m_2) = (60, 50)$, $(k_1, k_2) = (40, 30)$, $(T_1, T_2) = (5, 10)$, $R_1 = (0^{\star 59}, 40)$, and $R_2 = (0^{\star 49}, 10)$.
- **Scheme III:** $(n_1, n_2) = (100, 60)$, $(m_1, m_2) = (90, 58)$, $(k_1, k_2) = (80, 45)$, $(T_1, T_2) = (5, 10)$, $R_1 = (1^{\star 10}, 0^{\star 80})$, and $R_2 = (1^{\star 2}, 0^{\star 56})$.

The point and interval estimates obtained using classical and Bayesian approaches are presented in Tables 10 and 11, respectively. Additionally, the boot-p and boot-t interval estimates are reported in Table 12. We observed that Bayes estimates are quite close to the ML estimates based on the results reported in the tables. However, in case of interval estimates, it is noteworthy that the width of the Bayesian credible intervals is consistently smaller than the ACI across all three considered censoring schemes. The estimate of SSR indicates that the customer at bank A has higher chance to wait before customer service than at bank B. In the case of interval estimation based on real data, we get similar results as we have obtained in the simulation study for all considered methods. Further, the trace and posterior density plots for the considered real data set and specified schemes are given in Figure 2, 3, and 4 to show the fine mixing of the chain based on the MCMC samples.

7 Conclusion

This article explores point and interval estimation for model parameters and stress-strength reliability parameters (R) of the xgamma distribution under a generalized progressive hybrid censoring scheme in classical and Bayesian frameworks. For the Bayesian estimation, independent gamma distribution is considered as a prior distribution. The explicit solution of ML and Bayes estimates is not achievable, so we have used the Newton–Raphson iterative procedure in the classical approach and the MCMC technique in the Bayesian approach. We have also obtained 95% boot-p and boot-t confidence intervals. Based on the results obtained, we concluded that Bayes estimates are quite close to the ML estimates when we use non-informative priors under SELF and are better than ML when we use informative priors. For the interval estimation, we also observed that among all the considered methods ACIs, boot-p, and boot-t, boot-t perform more effectively than boot-p for the parameters θ_1 and θ_2, and ACIs outperform both based on their interval width. Furthermore, for stress-strength reliability, both boot-p and boot-t offered similar average lengths, which are shorter than ACIs. Among all the considered methods of interval estimation HPD credible interval performed best when we considered informative priors.

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Data Availability The data used in the simulation study was self-generated.

Declarations

Conflict of interest The authors declare that they have no conflict of interest.

Ethical statements This article does not contain any studies involving human participants performed by any of the authors.

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