DEPARTMENT OF MATHEMATICS
INDIAN INSTITUTE OF TECHNOLOGY GUWAHATI

(Deferred) Winter Semester of Academic Year 2022-2023

MA102 Mathematics II

*** Quiz 2 ***

Model Answers and Marking Scheme

Total Time: **60** minutes | Total Marks: **15** marks

Instructions

- This question paper has **five** questions.
- Answer **all** questions.
- Each question carries **three** marks.

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1. Solve $\left(y^4 + 2y\right)\ dx\ +\ \left(xy^3 + 2y^4 - 4x\right)\ dy\ =\ 0.$

Answer:

Step 1: Finding an Integrating Factor to reduce the Given Equation into Exact Equation

Set

$$M(x,y)\ =\ y^4 + 2y \qquad \text{and} \qquad N(x,y)\ =\ xy^3 + 2y^4 - 4x\ .$$

Then,

$$M_y\ =\ 4y^3 + 2 \qquad \text{and} \qquad N_x\ =\ y^3 - 4\ .$$

$$\frac{N_x - M_y}{M}\ =\ \frac{(-3)(y^3+2)}{(y^4+2y)}\ =\ \frac{-3}{y}\ =\ \phi(y)\ .$$

It is a function of y alone.

Therefore, an integrating factor can be

$$\mu(y) = e^{\int \phi(y)\ dy} = e^{\int \frac{(-3)}{y}\ dy} = \frac{1}{y^3}\ .$$

(1 mark)

Multiplying the given equation by this integrating factor $\mu(y)$, the given equation is transformed into the following exact equation.

$$\left(y\ +\ \frac{2}{y^2}\right)\ dx\ +\ \left(x\ +\ 2y\ -\ \frac{4x}{y^3}\right)\ dy\ =\ 0\ .$$

Step 2: Solving the Exact Equation

Set

$$M^*(x,y)\ =\ y\ +\ \frac{2}{y^2} \qquad \text{and} \qquad N^*(x,y)\ =\ x\ +\ 2y\ -\ \frac{4x}{y^3}\ .$$

We need to find $\phi : \mathbb{R}^2 \to \mathbb{R}$ such that $\frac{\partial \phi}{\partial x} = M^*(x,y)$ and $\frac{\partial \phi}{\partial y} = N^*(x,y)$. Then, $\phi(x,y) = c$ is the general solution of the exact equation as well as the original non-exact equation.

$$\frac{\partial \phi}{\partial x}\ =\ M^*(x,y)\ =\ y\ +\ \frac{2}{y^2}\ .$$

Integrating it partially with respect to x, we get

$$\phi(x,y) \;=\; \int \left(y \,+\, \frac{2}{y^2}\right)\,dx \;=\; xy \,+\, \frac{2x}{y^2} \,+\, h(y)\;,$$

where $h(y)$ is an arbitrary function of y only. **(1 mark)**

Now, differentiating the above equation partially with respect to y, we get

$$\frac{\partial\phi}{\partial y} \;=\; x \,-\, \frac{4x}{y^3} \,+\, h'(y)\;.$$

But

$$\frac{\partial\phi}{\partial y} \;=\; N^*(x,y) \;=\; x \,+\, 2y \,-\, \frac{4x}{y^3}\;.$$

This gives that

$$h'(y) = 2y\;.$$

$$\Longrightarrow \qquad h(y) = y^2 + k\;,$$

where k is an arbitrary real constant.

Therefore,

$$\phi(x,y) \;=\; xy \,+\, \frac{2x}{y^2} \,+\, y^2 \,+\, k\;.$$

Therefore, the implicit solution of given differential equation is given by

$$xy \,+\, \frac{2x}{y^2} \,+\, y^2 \;=\; c\;,$$

where c is an arbitrary real constant. **(1 mark)**

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2. By considering all different values/ cases for x_0 and y_0, discuss the existence and uniqueness of the solution of the initial value problem $(x^2 - 2x)\,\dfrac{dy}{dx} \;=\; 2(x-1)y$ and $y(x_0) = y_0$. In case of existence of solution(s), find the solution(s) to the IVP.

   Answer:

   Step 1: Analyzing continuity $f(x,y)$ and $\frac{\partial f}{\partial y}$

   $$\frac{dy}{dx} \;=\; f(x,y)\;, \qquad \text{where} \qquad f(x,y) = \frac{2(x-1)y}{x^2-2x}\;,$$
   $$y(x_0) = y_0\;.$$

   Observe that

   $$f(x,y) = \frac{2(x-1)y}{x^2-2x} \qquad \text{and} \qquad \frac{\partial f}{\partial y} = \frac{2(x-1)}{x^2-2x}$$

   are continuous if $x \neq 0$ and $x \neq 2$. **(0.5 mark)**

   Step 2: Solving the given ODE

   $$\frac{dy}{dx} \;=\; \frac{2(x-1)y}{x^2-2x} \;=\; \left[\frac{1}{(x-2)} + \frac{1}{x}\right]\,y$$

$$\Longrightarrow \quad \frac{dy}{y} = \left[\frac{1}{(x-2)} + \frac{1}{x}\right] dx$$

Integrating we get

$$\log(y) = \log(x-2) + \log(x) + \log(c) .$$

The general solution of the given equation is given by

$$y(x) = c\, x(x-2) .$$

(0.5 mark)

Step 3: $x_0 \neq 0$ and $x_0 \neq 2$

Assume that $x_0 \neq 0$ and $x_0 \neq 2$.

Consider the closed rectangle R given by

$$R = \{(x,y) \in \mathbb{R}^2 \ : \ |x - x_0| \leq a \quad \text{and} \quad |y - y_0| \leq b\} ,$$

where $a > 0$ and $b > 0$ and it is so chosen that $0 \notin [x_0 - a,\ x_0 + a]$ and $2 \notin [x_0 - a,\ x_0 + a]$.
Then, $f(x,y)$ and $\partial f/\partial y$ are continuous and bounded on the closed rectangle R.
By Picard's theorem, the given IVP has a unique solution in this case.
Applying the initial condition, we get the unique solution as

$$y(x) = \left(\frac{y_0}{x_0(x_0-2)}\right) x\,(x-2) \qquad \text{for} \qquad |x - x_0| \leq h .$$

Here y_0 can be any real number. **(1 mark)**

Step 4: $x_0 = 0$ or $x_0 = 2$

Assume that $x_0 = 0$ or $x_0 = 2$.
If $y_0 = 0$, then the given IVP has infinitely many solutions, namely, $y_c(x) = c\, x(x-2)$, where c is a real parameter. **(0.5 mark)**
If $y_0 \neq 0$, then the given IVP has NO solution, because it does not satisfy the initial condition.
(0.5 mark)

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3. Let $P(x)$, $Q(x)$ and $R(x)$ be given continuous functions in $[-1,\ 1]$. Let

$$L(y) = y'' + P(x)\, y' + Q(x)\, y .$$

Let x and $(x + e^x)$ be the solutions for the nonhomogeneous differential equation $L(y) = R$ in $[-1,\ 1]$. Suppose $W(y_1, y_2)(x) = e^{3x}$ is the Wronskian of two solutions y_1 and y_2 of $L(y) = 0$ in $[-1,\ 1]$. Determine a particular solution for $L(y) = 3\, e^{4x}$.

Answer:

Step 1: Finding a solution of homogeneous equation

Set $\phi(x) = x + e^x$ and $\psi(x) = x$.

Then, $L(\phi) = R$ and $L(\psi) = R$. This gives that $L(\phi - \psi) = R - R = 0$. Thus, $\phi(x) - \psi(x) = (x + e^x) - x = e^x$ is a solution of the homogeneous equation $L(y) = 0$. **(0.5 mark)**

Step 2: Determining $P(x)$

Given that $W(y_1, y_2)(x) = e^{3x}$.

We know that the Wronskian $W(x)$ of two solutions of the homogeneous equation $L(y) = y'' + P(x)\, y' + Q(x)\, y = 0$ satisfies the first order linear equation $W'(x) + P(x)W(x) = 0$.

Therefore,

$$3\, e^{3x} \;+\; P(x)\, e^{3x} \;=\; 0 \,.$$

Solving it, we get

$$P(x) = -3 \qquad \text{for all} \qquad x \in [-1,\ 1] \,.$$

(1 mark)

Step 3: Determining $Q(x)$

Take $y_1(x) = e^x$ from Step 1.

Then $y_1(x)$ satisfies $L(y) = y'' + P(x)\, y' + Q(x)\, y \;=\; 0$. That is,

$$e^x \;-\; 3\, e^x \;+\; Q(x)\, e^x \;=\; 0 \,.$$

This gives that

$$Q(x) \;=\; 2 \qquad \text{for all} \qquad x \in [-1,\ 1] \,.$$

(0.5 mark)

Step 4: Finding a particular solution $y_p(x)$ for $L(y) = 3\, e^{4x}$

In our case, $L \equiv P(D)$, where $P(D) = D^2 - 3D + 2$.

By the operator method, a particular solution to the nonhomogeneous equation $L(y) = 3\, e^{4x}$ can be obtained as

$$L(y_p) = \frac{1}{P(D)} 3\, e^{4x} \;=\; \frac{3\, e^{4x}}{P(4)} \;=\; \frac{3\, e^{4x}}{6} \;=\; \frac{e^{4x}}{2} \,.$$

(1 mark)

Therefore, the required particular solution is $y_p(x) = \frac{e^{4x}}{2}$.

Aliter for Steps 2 and 3: Determining $P(x)$ and $Q(x)$

Take $y_1(x) = e^x$ from Step 1. Then,

$$W(y_1, y_2)(x) \;=\; \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} \;=\; \begin{vmatrix} e^x & y_2(x) \\ e^x & y_2'(x) \end{vmatrix} \;=\; e^{3x} \,.$$

This gives that $e^x\, y_2'(x) \;-\; e^x\, y_2(x) \;=\; e^{3x}$. That is, $y_2(x)$ is a solution of $y' - y = e^{2x}$. Therefore, $y_2(x) = e^{2x}$. **(1 mark)**

Then, observe that $y_1(x) = e^x$ and $y_2(x) = e^{2x}$ are two linearly independent solutions of $y'' + P(x)y' + Q(x)y = y'' - 3y' + 2y = 0$. Therefore, $P(x) = -3$ for all $x \in [-1, 1]$ and $Q(x) = 2$ for all $x \in [-1, 1]$. **(0.5 mark)**

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4. We know that x and $(x^2 - 1)$ are two linearly independent solutions for the differential equation

$$(x^2 + 1)\, y'' \;-\; 2x\, y' \;+\; 2y \;=\; 0\,.$$

Use this information to find the solution for the following initial value problem:

$$(x^2 + 1)\, y'' \;-\; 2x\, y' \;+\; 2y \;=\; 6(x^2 + 1)^2\,, \qquad y(0) = 0 \quad \text{and} \quad y'(0) = 0\,.$$

Answer:

Step 1: Finding $W(y_1, y_2)(x)$

Given equation can be written as

$$y'' \;+\; P(x)\, y' \;+\; Q(x) \;=\; R(x)\,,$$

where

$$P(x) = \frac{-2x}{(x^2 + 1)}, \qquad Q(x) = \frac{2}{(x^2 + 1)}, \qquad \text{and} \qquad R(x) = 6(x^2 + 1)\,.$$

Set $y_1(x) = x$ and $y_2(x) = x^2 - 1$ for $x \in \mathbb{R}$. Given that $y_1(x)$ and $y_2(x)$ are two linearly independent solution of the associated homogeneous equation. Then,

$$W(y_1, y_2)(x) \;=\; \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} \;=\; \begin{vmatrix} x & (x^2 - 1) \\ 1 & 2x \end{vmatrix} \;=\; x^2 + 1 \qquad \text{for } x \in \mathbb{R}\,.$$

(0.5 mark)

Step 2: Determining $y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x)$

Since the given ODE has variable coefficients, we will find a particular solution $y_p(x)$ by the method of variation of parameters.

Let $y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x)$. We will determine $u_1(x)$ and $u_2(x)$.

$$u_1(x) \;=\; \int \frac{(-1)y_2(x)R(x)}{W(y_1, y_2)(x)}\, dx \;=\; \int \frac{(-1)(x^2 - 1)(6x^2 + 6)}{x^2 + 1}\, dx \;=\; (-6) \int (x^2 - 1)\, dx \;=\; 6x - 2x^3\,.$$

(1 mark)

$$u_2(x) \;=\; \int \frac{y_1(x)R(x)}{W(y_1, y_2)(x)}\, dx \;=\; \int \frac{x(6x^2 + 6)}{x^2 + 1}\, dx \;=\; 6 \int x\, dx \;=\; 3x^2\,.$$

(1 mark)

Therefore,

$$y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x) = (6x - 2x^3)x + (3x^2)(x^2 - 1) = x^4 + 3x^2\,.$$

Step 3: Applying the given initial conditions

The general solution to the given nonhomogenous ODE is given by

$$y(x) \;=\; y_c(x) + y_p(x) \;=\; c_1 x \;+\; c_2(x^2-1) \;+\; (x^4+3x^2)\;,$$

where c_1 and c_2 are arbitrary real constants.

Applying the initial condition $y(0) = 0$, we get $c_2 = 0$. **(0.25 mark)**

Applying the initial condition $y'(0) = 0$, we get $c_1 = 0$. **(0.25 mark)**

Therefore, the solution to the given initial value problem is $y(x) = x^4 + 3x^2$ which is valid in a neighborhood of 0.

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5. Find the general solution of

$$x^3\,\frac{d^3y}{dx^3} \;-\; x^2\,\frac{d^2y}{dx^2} \;-\; 6x\,\frac{dy}{dx} \;+\; 18\,y \;=\; 0 \quad \text{for} \quad x > 0\;.$$

Answer:

Step 1: Converting the given equation into an Equation with Constant Coefficients

Let us take the transformation $x = e^t$.

By the chain rule, we get

$$\begin{aligned} x\,\frac{dy}{dx} \;&=\; \frac{dy}{dt} \\ x^2\,\frac{d^2y}{dx^2} \;&=\; \frac{d^2y}{dt^2} \;-\; \frac{dy}{dt} \\ x^3\,\frac{d^3y}{dx^3} \;&=\; \frac{d^3y}{dt^3} \;-\; 3\,\frac{d^2y}{dt^2} \;+\; 2\,\frac{dy}{dt} \end{aligned}$$

The given equation will be transformed into

$$\left(\frac{d^3y}{dt^3} - 3\,\frac{d^2y}{dt^2} \;+\; 2\,\frac{dy}{dt}\right) \;-\; \left(\frac{d^2y}{dt^2} \;-\; \frac{dy}{dt}\right) \;-\; 6\,\left(\frac{dy}{dt}\right) \;+\; 18\,y \;=\; 0$$

$$\frac{d^3y}{dt^3} \;-\; 4\,\frac{d^2y}{dt^2} \;-\; 3\,\frac{dy}{dt} \;+\; 18\,y \;=\; 0$$

(1 mark)

Step 2: Solving the Transformed Equation

The auxiliary equation is $m^3 - 4m^2 - 3m + 18 = 0$.

$$m^3 - 4m^2 - 3m + 18 = (m-3)(m^2-m-6) = (m-3)(m-3)(m+2) = 0\;.$$

The roots of auxiliary equation are $m = 3$ with multiplicity 2 and $m = -2$ with multiplicity 1.

(0.5 mark)

Therefore, the general solution of the transformed equation is given by

$$y(t) \;=\; c_1\,t\,e^{3t} \;+\; c_2\,e^{3t} \;+\; c_3\,e^{-2t}\;,$$

where c_1, c_2 and c_3 are arbitrary real constants. **(1 mark)**

Step 3: Writing the General Solution of the Original Equation

Since $x = e^t$, we have $t = \ln(x)$.

The general solution of the original equation is given by

$$y(x) = c_1 \ln(x)\, x^3 + c_2\, x^3 + c_3\, x^{-2} = c_1 \ln(x)\, x^3 + c_2\, x^3 + c_3 \left(\frac{1}{x^2}\right),$$

where c_1, c_2 and c_3 are arbitrary real constants. **(0.5 mark)**

Aliter :

Step 1:

Assume that the solution to the given equation is of the form $y(x) = x^r$ in which r is to be determined.

Then,

$$r(r-1)(r-2)x^{r-3}x^3\frac{d^2y}{dx^2} - r(r-1)x^{r-2}x^2\frac{d^2y}{dx^2} - 6rx^{r-1}x\frac{dy}{dx} + 18x^r = 0 .$$

$$x^r\left[r(r-1)(r-2)\frac{d^2y}{dx^2} - r(r-1)\frac{d^2y}{dx^2} - 6r\frac{dy}{dx} + 18\right] = 0 .$$

$$r(r-1)(r-2)\,\frac{d^2y}{dx^2} - r(r-1)\,\frac{d^2y}{dx^2} - 6r\,\frac{dy}{dx} + 18 = 0 .$$

(1 mark)

Step 2: Wriing the Indicial/Auxiliary Equation and Finding its roots

The indicial/ auxiliary equation is $r(r-1)(r-2) - r(r-1) - 6r + 18 = 0$.

$$r^3 - 4r^2 - 3r + 18 = 0 .$$

$$(r-3)(r-3)(r+2) = 0 .$$

The roots of the indicial equation are $r = 3$ with multiplicity 2 and $r = -2$ with multiplicity 1.

(0.5 mark)

Step 3: Writing the General Solution of the Original Equation

The general solution of the original equation is given by

$$y(x) = c_1 \ln(x)\, x^3 + c_2\, x^3 + c_3\, x^{-2} = c_1 \ln(x)\, x^3 + c_2\, x^3 + c_3 \left(\frac{1}{x^2}\right),$$

where c_1, c_2 and c_3 are arbitrary real constants. **(1.5 mark)**

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