Mathematical Modelling of a Rotary Inverted Pendulum (Furuta)

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1. Definition

Generalized coordinates:

$\theta(t)$: arm yaw angle.

$\alpha(t)$: pendulum angle, with $\alpha = 0$ upright.

Constants:

L_r: arm length.

m_p: swinging mass.

l_p: distance from hinge to pendulum COM.

g: gravity.

τ: actuation torque about θ.

Define

$$K \equiv m_p L_r l_p, \qquad G \equiv m_p g l_p$$

Let

$\hat{J}_1$: yaw inertia about motor axis (arm-side).

$\hat{J}_2$: pendulum hinge inertia.

J_p: pendulum inertia about its center of mass, about an axis perpendicular to the swing plane.

$$\hat{J}_0 \triangleq \hat{J}_1 + m_p L_r^2$$

Rigid links; frictionless joints.

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2. Parameter evaluation

Given:

$$L_r = 0.19 \text{ m}, \quad m_r = 0.051 \text{ kg}$$

$$L_h = 0.17 \text{ m}, \quad L_v = 0.12 \text{ m}$$
$$m_{\text{rod,total}} = 10.3 \text{ g} = 0.0103 \text{ kg}$$
$$m_s = 7.7 \text{ g} = 0.0077 \text{ kg}, \quad x_s = 0.103 \text{ m}$$

Linear density:

$$\lambda = \frac{10.3 \text{ g}}{290 \text{ mm}} = 0.03552 \text{ g/mm}$$

Mass split:

$$m_h = 170\lambda = 6.04 \text{ g} = 0.006038 \text{ kg}$$
$$m_{\text{rod}} = 120\lambda = 4.26 \text{ g} = 0.004262 \text{ kg}$$

Swinging mass:

$$m_p = m_{\text{rod}} + m_s = 0.011962069 \text{ kg}$$

COM distance:

$$x_{\text{rod}} = \frac{L_v}{2} = 0.06 \text{ m}$$
$$l_p = \frac{m_{\text{rod}} x_{\text{rod}} + m_s x_s}{m_p}$$
$$l_p = \frac{(0.004262)(0.06) + (0.0077)(0.103)}{0.011962069}$$
$$l_p = \frac{2.5572 \times 10^{-4} + 7.931 \times 10^{-4}}{0.011962069} = 0.087679158 \text{ m}$$

Pendulum hinge inertia (rod about end + sphere point mass):

$$I_{\text{rod,piv}} = \frac{1}{3} m_{\text{rod}} L_v^2 = \frac{1}{3}(0.004262)(0.12^2) = 2.045793103 \times 10^{-5}$$
$$I_{\text{s,piv}} = m_s x_s^2 = (0.0077)(0.103^2) = 8.16893 \times 10^{-5}$$
$$\hat{J}_2 = I_{\text{rod,piv}} + I_{\text{s,piv}} = 1.021472310 \times 10^{-4} \text{ kg m}^2$$

Gravity constant:

$$G = m_p g l_p = (0.011962069)(9.81)(0.087679158) = 1.028896479 \times 10^{-2} \text{ N m}$$

Coupling constant:

$$K = m_p L_r l_p = (0.011962069)(0.19)(0.087679158) = 1.992765862 \times 10^{-4} \text{ kg m}^2$$

Arm-side yaw inertia:

$$\hat{J}_0 = \hat{J}_1 + m_p L_r^2$$

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3. Kinematics

Let $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ be an inertial orthonormal basis. Define the planar arm basis

$$\mathbf{e}_r \equiv \cos\theta\, \mathbf{i} + \sin\theta\, \mathbf{j}, \qquad \mathbf{e}_t \equiv -\sin\theta\, \mathbf{i} + \cos\theta\, \mathbf{j}$$

Arm hinge position:

$$\mathbf{r}_h = L_r \mathbf{e}_r$$

Pendulum COM relative to hinge:

$$\mathbf{r}_{p/h} = l_p(\sin\alpha\, \mathbf{e}_t + \cos\alpha\, \mathbf{k})$$

Pendulum COM position:

$$\mathbf{r}_p = \mathbf{r}_h + \mathbf{r}_{p/h}$$

Components $\mathbf{r}_p = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$:

$$\begin{aligned} x &= L_r \cos\theta - l_p \sin\alpha\, \sin\theta \\ y &= L_r \sin\theta + l_p \sin\alpha\, \cos\theta \\ z &= l_p \cos\alpha \end{aligned}$$

Derivatives:

$$\begin{aligned} \frac{d}{dt}(\cos\theta) = -\sin\theta\, \dot{\theta}, &\quad \frac{d}{dt}(\sin\theta) = \cos\theta\, \dot{\theta} \\ \frac{d}{dt}(\sin\alpha) = \cos\alpha\, \dot{\alpha}, &\quad \frac{d}{dt}(\cos\alpha) = -\sin\alpha\, \dot{\alpha} \end{aligned}$$

3.1 Velocity components

For x:

$$\dot{x} = \frac{d}{dt}(L_r \cos\theta) - l_p \frac{d}{dt}(\sin\alpha\ \sin\theta)$$
$$\frac{d}{dt}(L_r \cos\theta) = -L_r \sin\theta\,\dot{\theta}$$
$$\frac{d}{dt}(\sin\alpha\ \sin\theta) = (\cos\alpha\,\dot{\alpha})\sin\theta + \sin\alpha(\cos\theta\,\dot{\theta})$$
$$\dot{x} = -L_r \sin\theta\,\dot{\theta} - l_p \cos\alpha \sin\theta\,\dot{\alpha} - l_p \sin\alpha \cos\theta\,\dot{\theta}$$

For y:

$$\dot{y} = \frac{d}{dt}(L_r \sin\theta) + l_p \frac{d}{dt}(\sin\alpha\ \cos\theta)$$
$$\frac{d}{dt}(L_r \sin\theta) = L_r \cos\theta\,\dot{\theta}$$
$$\frac{d}{dt}(\sin\alpha\ \cos\theta) = (\cos\alpha\,\dot{\alpha})\cos\theta + \sin\alpha(-\sin\theta\,\dot{\theta})$$
$$\dot{y} = L_r \cos\theta\,\dot{\theta} + l_p \cos\alpha \cos\theta\,\dot{\alpha} - l_p \sin\alpha \sin\theta\,\dot{\theta}$$

For z:

$$\dot{z} = \frac{d}{dt}(l_p \cos\alpha) = -l_p \sin\alpha\,\dot{\alpha}$$

3.2 Speed squared

Compute $v^2 = \dot{x}^2 + \dot{y}^2 + \dot{z}^2$.

Define

$$A = L_r \sin\theta + l_p \sin\alpha \cos\theta, \quad B = l_p \cos\alpha \sin\theta$$
$$C = L_r \cos\theta - l_p \sin\alpha \sin\theta, \quad D = l_p \cos\alpha \cos\theta$$

Then

$$\dot{x} = -A\dot{\theta} - B\dot{\alpha}, \qquad \dot{y} = C\dot{\theta} + D\dot{\alpha}, \qquad \dot{z} = -l_p \sin\alpha\,\dot{\alpha}$$

Square:

$$\dot{x}^2 = A^2\dot{\theta}^2 + B^2\dot{\alpha}^2 + 2AB\dot{\theta}\dot{\alpha}$$
$$\dot{y}^2 = C^2\dot{\theta}^2 + D^2\dot{\alpha}^2 + 2CD\dot{\theta}\dot{\alpha}$$
$$\dot{z}^2 = l_p^2 \sin^2\alpha\,\dot{\alpha}^2$$

Collect $\dot{\theta}^2$:

$$A^2 + C^2 = (L_r \sin\theta + l_p \sin\alpha \cos\theta)^2 + (L_r \cos\theta - l_p \sin\alpha \sin\theta)^2$$

Expand:

$$\begin{aligned} A^2 &= L_r^2 \sin^2\theta + l_p^2 \sin^2\alpha \cos^2\theta + 2L_r l_p \sin\alpha \sin\theta \cos\theta, \\ C^2 &= L_r^2 \cos^2\theta + l_p^2 \sin^2\alpha \sin^2\theta - 2L_r l_p \sin\alpha \sin\theta \cos\theta. \end{aligned}$$

Add:

$$A^2 + C^2 = L_r^2(\sin^2\theta + \cos^2\theta) + l_p^2 \sin^2\alpha(\sin^2\theta + \cos^2\theta) = L_r^2 + l_p^2 \sin^2\alpha$$

Collect $\dot{\alpha}^2$:

$$B^2 + D^2 + l_p^2 \sin^2\alpha = l_p^2 \cos^2\alpha(\sin^2\theta + \cos^2\theta) + l_p^2 \sin^2\alpha = l_p^2$$

Collect $\dot{\theta}\dot{\alpha}$:

$$AB + CD = l_p \cos\alpha \Big[(L_r \sin\theta + l_p \sin\alpha \cos\theta)\sin\theta + (L_r \cos\theta - l_p \sin\alpha \sin\theta)\cos\theta\Big]$$

Expand bracket:

$$\begin{aligned} &(L_r \sin\theta + l_p \sin\alpha \cos\theta)\sin\theta + (L_r \cos\theta - l_p \sin\alpha \sin\theta)\cos\theta \\ &= L_r(\sin^2\theta + \cos^2\theta) + l_p \sin\alpha(\cos\theta \sin\theta - \sin\theta \cos\theta) = L_r. \end{aligned}$$

Thus

$$AB + CD = L_r l_p \cos\alpha$$

Therefore

$$v^2 = (L_r^2 + l_p^2 \sin^2\alpha)\dot{\theta}^2 + l_p^2 \dot{\alpha}^2 + 2L_r l_p \cos\alpha\, \dot{\theta}\dot{\alpha}$$

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4. Energies

Translational kinetic energy:

$$T_{\text{trans}} = \frac{1}{2} m_p v^2$$

Rotational kinetic energy of the pendulum about its COM (perpendicular axis):

$$T_{\text{rot}} = \frac{1}{2} J_p (\dot{\alpha}^2 + \sin^2 \alpha \, \dot{\theta}^2)$$

Parallel axis:

$$\hat{J}_2 = J_p + m_p l_p^2$$

Arm yaw kinetic energy:

$$T_{\text{arm}} = \frac{1}{2} \hat{J}_1 \dot{\theta}^2$$

Total kinetic energy:

$$T = T_{\text{arm}} + T_{\text{trans}} + T_{\text{rot}}$$

Collect terms.

Coefficient of $\dot{\theta}^2$:

$$\frac{1}{2} \left[\hat{J}_1 + m_p L_r^2 + (m_p l_p^2 + J_p) \sin^2 \alpha \right] \dot{\theta}^2 = \frac{1}{2} (\hat{J}_0 + \hat{J}_2 \sin^2 \alpha) \dot{\theta}^2$$

Coefficient of $\dot{\alpha}^2$:

$$\frac{1}{2} (m_p l_p^2 + J_p) \dot{\alpha}^2 = \frac{1}{2} \hat{J}_2 \dot{\alpha}^2$$

Cross term:

$$m_p L_r l_p \cos \alpha \, \dot{\theta} \dot{\alpha} = K \cos \alpha \, \dot{\theta} \dot{\alpha}$$

Thus

$$T = \frac{1}{2} (\hat{J}_0 + \hat{J}_2 \sin^2 \alpha) \dot{\theta}^2 + \frac{1}{2} \hat{J}_2 \dot{\alpha}^2 + K \cos \alpha \, \dot{\theta} \dot{\alpha}$$

Potential energy:

$$V = m_p g z = m_p g l_p \cos \alpha = G \cos \alpha$$

Lagrangian:

$$\mathcal{L} = T - V$$

5. Euler–Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right) - \frac{\partial \mathcal{L}}{\partial q} = Q$$

$$Q_\theta = \tau, \qquad Q_\alpha = 0$$

5.1 Equation in θ

$$\frac{\partial \mathcal{L}}{\partial \dot{\theta}} = (\hat{J}_0 + \hat{J}_2 \sin^2 \alpha)\dot{\theta} + K \cos\alpha\, \dot{\alpha}$$

Differentiate:

$$\frac{d}{dt}\left[(\hat{J}_0 + \hat{J}_2 \sin^2 \alpha)\dot{\theta}\right] = (\hat{J}_0 + \hat{J}_2 \sin^2 \alpha)\ddot{\theta} + \hat{J}_2 \frac{d}{dt}(\sin^2 \alpha)\dot{\theta}$$

$$\frac{d}{dt}(\sin^2 \alpha) = 2 \sin\alpha \cos\alpha\, \dot{\alpha} = \sin(2\alpha)\dot{\alpha}$$

$$\frac{d}{dt}(K \cos\alpha\, \dot{\alpha}) = K\left(-\sin\alpha\, \dot{\alpha}^2 + \cos\alpha\, \ddot{\alpha}\right)$$

Also $\partial \mathcal{L}/\partial \theta = 0$.

Thus

$$(\hat{J}_0 + \hat{J}_2 \sin^2 \alpha)\ddot{\theta} + K \cos\alpha\, \ddot{\alpha} + \hat{J}_2 \sin(2\alpha)\dot{\theta}\dot{\alpha} - K \sin\alpha\, \dot{\alpha}^2 = \tau$$

5.2 Equation in α

$$\frac{\partial \mathcal{L}}{\partial \dot{\alpha}} = \hat{J}_2 \dot{\alpha} + K \cos\alpha\, \dot{\theta}$$

Differentiate:

$$\frac{d}{dt}\left(\hat{J}_2 \dot{\alpha}\right) = \hat{J}_2 \ddot{\alpha}$$

$$\frac{d}{dt}(K \cos\alpha\, \dot{\theta}) = K\left(-\sin\alpha\, \dot{\alpha}\dot{\theta} + \cos\alpha\, \ddot{\theta}\right)$$

Partial derivative:

$$\frac{\partial \mathcal{L}}{\partial \alpha} = \frac{1}{2}\hat{J}_2 \frac{\partial}{\partial \alpha}(\sin^2 \alpha)\dot{\theta}^2 - K \sin\alpha\, \dot{\theta}\dot{\alpha} + G \sin\alpha$$

$$\frac{\partial}{\partial \alpha}(\sin^2 \alpha) = \sin(2\alpha)$$

Thus

$$\frac{\partial \mathcal{L}}{\partial \alpha} = \frac{1}{2}\hat{J}_2 \sin(2\alpha)\dot{\theta}^2 - K \sin\alpha\, \dot{\theta}\dot{\alpha} + G \sin\alpha$$

Euler–Lagrange ($Q_\alpha = 0$):

$$\hat{J}_2\ddot{\alpha} + K\cos\alpha\,\ddot{\theta} - K\sin\alpha\,\dot{\alpha}\dot{\theta} - \left(\frac{1}{2}\hat{J}_2\sin(2\alpha)\dot{\theta}^2 - K\sin\alpha\,\dot{\theta}\dot{\alpha} + G\sin\alpha\right) = 0$$

Cancel the $\pm K\sin\alpha\,\dot{\theta}\dot{\alpha}$ terms:

$$K\cos\alpha\,\ddot{\theta} + \hat{J}_2\ddot{\alpha} - \frac{1}{2}\hat{J}_2\sin(2\alpha)\dot{\theta}^2 - G\sin\alpha = 0$$

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6. Nonlinear model

$$(\hat{J}_0 + \hat{J}_2\sin^2\alpha)\ddot{\theta} + K\cos\alpha\,\ddot{\alpha} + \hat{J}_2\sin(2\alpha)\dot{\theta}\dot{\alpha} - K\sin\alpha\,\dot{\alpha}^2 = \tau$$

$$K\cos\alpha\,\ddot{\theta} + \hat{J}_2\ddot{\alpha} - \frac{1}{2}\hat{J}_2\sin(2\alpha)\dot{\theta}^2 - G\sin\alpha = 0$$

The derived equations are structurally consistent with classical Furuta pendulum models, with the distinction that actuator dynamics are treated at the acceleration level rather than torque level.

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7. Linearization about upright ($\alpha \approx 0$)

$$\sin\alpha \approx \alpha, \qquad \cos\alpha \approx 1, \qquad \sin(2\alpha) \approx 2\alpha$$

Neglect products of small quantities.

Linear model:

$$\hat{J}_0\ddot{\theta} + K\ddot{\alpha} = \tau$$
$$K\ddot{\theta} + \hat{J}_2\ddot{\alpha} - G\alpha = 0$$

Solve for $\ddot{\alpha}$:

$$\ddot{\alpha} = \frac{G}{\hat{J}_2}\alpha - \frac{K}{\hat{J}_2}\ddot{\theta}$$

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8. Numerical specialization

Let $J_0 \equiv \hat{J}_0$ and $J_1 \equiv \hat{J}_2$. Numerical values:

$$J_0 = 0.001104\ \mathrm{kg\,m^2}, \quad J_1 = 1.021 \times 10^{-4}\ \mathrm{kg\,m^2}, \quad K = 1.993 \times 10^{-4}\ \mathrm{kg\,m^2}, \quad G = 0.01029\ \mathrm{N\,m}$$

Linear equations:

$$J_0\ddot{\theta} + K\ddot{\alpha} = \tau$$
$$K\ddot{\theta} + J_1\ddot{\alpha} - G\alpha = 0$$

Reduced relation:

$$\ddot{\alpha} = \frac{G}{J_1}\alpha - \frac{K}{J_1}\ddot{\theta}$$

Numerically:

$$\ddot{\alpha} = 100.8\,\alpha - 1.952\,\ddot{\theta}$$

Open-loop fall rate ($\tau = 0$). From $\ddot{\theta} = -(K/J_0)\ddot{\alpha}$:

$$\left(J_1 - \frac{K^2}{J_0}\right)\ddot{\alpha} - G\alpha = 0$$

$$J_{eq} = J_1 - \frac{K^2}{J_0} = 0.0001021 - \frac{(0.0001993)^2}{0.001104} = 6.612 \times 10^{-5}$$

$$\ddot{\alpha} = \frac{G}{J_{eq}}\alpha, \qquad \lambda = \pm\sqrt{\frac{G}{J_{eq}}} = \pm 12.47\ \mathrm{rad/s}$$

Mass matrix inverse:

$$M = \begin{bmatrix} J_0 & K \\ K & J_1 \end{bmatrix}, \qquad M^{-1} = \frac{1}{J_0 J_1 - K^2}\begin{bmatrix} J_1 & -K \\ -K & J_0 \end{bmatrix}$$

$$J_0 J_1 - K^2 = 7.2998 \times 10^{-8}$$

$$M^{-1} = \begin{bmatrix} 1398.7 & -2730.2 \\ -2730.2 & 15123.7 \end{bmatrix}$$

Substituted form:

$$0.001104\,\ddot{\theta} + 0.0001993\,\ddot{\alpha} = \tau$$

$$0.0001993\,\ddot{\theta} + 0.0001021\,\ddot{\alpha} - 0.01029\,\alpha = 0$$

Nonlinear model with numerical constants:

$$(J_0 + J_1 \sin^2 \alpha)\ddot{\theta} + K \cos\alpha\,\ddot{\alpha} + J_1 \sin(2\alpha)\dot{\theta}\dot{\alpha} - K \sin\alpha\,\dot{\alpha}^2 = \tau$$

$$K \cos\alpha\,\ddot{\theta} + J_1\ddot{\alpha} - \frac{1}{2}J_1 \sin(2\alpha)\dot{\theta}^2 - G \sin\alpha = 0$$

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9. PD design on commanded arm acceleration

Plant:

$$\ddot{\alpha} = A\alpha - B\ddot{\theta}, \qquad A \equiv \frac{G}{J_1}, \quad B \equiv \frac{K}{J_1}$$

Controller:

$$\ddot{\theta}_{cmd} = -k_p\alpha - k_d\dot{\alpha}$$

Closed-loop characteristic equation:

$$\ddot{\alpha} - (Bk_d)\dot{\alpha} - (A + Bk_p)\alpha = 0$$

Match to

$$\ddot{\alpha} + 2\zeta\omega_c\dot{\alpha} + \omega_c^2\alpha = 0$$

Gains:

$$k_d = -\frac{2\zeta\omega_c}{B}, \qquad k_p = -\frac{A + \omega_c^2}{B}$$

With $\omega_c = 15,\ \zeta = 0.8,\ A = 100.8,\ B = 1.952$:

$$k_p = -166.9, \qquad k_d = -12.3$$

Negative gains arise from the sign convention $\alpha = 0$ upright; the controller provides restoring acceleration opposing the unstable gravitational term.

Stepper conversion ($N = 1600$ microsteps/rev):

$$\text{steps/rad} = \frac{N}{2\pi} = 254.65, \qquad \text{steps/deg} = \frac{N}{360} = 4.444$$

Degree variables ($\alpha_{\text{deg}} = \alpha\,180/\pi$):

$$\ddot{\theta}_{\text{steps}} = -(742\,\alpha_{\text{deg}} + 54.6\,\dot{\alpha}_{\text{deg}})$$

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10. State-space representation and LQR design

10.1 State definition

Define the state vector as

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} \theta \\ \alpha \\ \dot{\theta} \\ \dot{\alpha} \end{bmatrix}$$

The control input is chosen as the commanded arm angular acceleration

$$u \equiv \ddot{\theta}$$

10.2 Linearized state-space model

The state equations are

$$\dot{x}_1 = x_3$$
$$\dot{x}_2 = x_4$$
$$\dot{x}_3 = u$$
$$\dot{x}_4 = A\,x_2 - B\,u$$

where

$$A \equiv \frac{G}{J_1}, \qquad B \equiv \frac{K}{J_1}$$

Collecting terms, the state-space model is

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$$

with

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & A & 0 & 0 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -B \end{bmatrix}$$

Numerically,

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 100.8 & 0 & 0 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1.952 \end{bmatrix}$$

10.3 LQR formulation

A continuous-time linear quadratic regulator (LQR) is designed with the control law

$$u = -\mathbf{K}\mathbf{x}$$

minimizing the cost function

$$J = \int_0^\infty \left(\mathbf{x}^\top \mathbf{Q} \mathbf{x} + u^\top \mathbf{R} u \right) \, dt$$

10.4 LQR gain

The gain is computed by solving the continuous-time algebraic Riccati equation (CARE)

$$\mathbf{A}^\top \mathbf{P} + \mathbf{P}\mathbf{A} - \mathbf{P}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^\top \mathbf{P} + \mathbf{Q} = \mathbf{0}$$

and then

$$\mathbf{K} = \mathbf{R}^{-1}\mathbf{B}^\top \mathbf{P}.$$

For the numerical gain below, the weights are

$$\mathbf{Q} = \mathrm{diag}(0.5,\ 50.0,\ 0.05,\ 5.0), \qquad \mathbf{R} = \begin{bmatrix} 1.0 \end{bmatrix}.$$

This choice prioritizes pendulum angle stabilization (large weight on α) while penalizing control effort.

Numerical values are obtained by solving the CARE (e.g., via SciPy `solve_continuous_are`) and evaluating $\mathbf{K} = \mathbf{R}^{-1}\mathbf{B}^\top \mathbf{P}$.

$$\mathbf{K} = \begin{bmatrix} -0.70710678 & -117.18259227 & -1.3583044 & -11.86304115 \end{bmatrix}$$

The resulting control law is

$$u = -(-0.7071\,\theta - 117.183\,\alpha - 1.3583\,\dot{\theta} - 11.8630\,\dot{\alpha})$$

Stepper units using degree variables. Let

$$\mathbf{x}_{\text{deg}} = \begin{bmatrix} \theta_{\text{deg}} \\ \alpha_{\text{deg}} \\ \dot{\theta}_{\text{deg}} \\ \dot{\alpha}_{\text{deg}} \end{bmatrix}, \qquad \mathbf{x} = \frac{\pi}{180}\mathbf{x}_{\text{deg}}$$

Let $u_{\text{deg}} \equiv \ddot{\theta}_{\text{deg}}$ denote acceleration in deg/s^2, so that $u = (\pi/180)u_{\text{deg}}$. Then

$$\ddot{\theta}_{\text{steps}} = \frac{N}{2\pi}u = \frac{N}{360}u_{\text{deg}}$$

Then

$$\ddot{\theta}_{\text{steps}} = -\mathbf{K}_{\text{steps}}\mathbf{x}_{\text{deg}}$$

$$\mathbf{K}_{\text{steps}} = \left(\frac{N}{360}\right)\mathbf{K} = 4.444\,\mathbf{K} = \begin{bmatrix} -3.1424 & -520.76 & -6.0363 & -52.72 \end{bmatrix}$$

$$\ddot{\theta}_{\text{steps}} = -(-3.1424\,\theta_{\text{deg}} - 520.76\,\alpha_{\text{deg}} - 6.0363\,\dot{\theta}_{\text{deg}} - 52.72\,\dot{\alpha}_{\text{deg}})$$
