

TTK4551 Technical Cybernetics - Specialization Project

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Acronyms

ASV	Autonomous Surface Vessel
AUR Lab	Applied Underwater Robotics Lab
AUV	Autonomous Underwater Vehicle
COG	Center Of Gravity
COLAMD	Column Approximate Minimum Degree
DDS	Data Distribution Service
DOF	Degrees Of Freedom
ECEF	Earth-Centered, Earth-Fixed
EKF	Extended Kalman Filter
ERK	Explicit Runge-Kutta
ESC	Electronic Speed Controller
ESKF	Error State Kalman Filter
GNSS	Global Navigation Satellite System
GTSAM	Georgia Tech Smoothing And Mapping
HDPE	High Density Polyethylene
IMU	Inertial Measurement Unit
INS	Inertial Navigation System
iSAM	Iterative Smoothing And Mapping
KF	Kalman Filter
LAUV	Light Autonomous Underwater Vehicle
LiDAR	Light Detection And Ranging
LTI	Linear Time Invariant
LTV	Linear Time Varying
MAP	Maximum A Posteriori
MPC	Model Predictive Control
NED	North East Down
PPS	Pulse Per Second
PSM	Power Sense Module
PWM	Pulse Width Modulation
RL	Reinforcement Learning
ROS2	Robot Operating System 2
SBC	Single Board Computer
SE(3)	Special Euclidean Group in three dimensions
SLAM	Simultaneous Localization And Mapping
SLERP	Spherical Linear Interpolation
SO(3)	Special Orthogonal Group in three dimensions
SSS SLAM	Side Scan Sonar Simultaneous Localization And Mapping
UKF	Unscented Kalman Filter
UKF-M	Unscented Kalman Filter on Manifolds
UT	Unscented Transform
WGS84	World Geodetic System 1984
ZOH	Zero Order Hold

1 Introduction

1.1 Goal

Maybe I should resturcture the Introduction Goal and Motivation maybe?...

Also add the FN bærekrafts mål I suppose?...

This specialization project focuses on navigation and Simultaneous Localization and Mapping (SLAM) for marine robots, with an emphasis on Side Scan Sonar based SLAM. The main objective is to study, reimplement, and optimize the core components of modern SSS SLAM pipelines to achieve real-time performance on real world AUV datasets, such as those collected from NTNU's AUR Lab platforms (for example LAUV Harald).

The work builds upon the 2023 masters thesis by Haraldstad [1] and recent research on side scan sonar landmark detection [2], which are heavily inspired by the earlier work of Hogstad, Bjørnar Reitan [3]. The projects scope is to implement and benchmark the essential components required for real-time SSS SLAM operation, validate them on existing datasets, and demonstrate that the processing pipeline can operate at or above the sonars frame rate while maintaining high mapping accuracy.

A second phase of the project focuses on embedded deployment on a autonomous surface vessel (ASV). The goal is to port and integrate the validated core pipeline on ship borne hardware, assess side scan sonar performance in coastal waters, and shift the work from algorithm design to practical system integration. This phase will use NTNU's MicroAmpere ASV.

1.2 Motivation

Reliable maritime navigation requires onboard estimation and mapping when external positioning is weak or unavailable. This project focuses on side scan sonar based simultaneous localization and mapping, SSS SLAM. SSS SLAM uses side scan sonar to estimate the vehicle pose while building a seafloor map at the same time. In SSS SLAM, sonar images drive feature extraction and data association, loop closures correct drift, and the SLAM back end fuses all measurements into a consistent trajectory and map. Marine robots operate with limited access and often far from support. They must know where they are and what surrounds them to move safely and do useful work. Static charts help, but the ocean changes over time. Currents, waves, moving vessels, new structures, and shifting seabeds make static maps go out of date. GNSS is weak or unavailable underwater, and dead reckoning drifts. Even in coastal areas, terrain can block signals, and shallow water operations require safe margins to the bottom. These factors make SSS SLAM a practical path to robust navigation and mapping.

Prior work [1] presented a pipeline for SSS SLAM but did not reach real time performance. Field deployment needs real time operation on real data with measured accuracy and robustness. The first motivation is to deliver a lean real time SSS SLAM implementation and to quantify performance on the same dataset used previously, so the results are directly comparable.

The second motivation is to move from offline studies to reliable system behavior at sea. The plan is to measure accuracy, robustness, and runtime on recorded AUV data, then prepare the pipeline for embedded use on a autonomous surface vessel (ASV). This shifts effort from algorithm design to practical integration on real hardware, including time synchronization, calibration, and stable runtime.

1.3 Side Scan Sonar SLAM Architecture

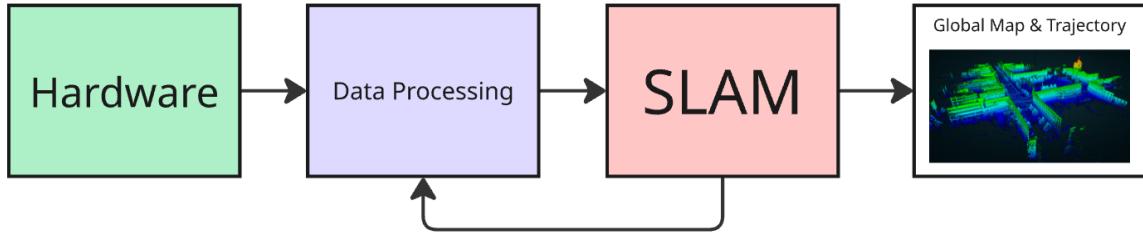


Figure 1: A simplified picture of SSS SLAM pipeline from raw sensor data to global map

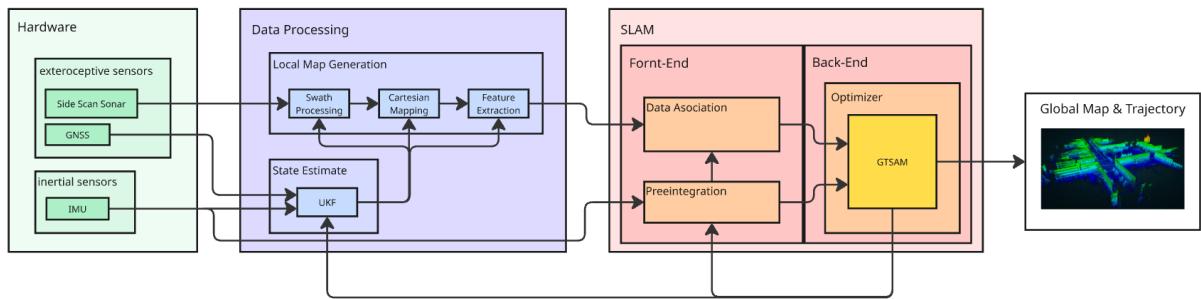


Figure 2: A picture of the whole SSS SLAM pipeline in detail from raw sensor data to global map

Some images here of basic overviews

Talk about SLAM

Then a picture of complex overview

Talk a bit more in depth on slam

2 Sonar Theory

Talk about acoustics

Talk about Sonar

Talk about camera and visual odometry stuff and how sonar is used

3 Hardware

3.1 Micro Ampere ASV



Figure 3: A picture of microAmpere Autonomous Surface Vessel (ASV). Picture taken from Luan Cao Vo Tran master thesis on microAmpere ASV.^[4]

This section focuses on the “*microAmpere Autonomous Surface Vessel (ASV)*” platform, which serves as the primary research and development base for this work. The information presented here is summarized from two detailed master theses, Luan Cao Vo Trans thesis on microAmpere hardware and integration [4], and Henrik Reimers thesis on control and system design [5]. More detailed technical descriptions can be found in the referenced theses.

In this section, only the most relevant aspects of the platform are presented, specifically those that are essential for integrating a side scan sonar payload and for enabling SSS SLAM. The focus is therefore on the hardware architecture, computing layout, and synchronization systems that directly affect sonar data acquisition and processing.

The microAmpere is an advanced Autonomous Surface Vessel (ASV) developed at NTNU as a modular platform for autonomous docking and maritime robotics research. It is part of the Autodocking25 project and represents the next evolution of the Blue Robotics BlueBoat platform. The vessel was redesigned to support high performance perception, navigation, and control tasks under realistic marine conditions. Its main purpose is to provide a flexible and open research platform for experimenting with autonomous operations such as docking, waypoint navigation, and sensor fusion at sea.

Physically, the vessel is based on the BlueBoat twin hull design, offering high stability and sufficient deck space for custom payloads. The hulls are constructed from High Density Polyethylene (HDPE), ensuring durability and resistance to saltwater corrosion. The superstructure houses one watertight enclosures containing the compute units, sensor interfaces, and power management systems. This enclosure is designed for quick maintenance and allows for modular upgrades for new hardware.

The system integrates a distributed computing architecture consisting of a “*Jetson Orin NX*” for GPU intensive perception tasks such as LiDAR and camera processing, and a “*LattePanda 3 Delta*” Single Board Computer (SBC) for real-time navigation, control, and communication handling. Both systems run ROS2 with synchronized clocks via GNSS and PPS hardware timing, achieving sub microsecond precision across devices. This allows accurate sensor timestamping and time alignment between the navigation, control, and perception subsystems.

The sensor suite includes dual GNSS antennas for accurate heading estimation, a high grade IMU for attitude and acceleration measurements, an “*Ouster OS1*” LiDAR for 3D environmental mapping, and “*StereoLabs ZED-X*” stereo cameras for visual perception and teleoperation. A dedicated power sensing board monitors voltage and current to ensure system safety, and an industrial “*Teltonika RUTX50*” router provides 4G/5G and GNSS connectivity, extending operational range far beyond line of sight limits. All components are interconnected through a structured internal wiring system with proper isolation, grounding, and surge protection to handle harsh marine environments.

The MicroAmperes software stack is fully containerized, combining real-time control loops with modular high level autonomy nodes. Each module communicates through standardized ROS2 topics and services, enabling independent algorithm development without system reconfiguration. This architecture also facilitates rapid iteration and field testing of advanced autonomy features such as Model Predictive Control (MPC), Reinforcement Learning (RL), and Simultaneous Localization and Mapping (SLAM).

In addition to autonomy research, the microAmpere platform is used for multi sensor data collection and algorithm benchmarking under controlled sea trials. The synchronized data from LiDAR, stereo vision, IMU, and GNSS enables high quality datasets for perception, sensor fusion, and state estimation studies. The vessels modular interfaces also make it suitable for integration with experimental payloads such as side scan sonar, underwater acoustic systems, or alternative control computers for specific research tasks.

From a systems engineering perspective, microAmpere demonstrates the effectiveness of distributed embedded computing in marine robotics. By separating perception and control workloads, developers can independently optimize performance for GPU heavy machine learning pipelines and deterministic control processes. The modular ROS2 setup also supports simulation-in-the-loop and hardware-in-the-loop testing, ensuring smooth transition from virtual models to field deployment.

According to Luan Cao Vo Tran master thesis on microAmpere ASV [4] and Henrik Reimers master thesis on microAmpere ASV [5], the platform has already undergone multiple field campaigns in Trondheim Fjord. These tests validated microAmpere ASVs hardware robustness, networking reliability, and autonomy stack. Its success has positioned it as a reference platform for NTNU’s future autonomous surface vessel research, bridging efforts between the Department of Cybernetics and the AUR Lab.

Overall, the microAmpere ASV serves as a compact but powerful research platform, bridging the gap between simulation and real world autonomous marine systems. Its modular design, distributed computation, and precise timing infrastructure make it ideal for experimentation with advanced autonomy, perception, and control algorithms in challenging maritime environments.

3.2 Specifications

The microAmpere ASV was designed as a flexible and modular research platform, combining reliable marine hardware with powerful embedded computing and sensing capabilities. Its specification focuses on providing stable, high precision navigation and perception for real-time autonomy experiments in nearshore and harbor environments. The systems modularity allows researchers to swap components and integrate additional sensors without structural modification. The vessels configuration and components are summarized in Table 2 down bellow.

Table 2: Technical specifications of the microAmpere ASV.

Category	Specification
Platform Base	Blue Robotics BlueBoat twin-hull (HDPE construction)
Dimensions	Length: 1.8 m Width: 1.1 m Draft: 0.25 m
Weight	Approx. 45 kg (fully equipped)
Propulsion	Dual Blue Robotics T200 thrusters with ESC control, differential thrust for steering
Power Supply	Dual 6S Li-Ion batteries (22.2 V nominal, 20 Ah each), hot-swappable configuration
Compute Units	NVIDIA Jetson Orin NX (perception and LiDAR processing) LattePanda 3 Delta (navigation, control, and communication)
Operating System	Ubuntu 24.04 LTS with ROS2 Jazzy Jalisco
Networking	Teltonika RUTX50 router (4G/5G, Wi-Fi, and GNSS) with Tailscale VPN
Localization Sensors	Dual u-blox ZED-F9P GNSS modules with RTK correction Xsens MTi-680G high-precision IMU
Perception Sensors	Ouster OS1-64 LiDAR StereoLabs ZED-X stereo camera pair
Timing and Synchronization	PPS signal distribution and PTP-based synchronization via custom timing board
Power Monitoring	Custom power sensing PCB (voltage, current, and temperature feedback)
Communication Interfaces	CAN, Ethernet, USB 3.0, Serial (RS232/RS485), GPIO
Software Capabilities	Real-time navigation, control, diagnostics, and data logging Modular autonomy framework for MPC, RL, and SLAM integration
Field Operation	Endurance: ~3 hours under typical load Max speed: ~2.5 m/s Remote or fully autonomous operation modes

The specifications highlight the microAmpere balance between portability and computational power. The distributed computing setup provides sufficient resources for advanced autonomy tasks while maintaining low latency in the control loop. The integrated sensors enable precise positioning and rich perception, making it an ideal testbed for advanced control, SLAM, and sensor fusion algorithms in dynamic maritime environments.

3.3 Layout

The microAmperes internal and external layout is designed to achieve an optimal balance between functionality, modularity, and serviceability. The vessel is divided into three main physical sections, consisting of the two hulls and the central watertight enclosure. Each section has a specific role in the system: propulsion and power are located in the hulls, computation and communication in the watertight enclosure, and perception sensors are distributed around the vessel. This structure simplifies cabling, improves center-of-gravity stability, and provides a clean and organized platform for integrating research equipment.

The port hull houses the port side battery pack and Electronic Speed Controller (ESC) that drive the left thruster, while the starboard hull contains the mirrored setup together with a Power Sense Module responsible for voltage and current monitoring. Both hulls are interconnected through a reinforced cross tube that carries power, Pulse Width Modulation (PWM), and Ethernet lines to the watertight enclosure. This cross connection ensures a robust and low latency interface between propulsion, control, and sensor systems, while maintaining mechanical symmetry and balance across the hulls.

On the top deck, the vessel supports a LiDAR platform and stereo camera mounts designed to provide an unobstructed 360° field of view. These sensors enable simultaneous environmental mapping and perception. Additional top mounted features include power switch, diagnostic status LEDs, and the antennas for GNSS, 5G, and radio communication. The overall sensor placement was chosen to ensure optimal visibility, minimal interference between radio and optical systems, and ease of access during maintenance or calibration.

Externally, all antennas and sensors are symmetrically arranged to guarantee signal balance and redundancy. The vessel is equipped with dual GNSS antennas mounted at the rear of each hull for precise position and heading estimation, a 5G antenna for long range communication, and a radio antenna for local telemetry. The perception suite consists of two stereo cameras and a LiDAR unit mounted on the LiDAR platform. Together, these sensors provide full 360° situational awareness and deliver high quality data for SLAM, obstacle detection, and perception algorithms.

The internal layout focuses on structured modularity, with all high power and propulsion related systems isolated within the hulls, and all sensitive electronics centralized in the watertight enclosure. This separation improves electromagnetic compatibility and enhances maintainability during field operations. The vessel's internal wiring is routed through marine grade cable glands, organized in cable harnesses with labeled connectors for quick service and component replacement.

In addition, active cooling fans are installed at the rear of the hulls and inside the watertight enclosure to maintain safe operating temperatures for both computation and power electronics. Proper thermal design and airflow direction have been validated through practical testing to ensure stability under extended mission durations and high computational loads.

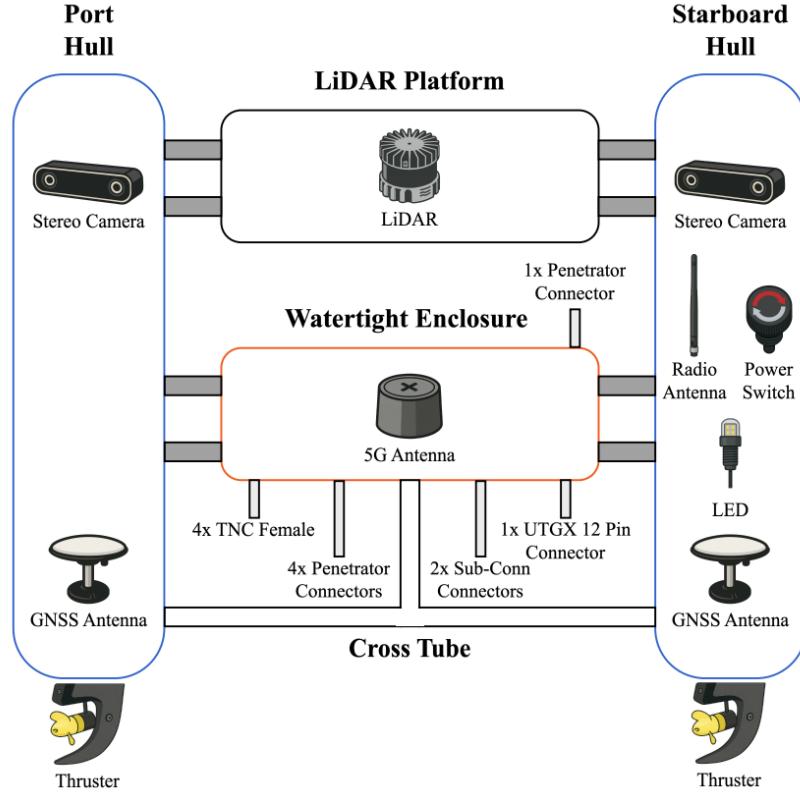


Figure 4: Top level overview of the microAmpere vessel showing the placement of cameras, antennas, thrusters, power switches, status LEDs, and LiDAR platform. Picture taken from Luan Cao Vo Tran master thesis on microAmpere ASV.^[4]

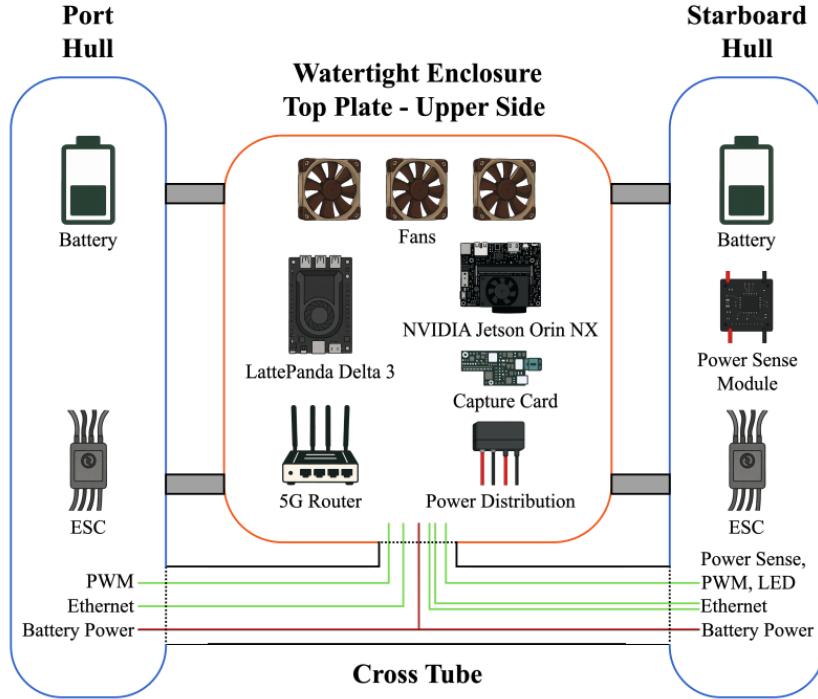


Figure 5: Bottom view of microAmpere showing the internal layout of batteries, ESCs, cooling fans, SBCs, capture card, and power distribution units for optimized performance and accessibility. Picture taken from Luan Cao Vo Tran master thesis on microAmpere ASV.^[4]

Focusing on the watertight enclosure, this section contains all critical computational, power conversion, and synchronization electronics of the microAmpere system. The internal structure follows a two layer plexiglass mounting design with a top and bottom plate. This setup maximizes thermal efficiency, ensures proper airflow, and allows easy access for maintenance or hardware upgrades.

The top plate upper side holds the main compute stack, including the “*NVIDIA Jetson Orin NX*”, “*LattePanda 3 Delta*”, and the “*Teltonika RUTX50 5G router*”. These components are mounted alongside a power distribution module and a set of “*Noctua 60 mm cooling fans*” for optimal airflow and thermal stability during intensive computational loads. The placement of these units also provides clear cable routing paths and short communication links between processing and networking hardware.

The underside of the top plate contains the “*DC/DC converters*” rated at 30 W and 150 W, a “*fuse board*”, and the fan bracket assembly. These converters supply stable 12 V and 5 V lines to different subsystems including the compute boards, router, Sentiboard, and auxiliary electronics. The components are arranged for efficient heat dissipation and quick replacement in case of hardware maintenance.

The bottom plate top side integrates the primary navigation and synchronization modules. It includes the “*Xsens MTi-680G IMU*”, dual “*u-blox ZED-F9P GNSS receivers*”, and the custom “*Sentiboard*” used for precise hardware timestamping and signal synchronization. The stacked layout minimizes wiring complexity, reduces signal interference, and maintains clear access for diagnostics and firmware updates.

Overall, the watertight layout emphasizes modularity, cooling efficiency, and reliability. By separating compute, power, and navigation layers, the design reduces thermal coupling between components and improves system robustness during long duration field operations. This configuration allows the microAmpere to operate safely under variable weather and computational conditions while maintaining high data integrity across all subsystems.

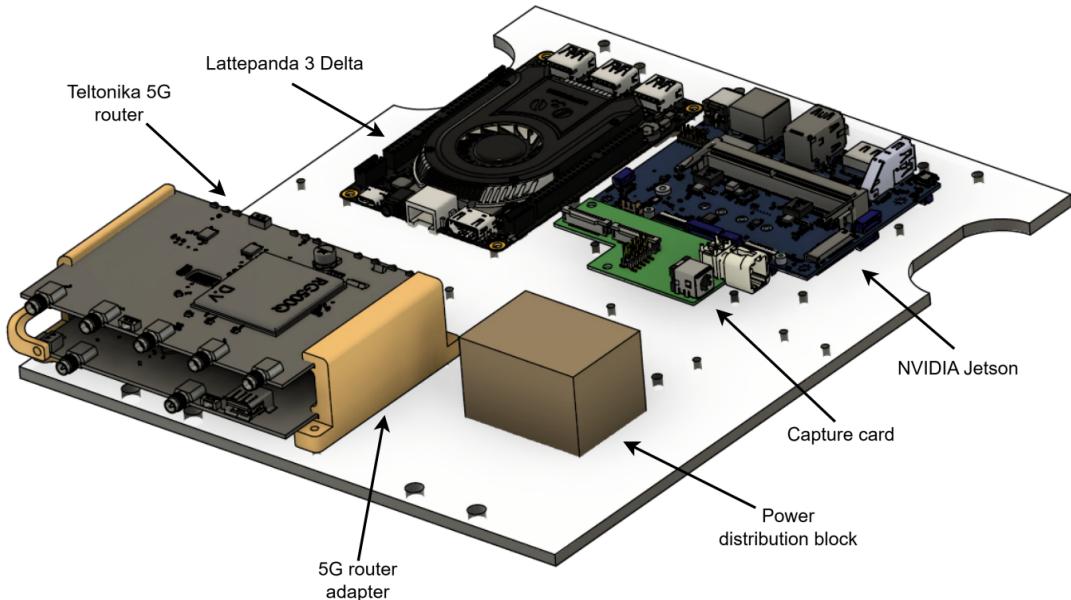


Figure 6: Watertight enclosure top plate (upper side) showing the main compute stack with Jetson Orin NX, LattePanda 3 Delta, and Teltonika RUTX50 router. Picture taken from Henrik Reimers master thesis on microAmpere ASV.^[4]

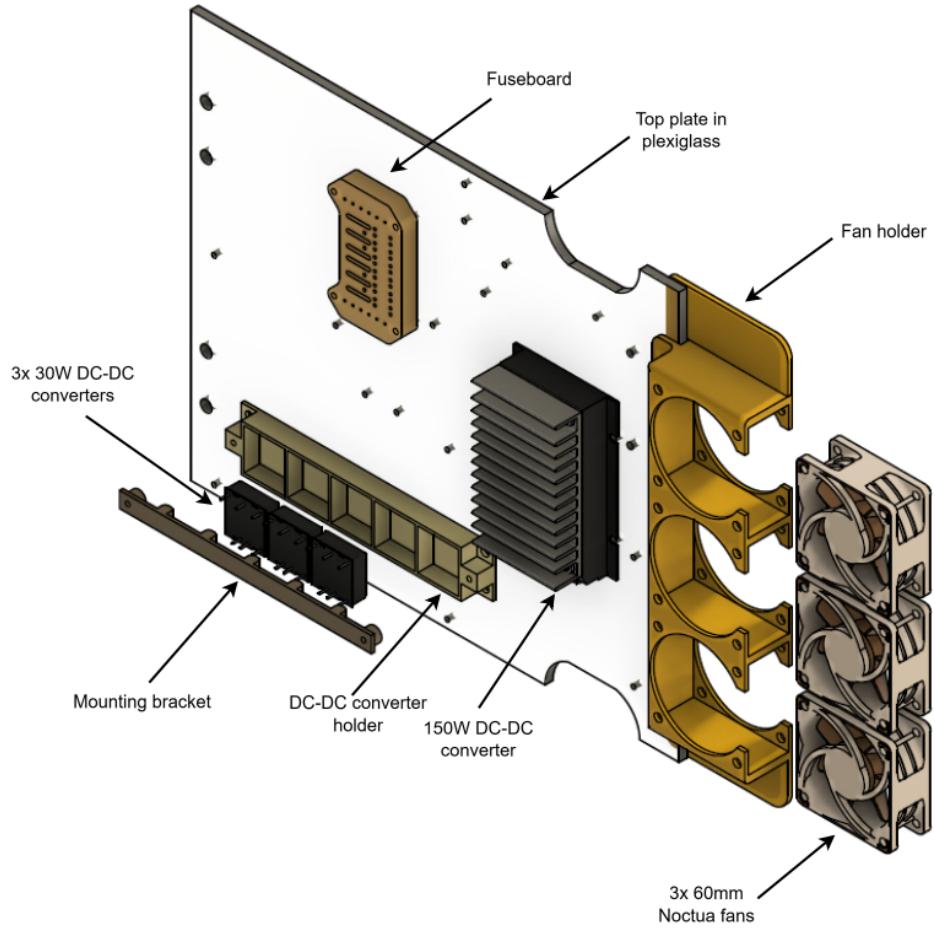


Figure 7: Underside of the top plate showing the DC/DC converters, fuse board, and cooling fans for power regulation and heat management. Picture taken from Henrik Reimers master thesis on microAmpere ASV.^[4]

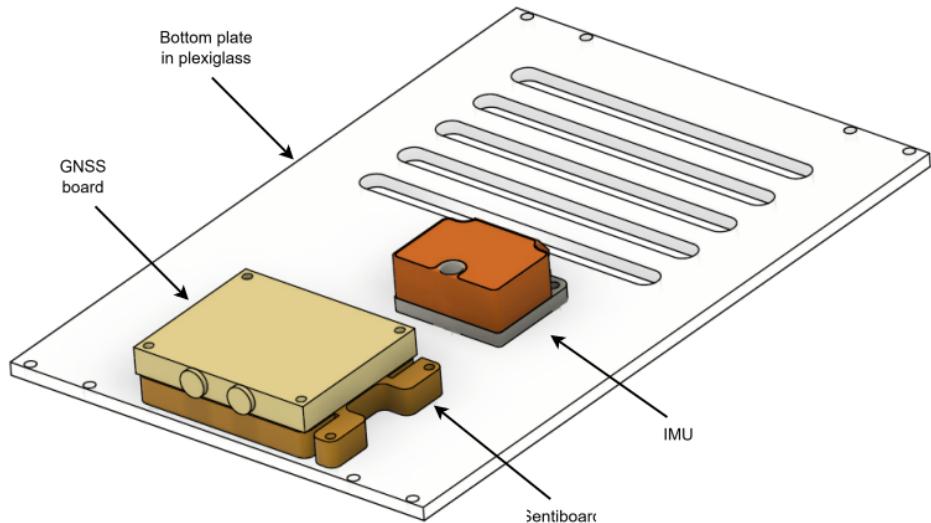


Figure 8: Bottom plate layout containing the IMU, Sentiboard, and GNSS receivers used for precise navigation and time synchronization. Picture taken from Henrik Reimers master thesis on microAmpere ASV.^[4]

3.4 Sensors

3.4.1 Introduction

The microAmpere platform is equipped with a wide range of sensors supporting navigation, perception, and control tasks. For this work, the focus lies on the integration of a Side Scan Sonar and the development of an SSS SLAM. Two of the existing sensors play a key role in enabling this, the Inertial Measurement Unit (IMU) and the Global Navigation Satellite System (GNSS). These sensors together provide accurate state estimation and position tracking, forming the foundation required for precise sonar data alignment, georeferencing, and mapping.

3.4.2 Inertial Measurement Unit (IMU) for Navigation and Pose Estimation

The “*Sensoror STIM300*” is the primary Inertial Measurement Unit (IMU) onboard microAmpere. It is a tactical grade sensor providing precise angular rate and linear acceleration measurements for dead reckoning, attitude estimation, and control stabilization. Its compact and rugged design makes it well suited for marine robotics, where vibration, temperature variation, and magnetic disturbances are common. The unit integrates three high precision gyroscopes and three accelerometers, enabling full 3-axis motion sensing with excellent mechanical and thermal stability.

The STIM300 offers a $\pm 400 \text{ }^{\circ}/\text{s}$ gyroscope range and $\pm 10 \text{ g}$ accelerometer range with 24-bit resolution, ensuring low noise and stable long term performance. It communicates via an RS-422 interface, operates from a 5 V supply, and supports sampling rates up to 2 kHz with deterministic timing. Its low bias instability and minimal drift allow reliable short term dead reckoning when GNSS signals are unavailable, making it a key component for autonomous control and future side scan sonar-based SLAM integration.

An overview of the IMU is shown in Figure 9, and its key specifications are summarized in Table 3.



Figure 9: Sensoror STIM300 high-performance IMU used for internal navigation and pose estimation. Picture taken from Henrik Reimers master thesis on microAmpere ASV.^[4]

Table 3: Sensoror STIM300 IMU specifications summary.^[6]

Parameter	Value
Gyroscope range	$\pm 400 \text{ }^{\circ}/\text{s}$
Accelerometer range	$\pm 10 \text{ g}$
Resolution	24-bit
Max data rate	2 kHz
Interface	RS-422
Input voltage	5 V
Weight	55 g

3.4.3 GNSS for Positioning and Heading Estimation

For accurate positioning and heading estimation, microAmpere employs two “*u-blox ZED-F9P*” GNSS modules integrated on a Dual GNSS card from SentiSystems. These modules support Real-Time Kinematic (RTK) corrections for centimeter level positioning accuracy and operate in a moving base configuration to estimate heading from the relative position of two GNSS antennas.

The external “*SignalPlus*” GNSS antennas provide reliable satellite reception and minimize multipath effects common in harbor environments. With a baseline distance of approximately 0.7m between antennas, the system achieves heading accuracy of around 0.5°, while maintaining position accuracy down to 1.5m, or 0.01m with RTK corrections. This configuration ensures precise navigation and robust performance even under challenging marine conditions.

Together with the onboard IMU, the dual GNSS system forms the foundation for the vessels navigation, control, and future SSS SLAM integration. The Dual GNSS card and antennas are shown in Figure 10, and the system specifications are summarized in Table 4.



Figure 10: (Left) SentiSystems Dual GNSS card with two u-blox ZED-F9P modules. (Right) SignalPlus GNSS antennas used for position and heading estimation. Picture taken from Henrik Reimers master thesis on microAmpere ASV.^[4]

Table 4: GNSS module specifications summary.^[7]

Parameter	ZED-F9P
Position accuracy (CEP)	1.5 m (0.01 m with RTK)
Heading accuracy	0.4° (baseline 1.8 m)
RTK capability	Base + Moving Base
Satellite bands	L1, L2, L5
Max navigation rate	25 Hz (5 Hz moving base)

3.4.4 Side Scan Sonar for Environmental Perception and SLAM

The “Deep Vision OSM Ethernet Sonar System” is the side scan sonar used onboard the microAmpere platform for seabed imaging, mapping, and SLAM development. Developed by DeepVision AB, this system operates with dual 680 kHz transducers utilizing chirp modulation to produce high resolution acoustic imagery. It supports a maximum operating depth of 100 m and can achieve image resolutions down to 1 cm depending on configuration settings.

The OSM sonar was previously evaluated by Hogstad, Bjørnar Reitan in his masters thesis at NTNU, where it was integrated on a BlueROV2 platform with an Error State Kalman Filter (ESKF) for autonomous navigation and acoustic mapping [3]. In that setup, the transducers were mounted symmetrically and angled downward at $\theta = 45^\circ$, providing a vertical beam width of $\alpha = 60^\circ$ and a horizontal beam width of $\phi = 0.7^\circ$. This configuration allowed both wide area coverage and high spatial resolution across the seafloor.

The sonar connects via Ethernet to the onboard computer and is automatically recognized by the “DeepView LT” software for real-time imaging, control, and data logging. Its high operating frequency and narrow horizontal beam enable precise seafloor mapping, while the wide vertical beam ensures efficient coverage for environmental perception. This makes it highly suitable for integration into the microAmpere ASVs perception stack, supporting the development of SSS SLAM and environment aware autonomy.

An overview of the sonar system is shown in Figure 11, and its main specifications are summarized in Table 5.



Figure 11: DeepVision OSM Ethernet Side-Scan Sonar system with dual 680 kHz transducers used for seabed imaging and SLAM development. Picture adapted from Hogstads master thesis on BlueROV2 sonar integration.[3]

Table 5: DeepVision OSM Ethernet Sonar System specifications summary.[3]

Parameter	Value
Operating frequency	680 kHz (Chirp signal)
Resolution	Down to 1 cm
Maximum operating depth	100 m
Vertical beam width (α)	60°
Horizontal beam width (ϕ)	0.7°
Mounting angle (θ)	45°
Swath range	Up to 50 m per side (100 m total)
Interface	Ethernet
Software	DeepView LT
Manufacturer	DeepVision AB

3.5 Software Architecture

Although not strictly part of the hardware system, the software architecture is what connects, manages, and coordinates all onboard components. It defines how the sensors, actuators, and computational units communicate to form a complete autonomous system. The design follows a modular and distributed approach, separating computational loads across multiple processors to improve reliability, maintainability, and scalability. This structure also allows new hardware, such as the side scan sonar, to be integrated seamlessly without major reconfiguration of existing modules.

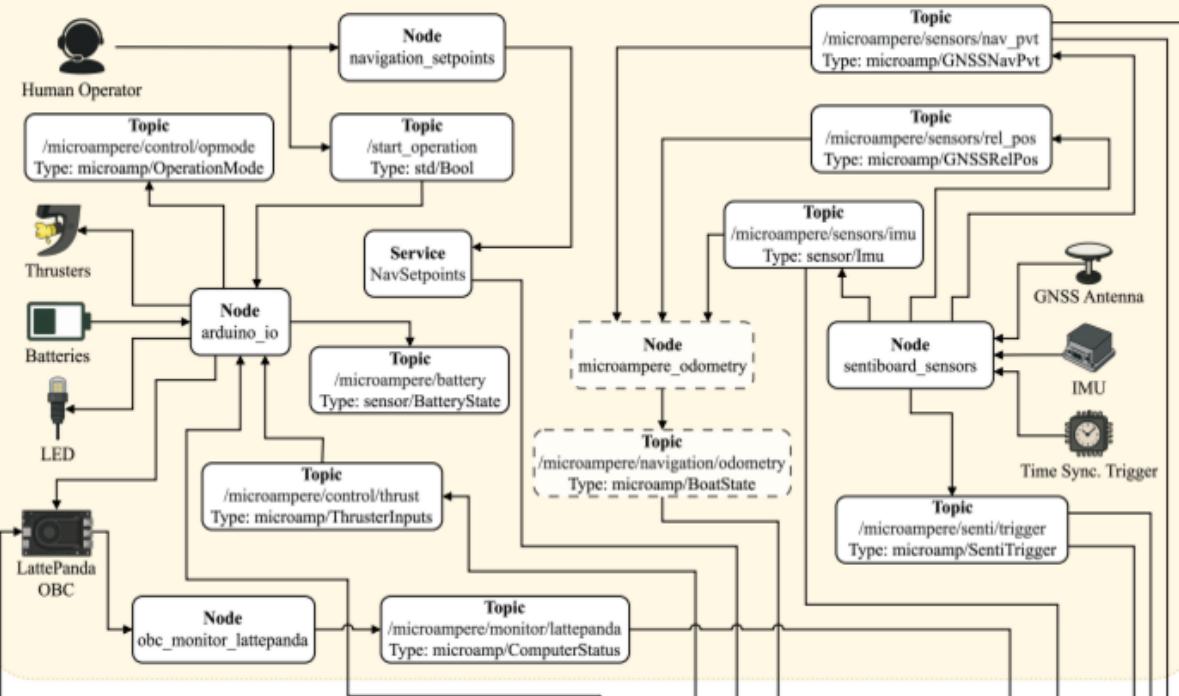
Each subsystem is assigned a specific computational domain with a well-defined purpose. The “*Navigation Unit*” on the “*LattePanda 3 Delta*” is responsible for low-level control, sensor fusion, and actuator management, interfacing directly with the IMU, GNSS, and thruster controllers. The “*Perception Unit*” on the “*NVIDIA Jetson Orin NX*” handles high bandwidth sensors such as LiDAR and stereo cameras, performing real-time perception, object detection, and mapping. The “*Ground Control Station*” provides remote supervision, telemetry visualization, and manual override when required. Together, these systems operate in a synchronized local Ethernet network, exchanging data through a standardized message layer.

The software stack runs on “*Ubuntu 24.04 LTS*” with the “*ROS2 Jazzy Jalisco*” distribution. ROS2 acts as the communication backbone, providing a real-time publish/subscribe framework through the Data Distribution Service (DDS) middleware. Each functional block in the system is implemented as an independent ROS2 node that communicates via topics, services, and actions. This de-coupled design simplifies debugging, system upgrades, and integration of experimental modules for research.

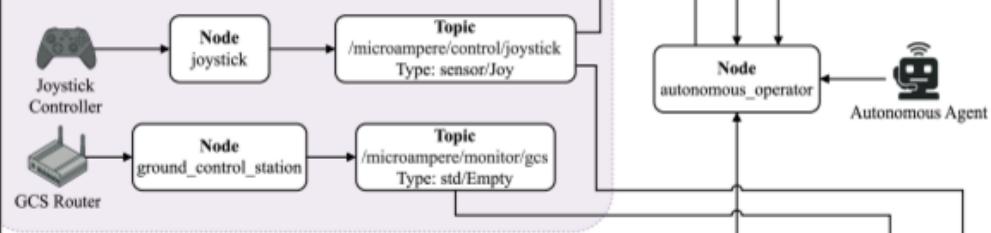
A key part of the architecture is precise time synchronization between all onboard computers. The system uses a hardware signal called Pulse Per Second (PPS) from the GNSS modules to align the internal clocks of each computer. This signal marks the exact start of every second and is combined with the GNSS time data to keep all systems synchronized. A background process running in Linux automatically adjusts the system clocks based on this information, ensuring that all sensor data are timestamped with the same reference time. This sub-microsecond synchronization is essential for accurate sensor fusion, navigation, and SLAM.

An overview of the complete software architecture and node interconnections is shown in Figure 12 down below on the next page.

LattePanda Delta 3 - Navigation Unit



Khadas VIM4 - Ground Control Station



NVIDIA Jetson Orin NX - Perception Unit

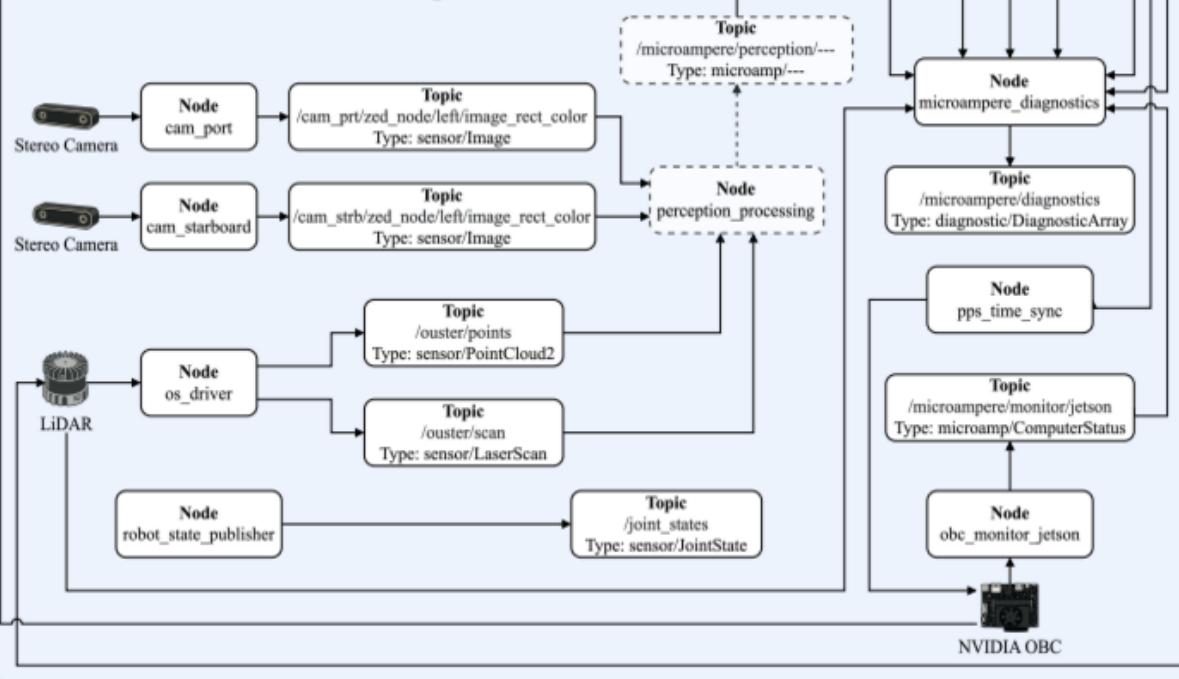


Figure 12: Overview of the software architecture for the microAmpere ASV, showing the distributed communication between perception, navigation, and control units. Picture taken from Henrik Reimers master thesis on microAmpere ASV.^[4]

3.6 ROS2

The Robot Operating System 2 (ROS2) is a middleware framework designed for distributed robotic systems. It provides a modular communication layer that connects sensors, actuators, and computational nodes through a unified publish and subscribe architecture. ROS2 extends the original ROS1 with support for real time communication, security, and multi platform operation, making it suitable for both research and industrial use.

The structure of ROS2 is organized in several layers as shown in Figure 13.

The “*Operating System Layer*” provides hardware abstraction and supports multiple systems such as Linux, Windows, and macOS.

The “*ROS Middleware Layer (RMW)*” implements communication based on the Data Distribution Service (DDS) standard. DDS enables decentralized peer to peer communication without a central master, providing real time capabilities, reliability control, and quality of service policies for each data connection.

The “*ROS Client Library (RCL)*” defines the core functionality of ROS2 nodes in C, with language specific APIs such as `rclcpp` for C++ and `rclpy` for Python. This allows consistent behavior across different programming languages.

The “*User Code Layer*” contains the application specific nodes implemented by the developer. Each node handles one task such as sensor data processing, control, or estimation, and communicates with other nodes using topics, services, or actions. This modular approach improves scalability, debugging, and code reuse.

Overall, ROS2 provides a flexible communication framework that enables distributed systems to exchange data deterministically, synchronize timing, and scale across multiple computing units in real time applications.

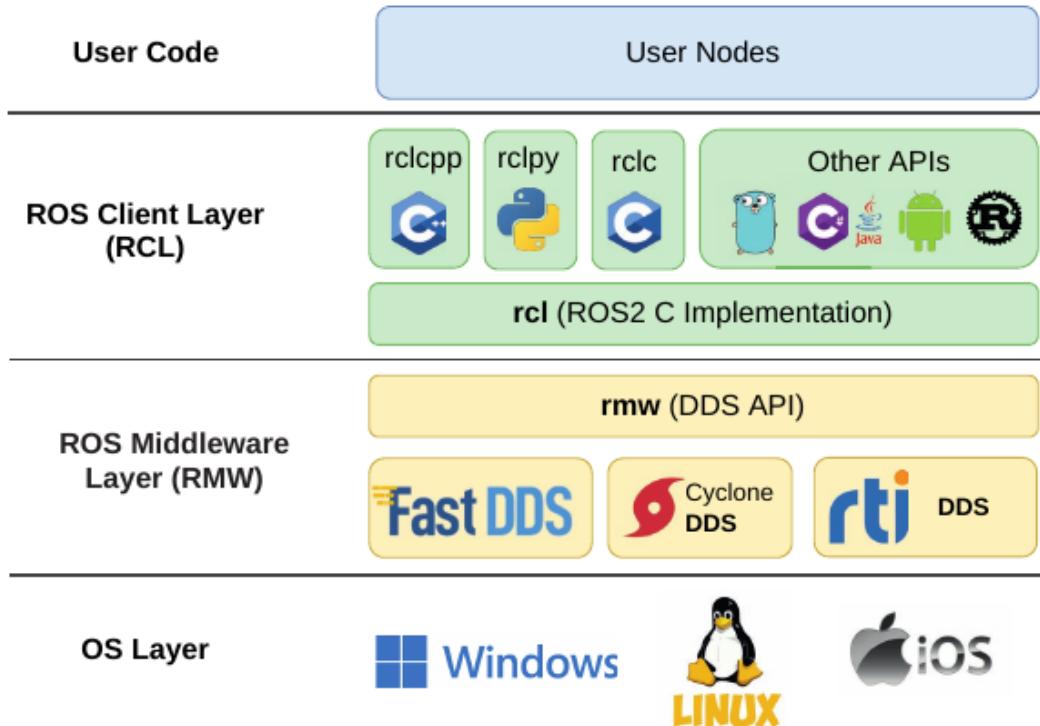


Figure 13: Layered structure of ROS2 showing the relation between the OS, middleware, client library, and user application layers.^[8]

ROS2 is used as the main communication framework because it provides a decentralized, reliable, and real time capable architecture for robotic systems. Unlike ROS1, which depended on a single master node to manage communication, ROS2 uses the DDS middleware to automatically discover and connect nodes on the same network. This removes single points of failure and enables multiple computers to share data seamlessly. Each node in ROS2 can publish, subscribe, or request services without a central broker, which makes it ideal for distributed systems such as autonomous vehicles where perception, navigation, and control units must operate independently but still exchange synchronized information.

Communication in ROS2 is built on a publish and subscribe model. Nodes publish data on topics, and any node subscribed to that topic receives the data in real time. This approach decouples software components so that each one can be modified, restarted, or replaced without affecting others. For request and response type communication, ROS2 uses services, and for long running or feedback based tasks, it uses actions. Together these communication types allow flexible interaction between nodes, supporting both high frequency data streams such as IMU readings and low rate commands such as mission updates.

The middleware, based on DDS, handles message serialization, transport, and discovery automatically. DDS offers configurable Quality of Service (QoS) settings, allowing developers to control how data are delivered, stored, and synchronized. For example, high priority sensor data can be transmitted using reliable communication, while large perception messages can be sent using best effort mode to reduce latency. This control is essential in real time systems where timing and determinism are critical.

Time synchronization and deterministic execution are further supported through the ROS2 clock and timestamping mechanisms. Each message carries a precise time reference, which enables accurate sensor fusion and replay of recorded data. The system can integrate with external synchronization sources such as the GNSS Pulse Per Second (PPS) signal to align clocks between distributed nodes. This ensures that all data within the system share the same temporal reference, which is necessary for consistent estimation, control, and SLAM operations.

Another advantage of ROS2 is its modularity and reusability. Nodes are grouped into packages that can be developed, built, and deployed independently. The use of launch files written in Python allows flexible startup sequences, parameter configuration, and automatic interconnection of nodes. This makes ROS2 highly maintainable and adaptable to future upgrades, such as adding new sensors or switching computing hardware without rewriting the entire system.

Overall, ROS2 provides a robust, standardized, and industry ready framework for building distributed robotic systems. It forms the backbone of modern autonomous platforms by managing data flow, synchronization, and process isolation while maintaining real time performance and flexibility for research and development.

4 System Modeling

4.1 Introduction

Reliable navigation and mapping for an autonomous vessel depend on accurate mathematical models that describe how the vessel moves and how its sensors perceive the surrounding environment. These models form the foundation for state estimation, control, and SLAM algorithms, ensuring that all physical quantities such as position, velocity, and orientation are consistently defined and propagated over time.

The modeling framework is divided into two complementary parts, kinematics and dynamics. Kinematics focuses on the geometric description of motion without considering the forces that cause it. It defines how position, velocity, and orientation are represented and transformed between coordinate frames. Global reference frames such as the World Geodetic System 1984 (WGS84) and the Earth Centered Earth Fixed (ECEF) frame describe the Earth geometry and provide absolute positioning. Local frames such as the North East Down (NED) and Body frames define the vessel motion relative to its own orientation and surroundings. The transformation chain between these frames ensures that all measured and estimated quantities are consistently expressed and can be converted between global and local coordinates.

Dynamics extend the kinematic framework by describing how forces and moments act on the vessel to generate motion. This relationship forms the basis of the motion models used in navigation and estimation. Two main formulations are considered. The Marine Craft Model by Thor Inge Fossen [9] provides a comprehensive six degree of freedom (6 DOF) description that captures hydrodynamic effects, external disturbances, and control inputs, making it ideal for simulation and model based control. The Inertial Navigation System (INS) model, derived from the work of Edmund Brekke in Fundamentals of Sensor Fusion [10], offers a simplified sensor driven formulation that relies directly on IMU and GNSS data. This approach is better suited for real-time operation and SLAM applications, where computational efficiency and robustness to environmental uncertainty are prioritized over detailed hydrodynamic accuracy.

The complete mathematical framework connects these kinematic and dynamic representations into a unified structure. It defines how vessel states are represented, transformed, and propagated in time using rotation formulations such as Euler angles, quaternions, and Lie group representations. It establishes consistent conventions between global and local reference frames and expresses the relationships between linear and angular motion necessary for deriving velocities, accelerations, and time derivatives of position and orientation.

To achieve stable and accurate temporal evolution of the state, numerical solvers are used to integrate the underlying differential equations. Classical integration schemes such as Newton-Euler and Runge-Kutta methods are employed to propagate the vessel dynamics in discrete time while minimizing numerical drift and maintaining physical consistency. The choice of solver has a significant impact on model accuracy, especially in real-time applications where integration stability and computational efficiency determine overall system performance.

Together, these formulations provide a rigorous mathematical foundation for navigation and mapping in autonomous surface vessels. They enable consistent state representation, accurate propagation, and reliable fusion of multi-sensor information, which are all essential for achieving robust autonomy and for the integration of advanced perception systems such as SSS SLAM.

4.2 Orientation Representations

4.2.1 Euler Angles

Euler angles provide a simple and intuitive method for representing three dimensional orientation through a sequence of rotations about the principal axes. The orientation of a rigid body is described by three angles corresponding to yaw, pitch, and roll, which define the body's rotation relative to a fixed reference frame. These angles are widely used in navigation and control applications due to their clear geometric interpretation and direct relationship to measurable quantities such as heading and inclination.

In the North East Down (NED) convention commonly used for marine and aerial vehicles, the rotation sequence is typically defined as a rotation about the z -axis (yaw, ψ), followed by the y -axis (pitch, θ), and finally the x -axis (roll, ϕ). This corresponds to the rotation matrix:

$$R = R_x(\phi)R_y(\theta)R_z(\psi)$$

$$R_x(\phi) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \phi & -\sin \phi \\ 0 & \sin \phi & \cos \phi \end{bmatrix}, \quad R_y(\theta) = \begin{bmatrix} \cos \theta & 0 & \sin \theta \\ 0 & 1 & 0 \\ -\sin \theta & 0 & \cos \theta \end{bmatrix}, \quad R_z(\psi) = \begin{bmatrix} \cos \psi & -\sin \psi & 0 \\ \sin \psi & \cos \psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Combining these gives the full NED transformation from the body frame to the navigation frame:

$$R_b^n = \begin{bmatrix} c_\theta c_\psi & s_\phi s_\theta c_\psi - c_\phi s_\psi & c_\phi s_\theta c_\psi + s_\phi s_\psi \\ c_\theta s_\psi & s_\phi s_\theta s_\psi + c_\phi c_\psi & c_\phi s_\theta s_\psi - s_\phi c_\psi \\ -s_\theta & s_\phi c_\theta & c_\phi c_\theta \end{bmatrix}$$

where $s_\bullet = \sin(\bullet)$ and $c_\bullet = \cos(\bullet)$ for brevity. The NED frame transformation can be visualized in Figure 14, which illustrates the intrinsic yaw-pitch-roll rotation sequence following the NED convention. This sequence shows how the body frame is successively rotated about the global z -axes, y -axes, and x -axes to achieve the desired orientation.

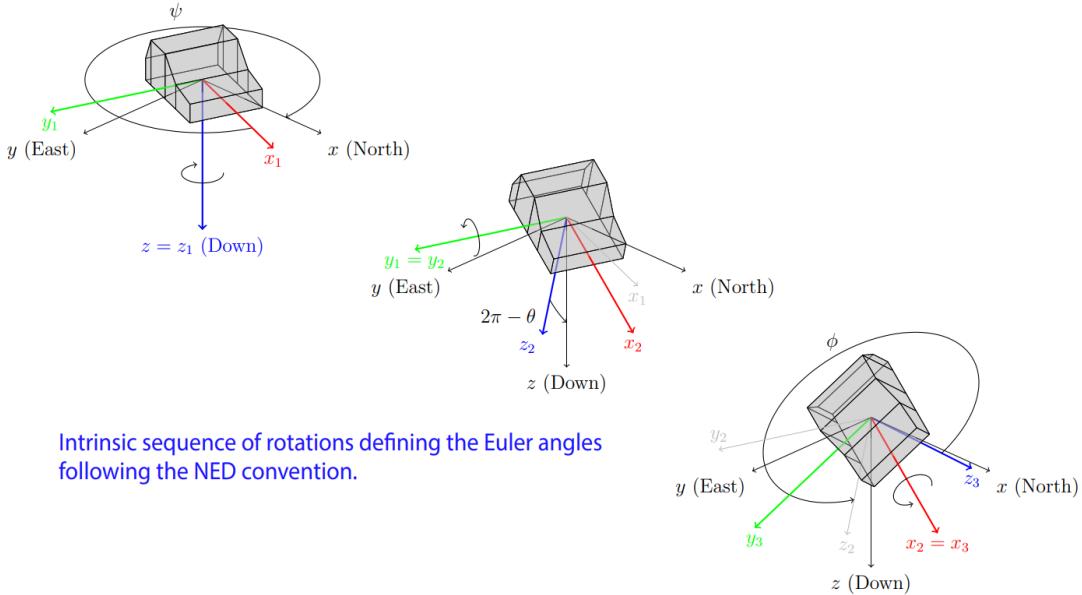


Figure 14: Intrinsic sequence of rotations defining the Euler angles following the NED convention. Figure taken from Edmund Brekke book on Fundamentals of Sensor Fusion.^[10]

The main advantages of Euler angles are their simplicity, minimal parameter count, and intuitive physical meaning. Each angle directly corresponds to an observable rotational motion, which simplifies both system design and debugging. They also integrate naturally with classical control systems, where angular rates are easily expressed as time derivatives of Euler angles.

However, the Euler representation suffers from a singularity problem known as “gimbal lock”, which in the NED convention occurs when the pitch angle approaches $\theta = \pm 90^\circ$. At this configuration,

the yaw and roll axes align, resulting in the loss of one rotational degree of freedom. In this state, small changes in pitch can cause disproportionately large or undefined changes in yaw and roll rates. Mathematically, the transformation matrix that relates Euler angle rates to body angular velocities becomes ill-conditioned, leading to singularities where the angular rate terms tend toward infinity. This introduces numerical instability and unreliable attitude estimates, which are particularly problematic for navigation and control systems relying on smooth and continuous angular motion.

The relationship between the body angular velocity vector $\omega_b = [p \ q \ r]^T$ and the Euler angle rate vector $\dot{\Theta} = [\dot{\phi} \ \dot{\theta} \ \dot{\psi}]^T$ for the NED yaw-pitch-roll convention is expressed as:

$$\begin{bmatrix} p \\ q \\ r \end{bmatrix} = \begin{bmatrix} 1 & 0 & -\sin \theta \\ 0 & \cos \phi & \sin \phi \cos \theta \\ 0 & -\sin \phi & \cos \phi \cos \theta \end{bmatrix} \begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix}$$

When the pitch angle approaches $\theta = \pm 90^\circ$, the cosine term $\cos \theta$ tends to zero, causing the matrix to lose rank and become singular. As a result, any attempt to compute Euler rate derivatives or integrate attitude dynamics at this configuration leads to undefined or infinite angular velocities.

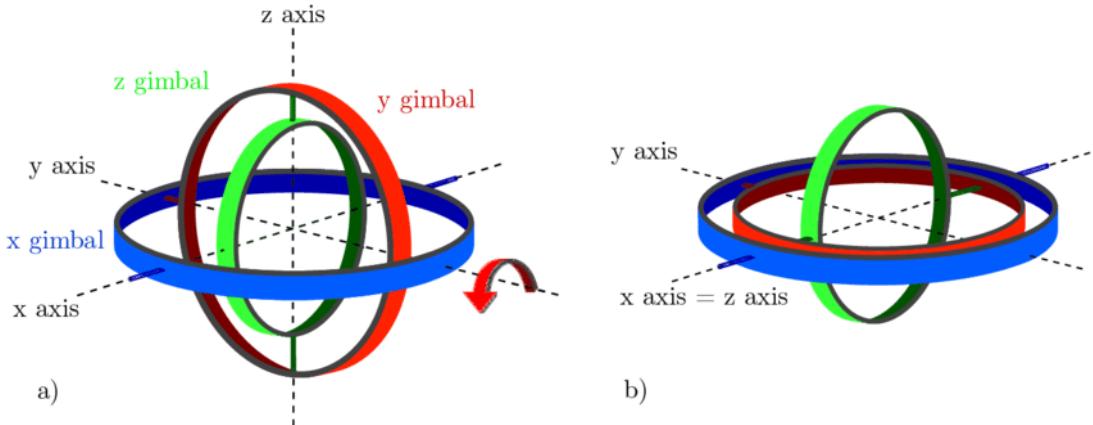


Figure 15: Illustration of gimbal lock where the yaw and roll axes coincide, causing loss of one rotational degree of freedom. Picture taken from a research paper that explains gimbal lock with illustrations.^[11]

Despite this limitation, Euler angles remain a practical choice for surface and marine robotics, where pitch and roll angles typically remain small. Their intuitive interpretation and computational simplicity make them well suited for real-time estimation and control near level operating conditions. In the case of an ASV, large attitude excursions are highly unlikely, as excessive roll or pitch would indicate a loss of stability or complete capsizing. Under such operational constraints, the singularity at $\theta = \pm 90^\circ$ is never encountered, making the Euler representation both sufficient and efficient for navigation and SLAM applications.

4.2.2 Quaternions

Quaternions provide a compact and singularity free alternative to Euler angles for representing three dimensional orientations. They extend complex numbers into four dimensions and are defined as a scalar and a three component vector

$$\mathbf{q} = \begin{bmatrix} q_w \\ q_x \\ q_y \\ q_z \end{bmatrix} = \begin{bmatrix} \cos \frac{\theta}{2} \\ \hat{u}_x \sin \frac{\theta}{2} \\ \hat{u}_y \sin \frac{\theta}{2} \\ \hat{u}_z \sin \frac{\theta}{2} \end{bmatrix}$$

where θ is the rotation angle and $\hat{u} = [\hat{u}_x \hat{u}_y \hat{u}_z]^T$ is the unit vector defining the rotation axis. The quaternion \mathbf{q} therefore represents a rotation of θ radians about \hat{u} . To ensure that it encodes a valid rotation, it must satisfy the unit norm constraint

$$\|\mathbf{q}\| = \sqrt{q_w^2 + q_x^2 + q_y^2 + q_z^2} = 1$$

Normalization is typically enforced after each update step in numerical integration or filtering to avoid drift due to floating point errors. This is done by dividing the quaternion by its magnitude:

$$\mathbf{q}_{\text{normalized}} = \frac{\mathbf{q}}{\|\mathbf{q}\|} = \frac{1}{\sqrt{q_w^2 + q_x^2 + q_y^2 + q_z^2}} \begin{bmatrix} q_w \\ q_x \\ q_y \\ q_z \end{bmatrix}$$

A geometric interpretation of quaternions can be seen in Figure 16. Quaternions can be viewed as points on the unit four dimensional hypersphere S^3 , composed of a real component and a three dimensional imaginary subspace spanned by (i, j, k) . The real component represents the cosine of half the rotation angle, while the imaginary vector component encodes the rotation axis scaled by the sine of half the angle. Together, they define a rotation in three dimensional space through the relation $\mathbf{q} = \cos \frac{\theta}{2} + \hat{u} \sin \frac{\theta}{2}$. Each point on this hypersphere corresponds to a unique orientation of a rigid body, and continuous motion along its surface represents a smooth change in attitude without encountering any discontinuities or singularities.

When a vector \mathbf{v} is rotated using \mathbf{qvq}^{-1} , the operation effectively applies a double rotation, first through \mathbf{q} and then through its conjugate \mathbf{q}^{-1} , producing a single pure 3D rotation about the axis \hat{u} by an angle θ . The scalar part $\cos(\theta/2)$ defines the rotation magnitude, while the vector part $\sin(\theta/2)\hat{u}$ specifies the rotation axis in the imaginary (i, j, k) space. This representation forms the basis for quaternion based attitude kinematics used in estimation and control.

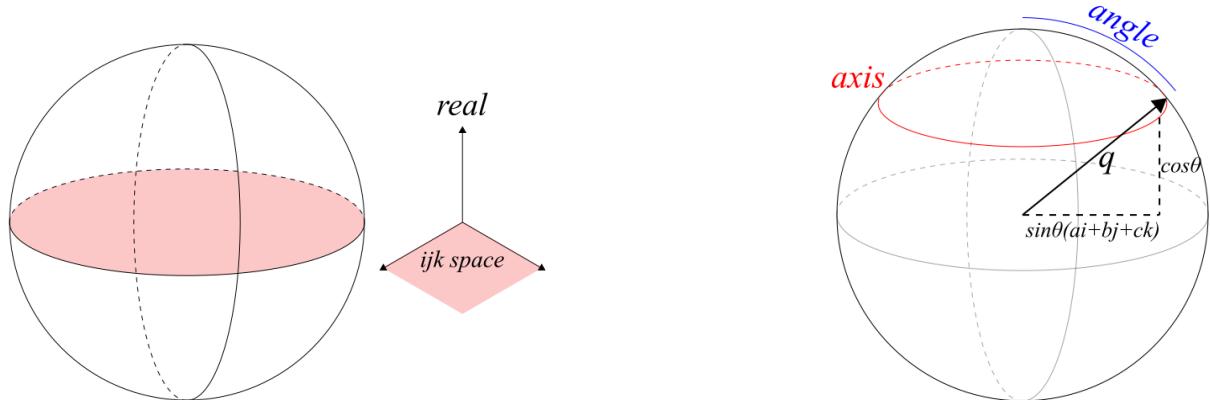


Figure 16: Geometric interpretation of quaternions on the unit hypersphere S^3 . (Left) Decomposition into the real axis and imaginary (i, j, k) subspace. (Right) Axis-angle representation showing how a quaternion encodes rotation by θ about the unit axis \hat{u} . Picture taken from article that explains quaternion rotation in a intuitive way.^[12]

Rotations using quaternions are performed through quaternion multiplication, which is associative but not commutative. Given two orientations \mathbf{q}_1 and \mathbf{q}_2 , their combined rotation is expressed as

$$\mathbf{q}_{\text{combined}} = \mathbf{q}_2 \otimes \mathbf{q}_1,$$

where \otimes denotes the quaternion product. The rotation of a vector \mathbf{v} in three dimensional space can then be performed as

$$\mathbf{v}' = \mathbf{q} \otimes \mathbf{v}_q \otimes \mathbf{q}^{-1},$$

where $\mathbf{v}_q = [0 \ v_x \ v_y \ v_z]^T$ is the quaternion form of the vector \mathbf{v} . This operation applies the rotation represented by \mathbf{q} to \mathbf{v} in a smooth and continuous manner without encountering singularities.

For smooth orientation transitions, quaternions can be interpolated using Spherical Linear Interpolation (SLERP). Instead of blending components linearly, SLERP moves along the great circle that connects two orientations on the unit quaternion sphere S^3 . This ensures that the interpolated motion follows a constant angular velocity and remains at a fixed distance from the sphere center, preserving rotation smoothness and avoiding distortion.

Given two normalized quaternions \mathbf{q}_1 and \mathbf{q}_2 , the interpolation for $t \in [0, 1]$ is defined as

$$\text{SLERP}(\mathbf{q}_1, \mathbf{q}_2; t) = \frac{\sin((1-t)\Omega)}{\sin(\Omega)} \mathbf{q}_1 + \frac{\sin(t\Omega)}{\sin(\Omega)} \mathbf{q}_2,$$

where $\Omega = \cos^{-1}(\mathbf{q}_1 \cdot \mathbf{q}_2)$ represents the angle between the two quaternions. When $t = 0$, the result is \mathbf{q}_1 , and when $t = 1$, the result is \mathbf{q}_2 . Intermediate values of t trace the shortest rotation path between the two orientations.

This method is widely used in robotics, computer graphics, and navigation to generate smooth transitions between different orientations. SLERP ensures that intermediate orientations are evenly spaced in angular distance, resulting in constant rotational velocity, a property important for stable attitude control and estimation.

To ensure interpolation follows the shortest possible rotation path, many implementations adjust the sign of one quaternion when their dot product is negative. This avoids interpolation along the long arc (greater than 180°) and guarantees smooth motion along the minimal angular distance. Figure 17 illustrates this process, showing both the “short” and the “natural” interpolation paths along the quaternion sphere.

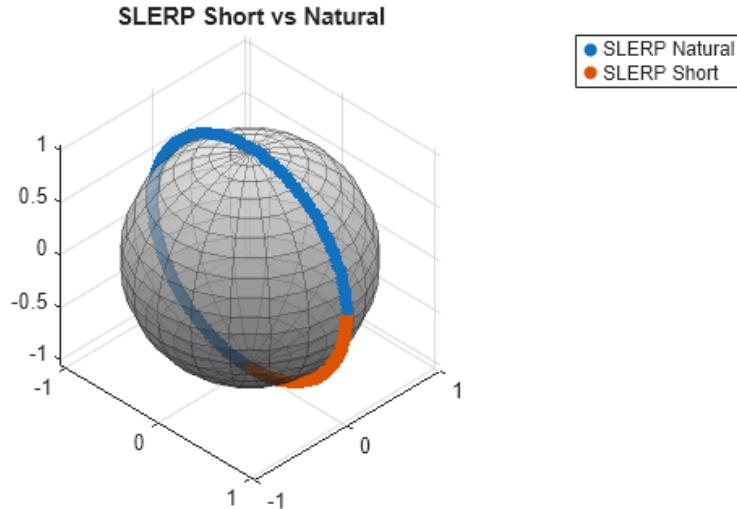


Figure 17: Spherical linear interpolation (SLERP) between two orientations \mathbf{q}_1 and \mathbf{q}_2 on the unit quaternion sphere. The “short” path minimizes angular distance, while the “natural” path follows the original direction without sign correction. Figure taken from MATLAB documentation for illustration purposes.^[13]

In practical applications, quaternions are widely used in estimators and sensors where singularity free orientation handling is critical. The IMU on the microAmpere platform outputs attitude in quaternion form, which integrates naturally with sensor fusion algorithms such as the Error State Kalman Filter

(ESKF) discussed later in the report. These algorithms rely on quaternion algebra to represent small attitude perturbations efficiently and avoid numerical instability associated with Euler angle singularities.

The main advantages of quaternions include their smooth and continuous representation of orientation, absence of gimbal lock, and computational efficiency in rotation composition and interpolation. They maintain numerical stability under integration and are well suited for optimization and estimation algorithms. However, quaternions require one additional parameter compared to Euler angles and lack intuitive physical meaning, as their four components do not directly correspond to measurable angular quantities. Furthermore, they are less straightforward to interpret and manipulate mathematically, often requiring specialized operations such as normalization, conjugation, and noncommutative multiplication. Despite these challenges, their robustness and compatibility with modern sensor fusion and control frameworks make quaternions a preferred internal representation for real-time navigation and estimation systems.

4.2.3 Lie Groups and Manifolds

Lie groups provide a mathematical framework for representing continuous and smooth transformations such as rotation and translation in three dimensional space. They combine the properties of algebraic groups and differentiable manifolds, allowing both analytical manipulation and geometric interpretation of motion. In navigation, control, and robotics, Lie groups enable consistent handling of orientation and pose without the discontinuities or ambiguities present in minimal representations such as Euler angles.

The rotation group $\text{SO}(3)$, called the Special Orthogonal Group, represents all possible 3D rotations. Each rotation is described by a 3×3 matrix R that satisfies

$$R^T R = I, \quad \det(R) = 1$$

This group is smooth and continuous, meaning that small changes in orientation correspond to small movements on the manifold. To describe small or incremental rotations, $\text{SO}(3)$ is associated with its tangent space, the Lie algebra $\mathfrak{so}(3)$.

Elements of $\mathfrak{so}(3)$ are skew-symmetric matrices that encode infinitesimal rotations. A rotation vector $\boldsymbol{\omega} = [\omega_x \ \omega_y \ \omega_z]^T$ can be written as

$$\boldsymbol{\omega}^\times = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix}$$

This matrix form is simply another way of expressing the cross product $\boldsymbol{\omega} \times \mathbf{v}$, which rotates a vector \mathbf{v} by a small angle.

The exponential map connects this local representation to an actual finite rotation on $\text{SO}(3)$:

$$R = \exp(\boldsymbol{\omega}^\times) \quad (\text{Tangent} \rightarrow \text{Manifold}) \quad (1)$$

And the inverse operation (the logarithmic map) retrieves the corresponding small rotation from R :

$$\boldsymbol{\omega} = \log(R) \quad (\text{Tangent} \leftarrow \text{Manifold}) \quad (2)$$

These two functions allow smooth transitions between local angular velocity representations and full 3D orientations, which is highly useful for filters and optimizers for SLAM.

To include translation as well as rotation, the concept extends to $\text{SE}(3)$, the so called Special Euclidean Group, which describes full 3D rigid body motion:

$$T = \begin{bmatrix} R & \mathbf{p} \\ 0 & 1 \end{bmatrix}, \quad T \in \text{SE}(3) \quad (3)$$

Here R is orientation and \mathbf{p} is position. This group provides a consistent mathematical way to represent a vehicle's full pose and combine both rotational and translational motion.

In robotics and navigation, $\text{SO}(3)$ and $\text{SE}(3)$ are used to describe smooth and continuous motion in three dimensional space. $\text{SO}(3)$ represents pure rotations, while $\text{SE}(3)$ extends this to include both rotation and translation, forming the full rigid body pose. These groups define motion directly on the manifold, ensuring mathematically consistent and globally valid transformations without singularities.

The curved surface of the $\text{SE}(3)$ manifold represents all possible poses of a rigid body in space. The tangent plane at the identity, denoted $\mathfrak{se}(3)$, represents small local motions that can be combined and integrated into full poses using the exponential map $\exp(\cdot)$. Conversely, the logarithmic map $\log(\cdot)$ converts a pose difference back into this local space. Together, these mappings allow smooth transitions between small local displacements and full 3D transformations, which is essential for analyzing and composing motion in robotics and control.

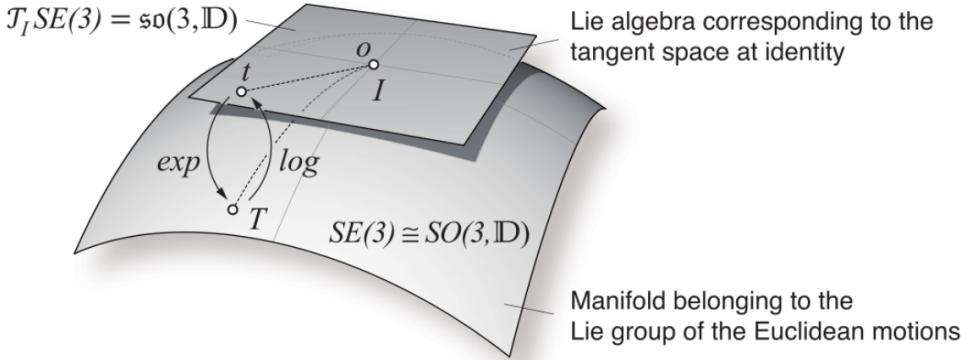


Figure 18: Visualization of the $\text{SE}(3)$ manifold and its tangent space $\mathfrak{se}(3)$. Small motions are represented in the tangent space and mapped to the manifold through the exponential map $\exp(\cdot)$, while the logarithmic map $\log(\cdot)$ projects manifold elements back to the local linear space. Figure taken from Nikolay Atanasov lecture notes on Sensing & Estimation in Robotics.^[14]

Lie group formulations are fundamental in modern SLAM systems because they provide a mathematically consistent way to represent and manipulate poses in three dimensions. When building and updating a map, each robot or camera pose is an element of $\text{SE}(3)$, combining both rotation and translation into a single compact structure. This allows pose composition, inversion, and differentiation to be performed directly on the manifold without approximations or singularities. In practice, this means that motion updates, sensor transformations, and loop closure corrections can all be handled through clean matrix operations that are both efficient and numerically stable.

Using Lie groups also enables fast and scalable computation, a necessity when optimizing over thousands of poses and landmarks in large scale SLAM problems. The same mathematical principles are applied in computer graphics and 3D rendering, where $\text{SE}(3)$ transformations are used to efficiently manipulate hundreds or thousands of objects and vertices in real time. By working on the manifold, transformations remain consistent regardless of scale or complexity, ensuring that rotations and translations compose correctly without distortion.

Overall, Lie group representations allow SLAM and mapping algorithms to combine geometry, motion, and optimization within a single unified framework. They ensure that the estimated trajectory and map remain globally consistent, even after many iterations of motion and correction, making them the foundation of accurate and robust 3D perception in robotics.

4.2.4 Handy Conversions

The following conversion formulas are taken directly from Edmund Brekkes book on “*Fundamentals of Sensor Fusion*” [10]. They are commonly used to convert between different orientation representations for navigation, control, and estimation.

Euler to Quaternion

$$\mathbf{q} = \begin{bmatrix} q_w \\ q_x \\ q_y \\ q_z \end{bmatrix} = \begin{bmatrix} \cos \frac{\phi}{2} \cos \frac{\theta}{2} \cos \frac{\psi}{2} + \sin \frac{\phi}{2} \sin \frac{\theta}{2} \sin \frac{\psi}{2} \\ \sin \frac{\phi}{2} \cos \frac{\theta}{2} \cos \frac{\psi}{2} - \cos \frac{\phi}{2} \sin \frac{\theta}{2} \sin \frac{\psi}{2} \\ \cos \frac{\phi}{2} \sin \frac{\theta}{2} \cos \frac{\psi}{2} + \sin \frac{\phi}{2} \cos \frac{\theta}{2} \sin \frac{\psi}{2} \\ \cos \frac{\phi}{2} \cos \frac{\theta}{2} \sin \frac{\psi}{2} - \sin \frac{\phi}{2} \sin \frac{\theta}{2} \cos \frac{\psi}{2} \end{bmatrix}$$

Quaternion to Euler

$$\begin{bmatrix} \phi \\ \theta \\ \psi \end{bmatrix} = \begin{bmatrix} \text{atan2}(2(q_w q_x + q_y q_z), 1 - 2(q_x^2 + q_y^2)) \\ \text{asin}(2(q_w q_y - q_z q_x)) \\ \text{atan2}(2(q_w q_z + q_x q_y), 1 - 2(q_y^2 + q_z^2)) \end{bmatrix}$$

Euler to SO(3)

$$R_{from}^{to} = R_b^n = R_x(\phi) R_y(\theta) R_z(\psi)$$

$$R_x(\phi) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \phi & -\sin \phi \\ 0 & \sin \phi & \cos \phi \end{bmatrix}, \quad R_y(\theta) = \begin{bmatrix} \cos \theta & 0 & \sin \theta \\ 0 & 1 & 0 \\ -\sin \theta & 0 & \cos \theta \end{bmatrix}, \quad R_z(\psi) = \begin{bmatrix} \cos \psi & -\sin \psi & 0 \\ \sin \psi & \cos \psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Quaternion to SO(3)

$$R_{from}^{to} = R_b^n = \begin{bmatrix} 1 - 2(q_y^2 + q_z^2) & 2(q_x q_y - q_z q_w) & 2(q_x q_z + q_y q_w) \\ 2(q_x q_y + q_z q_w) & 1 - 2(q_x^2 + q_z^2) & 2(q_y q_z - q_x q_w) \\ 2(q_x q_z - q_y q_w) & 2(q_y q_z + q_x q_w) & 1 - 2(q_x^2 + q_y^2) \end{bmatrix}$$

These relationships provide a convenient reference for converting between orientation representations depending on the application, whether for computation, visualization, or integration with sensor data.

4.3 Reference Frames and Transformations

4.3.1 Basic Translation and Rotation Transformations

Rigid body motion in three dimensions can be expressed compactly using the homogeneous transformation matrix $T \in \text{SE}(3)$ defined in Equation 3. These matrices combine rotation and translation into a single representation and are used to map coordinates between frames such as the navigation (NED), body, and sensor frames.

The pose of the body frame relative to the navigation frame is written as

$$T_b^n = \begin{bmatrix} R_b^n & \mathbf{p}_b^n \\ 0 & 1 \end{bmatrix}$$

where R_b^n is the rotation matrix from the body to the NED frame, and \mathbf{p}_b^n is the position of the body origin expressed in NED coordinates.

A point \mathbf{x}_b^b expressed in the body frame can be transformed to the NED frame as:

$$\mathbf{x}_b^n = T_b^n \begin{bmatrix} \mathbf{x}_b^b \\ 1 \end{bmatrix} = R_b^n \mathbf{x}_b^b + \mathbf{p}_b^n$$

Composing multiple transformations is performed through matrix multiplication. For instance, if T_s^b defines the transformation from a sensor frame to the body frame, then the sensor pose in the NED frame becomes

$$T_s^n = T_b^n T_s^b$$

The inverse transformation, which converts coordinates from NED back to the body frame, is given by

$$T_n^b = (T_b^n)^{-1} = \begin{bmatrix} (R_b^n)^T & -(R_b^n)^T \mathbf{p}_b^n \\ 0 & 1 \end{bmatrix}$$

This matrix formulation provides a clean and consistent way to represent spatial relationships between the navigation, body, and sensor frames. It is fundamental in robotics, navigation, and control, where accurate frame alignment and transformation chaining are essential for pose estimation and sensor fusion.

These homogeneous transformations form the foundation for defining global and local reference frames such as WGS84, ECEF, NED, and Body, which are described in the following sections

4.3.2 Global Reference Frames

Global reference frames provide the foundation for representing absolute positions on Earth and are essential for all GNSS based navigation and mapping systems. Since GNSS receivers provide position estimates in a global Earth fixed frame, while navigation and control systems typically operate in local frames such as NED or body coordinates, it becomes necessary to define consistent global reference models to enable accurate transformations between these coordinate systems.

The two main reference systems used for this purpose are the World Geodetic System 1984 (WGS84) geodetic model, which defines the Earth's ellipsoidal shape and geodetic coordinates (latitude, longitude, altitude), and the Earth-Centered Earth-Fixed (ECEF) Cartesian frame, which expresses these same positions as 3D Cartesian coordinates centered at the Earth's center of mass. These systems provide the common link between GNSS measurements and local navigation frames used in estimation and control.

WGS84

The World Geodetic System 1984 (WGS84) is the global geodetic reference used by all major GNSS systems. It defines an Earth-fixed ellipsoidal model that closely approximates the planet's mean shape, accounting for the equatorial bulge caused by rotation. Although regional realizations such as "ETRS89" (European Terrestrial Reference System 1989) are used in Europe to reduce tectonic drift, the WGS84 reference remains the standard for GNSS based navigation in Norway and most marine applications, with differences between the two systems being only a few decimeters.

The WGS84 model defines the Earth as an oblate ellipsoid, flattened at the poles and expanded at the equator. It serves as the global geodetic reference for GPS and most modern satellite navigation systems. The ellipsoid parameters are defined as $a = 6378137.0$ m and $f = 1/298.257223563$, where a is the semi major axis and f is the flattening factor. The semi minor axis can then be computed as $b = a(1-f)$.

A position on Earth is defined by three geodetic coordinates. First the latitude φ , which is the angle between the equatorial plane and the ellipsoid normal. Then the longitude λ , which is the angle between the Greenwich meridian and the projection of the point onto the equatorial plane. Finally the altitude h , which is the height above the ellipsoidal surface. These coordinates describe a points global position and are the standard output format from all GNSS receivers.

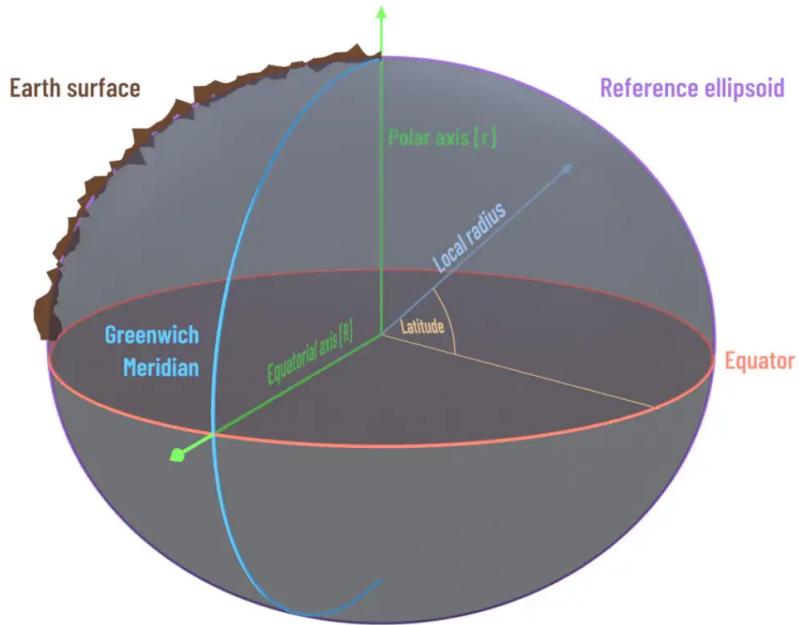


Figure 19: WGS84 ellipsoidal Earth model showing the relationship between latitude, longitude, and altitude. Figure taken from GeneSys Elektronik GmbH documentation.^[15]

ECEF

The Earth-Centered Earth-Fixed (ECEF) frame is a three dimensional Cartesian coordinate system with its origin at the Earths center of mass. The x -axis passes through the intersection of the equator and the Greenwich meridian, the y -axis lies along the equator 90° east of the x -axis, and the z -axis points toward the North Pole. The ECEF frame rotates with the Earth and is commonly used for expressing GNSS positions in Cartesian form.

The transformation from geodetic coordinates WGS84 (φ, λ, h) to ECEF coordinates (x, y, z) is given by:

$$\begin{aligned} e^2 &= 2f - f^2, \\ N(\varphi) &= \frac{a}{\sqrt{1 - e^2 \sin^2 \varphi}}, \\ x &= (N(\varphi) + h) \cos \varphi \cos \lambda, \\ y &= (N(\varphi) + h) \cos \varphi \sin \lambda, \\ z &= [(1 - e^2)N(\varphi) + h] \sin \varphi \end{aligned}$$

where $N(\varphi)$ is the prime vertical radius of curvature and e is the first eccentricity of the ellipsoid. The

inverse conversion from ECEF to geodetic coordinates is typically solved iteratively due to the nonlinear nature of the equations.

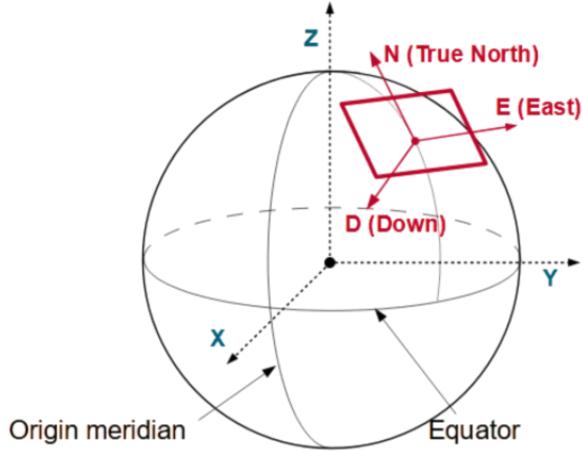


Figure 20: Earth-Centered Earth-Fixed (ECEF) coordinate system with origin at Earth's center and axes aligned with the equator and rotational axis. Figure taken from open source GNSS reference material.^[16]

The WGS84 and ECEF frames together form the foundation of all global navigation and positioning systems. While WGS84 defines the geometric reference ellipsoid used by GNSS measurements, ECEF provides a Cartesian coordinate representation that is better suited for numerical computation, sensor fusion, and integration with local navigation frames such as NED or body fixed coordinates.

4.3.3 Local Reference Frames

Local reference frames are used to describe the motion and orientation of vehicles in a way that is intuitive and convenient for navigation, estimation, and control. In marine and aerial systems, the two most common local frames are the North-East-Down (NED) frame and the Body frame. These provide a consistent way to represent vehicle states such as position, velocity, and attitude relative to the Earth.

The NED frame is a local tangent frame fixed to a point on the Earth's surface, with the x -axis pointing toward geographic north, the y -axis pointing east, and the z -axis pointing downward, perpendicular to the ellipsoid. This convention is widely used in navigation because it aligns naturally with compass directions and simplifies interpretation of GNSS and IMU data. The origin of the NED frame is typically defined at a reference point near the vehicle's initial position, tangent to the Earth at that location.

The Body frame is attached to the vehicle itself, with its origin at the vehicle's center of gravity or another defined point. The x -axis points forward along the vehicle's longitudinal direction, the y -axis points to the right, and the z -axis points downward. All onboard sensor measurements, such as accelerations, angular velocities, and forces, are naturally expressed in this frame.

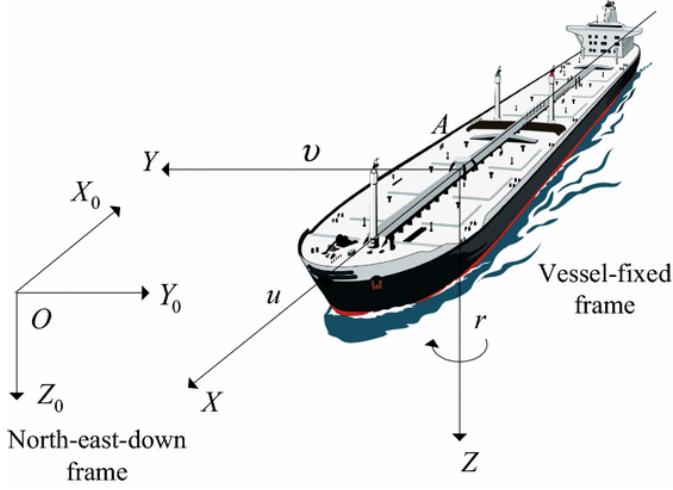


Figure 21: NED coordinate system attached to a marine vehicle. The illustration shows the local navigation axes and their relation to the body frame. Figure taken from a report on robust nonlinear control design for dynamic positioning of marine vessels.^[17]

The relationship between the global and local reference frames follows the transformation chain:

$$\text{WGS84 (geodetic)} \rightarrow \text{ECEF (Cartesian)} \rightarrow \text{NED (local tangent)} \rightarrow \text{Body (vehicle-fixed)}$$

GNSS receivers typically provide positions in the WGS84 frame, expressed as geodetic coordinates (φ, λ, h) , where φ is latitude, λ is longitude, and h is altitude above the reference ellipsoid. For computational purposes, these coordinates are first converted to the ECEF frame, which represents positions in Cartesian coordinates (x, y, z) relative to the Earth's center of mass.

To obtain a local navigation frame, the ECEF position is then rotated and translated into the NED frame, which defines a tangent plane fixed at a chosen reference point (φ_0, λ_0) . The rotation from ECEF to NED is given by

$$R_e^n = \begin{bmatrix} -\sin \varphi_0 \cos \lambda_0 & -\sin \varphi_0 \sin \lambda_0 & \cos \varphi_0 \\ -\sin \lambda_0 & \cos \lambda_0 & 0 \\ -\cos \varphi_0 \cos \lambda_0 & -\cos \varphi_0 \sin \lambda_0 & -\sin \varphi_0 \end{bmatrix}$$

where φ_0 and λ_0 are the latitude and longitude of the local origin. The superscript n and subscript e indicate that this matrix transforms a vector expressed in the ECEF frame $\{e\}$ into the NED frame $\{n\}$.

Given the ECEF position of the vehicle $\mathbf{p}_{b/e}^e$ and the ECEF position of the NED origin $\mathbf{p}_{O/e}^e$, the vehicles position in NED coordinates is computed as

$$\mathbf{p}_{b/O}^n = \mathbf{p}_{b/e}^e + \mathbf{p}_{e/O}^n = R_e^n(\mathbf{p}_{b/e}^e - \mathbf{p}_{O/e}^e)$$

For onboard sensors, the position of a sensor $\{s\}$ relative to the vehicle body origin, expressed in the NED frame, is given by

$$\mathbf{p}_{s/b}^n = R_b^n \mathbf{p}_{s/b}^b$$

The absolute position of the sensor relative to the NED origin can then be found as

$$\mathbf{p}_{s/O}^n = \mathbf{p}_{s/b}^n + \mathbf{p}_{b/O}^n$$

where $\mathbf{p}_{s/b}^b$ is the known position of the sensor in the body frame, and R_b^n is the rotation matrix representing the vehicles attitude from the Body to the NED frame.

This formulation relates the sensor position to the global reference frame by first rotating the sensor offset from the body frame into the navigation frame, then translating it using the vehicles global position. The rotation matrix R_b^n is obtained from the vehicles orientation representation, typically parameterized using Euler angles or a unit quaternion, ensuring consistent alignment between the sensor,

body, and navigation coordinate frames.

Maintaining a consistent chain of transformations between these frames is essential for reliable navigation and estimation. In practice, GNSS delivers global positions in WGS84 or ECEF coordinates, while onboard IMU sensors provide measurements in the Body frame. Sensor fusion algorithms such as the Extended Kalman Filter (EKF) and Error State Kalman Filter (ESKF) depend on these transformations to express all quantities consistently within the NED frame used for state estimation and control.

4.4 Rigid Body Kinematics

4.4.1 Relevance for Navigation and Modeling

Rigid body kinematics provides the mathematical foundation for expressing motion, orientation, and acceleration consistently across coordinate frames. In navigation systems, these relationships allow the integration of measurements from inertial sensors, GNSS, and other sources within a unified dynamic model.

The transformation equations for position, velocity, and acceleration ensure that sensor data expressed in the body frame can be accurately related to the navigation frame, enabling correct estimation of the vehicle's state. This consistency is essential for inertial navigation, attitude estimation, and sensor fusion, where body fixed measurements such as angular velocity and specific force must be mapped into global coordinates for integration and correction.

Moreover, the rigid body framework is fundamental for simulation and control system design. It allows modeling of vehicle dynamics, actuator response, and sensor placement with precise spatial relationships. The kinematic expressions derived in this section thus serve as the basis for dynamic modeling, state estimation, and motion prediction used in autonomous and navigation systems.

4.4.2 Position and Orientation

The position of a rigid body is represented by the vector $\mathbf{p}_{b/O}^n$, which denotes the location of the body frame origin expressed in the navigation (NED) frame. The orientation of the body is represented by the rotation matrix $R_b^n \in \text{SO}(3)$, which maps a vector from the body frame $\{b\}$ to the navigation frame $\{n\}$.

The combined rigid body pose is described by the homogeneous transformation matrix $T_{b/O}^n \in \text{SE}(3)$:

$$T_{b/O}^n = \begin{bmatrix} R_b^n & \mathbf{p}_{b/O}^n \\ 0 & 1 \end{bmatrix}$$

The kinematic relationship between the time derivative of position and the linear velocity is then given by

$$\dot{\mathbf{p}}_{b/O}^n = \mathbf{v}_{b/O}^n$$

where $\mathbf{v}_{b/O}^n$ is the velocity of the body origin expressed in the navigation frame.

4.4.3 Angular Velocity and Euler Angle Relationship

The angular velocity vector $\boldsymbol{\omega}_{b/O}^b = [p, q, r]^\top$ represents the instantaneous rotation rate of the body about its own axes, roll rate p about the x_b -axis (North), pitch rate q about the y_b -axis (East), and yaw rate r about the z_b -axis (Down).

When the body orientation is represented by ZYX Euler angles (yaw ψ , pitch θ , roll ϕ), the time derivative of the Euler angles relates to the body angular velocity through

$$\begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} = T(\phi, \theta) \boldsymbol{\omega}_{b/O}^b$$

where the transformation matrix $T(\phi, \theta)$ for the NED convention is defined as

$$T(\phi, \theta) = \begin{bmatrix} 1 & \sin \phi \tan \theta & \cos \phi \tan \theta \\ 0 & \cos \phi & -\sin \phi \\ 0 & \sin \phi / \cos \theta & \cos \phi / \cos \theta \end{bmatrix}$$

This maps the body angular velocity to the Euler angle rates according to the NED convention.

The inverse relationship, expressing body angular velocity from Euler angle derivatives, is given by

$$\boldsymbol{\omega}_{b/O}^b = \begin{bmatrix} p \\ q \\ r \end{bmatrix} = T^{-1}(\phi, \theta) \begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix}$$

where

$$T^{-1}(\phi, \theta) = \begin{bmatrix} 1 & 0 & -\sin \theta \\ 0 & \cos \phi & \sin \phi \cos \theta \\ 0 & -\sin \phi & \cos \phi \cos \theta \end{bmatrix}$$

This formulation aligns with the NED coordinate convention and ensures correct mapping between angular velocities and Euler angle derivatives. The transformation matrix becomes singular at $\theta = \pm 90^\circ$, corresponding to gimbal lock, where $\tan \theta$ diverges.

4.4.4 Angular Velocity and Quaternion Relationship

Quaternions provide a compact and singularity free representation of rotation. A unit quaternion $q = [q_0, q_1, q_2, q_3]^\top$ consists of one scalar and three vector components, often written as

$$q = \begin{bmatrix} q_0 \\ \mathbf{q}_v \end{bmatrix} = \begin{bmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{bmatrix}$$

where q_0 is the scalar part and $\mathbf{q}_v = [q_1, q_2, q_3]^\top$ is the vector part. The quaternion represents a rotation of angle θ around the unit axis $\hat{\mathbf{u}}$:

$$q = \begin{bmatrix} \cos \frac{\theta}{2} \\ \hat{\mathbf{u}} \sin \frac{\theta}{2} \end{bmatrix}$$

The time evolution of the quaternion is governed by the angular velocity vector $\boldsymbol{\omega}_{b/O}^b = [p, q, r]^\top$ measured in the body frame. The quaternion kinematics are given by

$$\dot{q} = \frac{1}{2}\Omega(\boldsymbol{\omega}_{b/O}^b)q$$

where $\Omega(\boldsymbol{\omega}_{b/O}^b)$ is the quaternion rate matrix:

$$\Omega(\boldsymbol{\omega}_{b/O}^b) = \begin{bmatrix} 0 & -p & -q & -r \\ p & 0 & r & -q \\ q & -r & 0 & p \\ r & q & -p & 0 \end{bmatrix}$$

Given the quaternion $q = [q_0, \mathbf{q}_v]^\top$ and its time derivative $\dot{q} = [\dot{q}_0, \dot{\mathbf{q}}_v]^\top$, the body angular velocity can be reconstructed directly as

$$\boldsymbol{\omega}_{b/O}^b = 2(q_0 \mathbf{q}_v - \dot{q}_0 \mathbf{q}_v - \mathbf{q}_v \times \dot{\mathbf{q}}_v).$$

Here, q_0 is the scalar part of the quaternion and $\mathbf{q}_v \times \dot{\mathbf{q}}_v$ its vector part. This relation provides a direct and numerically stable way to compute the instantaneous angular velocity from quaternion derivatives while maintaining full consistency with rigid body rotational kinematics in the body frame. This formulation complements the quaternion rate equation which integrates angular velocity to update attitude over time. The quaternion must remain normalized, i.e. $|q| = 1$, to represent a valid rotation, and is therefore periodically renormalized during numerical integration as

$$q = \frac{q}{|q|}.$$

The corresponding rotation matrix from body to navigation frame is obtained as

$$R_b^n(q) = \begin{bmatrix} 1 - 2(q_2^2 + q_3^2) & 2(q_1 q_2 - q_0 q_3) & 2(q_1 q_3 + q_0 q_2) \\ 2(q_1 q_2 + q_0 q_3) & 1 - 2(q_1^2 + q_3^2) & 2(q_2 q_3 - q_0 q_1) \\ 2(q_1 q_3 - q_0 q_2) & 2(q_2 q_3 + q_0 q_1) & 1 - 2(q_1^2 + q_2^2) \end{bmatrix}$$

This rotation matrix provides the mapping between the body and navigation frames, analogous to R_b^n obtained from Euler angles, but without the gimbal lock problem.

In practice, quaternion based orientation integration using $\dot{q} = \frac{1}{2}\Omega(\boldsymbol{\omega}_{b/O}^b)q$ is preferred for real-time attitude propagation, while the equivalent rotation matrix or Euler angles are extracted only when needed for visualization or control.

4.4.5 Linear Velocity Relationship

The linear velocity of any point P fixed on a rigid body can be related to the velocity of a reference point O on the same body as

$$\mathbf{v}_{P/O}^b = \mathbf{v}_{b/O}^b + \mathbf{v}_{P/b}^b + \boldsymbol{\omega}_{b/O}^b \times \mathbf{p}_{P/b}^b$$

Here, $\mathbf{v}_{b/O}^b$ is the translational velocity of the body origin expressed in the body frame, $\mathbf{v}_{P/b}^b$ is the velocity of point P relative to the body (zero for fixed sensors), $\boldsymbol{\omega}_{b/O}^b$ is the body angular velocity, and $\mathbf{p}_{P/b}^b$ is the position of P relative to the body origin, both expressed in the body frame.

The cross product term $\boldsymbol{\omega}_{b/O}^b \times \mathbf{p}_{P/b}^b$ represents the additional linear velocity experienced by point P due to the rotational motion of the body. This term becomes more significant for points located farther away from the rotation axis, such as antennas or sensors mounted away from the vessel's center of gravity.

In many applications, it is necessary to express all velocities in a common reference frame. The velocity of the body origin expressed in the navigation (NED) frame is obtained by rotating the body frame velocity using the rotation matrix R_b^n :

$$\mathbf{v}_{b/O}^n = R_b^n \mathbf{v}_{b/O}^b.$$

Conversely, the body frame velocity can be obtained from the navigation frame velocity by

$$\mathbf{v}_{b/O}^b = (R_b^n)^\top \mathbf{v}_{b/O}^n,$$

since $(R_b^n)^\top = R_n^b$. The rotation matrix R_b^n is derived from the vehicle's orientation, represented either by Euler angles or a unit quaternion.

Similarly, for a sensor or point P rigidly fixed to the vehicle (i.e., $\mathbf{v}_{P/b}^b = 0$), the velocity expressed in the navigation frame simplifies to

$$\mathbf{v}_{P/O}^n = R_b^n \left(\mathbf{v}_{b/O}^b + \boldsymbol{\omega}_{b/O}^b \times \mathbf{p}_{P/b}^b \right),$$

where $\mathbf{p}_{P/b}^b$ defines the sensor's position in the body frame. This formulation provides a consistent way to compute the motion of any fixed point on the vehicle in both body and navigation frames, ensuring proper alignment between inertial and navigation states.

4.4.6 Angular Acceleration in Euler Representation

The angular acceleration $\boldsymbol{\alpha}_{b/O}^b$ describes the rate of change of the body's angular velocity vector $\boldsymbol{\omega}_{b/O}^b = [p, q, r]^\top$, which represents the instantaneous rotation of the body about its own x , y , and z axes. Physically, $\boldsymbol{\alpha}_{b/O}^b$ captures how fast the rotational motion itself is changing, and it directly relates to the torques acting on the body through the rotational dynamics equations.

When the body orientation is represented using Euler angles (ϕ, θ, ψ) , the angular acceleration can be obtained by differentiating the Euler angle to angular velocity relationship:

$$\boldsymbol{\alpha}_{b/O}^b = T^{-1}(\phi, \theta, \psi) \ddot{\Theta} + \dot{T}^{-1}(\phi, \theta, \psi) \dot{\Theta}$$

where $\dot{\Theta} = [\dot{\phi}, \dot{\theta}, \dot{\psi}]^\top$ and $\ddot{\Theta} = [\ddot{\phi}, \ddot{\theta}, \ddot{\psi}]^\top$ are the first and second derivatives of the Euler angles. The first term represents the direct contribution from angular rate changes, while the second term captures coupling effects caused by the nonlinearity of rotational kinematics, for instance, when pitch or roll rates influence yaw acceleration.

In differential form, the time derivative of angular velocity is expressed as

$$\dot{\boldsymbol{\omega}}_{b/O}^b = \boldsymbol{\alpha}_{b/O}^b$$

which defines angular acceleration in the same frame as $\boldsymbol{\omega}_{b/O}^b$. To express angular acceleration in another reference frame, such as the navigation frame, it can be transformed using the rotation matrix:

$$\boldsymbol{\alpha}_{b/O}^n = R_b^n \boldsymbol{\alpha}_{b/O}^b$$

where R_b^n is the rotation from the body frame to the navigation (NED) frame.

In practical navigation and control systems, the angular acceleration $\alpha_{b/O}^b$ is rarely measured directly. It is usually approximated by differentiating the body angular velocity $\omega_{b/O}^b$ obtained from gyroscopes or inferred from rigid body dynamics. Maintaining consistent frame alignment between body and navigation frames is essential for accurate attitude propagation and torque computation.

4.4.7 Angular Acceleration in Quaternion Representation

The quaternion representation provides a singularity free alternative to Euler angles for describing rotational motion. The unit quaternion $q = [q_0, q_1, q_2, q_3]^\top$ defines the orientation of the body frame relative to the navigation frame, where q_0 is the scalar part and $[q_1, q_2, q_3]^\top$ is the vector part.

The time evolution of the quaternion is governed by the rotational kinematics equation:

$$\dot{q} = \frac{1}{2}\Omega(\omega_{b/O}^b)q$$

where $\omega_{b/O}^b = [p, q, r]^\top$ is the angular velocity in the body frame and $\Omega(\omega_{b/O}^b)$ is the quaternion rate matrix:

$$\Omega(\omega_{b/O}^b) = \begin{bmatrix} 0 & -p & -q & -r \\ p & 0 & r & -q \\ q & -r & 0 & p \\ r & q & -p & 0 \end{bmatrix}$$

Differentiating this expression gives the quaternion based angular acceleration:

$$\ddot{q} = \frac{1}{2}\Omega(\alpha_{b/O}^b)q + \frac{1}{2}\Omega(\omega_{b/O}^b)\dot{q}$$

which captures both the direct contribution of angular acceleration $\alpha_{b/O}^b$ and the coupling effect from the current angular velocity.

In vector form, the time derivative of angular velocity is expressed as

$$\dot{\omega}_{b/O}^b = \alpha_{b/O}^b$$

which defines angular acceleration in the same frame as ω^b . To express angular acceleration in another reference frame, such as the navigation frame, it can be transformed using the rotation matrix:

$$\alpha_{b/O}^n = R_b^n \alpha_{b/O}^b$$

where R_b^n is the rotation matrix from body to navigation frame.

To ensure q remains a valid rotation, it must satisfy the unit norm constraint:

$$\|q\| = 1$$

which is typically enforced by periodic normalization during numerical integration like so:

$$q = \frac{q}{\|q\|}$$

This quaternion based formulation is computationally efficient and avoids the gimbal lock issue inherent in Euler angle representations. For this reason, quaternion propagation is widely preferred for representing orientation and modeling angular dynamics in navigation and control systems.

4.4.8 Linear Acceleration

The linear acceleration of a point P on a rigid body consists of translational, tangential, centripetal, and Coriolis components. It is expressed in the body frame as

$$\mathbf{a}_{P/O}^b = \mathbf{a}_{b/O}^b + \mathbf{a}_{P/b}^b + \alpha_{b/O}^b \times \mathbf{p}_{P/b}^b + \omega_{b/O}^b \times (\omega_{b/O}^b \times \mathbf{p}_{P/b}^b) + 2\omega_{b/O}^b \times \mathbf{v}_{P/b}^b$$

Here, $\mathbf{a}_{b/O}^b$ is the translational acceleration of the body origin, $\mathbf{a}_{P/b}^b$ is acceleration of point P on sensor relative to body, $\boldsymbol{\alpha}_{b/O}^b$ is the angular acceleration, $\boldsymbol{\omega}_{b/O}^b$ is the angular velocity, $\mathbf{p}_{P/b}^b$ is the position of point P relative to the body origin, and $\mathbf{v}_{P/b}^b$ is the velocity of P relative to the body frame.

Each term represents a specific physical contribution to the total acceleration. The first term $\mathbf{a}_{b/O}^b$ describes the translational acceleration of the body origin. Second term $\mathbf{a}_{P/b}^b$ denotes the local acceleration of P relative to the body frame, which becomes zero for fixed sensors, while $\boldsymbol{\alpha}_{b/O}^b \times \mathbf{p}_{P/b}^b$ represents the tangential acceleration caused by changes in rotational rate. The term $\boldsymbol{\omega}_{b/O}^b \times (\boldsymbol{\omega}_{b/O}^b \times \mathbf{p}_{P/b}^b)$ accounts for the centripetal acceleration directed toward the axis of rotation, and $2\boldsymbol{\omega}_{b/O}^b \times \mathbf{v}_{P/b}^b$ captures the Coriolis acceleration arising from relative motion within the rotating body.

For a sensor rigidly mounted on the body, where $\mathbf{v}_{P/b}^b = 0$ and $\mathbf{a}_{P/b}^b = 0$, the equation simplifies to

$$\mathbf{a}_{P/O}^b = \mathbf{a}_{b/O}^b + \boldsymbol{\alpha}_{b/O}^b \times \mathbf{p}_{P/b}^b + \boldsymbol{\omega}_{b/O}^b \times (\boldsymbol{\omega}_{b/O}^b \times \mathbf{p}_{P/b}^b).$$

This formulation is widely used to compute the acceleration at sensor locations such as IMUs or GNSS antennas mounted away from the vehicles center of mass.

To express the acceleration in the navigation (NED) frame, the transformation is performed using the rotation matrix R_b^n :

$$\mathbf{a}_{P/O}^n = R_b^n \mathbf{a}_{P/O}^b,$$

and conversely,

$$\mathbf{a}_{P/O}^b = (R_b^n)^\top \mathbf{a}_{P/O}^n.$$

In inertial navigation systems, accelerometers measure the specific force, which is the total acceleration excluding gravity:

$$\mathbf{f}^b = \mathbf{a}_{b/O}^b - R_n^b \mathbf{g}^n,$$

where $\mathbf{g}^n = [0, 0, g]^\top$ is the gravity vector in the NED frame. The specific force \mathbf{f}^b is the quantity directly measured by IMUs and serves as the basis for estimating velocity and position through numerical integration.

4.5 ASV Motion and Measurement Models

4.5.1 Overview

This chapter presents the mathematical framework used to model the motion dynamics and sensor measurements of the ASV. Two main modeling approaches are discussed, the nonlinear 6 DOF marine craft model developed by Fossen [9], and the Inertial Navigation System (INS) based kinematic model described in Edmund Brekke book on Sensor Fusion [10]. Both formulations provide complementary perspectives on vehicle motion, Fossen's model emphasizes a physically consistent hydrodynamic representation, while the INS model focuses on inertial propagation and sensor fusion suitable for real-time navigation.

The chapter begins by introducing Fossen's marine craft model, which captures the rigid body and hydrodynamic behavior of marine vehicles through a compact matrix vector formulation that includes added mass, Coriolis, damping, and restoring effects. The INS based kinematic model is then presented as the practical implementation used in this work, describing the evolution of position, velocity, and attitude from inertial data while modeling sensor biases and stochastic errors. Aiding measurements from a dual antenna GNSS system are incorporated to correct drift and estimate heading. Although Fossen's model provides higher physical fidelity, it requires extensive parameter identification, whereas the INS model achieves sufficient accuracy for SLAM and local navigation when aided by GNSS. Thus, the INS based approach is chosen as the optimal balance between simplicity, computational efficiency, and practical performance.

4.5.2 Kinetic Motion Model

The kinetic motion model employed in this work is based on Fossen's nonlinear 6 DOF marine craft formulation, which provides a unified mathematical framework for describing the dynamic behavior of marine vehicles [9]. This model captures both rigid body and hydrodynamic effects using a compact matrix vector representation, making it suitable for analysis, control, and simulation of ships, underwater vehicles, and surface vessels. By bridging classical rigid body mechanics with hydrodynamic theory, the formulation offers a complete description of marine craft dynamics while preserving key physical properties such as symmetry, passivity, and energy consistency.

The general 6 DOF equations of motion can be written as

$$\mathbf{M}\dot{\boldsymbol{\nu}} + \mathbf{C}(\boldsymbol{\nu})\boldsymbol{\nu} + \mathbf{D}(\boldsymbol{\nu})\boldsymbol{\nu} + \mathbf{g}(\boldsymbol{\eta}) = \boldsymbol{\tau}$$

where $\boldsymbol{\eta} = [x, y, z, \phi, \theta, \psi]^\top$ represents position and orientation, and $\boldsymbol{\nu} = [u, v, w, p, q, r]^\top$ denotes linear and angular velocities in the body frame. The vector $\boldsymbol{\tau}$ contains external forces and moments acting on the vessel from propulsion, wind, waves, and current. This formulation expresses the total balance between inertia, Coriolis, damping, and restoring forces on the left hand side, and all external excitations on the right hand side, forming the foundation for the dynamics of any marine craft.

The inertia matrix \mathbf{M} consists of both rigid body and added mass contributions,

$$\mathbf{M} = \mathbf{M}_{RB} + \mathbf{M}_A$$

where \mathbf{M}_{RB} represents the rigid body mass and inertia, while \mathbf{M}_A captures the hydrodynamic added mass caused by the acceleration of surrounding water as the hull moves. The added mass terms are particularly important for underwater and high speed vehicles, as they significantly influence acceleration response and stability. This total inertia matrix determines how the craft resists changes in motion, acting as a coupling term between linear and angular accelerations.

The Coriolis and centripetal effects are similarly expressed as

$$\mathbf{C}(\boldsymbol{\nu}) = \mathbf{C}_{RB}(\boldsymbol{\nu}) + \mathbf{C}_A(\boldsymbol{\nu})$$

with \mathbf{C}_{RB} and \mathbf{C}_A representing the rigid body and added mass contributions, respectively. These matrices account for dynamic coupling between translational and rotational motion, such as how a turn induces sway or roll. The Coriolis matrix is skew-symmetric, ensuring that it does not contribute to net energy gain or loss, only redistributing kinetic energy among the motion axes.

The matrix $\mathbf{D}(\boldsymbol{\nu})$ models hydrodynamic damping from viscous drag, wave radiation, and flow separation effects. It typically includes linear damping terms valid for small velocities and nonlinear or quadratic terms that dominate at higher speeds. The damping effects are always dissipative, converting kinetic energy into heat or wave energy, thus stabilizing the system over time.

The restoring forces and moments $\mathbf{g}(\boldsymbol{\eta})$ describe gravitational and buoyancy effects, depending on the vehicle's geometry and displacement. These terms act to return the vessel to equilibrium when displaced, defining roll, pitch, and heave stability through hydrostatic stiffness. For surface vessels, the restoring forces are strongly dependent on metacentric heights and the waterplane area, while for submerged vehicles, they depend mainly on the distance between the centers of gravity and buoyancy.

The kinematic relationships between the body and NED frames are given by

$$\dot{\boldsymbol{\eta}} = \mathbf{J}(\boldsymbol{\eta})\boldsymbol{\nu}$$

where $\mathbf{J}(\boldsymbol{\eta})$ is a block diagonal transformation matrix containing the rotation matrix R_b^n for translational motion and $T(\phi, \theta)$ for angular motion. This mapping relates the velocity expressed in the body frame to the rate of change of position and attitude in the navigation frame, and can be implemented using either Euler angles or quaternions to avoid singularities.

Environmental effects such as ocean currents, wind, and wave disturbances can be included by introducing the relative velocity $\boldsymbol{\nu}_r = \boldsymbol{\nu} - \boldsymbol{\nu}_c$, where $\boldsymbol{\nu}_c$ is the current velocity expressed in the body frame. This modifies the hydrodynamic forces and damping terms to account for the vehicle's motion relative to the surrounding water, which is essential for accurate modeling of drift and current induced forces.

For surface vessels, the equations can be simplified to the planar 3 DOF surge, sway, and yaw model:

$$\mathbf{M}\dot{\boldsymbol{\nu}} + \mathbf{C}(\boldsymbol{\nu})\boldsymbol{\nu} + \mathbf{D}(\boldsymbol{\nu})\boldsymbol{\nu} = \boldsymbol{\tau}$$

where $\boldsymbol{\nu} = [u, v, r]^\top$. This reduced model captures the dominant motion in the horizontal plane and is widely used for uncrewed surface vehicles and control applications where roll, pitch, and heave motions are negligible.

The full nonlinear state space representation of Fossen's marine craft model can be written as

$$\dot{\mathbf{x}} = f(\mathbf{x}, \mathbf{u}) = \begin{bmatrix} \dot{\boldsymbol{\eta}} \\ \dot{\boldsymbol{\nu}} \end{bmatrix} = \begin{bmatrix} \mathbf{J}(\boldsymbol{\eta})\boldsymbol{\nu} \\ \mathbf{M}^{-1}(\boldsymbol{\tau} - \mathbf{C}(\boldsymbol{\nu})\boldsymbol{\nu} - \mathbf{D}(\boldsymbol{\nu})\boldsymbol{\nu} - \mathbf{g}(\boldsymbol{\eta})) \end{bmatrix}$$

where

$$\mathbf{x} = [\boldsymbol{\eta} \quad \boldsymbol{\nu}]^\top = [x, y, z, \phi, \theta, \psi, u, v, w, p, q, r]^\top, \quad \mathbf{u} = \boldsymbol{\tau}$$

Here, the first part of the function $f(\mathbf{x}, \mathbf{u})$ describes the kinematic mapping from body frame velocities to the time derivative of position and attitude through the transformation matrix $\mathbf{J}(\boldsymbol{\eta})$, while the second part represents the vehicle kinetics defined by Newton-Euler dynamics. The term \mathbf{M}^{-1} acts as an inverse inertia operator, mapping the net generalized forces and moments (after accounting for Coriolis, damping, and restoring effects) to the acceleration in the body frame.

This formulation provides a complete and compact representation of the vessel's motion, directly linking control inputs and environmental disturbances to the time evolution of the vehicle's state.

Fossen's formulation exhibits several key properties, the mass matrix \mathbf{M} is symmetric and positive definite, the Coriolis matrix $\mathbf{C}(\boldsymbol{\nu})$ is skew-symmetric, and the damping matrix $\mathbf{D}(\boldsymbol{\nu})$ is positive definite. These properties guarantee passivity and energy conservation, ensuring that the system dissipates energy over time and remains physically consistent. This structure also provides a strong foundation for control system design and stability analysis, as it aligns with fundamental principles of energy based modeling and Lyapunov stability.

4.5.3 Kinematic Motion Model

The kinematic motion model adopted in this work is based on the Inertial Navigation System (INS) formulation, which describes the time evolution of a vehicle's position, velocity, and attitude as functions

of inertial sensor measurements [10]. Unlike hydrodynamic models that depend on physical parameters such as mass, damping, or buoyancy, the INS model is purely kinematic and relies solely on accelerometer and gyroscope data. This independence from the vehicles geometry, operating environment, and fluid interaction makes it well suited for generic navigation and SLAM applications, where a simplified yet robust representation of motion is sufficient for accurate state propagation between aiding measurements.

IMUs typically represent attitude using quaternions rather than Euler angles. This avoids singularities and ensures smooth rotational updates even under large orientation changes, making quaternion based propagation the standard approach for INS implementations.

The continuous time state vector is defined as

$$\mathbf{x} = \begin{bmatrix} \mathbf{p}_{b/O}^n & \mathbf{v}_{b/O}^n & \mathbf{q} & \mathbf{b}_a & \mathbf{b}_g \end{bmatrix}^\top$$

where $\mathbf{p}_{b/O}^n = [x, y, z]^\top$ is the position of the body origin relative to the navigation frame, $\mathbf{v}_{b/O}^n = [v_x, v_y, v_z]^\top$ is the velocity in the same frame, \mathbf{q} is the quaternion representing the rotation from body to navigation frame, and \mathbf{b}_a and \mathbf{b}_g are the accelerometer and gyroscope biases, respectively.

The position and velocity states describe translational motion in the navigation frame, while the quaternion captures attitude evolution through rotational kinematics. The bias states account for the slowly varying sensor offsets that develop over time due to temperature changes, component aging, and other environmental factors. Together, these quantities form a complete minimal representation of the navigation state used in inertial systems, forming the basis for the continuous time kinematic model that governs their time evolution.

The inertial navigation model therefore describes the kinematic evolution of the vehicles position, velocity, and attitude as driven by inertial sensor measurements. The foundation of this formulation is the deterministic, noise free relationship between translational and rotational motion, which forms the core of all INS propagation models.

In the ideal case without any sensor errors, the motion of the sensor can be expressed as

$$\begin{aligned} \dot{\mathbf{p}}_{s/O}^n &= \mathbf{v}_{s/O}^n \\ \dot{\mathbf{v}}_{s/O}^n &= R_b^n(\mathbf{q}) (R_s^b \mathbf{a}_m) + \mathbf{g}^n \\ \dot{\mathbf{q}} &= \frac{1}{2} \mathbf{q} \otimes (R_s^b \boldsymbol{\omega}_m) \end{aligned}$$

where $\mathbf{p}_{s/O}^n$ and $\mathbf{v}_{s/O}^n$ are the position and velocity of the sensor with respect to an inertial reference frame like NED, $R_b^n(\mathbf{q})$ is the rotation matrix transforming vectors from the body frame to the navigation frame, and R_s^b is the rotation matrix from the sensor to body frame. The measured quantities \mathbf{a}_m and $\boldsymbol{\omega}_m$ are the specific force and angular rate readings provided by the IMU, expressed in the sensors own coordinate frame. These measurements are therefore related to the true body frame motion through the sensors fixed mounting orientation, such that $\mathbf{a}_m = \mathbf{a}_{s/b}^s$ and $\boldsymbol{\omega}_m = \boldsymbol{\omega}_{s/b}^s$.

In practice, the inertial sensors are rarely placed exactly at the vehicles center of gravity (COG). When the sensor is offset by a position vector $\mathbf{r}_{s/b}^b$ from the body frame origin usually defined at the COG, the accelerations measured by the IMU no longer correspond directly to the translational accelerations of the vehicle. Instead, the rotational motion of the body introduces additional terms due to centripetal and tangential accelerations. The total acceleration of the sensor can then be expressed as

$$\mathbf{a}_{s/O}^b = \mathbf{a}_{b/O}^b + \boldsymbol{\alpha}_{b/O}^b \times \mathbf{r}_{s/b}^b + \boldsymbol{\omega}_{b/O}^b \times (\boldsymbol{\omega}_{b/O}^b \times \mathbf{p}_{s/b}^b)$$

where $\mathbf{a}_{b/O}^b$ is the linear acceleration of the body origin, $\boldsymbol{\omega}_{b/O}^b$ is the angular velocity of the body, and $\boldsymbol{\alpha}_{b/O}^b$ its angular acceleration. The second term represents the tangential acceleration component caused by rotational acceleration, while the third term corresponds to the centripetal acceleration arising from constant rotation. These effects are collectively known as the “lever arm effect”, and they cause the IMU to measure accelerations that differ from those experienced at the COG.

Accurately compensating for the lever arm effect requires knowledge of the exact mounting offset $\mathbf{p}_{s/b}^b$ and precise estimates of both angular velocity and angular acceleration. The latter is particularly difficult to obtain in real time, as most MEMS-based IMU (like microAmpere IMU) do not measure angular acceleration directly. Numerical differentiation of gyroscope signals amplifies measurement noise and introduces additional uncertainty, making real-time compensation computationally challenging and often unreliable.

Therefore, in this master thesis work, it is assumed that the inertial sensors are mounted close to the vehicles center of gravity. For the microAmpere ASV, this assumption is valid since the IMU is physically located near the COG within the central electronics housing. As a result, the lever arm terms become negligible, and the simplified kinematic model given above provides an accurate and practical representation of the vehicle motion. This assumption greatly simplifies the mathematical model while retaining sufficient fidelity for navigation and SLAM applications.

Under this assumption, the IMU can be considered approximately coincident with the body frame origin, effectively treating the sensor as rigidly aligned with the vehicles center of gravity. The motion model can then be expressed directly in terms of the body frame as

$$\begin{aligned}\dot{\mathbf{p}}_{b/O}^n &= \mathbf{v}_{b/O}^n \\ \dot{\mathbf{v}}_{b/O}^n &= R_b^n(\mathbf{q}) \mathbf{a}_m + \mathbf{g}^n \\ \dot{\mathbf{q}} &= \frac{1}{2} \mathbf{q} \otimes \boldsymbol{\omega}_m\end{aligned}$$

which defines the nominal, noise free kinematic model used in this work. It captures the translational and rotational motion of the ASV based solely on the IMUs accelerometer and gyroscope measurements, with attitude expressed through quaternion integration for smooth and singularity free orientation propagation.

In practice, the accelerometer and gyroscope measurements are not perfect and are affected by sensor biases and measurement noise. These imperfections cause the estimated position and attitude to drift over time, as small integration errors accumulate during motion. The measured sensor outputs can therefore be modeled as

$$\begin{aligned}\mathbf{a}_m &= R^T(q_t)(\mathbf{a}_t - \mathbf{g}) + \mathbf{a}_{bt} + \mathbf{a}_n \\ \boldsymbol{\omega}_m &= \boldsymbol{\omega}_t + \boldsymbol{\omega}_{bt} + \boldsymbol{\omega}_n\end{aligned}$$

where \mathbf{a}_m and $\boldsymbol{\omega}_m$ are the measured accelerations and angular rates, \mathbf{a}_{bt} and $\boldsymbol{\omega}_{bt}$ denote slowly varying sensor biases, and \mathbf{a}_n and $\boldsymbol{\omega}_n$ are zero mean Gaussian noise processes representing measurement uncertainty. The reason for choosing zero mean Gaussian noise processes representation is just to make calculations easier to manage, and usually inertial sensor noise can be approximated to a pure gaussian.

While the previous kinematic model assumed ideal, noise free measurements, real inertial sensors inherently produce signals corrupted by both bias and random noise. These effects introduce long term drift in velocity, position, and orientation estimates if uncorrected. To account for this, the noise and bias terms are explicitly included in the dynamic equations.

By substituting the noisy sensor models into the noise free kinematics, we obtain the full continuous time INS motion model that represents the true system behavior under realistic sensor conditions:

$$\begin{aligned}\dot{\mathbf{p}}_{b/O}^n &= \mathbf{v}_{b/O}^n \\ \dot{\mathbf{v}}_{b/O}^n &= R_b^n(\mathbf{q})(\mathbf{a}_m - \mathbf{a}_{bt} - \mathbf{a}_n) + \mathbf{g}^n \\ \dot{\mathbf{q}} &= \frac{1}{2} \mathbf{q} \otimes (\boldsymbol{\omega}_m - \boldsymbol{\omega}_{bt} - \boldsymbol{\omega}_n)\end{aligned}$$

where $R_b^n(\mathbf{q})$ transforms body frame accelerations into the navigation frame according to the current orientation quaternion.

This formulation captures the complete inertial motion propagation process, accounting for both deterministic motion and stochastic sensor effects. It forms the basis for all subsequent modeling of bias dynamics and sensor noise processes.

Inertial measurement systems such as accelerometers and gyroscopes inherently suffer from drift over time, as they measure relative motion rather than absolute quantities. Small integration errors

in acceleration and angular rate accumulate, causing increasing uncertainty in position and attitude estimates. To account for this phenomenon, bias terms are introduced to represent slow, time varying offsets in the sensor readings. Accurate bias modeling is essential for achieving reliable and stable navigation performance, particularly for low cost MEMS-based sensors where bias instability is a dominant error source.

Sensor biases are typically represented as stochastic processes that evolve gradually over time. Several modeling approaches exist, each offering a trade off between realism and complexity. The simplest approach is the constant bias model,

$$\dot{\mathbf{b}} = 0$$

which assumes a fixed, time invariant offset in the sensor readings. Although straightforward, this model is generally insufficient for practical applications since real world sensors exhibit continuous drift due to thermal, mechanical, and electronic variations.

A more flexible formulation is the Wiener process, or better known as random walk model,

$$\dot{\mathbf{b}} = \mathbf{b}_n$$

where \mathbf{b}_n is a zero mean white noise process. This allows the bias to evolve stochastically over time, capturing unmodeled variations in sensor output. However, this model tends to be overly conservative, as the bias uncertainty grows without bound, making it unsuitable for long duration navigation where bounded behavior is required.

A more realistic representation for modern inertial sensors is the first-order Gauss-Markov process,

$$\dot{\mathbf{b}} = -\frac{1}{\tau} \mathbf{b} + \mathbf{b}_n$$

where τ is the correlation time constant that defines how quickly the bias decays toward zero. This model captures both short term fluctuations and long term drift, providing a balanced compromise between physical accuracy and numerical stability. It ensures that bias uncertainty remains bounded while reflecting the natural stochastic variation of real IMU behavior.

Accordingly, the accelerometer and gyroscope biases are modeled as first order Gauss-Markov processes:

$$\begin{aligned}\dot{\mathbf{a}}_{bt} &= -p_{ab} I \mathbf{a}_{bt} + \mathbf{a}_w \\ \dot{\omega}_{bt} &= -p_{\omega b} I \omega_{bt} + \omega_w\end{aligned}$$

where $p_{ab} = 1/\tau_a$ and $p_{\omega b} = 1/\tau_g$ denote the inverse correlation times for the accelerometer and gyroscope bias models, respectively. The terms \mathbf{a}_w and ω_w represent zero mean white noise processes driving the bias evolution.

This bias modeling framework provides a physically consistent and numerically stable representation of inertial sensor drift. By incorporating these bias dynamics, the INS model can accurately describe both the deterministic vehicle motion and the stochastic effects introduced by the inertial sensors.

With the inclusion of bias dynamics, the complete continuous time true state kinematic model of the system is expressed as

$$\begin{aligned}\dot{\mathbf{p}}_{b/O}^n &= \mathbf{v}_{b/O}^n \\ \dot{\mathbf{v}}_{b/O}^n &= R_b^n(\mathbf{q}) (\mathbf{a}_m - \mathbf{a}_{bt} - \mathbf{a}_n) + \mathbf{g}^n \\ \dot{\mathbf{q}} &= \frac{1}{2} \mathbf{q} \otimes (\boldsymbol{\omega}_m - \boldsymbol{\omega}_{bt} - \boldsymbol{\omega}_n) \\ \dot{\mathbf{a}}_{bt} &= -p_{ab} I \mathbf{a}_{bt} + \mathbf{a}_w \\ \dot{\omega}_{bt} &= -p_{\omega b} I \omega_{bt} + \omega_w\end{aligned}$$

This formulation constitutes the complete nonlinear inertial motion model, capturing both the deterministic kinematics of the vehicle and the stochastic processes that characterize real sensor behavior.

The complete INS model previously presented captures the full physical and stochastic behavior of the vehicle, incorporating both deterministic motion and random processes such as measurement noise and bias drift. While this representation accurately reflects real sensor behavior, it is often desirable to

isolate the deterministic part of the dynamics to describe the nominal system behavior. This simplified formulation, known as the nominal state kinematic model, represents how the system would evolve in the absence of stochastic disturbances. It provides a clear and noise-free description of the vehicles motion based purely on inertial measurements, forming the foundation for state propagation in navigation and estimation frameworks.

The nominal model assumes that all noise components are zero, leaving only the essential deterministic relationships between position, velocity, attitude, and sensor biases. In this formulation, the accelerometer and gyroscope measurements are treated as control inputs driving the systems motion. This perspective simplifies the dynamics while preserving the physical structure of the original model, allowing efficient and accurate propagation of the navigation states between external aiding updates.

By removing the stochastic noise terms from the true state equations, the nominal state kinematic model becomes

$$\dot{\mathbf{x}} = f(\mathbf{x}, \mathbf{u}) = \begin{bmatrix} \dot{\mathbf{p}}_{b/O}^n \\ \dot{\mathbf{v}}_{b/O}^n \\ \dot{\mathbf{q}} \\ \dot{\mathbf{a}}_b \\ \dot{\omega}_b \end{bmatrix} = \begin{bmatrix} \mathbf{v}_{b/O}^n \\ R_b^n(\mathbf{q})(\mathbf{a}_m - \mathbf{a}_b) + \mathbf{g}^n \\ \frac{1}{2}\mathbf{q} \otimes (\boldsymbol{\omega}_m - \boldsymbol{\omega}_b) \\ -p_{ab} I \mathbf{a}_b \\ -p_{\omega b} I \boldsymbol{\omega}_b \end{bmatrix} \quad (4)$$

where

$$\mathbf{x} = [\mathbf{p}_{b/O}^n \ \mathbf{v}_{b/O}^n \ \mathbf{q} \ \mathbf{b}_a \ \mathbf{b}_g]^\top, \quad \mathbf{u} = [\mathbf{a}_m \ \boldsymbol{\omega}_m]^\top \quad (5)$$

Here, $\mathbf{p}_{b/O}^n$ and $\mathbf{v}_{b/O}^n$ represent the vehicles position and velocity expressed in the navigation frame, \mathbf{q} denotes the attitude quaternion defining the rotation from the body to the navigation frame, and \mathbf{b}_a and \mathbf{b}_g represent the accelerometer and gyroscope biases, respectively. The parameters $p_a = 1/\tau_a$ and $p_g = 1/\tau_g$ correspond to the inverse correlation times governing the exponential decay rates of the bias dynamics.

This nominal model represents the idealized, noise free motion of the vehicle as inferred from IMU measurements. It captures the deterministic propagation of position, velocity, attitude, and bias under perfect sensor conditions, serving as the baseline dynamic model for integrated navigation systems.

By separating deterministic kinematics from stochastic sensor effects, the INS formulation provides a structured foundation for real-time state estimation. This nominal model is particularly suited for use within an Error State Kalman Filter (ESKF), where it defines the expected system evolution between measurement updates and enables correction using external aiding sources such as GNSS or magnetometers.

Overall, the INS based motion model offers a compact and computationally efficient representation of the ASVs dynamics. While it omits detailed hydrodynamic effects, its balance of simplicity, generality, and compatibility with sensor fusion frameworks makes it a robust and practical choice for autonomous surface vehicle navigation and SLAM applications.

4.5.4 Aiding Measurement Model

The aiding measurements are obtained from a dual antenna GNSS system that provides absolute position and heading information in the NED frame. The two antennas are rigidly mounted on the bow, one on the port side and one on the starboard side, at known offsets from the vessels COG. This setup enables direct estimation of both the vessels global position and its yaw orientation, offering an absolute reference that continuously corrects the drift accumulated in the INS.

Accurate aiding of position and attitude is crucial for maintaining reliable long term navigation performance. While the INS can propagate the motion of the vessel accurately over short time spans, it inevitably accumulates errors due to sensor noise and bias drift. The GNSS updates counteract this drift by providing absolute position corrections, while the dual antenna baseline yields a precise heading measurement that stabilizes the yaw estimate, one of the most critical states for surface vessel guidance and control. Reliable yaw information directly improves trajectory tracking, waypoint following, and steering stability.

For autonomous surface vessels like the microAmpere, roll and pitch angles can be neglected due to the platforms inherently stable catamaran style hull. The dual pontoons provide strong passive stability and a nearly constant deck attitude even under moderate disturbances. Consequently, the vessel motion is effectively modeled in two dimensions, considering surge, sway, and yaw, while the vertical position z is retained from GNSS for completeness. This simplification reduces system complexity without sacrificing accuracy and aligns well with the physical behavior and operational characteristics of small, inherently stable surface craft.

Each GNSS receiver provides a position measurement expressed in the navigation frame as

$$\mathbf{z}_{p,i} = \mathbf{p}_{\text{GNSS}_i/O}^n = \mathbf{p}_{b/O}^n + R_b^n(\mathbf{q}) \mathbf{p}_{\text{GNSS}_i/b}^b + \mathbf{n}_{p,i}, \quad i \in \{1, 2\}$$

where $\mathbf{p}_{b/O}^n$ denotes the body origin (COG) position in the navigation frame, $\mathbf{p}_{\text{GNSS}_i/b}^b$ are the known lever arm offsets of each antenna expressed in the body frame, and $\mathbf{n}_{p,i}$ represents zero mean Gaussian measurement noise. Since both antennas are positioned forward of the COG, this transformation is necessary to align the raw GNSS positions with the INS state representation.

From these two absolute position measurements, the relative vector between the antennas can be expressed in the navigation frame as

$$\mathbf{r}_{1/2}^n = \mathbf{r}_{\text{GNSS}_1/\text{GNSS}_2}^n = \mathbf{p}_{\text{GNSS}_2/O}^n - \mathbf{p}_{\text{GNSS}_1/O}^n$$

This baseline vector defines the spatial relationship between the antennas and allows extraction of the vessels yaw angle ψ through its horizontal components as

$$\psi = \text{atan2}(r_{1/2,y}^n, r_{1/2,x}^n)$$

Here, ψ represents the heading of the line connecting the port and starboard antennas relative to true north. The accuracy of the heading estimate depends on the baseline length, where a longer lateral separation provides greater sensitivity and higher precision. The dual antenna setup therefore yields a robust and drift free heading measurement that complements the attitude propagation performed by the INS.

The two GNSS position measurements and the derived yaw angle are then combined into a single aiding measurement vector that provides both global position and orientation information. The complete GNSS measurement is expressed as

$$\mathbf{z}_{\text{GNSS}} = \begin{bmatrix} \mathbf{p}_{b/O}^n \\ \psi \end{bmatrix} = h(\mathbf{x}) + \mathbf{n}_{\text{GNSS}}$$

where the nonlinear measurement function $h(\mathbf{x})$ relates the navigation state

$$\mathbf{x} = \begin{bmatrix} \mathbf{p}_{b/O}^n & \mathbf{v}_{b/O}^n & \mathbf{q} & \mathbf{b}_a & \mathbf{b}_g \end{bmatrix}^\top$$

to the observed quantities as

$$h(\mathbf{x}) = \begin{bmatrix} \mathbf{p}_{b/O}^n + R_b^n(\mathbf{q}) \mathbf{p}_{\text{GNSS}_1/b}^b \\ \text{atan2}((R_b^n(\mathbf{q}) \mathbf{r}_{1/2}^b)_y, (R_b^n(\mathbf{q}) \mathbf{r}_{1/2}^b)_x) \end{bmatrix} \quad (6)$$

In this expression, the first component corresponds to the transformed GNSS position measurement, while the second term computes the expected yaw angle ψ from the vehicle attitude quaternion \mathbf{q} and the known baseline vector $\mathbf{r}_{1/2}^b$ between antennas in the body frame. The term \mathbf{n}_{GNSS} represents zero mean Gaussian noise, combining both position and heading measurement uncertainties, with covariance $\mathbf{R}_{\text{GNSS}} = \text{diag}(\mathbf{R}_p, \sigma_\psi^2)$.

For small autonomous surface vessels such as the MicroAmpere, the platform demonstrates inherently stable roll and pitch dynamics due to its catamaran style pontoon design. This hydrodynamic stability allows the assumption that roll and pitch angles remain close to zero during nominal operation, effectively reducing the attitude representation to a single dominant rotation about the vertical axis, corresponding to the yaw angle. With this simplification, the GNSS aiding measurements primarily correct horizontal position and heading, while small vertical displacements and tilting motions are

neglected, as they have minimal impact on overall navigation accuracy.

The dual antenna GNSS configuration therefore provides a compact and robust aiding source that supplies both absolute position and heading information in the navigation frame. By constraining the inertial navigation solution with globally referenced measurements, it compensates for the natural drift of the INS and ensures long-term stability, accuracy, and global consistency in the estimated vehicle pose. This makes the aiding model particularly suitable for marine environments where attitude variations are small and precise horizontal localization is of primary importance.

4.5.5 Model Comparison and Selection

The Fossen marine craft kinetic model and the INS kinematic model represent two complementary approaches to describing marine vehicle motion, positioned at opposite ends of the modeling spectrum between kinetic and kinematic formulations. The kinetic, or dynamic, model proposed by Fossen provides a complete physical representation of the vessels motion by explicitly accounting for rigid body dynamics, hydrodynamic effects, added mass, damping, and restoring forces. This yields a high fidelity model suitable for control design, simulation, and performance analysis under varying sea conditions. However, this level of detail comes with considerable complexity, as the model depends on numerous hydrodynamic and inertial parameters that must be carefully identified or experimentally measured.

Conversations with experts in system identification, such as Shenglun Yi from the University of Padova, highlight that while partial parameter identification using recursive or adaptive methods is feasible, obtaining a complete and accurate hydrodynamic parameter set is extremely difficult in practice. Only a few parameters, such as surge damping or longitudinal added mass, can be reliably identified through simplified tests, while the majority require extensive experimentation and highly controlled conditions. As a result, performing full hydrodynamic system identification for small autonomous vessels like the microAmpere would demand substantial effort with limited improvement in navigation performance, making it impractical for this application.

On the other hand, the INS based kinematic model provides a simplified but effective alternative. It represents the vessels motion through inertial sensor data, accelerations and angular rates, without explicitly modeling the underlying dynamics or environmental interactions. This makes it computationally efficient and easily implementable, while still providing accurate short term state propagation suitable for SLAM, navigation, and sensor fusion tasks. The model inherently focuses on position, velocity, and attitude evolution, allowing seamless integration with aiding sensors such as GNSS or vision based systems.

The INS formulation, however, also introduces certain assumptions. One key simplification adopted in this work is that the IMU is positioned close to the vessels COG, effectively neglecting the lever arm effects that arise from angular accelerations acting on sensors offset from the COG. For the microAmpere, this assumption is valid since the IMU is mounted sufficiently close to the COG, minimizing the induced linear acceleration errors and simplifying the overall model structure.

In conclusion, the INS based kinematic model occupies a balanced position between dynamic completeness and modeling simplicity. It omits the complex hydrodynamic modeling of the kinetic approach but retains sufficient fidelity for accurate state estimation and SLAM applications. For small, stable surface vessels, this trade off provides the optimal compromise between accuracy, computational efficiency, and practical feasibility, making the INS based formulation the preferred choice for this thesis.

4.6 Numerical Solvers

4.6.1 Implicit vs Explicit Solvers

Numerical solvers are used to integrate the system dynamics over time, approximating the continuous evolution of a state $\dot{x} = f(t, x)$ through discrete time steps Δt . Depending on how the derivative is evaluated, integration schemes are categorized as either “*explicit*” or “*implicit*”.

In an “*explicit*” method, the state update is computed directly from known quantities at the current time step as such

$$x_{k+1} = x_k + \Delta t f(t_k, x_k)$$

Making it simple and computationally efficient. However, explicit methods are only “*conditionally stable*”, meaning the chosen time step must be sufficiently small to maintain numerical stability. In contrast, an “*implicit*” method evaluates the derivative at the next time step,

$$x_{k+1} = x_k + \Delta t f(t_{k+1}, x_{k+1})$$

This requires solving a nonlinear equation for x_{k+1} , often using iterative techniques such as the Newton-Raphson method. Implicit solvers are generally “*unconditionally stable*”, allowing larger time steps and better handling of stiff systems, but at the cost of significantly higher computational complexity.

Figure 22 illustrates the stability regions of several explicit Runge-Kutta (ERK) methods of increasing order. As the order increases (from ERK1 to ERK4), the stability region expands, allowing larger integration time steps before instability occurs. However, this improvement comes with additional intermediate evaluations of $f(t, x)$ per time step, increasing computational cost. Advanced explicit techniques, such as Adaptive Runge-Kutta methods (e.g. ARK4(5) or Dormand-Prince), further optimize stability by dynamically adjusting the time step, whereas implicit formulations like Gauss-Legendre Runge-Kutta (GLRK) offer superior stability at the expense of computational simplicity.

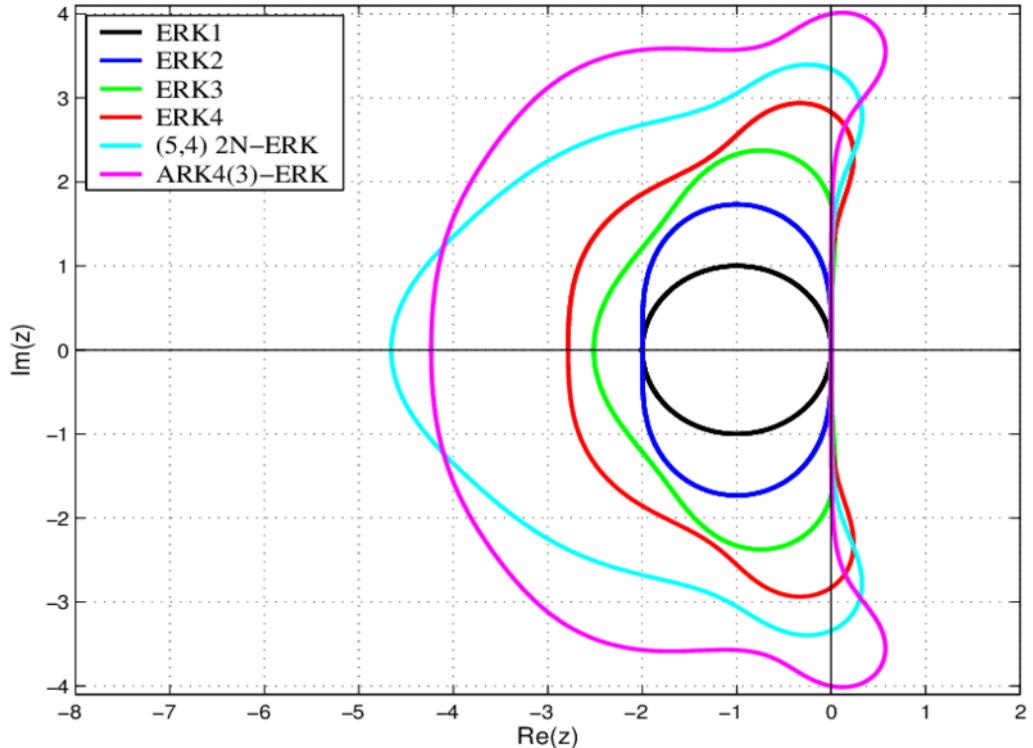


Figure 22: Stability regions of various explicit Runge-Kutta methods. Higher order methods support larger integration time steps before instability occurs, at the cost of increased computational effort. Image taken from research paper on explicit and implicit numerical solvers.^[18]

In real-time navigation and estimation applications, such as the INS model implemented in this work, explicit solvers provide the optimal balance between accuracy and efficiency. Although implicit solvers are preferred for offline simulation or highly stiff systems, their iterative nature and computational load make them impractical for embedded execution. Moreover, any minor numerical drift introduced by the explicit integration scheme is compensated during sensor fusion, where aiding measurements (e.g. GNSS) correct accumulated errors at each update step.

Overall, ERK methods, particularly the 4th order ERK4 formulation, offer an efficient and sufficiently accurate approach for forward state propagation in real-time marine navigation systems.

4.6.2 Newton-Euler Method

The Newton-Euler method represents the simplest and most direct approach to numerically integrating a system's dynamics over time. It forms the foundation for explicit state propagation in real-time navigation and control systems, providing a fast and straightforward way to approximate the continuous evolution of the system state.

The general continuous time system is described by

$$\dot{\mathbf{x}} = f(\mathbf{x}, \mathbf{u})$$

where \mathbf{x} denotes the state vector, \mathbf{u} represents the system inputs, and $f(\mathbf{x}, \mathbf{u})$ defines the continuous time motion model.

Using the explicit or forward Euler discretization, the system is approximated over discrete time intervals Δt as

$$\mathbf{x}_{k+1} = \mathbf{x}_k + \Delta t f(\mathbf{x}_k, \mathbf{u}_k)$$

This update relies only on the current state and input, making the method fully explicit with no need for iteration or matrix inversion. Its direct formulation ensures minimal computational overhead, which is particularly important for real-time embedded execution.

The Newton-Euler method is explicit, first order accurate, and computationally efficient. Its simplicity makes it ideal for systems where high update rates and frequent measurement corrections are available. However, it is conditionally stable, and numerical errors can accumulate for large time steps or rapidly changing dynamics. In aided estimation systems, these effects are typically compensated by sensor fusion corrections such as GNSS or magnetometer updates.

Although higher order explicit methods like Runge-Kutta improve numerical accuracy, they require additional evaluations of $f(\mathbf{x}, \mathbf{u})$ and therefore increase computational cost. For real-time navigation and control, the Newton-Euler method provides an optimal trade-off between computational efficiency and integration accuracy, making it well suited for embedded state propagation in this work.

4.6.3 Runge-Kutta Methods (RK4)

The Runge-Kutta family of methods generalizes the explicit Newton-Euler integration scheme by combining several intermediate evaluations of the system dynamics to achieve higher accuracy. The general continuous time system is written as

$$\dot{\mathbf{x}} = f(\mathbf{x}, \mathbf{u})$$

where \mathbf{x} represents the state vector and \mathbf{u} the control inputs.

The Runge-Kutta approach evaluates $f(\mathbf{x}, \mathbf{u})$ at multiple points within each time step Δt , and combines these intermediate slopes using a set of weighting coefficients defined by the Runge-Kutta formulation. This structure can be compactly represented using the Butcher tableau:

c_1			
c_2	a_{21}		
c_3	a_{31}	a_{32}	
c_4	a_{41}	a_{42}	a_{43}
	b_1	b_2	b_3
			b_4

Each constant defines how intermediate stages are calculated and combined.

For the classical 4th order Explicit Runge-Kutta (ERK4) method, the coefficients are given by:

$$\begin{array}{c|cccc} & 0 & & & \\ \frac{1}{2} & \frac{1}{2} & & & \\ \frac{1}{2} & 0 & \frac{1}{2} & & \\ \hline 1 & 0 & 0 & 1 & \\ & \frac{1}{6} & \frac{1}{3} & \frac{1}{3} & \frac{1}{6} \end{array}$$

Using these coefficients, the RK4 method is computed as

$$\begin{aligned} k_1 &= f(\mathbf{x}_k, \mathbf{u}_k) \\ k_2 &= f(\mathbf{x}_k + \frac{\Delta t}{2} k_1, \mathbf{u}_k) \\ k_3 &= f(\mathbf{x}_k + \frac{\Delta t}{2} k_2, \mathbf{u}_k) \\ k_4 &= f(\mathbf{x}_k + \Delta t k_3, \mathbf{u}_k) \\ \mathbf{x}_{k+1} &= \mathbf{x}_k + \Delta t \sum_{i=1}^4 b_i k_i \end{aligned}$$

Each k_i represents an intermediate slope that estimates the systems rate of change at different points inside the time step. The final update is a weighted average of these slopes, providing a more accurate approximation of the state evolution compared to the single slope Newton-Euler approach.

The RK4 method achieves a global truncation error of $\mathcal{O}(\Delta t^4)$, making it more accurate for smooth nonlinear systems. However, this improvement requires four evaluations of $f(\mathbf{x}, \mathbf{u})$ per integration step, which increases computational cost and memory usage.

The Newton-Euler method can be viewed as the 1st order Explicit Runge-Kutta scheme (ERK1), where only a single evaluation of $f(\mathbf{x}, \mathbf{u})$ is used. While ERK4 provides higher numerical precision, its added complexity offers minimal benefit in real-time embedded systems where aiding sensors, such as GNSS or magnetometers, continuously correct small integration errors.

In practical navigation and estimation systems, especially on embedded platforms, the Newton-Euler (ERK1) method remains the preferred choice due to its low computational load and sufficient accuracy. ERK4 is mainly reserved for offline simulation or analysis, where increased precision justifies the additional computational effort.

5 State Estimation

5.1 Introduction

Estimation theory provides the mathematical foundation for inferring unknown quantities from noisy sensor data. In general, two main paradigms are used in estimation and signal processing, the Bayesian and the Frequency domain approaches. Each framework offers different strengths and assumptions depending on the type of system being analyzed.

The frequency domain approach is commonly applied in classical signal processing, where the primary goal is to extract information from stationary or periodic signals. Techniques such as spectral analysis, transfer function estimation, and classical filtering operate efficiently when system dynamics are linear and time invariant. However, these methods struggle to represent the nonlinear, time varying behavior typical of real-world dynamic systems such as autonomous vehicles or navigation platforms. Their focus on steady state frequency content limits their ability to perform recursive state estimation in systems subject to stochastic disturbances and nonlinear motion.

The Bayesian approach, in contrast, provides a probabilistic framework for representing and updating uncertainty over time. Instead of working in the frequency domain, Bayesian estimation directly models the probability distribution of the system state given all available measurements. This allows the estimation process to incorporate both process dynamics and measurement models, enabling consistent fusion of heterogeneous sensor data. The Bayesian framework forms the foundation for modern state estimation algorithms such as the Kalman Filter, Extended Kalman Filter, and Unscented Kalman Filter.

The core idea of Bayesian estimation is recursive probabilistic inference. At each time step, the state distribution is first “*predicted*” forward in time using the motion model, and then “*corrected*” using the latest aiding sensor measurements. This recursive structure enables efficient real-time updates of the system state without requiring storage of the entire measurement history.

The recursive form of the Bayes filter can be expressed in two steps [10]. First, the “*prediction step*” propagates the state distribution forward in time:

$$p(\mathbf{x}_k | \mathbf{z}_{1:k-1}) = \int p(\mathbf{x}_k | \mathbf{x}_{k-1}) p(\mathbf{x}_{k-1} | \mathbf{z}_{1:k-1}) d\mathbf{x}_{k-1}$$

Then, the “*update step*” incorporates the new measurement:

$$p(\mathbf{x}_k | \mathbf{z}_{1:k}) = \frac{p(\mathbf{z}_k | \mathbf{x}_k) p(\mathbf{x}_k | \mathbf{z}_{1:k-1})}{\int p(\mathbf{z}_k | \mathbf{x}_k) p(\mathbf{x}_k | \mathbf{z}_{1:k-1}) d\mathbf{x}_k}$$

Together, these equations define the recursive Bayesian estimation framework used throughout this thesis.

In summary, the Bayesian paradigm offers a flexible and mathematically consistent method for estimating nonlinear system states under uncertainty, making it particularly suitable for navigation and sensor fusion applications. The frequency domain approach remains valuable for analyzing stationary signals or identifying system dynamics, but it is less effective for recursive, model-based estimation of dynamic processes.

The theoretical concepts and formulations presented in this section are based on the work by Edmund Brekke in “*Fundamentals of Sensor Fusion*” [10]. The discussion and implementations of the Unscented Kalman Filter and its manifold based extension are supported by the research papers [19] and [20].

5.2 Bayes Filter

The Bayes filter provides the probabilistic foundation for recursive state estimation. It describes how a system's state distribution evolves over time as new measurements are received, based on the principles of Bayesian inference. The filter assumes that the process and measurement models form a Markov chain, where the current state \mathbf{x}_k depends only on the previous state \mathbf{x}_{k-1} and control input \mathbf{u}_{k-1} , and each measurement \mathbf{z}_k depends only on the current state.

This structure is known as a Hidden Markov Model, where the underlying system state is hidden and only indirectly observed through noisy sensor measurements. The process and measurement noise are assumed to be white, zero mean, and statistically independent, ensuring that all uncertainty can be modeled probabilistically within the Bayesian framework.

The Bayes filter operates recursively in two stages: prediction and update. The “*prediction step*” propagates the previous state estimate forward in time using the process model:

$$p(\mathbf{x}_k | \mathbf{z}_{1:k-1}) = \int p(\mathbf{x}_k | \mathbf{x}_{k-1}) p(\mathbf{x}_{k-1} | \mathbf{z}_{1:k-1}) d\mathbf{x}_{k-1}$$

This step accounts for system dynamics and process noise, providing a prior belief of the current state before any new measurement is considered.

The “*update step*” then incorporates the new measurement \mathbf{z}_k to refine the estimate:

$$p(\mathbf{x}_k | \mathbf{z}_{1:k}) = \frac{p(\mathbf{z}_k | \mathbf{x}_k) p(\mathbf{x}_k | \mathbf{z}_{1:k-1})}{\int p(\mathbf{z}_k | \mathbf{x}_k) p(\mathbf{x}_k | \mathbf{z}_{1:k-1}) d\mathbf{x}_k}$$

Here, the numerator represents the likelihood of observing the measurement given a state hypothesis, multiplied by the predicted prior, while the denominator ensures normalization of the resulting probability distribution.

This recursive formulation captures the fundamental idea of probabilistic state estimation, where one combines prior knowledge with new information to obtain a posterior estimate. In practice, however, exact computation of these integrals is rarely feasible due to nonlinear process models, non Gaussian noise, and the high dimensionality of the state space. Evaluating the full probability distributions quickly becomes computationally intractable for real-time systems.

For this reason, practical implementations rely on approximate Bayesian filters that make simplifying assumptions to achieve tractable computation. The Kalman Filter assumes linear models with Gaussian noise, while the Extended Kalman Filter and Unscented Kalman Filter extend these ideas to nonlinear systems using local linearization or deterministic sampling. These methods retain the core Bayesian structure but operate with compact parametric representations of uncertainty, typically through the state mean and covariance.

A simplified overview of the process and measurement relationships is shown in Figure 23. The figure illustrates how control inputs \mathbf{u}_k influence the system state \mathbf{x}_k , which in turn generates measurements \mathbf{z}_k through a noisy observation model.

The formulation and structure presented here follow the conventions introduced by Edmund Brekke in “*Fundamentals of Sensor Fusion*” [10], which form the theoretical foundation for the estimation framework implemented in this thesis.

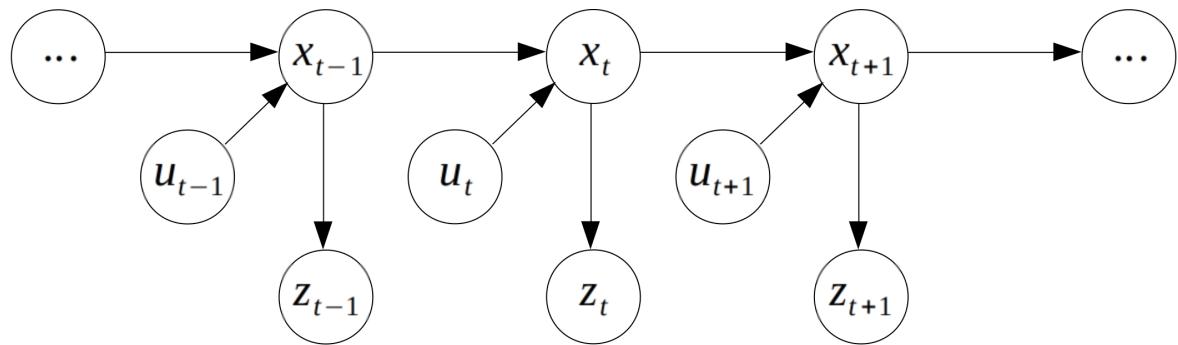


Figure 23: Hidden Markov Model representation of the Bayes filter, showing the relationships between state \mathbf{x} , measurement \mathbf{z} , and control input \mathbf{u} . Image sourced from Lei Mao blog on Bayes Filters.^[21]

5.3 Kalman Filter

Because the full Bayesian filter formulation is computationally intractable for most practical systems, the Kalman filter provides an efficient analytical solution under the assumption of linear system dynamics and Gaussian noise. It represents a special case of the Bayesian recursion where both the process and measurement models are linear, and all uncertainties are modeled as zero mean white Gaussian noise.

The discrete time linear Gaussian system is defined as

$$\mathbf{x}_{k+1} = F\mathbf{x}_k + G\mathbf{u}_k + \mathbf{w}_k, \quad \mathbf{z}_k = H\mathbf{x}_k + \mathbf{v}_k$$

where \mathbf{x}_k is the state vector, \mathbf{u}_k is the control input, and \mathbf{z}_k is the measurement vector. The matrices F , G , and H define the system dynamics, control influence, and measurement relationship, respectively. The process noise $\mathbf{w}_k \sim \mathcal{N}(0, Q)$ and measurement noise $\mathbf{v}_k \sim \mathcal{N}(0, R)$ are assumed white, uncorrelated, and Gaussian distributed.

In practical systems, the continuous time model $\dot{\mathbf{x}} = f(\mathbf{x}, \mathbf{u})$ must be converted into a discrete time form suitable for digital implementation. For a linear system of the form

$$\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u}$$

the equivalent discrete time model is expressed as

$$\mathbf{x}_{k+1} = F\mathbf{x}_k + G\mathbf{u}_k$$

where F and G represent the discrete state transition and control matrices, respectively.

The most accurate way to compute these matrices is through the exact exponential discretization,

$$F = e^{A\Delta t}, \quad G = \int_0^{\Delta t} e^{A\tau} B d\tau$$

where Δt is the sampling period. This formulation comes directly from the analytical solution of the linear time invariant (LTI) differential equation and preserves the systems true dynamics. It is commonly used in simulation and offline analysis where computational cost is less critical.

For real-time embedded systems, a simpler and computationally efficient approximation is usually applied. When the sampling interval Δt is small, the matrix exponential can be approximated using a first order Taylor expansion,

$$F \approx I + A\Delta t, \quad G \approx B\Delta t.$$

This approach, known as the forward Euler or Newton-Euler discretization, provides sufficient accuracy for high-rate estimation and control applications while requiring minimal computation. The resulting discrete system maintains stability and precision as long as Δt remains small relative to the systems dominant time constants.

Higher order integration methods such as ERK schemes can also be used to improve numerical accuracy. These methods evaluate the process function multiple times within each integration step, producing a more accurate approximation of the continuous dynamics. However, in most practical navigation and estimation systems, such increased precision provides limited benefit. Since the filter state is frequently corrected by external sensor updates, the integration error accumulated between prediction steps remains small. Therefore, the forward Euler discretization is generally preferred for real-time embedded implementations due to its simplicity, deterministic execution time, and sufficient numerical accuracy.

The Kalman filter operates recursively through two main stages that alternate over time. The “*prediction step*” advances the state estimate and its associated uncertainty based on the process model, using the latest available control input. This step provides a prior estimate of the system state before any new measurement is processed.

The predicted mean and covariance are computed as

$$\begin{aligned} \hat{\mathbf{x}}_{k|k-1} &= F\hat{\mathbf{x}}_{k-1|k-1} + G\mathbf{u}_{k-1}, \\ P_{k|k-1} &= FP_{k-1|k-1}F^\top + Q, \end{aligned}$$

where $\hat{\mathbf{x}}_{k|k-1}$ and $P_{k|k-1}$ represent the predicted state and covariance, F and G are the discrete system matrices, \mathbf{u}_{k-1} is the control input, and Q is the process noise covariance capturing model uncertainty.

The “*update step*” incorporates the most recent measurement to refine the predicted estimate. This is achieved by computing the Kalman gain, which determines the optimal weighting between the predicted state and the new measurement:

$$\begin{aligned} K &= P_{k|k-1} H^\top (H P_{k|k-1} H^\top + R)^{-1}, \\ \hat{\mathbf{x}}_{k|k} &= \hat{\mathbf{x}}_{k|k-1} + K(\mathbf{z}_k - H\hat{\mathbf{x}}_{k|k-1}), \\ P_{k|k} &= (I - KH)P_{k|k-1}. \end{aligned}$$

Here, H is the measurement model matrix, R is the measurement noise covariance, and \mathbf{z}_k is the received measurement. The innovation term $(\mathbf{z}_k - H\hat{\mathbf{x}}_{k|k-1})$ represents the discrepancy between the predicted measurement and the actual observation. The Kalman gain K determines how much influence the new measurement has relative to the predicted state, optimally balancing model and sensor uncertainty.

By iterating the prediction and update steps, the filter continuously refines the state estimate to provide an optimal and unbiased estimate of the true system state under linear and Gaussian noise assumptions.

The Kalman filter is optimal only under the assumptions of a linear system, Gaussian noise, and white uncorrelated process and measurement disturbances. For nonlinear systems, such as those encountered in INS motion model, these assumptions no longer hold, and extensions like the Extended Kalman Filter (EKF) or Unscented Kalman Filter (UKF) are required. Nonetheless, the Kalman filter remains the conceptual foundation of all modern recursive estimation algorithms.

Figure 24 illustrates the core idea of the Kalman filter, showing how the predicted and measured probability distributions are combined to yield an optimal state estimate with minimum variance.

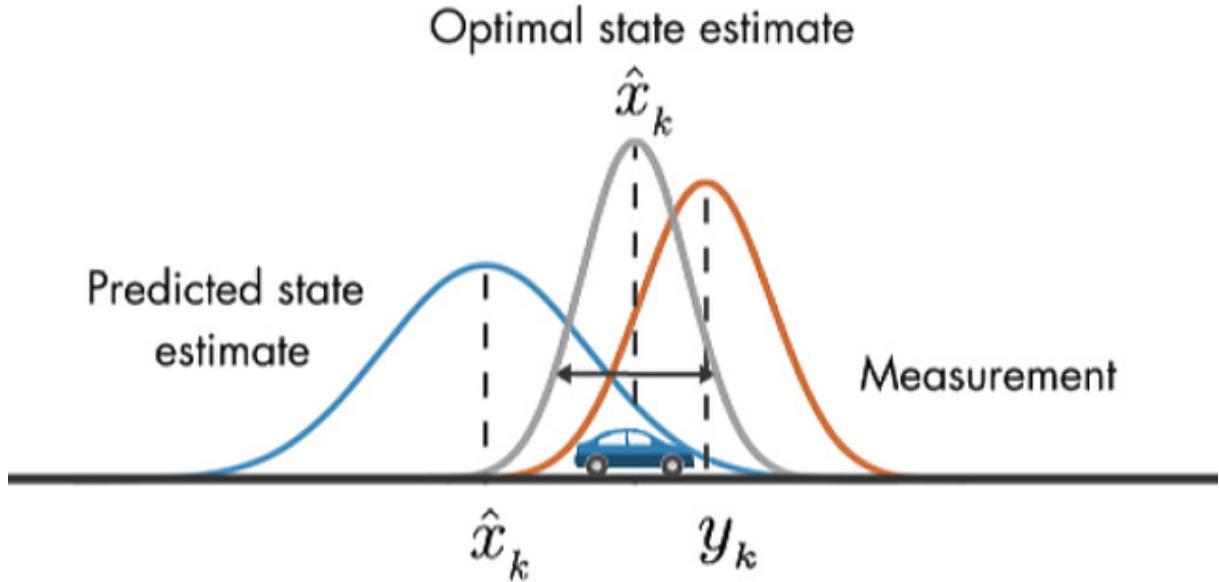


Figure 24: Kalman filter estimation principle combining prediction and measurement distributions to produce the optimal state estimate $\hat{\mathbf{x}}_k$. Image taken from MATLAB video on Kalman Filter.^[22]

5.4 Extended Kalman Filter

The Kalman filter provides optimal state estimation for linear systems under Gaussian noise assumptions. However, most real world systems such as those involving vehicle dynamics, navigation, or orientation estimation are inherently nonlinear. To handle such systems, the Extended Kalman Filter (EKF) generalizes the Kalman filter framework by linearizing the nonlinear system around the current state estimate. This allows the Kalman filter formulation to be applied locally in a linearized form, maintaining recursive estimation in nonlinear environments.

The nonlinear system and measurement models are expressed as

$$\mathbf{x}_{k+1} = f(\mathbf{x}_k, \mathbf{u}_k) + \mathbf{w}_k, \quad \mathbf{z}_k = h(\mathbf{x}_k) + \mathbf{v}_k$$

where \mathbf{x}_k is the system state, \mathbf{u}_k is the control input, and \mathbf{z}_k is the measurement vector. The process and measurement noises \mathbf{w}_k and \mathbf{v}_k are assumed to be zero mean, white, and Gaussian distributed with covariances Q and R , respectively.

To apply the Kalman filter framework, the nonlinear functions $f(\cdot)$ and $h(\cdot)$ must be approximated by their first order Taylor expansion around the current state estimate $\hat{\mathbf{x}}_{k|k-1}$. This yields the linearized system matrices

$$F_k = \left. \frac{\partial f}{\partial \mathbf{x}} \right|_{\hat{\mathbf{x}}_{k|k-1}, \mathbf{u}_k}, \quad H_k = \left. \frac{\partial h}{\partial \mathbf{x}} \right|_{\hat{\mathbf{x}}_{k|k-1}}$$

The Jacobian matrices F_k and H_k describe how small perturbations in the state affect the predicted dynamics and the expected measurements. This linearization step is local and valid only within the vicinity of the current state estimate, as illustrated in Figure 25.

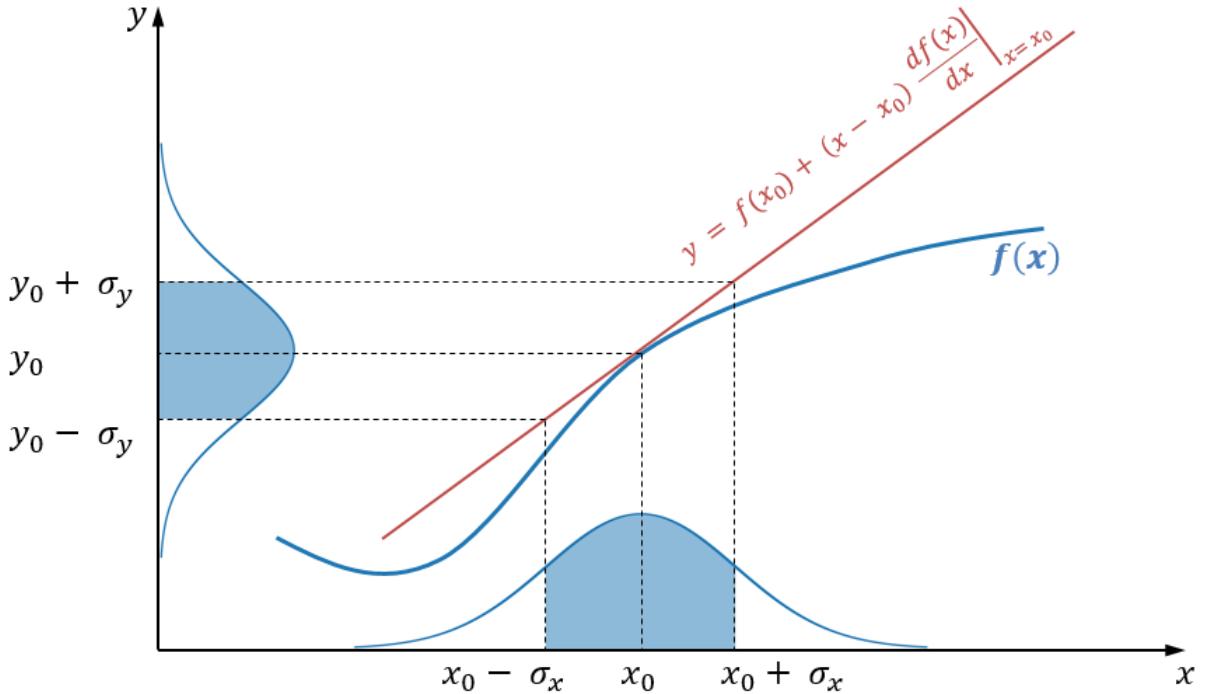


Figure 25: Illustration of local linearization in the Extended Kalman Filter. The nonlinear function $f(x)$ is approximated by its tangent around the current estimate, allowing local linear propagation. Image taken from Alex Becker's "Introduction to Kalman Filtering".^[23]

An accurate discretization of the continuous time dynamics is essential to ensure consistent state propagation. In this work, the continuous model $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$ is discretized using the "Zero-Order Hold (ZOH)" method, which assumes that the control input remains constant within each sampling interval $[k\Delta t, (k+1)\Delta t]$.

The resulting discrete time model is given by

$$\mathbf{x}_{k+1} = F_k \mathbf{x}_k + G_k \mathbf{u}_k$$

where

$$F_k = e^{A\Delta t}, \quad G_k = \int_0^{\Delta t} e^{A\tau} B d\tau$$

This can be computed compactly using the augmented matrix exponential

$$e^{\begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix} \Delta t} = \begin{bmatrix} F_k & G_k \\ 0 & I \end{bmatrix} \quad (7)$$

The ZOH method preserves both the stability and dynamic behavior of the original continuous time system exactly for LTI models. It provides a numerically stable and accurate discretization, making it particularly suitable for embedded real-time implementations.

In digital systems, the matrix exponential $e^{A\Delta t}$ cannot be computed analytically and must be approximated numerically. A reliable and efficient method is the “*Padé approximation with scaling and squaring*”, which provides a good balance between computational speed and numerical accuracy.

The Padé approximation expresses the exponential as a rational function:

$$e^A \approx \left(I - \frac{A}{2} + \frac{A^2}{12} - \frac{A^3}{120} + \dots \right)^{-1} \left(I + \frac{A}{2} + \frac{A^2}{12} + \frac{A^3}{120} + \dots \right) \quad (8)$$

where higher order terms improve accuracy at the cost of computation. To avoid instability for large $\|A\|$, the matrix is first scaled down by a power of two, computed as

$$A_s = \frac{A}{2^s} \quad (9)$$

where s is chosen such that $\|A_s\|$ is below a defined threshold. The exponential of A_s is then computed using the Padé rational form, and the final result is obtained by repeated squaring:

$$e^A = (e^{A_s})^{2^s} \quad (10)$$

This method ensures stable and efficient computation of matrix exponentials even for high order systems, making it practical for embedded discretization of continuous time models.

When using nonlinear dynamics, the continuous time system is generally expressed as

$$\dot{\mathbf{x}} = f(\mathbf{x}, \mathbf{u})$$

To implement the continuous time dynamics $\dot{\mathbf{x}} = f(\mathbf{x}, \mathbf{u})$ on a digital system, the time derivative must be approximated numerically. This is achieved through time discretization, where the continuous evolution of the state is expressed as a difference equation over discrete sampling intervals.

One of the simplest and most widely used discretization schemes is the forward (Newton-Euler) method.

$$\mathbf{x}_{k+1} \approx \mathbf{x}_k + \Delta t f(\mathbf{x}_k, \mathbf{u}_k)$$

In this discrete form, the function $f(\mathbf{x}, \mathbf{u})$ used within the EKF no longer represents the instantaneous derivative $\dot{\mathbf{x}}$, but rather the integrated or “*propagated*” state over one time step. For notational simplicity, many EKF formulations redefine the discrete transition function as

$$f_d(\mathbf{x}_k, \mathbf{u}_k) = \mathbf{x}_k + \Delta t f(\mathbf{x}_k, \mathbf{u}_k)$$

where $f_d(\cdot)$ implicitly includes the time step integration. This convention is used throughout this work to maintain concise notation while ensuring consistency with the underlying continuous time model.

The EKF prediction and update steps then follow directly from the linearized system. During the “*prediction step*”, the nonlinear process model is propagated as

$$\hat{\mathbf{x}}_{k|k-1} = f_d(\hat{\mathbf{x}}_{k-1|k-1}, \mathbf{u}_{k-1})$$

and the covariance prediction is computed using the linearized Jacobian:

$$P_{k|k-1} = F_k P_{k-1|k-1} F_k^\top + Q$$

The “*update step*” corrects the predicted estimate using the measurement model:

$$\begin{aligned} K &= P_{k|k-1} H^\top (H P_{k|k-1} H^\top + R)^{-1} \\ \hat{\mathbf{x}}_{k|k} &= \hat{\mathbf{x}}_{k|k-1} + K[\mathbf{z}_k - h(\hat{\mathbf{x}}_{k|k-1})] \\ P_{k|k} &= (I - K H) P_{k|k-1} \end{aligned}$$

After each update, attitude states represented by quaternions are normalized to maintain a valid unit quaternion. This step is crucial in inertial navigation systems, as numerical drift in the quaternion magnitude can lead to significant orientation errors over time. The normalization ensures that the quaternion remains on the unit hypersphere, preserving a valid rotation representation:

$$\mathbf{q}_{k|k} \leftarrow \frac{\mathbf{q}_{k|k}}{\|\mathbf{q}_{k|k}\|}.$$

Here, $\mathbf{q}_{k|k}$ denotes the updated quaternion after the correction step, and $\|\mathbf{q}_{k|k}\|$ is its Euclidean norm. This operation enforces the constraint $\|\mathbf{q}\| = 1$, guaranteeing that the quaternion continues to represent a proper rotation without scale distortion.

The EKF provides a computationally efficient framework for nonlinear estimation. However, it relies heavily on the validity of local linearization, which may lead to divergence if the system dynamics are strongly nonlinear or the initial state uncertainty is large. In particular, when dealing with orientation states represented by quaternions or with highly coupled nonlinear motion models such as those in INS, the EKFs 1st order approximation may fail to capture the true system behavior.

Despite these limitations, the EKF remains widely used in embedded navigation systems due to its simplicity, real-time efficiency, and well understood theoretical foundation. An important improvement to the EKF is the Error State Kalman Filter (ESKF), which reformulates the estimation problem by linearizing not around the full nominal states but around the small error states instead. These error variables tend to behave more linearly and remain bounded, resulting in improved numerical stability and consistency, especially for systems with rotational dynamics such as those using quaternions. Essentially, the ESKF provides a smarter way to linearize highly nonlinear systems by focusing on the deviation from the nominal trajectory rather than the absolute state itself. For applications requiring even higher robustness to nonlinearity, alternative methods such as the Unscented Kalman Filter (UKF) or manifold based approaches can provide improved performance at the cost of increased computational complexity.

5.5 Error State Kalman Filter

5.5.1 Concept and Motivation

While the Extended Kalman Filter provides a powerful framework for nonlinear estimation, it still linearizes directly around the full nominal state. For systems such as the INS, where the state includes both position, velocity, and attitude (represented by quaternions), this approach can lead to numerical inconsistencies, unstable orientation updates, and loss of orthogonality in rotation representations.

To address these issues, the Error State Kalman Filter (ESKF) reformulates the estimation problem by separating the system into two parts, a “*nominal state*” \mathbf{x} and a “*small error state*” $\delta\mathbf{x}$. The nominal state follows the nonlinear dynamics of the INS, while the error state captures small deviations from this nominal trajectory. This approach allows the filter to maintain accurate nonlinear propagation of the nominal state while estimating only small, approximately linear error quantities.

5.5.2 Error State Formulation

The nominal INS model is already defined earlier in this work in Equation 4 with nominal states Equation 5.

From the nominal INS model, the corresponding error state INS model is obtained by linearizing the nonlinear equations around the current nominal trajectory. This process isolates the small deviations between the true state and the estimated nominal state, allowing the linearized error dynamics to be described by a 1st order system. Following the formulation by Edmund Brekke in “*Fundamentals of Sensor Fusion*” [10], the continuous time linear time varying (LTV) error state dynamics are expressed as

$$\delta\mathbf{x} = \begin{bmatrix} \delta\mathbf{p}_{b/O}^n \\ \delta\mathbf{v}_{b/O}^n \\ \delta\theta \\ \delta\mathbf{a}_b \\ \delta\omega_b \end{bmatrix}$$

$$\dot{\delta\mathbf{x}} = A(\mathbf{x})\delta\mathbf{x} + G(\mathbf{x})\mathbf{n}$$

Here, $\delta\mathbf{x}$ represents the small deviation between the true and nominal state vectors, and \mathbf{n} is the continuous time process noise, typically modeled as zero mean Gaussian white noise. The system matrices $A(\mathbf{x})$ and $G(\mathbf{x})$ depend on the current nominal state and describe how these small deviations evolve in time.

The attitude error is represented using the small angle quaternion approximation, which linearizes the rotation error between the estimated and true attitudes. This approximation holds for small orientation deviations and is expressed as

$$\delta\mathbf{q} \approx \begin{bmatrix} 1 \\ \frac{\delta\theta}{2} \end{bmatrix}$$

This relation ensures that the attitude error $\delta\theta$ can be treated as a 3D vector in the local tangent space of the unit quaternion manifold, simplifying the linearization of rotational dynamics while maintaining geometric consistency.

5.5.3 Process Model and Jacobians

The linearized system matrices $A(\mathbf{x})$ and $G(\mathbf{x})$ are given by

$$A(\mathbf{x}) = \begin{bmatrix} \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -R_b^n(\mathbf{q})[\mathbf{a}_m - \mathbf{a}_b]_{\times} & -R_b^n(\mathbf{q}) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -[\boldsymbol{\omega}_m - \boldsymbol{\omega}_b]_{\times} & \mathbf{0} & -\mathbf{I} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathbf{p}_{ab}\mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathbf{p}_{\omega b}\mathbf{I} \end{bmatrix}, \quad G(\mathbf{x}) = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ -\mathbf{R}_b^n(\mathbf{q}) & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -\mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} \end{bmatrix}$$

In this representation, $R_b^n(\mathbf{q})$ is the rotation matrix from the body frame to the navigation frame obtained from the current quaternion estimate \mathbf{q} . The notation $[\cdot]_{\times}$ denotes the skew-symmetric operator, which converts a 3D vector into a matrix form representing the cross product.

The matrix $A(\mathbf{x})$ captures how small perturbations in position, velocity, attitude, and sensor biases evolve over time. The upper rows model the kinematic coupling between position and velocity, while the middle rows describe how accelerometer and gyroscope bias errors influence the navigation solution through attitude and velocity. The matrix $G(\mathbf{x})$ maps the continuous time process noise into the error dynamics, distributing the effect of IMU sensor noise, bias noise, and other stochastic disturbances into their corresponding error states.

This linearized form provides the foundation for propagating the error state covariance in the ESKF. It allows the nonlinear nominal INS to run independently, while the filter maintains a linear and well behaved model of how uncertainty evolves, a key advantage that improves both numerical stability and estimation consistency.

5.5.4 Discretization of Error Dynamics

Discretization of the continuous error dynamics is critical for digital implementation. In theory, the discrete time transition matrix is given by

$$F_k = e^{A(\mathbf{x})\Delta t}, \quad Q_k = \int_0^{\Delta t} e^{A(\mathbf{x})\tau} G(\mathbf{x}) Q G(\mathbf{x})^\top e^{A(\mathbf{x})^\top \tau} d\tau$$

However, directly computing these expressions is infeasible in real-time embedded systems due to the computational cost of evaluating the matrix exponential and noise covariance integral at every time step. Several discretization strategies exist to approximate this process efficiently while maintaining filter stability and accuracy.

The “*Cayley’Hamilton series*” offers an exact polynomial expansion of the matrix exponential using the characteristic polynomial of $A(\mathbf{x})$. This approach is mathematically elegant and often employed for symbolic or algebraic analysis, particularly in time varying systems where analytic expressions for $A(\mathbf{x})$ are desired. However, in practical implementations, this method is highly inefficient. The repeated evaluation of high order polynomial terms and matrix powers makes it unsuitable for real-time embedded estimation where deterministic computational performance is required.

A more practical and robust method is the ZOH discretization, introduced previously in Equation 7. The ZOH assumes that both the control inputs and process noise remain constant within each sampling interval. Under this assumption, it preserves the exact dynamics of linear time invariant systems and ensures numerical stability of the discrete time model. In practice, the ZOH discretization is implemented using an augmented matrix exponential containing the continuous time system matrices $A(\mathbf{x})$ and $G(\mathbf{x})$, which yields the discrete time transition matrix F_k and the process noise covariance Q_k . This approach provides an accurate mapping from the continuous error dynamics to their discrete time equivalent while avoiding the instability of low order approximations.

While theoretically precise, the ZOH can still be computationally demanding for real-time embedded systems, especially when matrix dimensions are large or $A(\mathbf{x})$ varies significantly over time. To address this, the “*Padé approximation with scaling and squaring*”, defined in Equations 8-10, is used in this work to efficiently approximate the matrix exponential. The Padé method offers an excellent trade off between numerical accuracy and computational speed. It remains stable even for ill conditioned matrices and maintains consistent results across varying sampling periods. This combination of ZOH based discretization and Padé based exponential approximation forms the foundation for robust real-time implementation of the ESKF.

Finally, it is important to emphasize that crude 1st order approximations, such as assuming $F_k \approx I + A(\mathbf{x})\Delta t$, should be strictly avoided. Although computationally inexpensive, these approximations can introduce significant numerical errors, degrade covariance consistency, and cause filter divergence under high dynamic motion or long sampling intervals.

5.5.5 Measurement Model and Jacobian

Before defining the discrete time ESKF stages, the measurement Jacobian structure must be introduced. The total Jacobian used in the update step is expressed as

$$H = H_{\mathbf{x}} X_{\delta \mathbf{x}}.$$

Here, $H_{\mathbf{x}}$ is the Jacobian of the measurement function with respect to the nominal state, obtained by linearizing the measurement model in Equation 6. Since this measurement function provides position and yaw observations, $H_{\mathbf{x}}$ captures how small variations in the nominal position and attitude affect the predicted GNSS position and yaw outputs.

However, because the ESKF performs estimation in the error state domain, the measurement sensitivity must be expressed with respect to the error state $\delta\mathbf{x}$. The matrix $X_{\delta\mathbf{x}}$ provides this mapping by relating perturbations in the nominal state to those in the error state. It has a fixed block diagonal structure:

$$X_{\delta\mathbf{x}} = \begin{bmatrix} I_6 & 0 & 0 \\ 0 & Q_{\delta\theta} & 0 \\ 0 & 0 & I_6 \end{bmatrix}$$

where $Q_{\delta\theta}$ relates small attitude errors $\delta\theta$ to quaternion perturbations and is derived from the differential of $\mathbf{q} \otimes \delta\mathbf{q}$ evaluated at $\delta\theta = 0$, where $\mathbf{q} = [\eta \epsilon_1 \epsilon_2 \epsilon_3]^T$:

$$Q_{\delta\theta} \approx \frac{1}{2} \begin{bmatrix} -\epsilon_1 & -\epsilon_2 & -\epsilon_3 \\ \eta & -\epsilon_3 & \epsilon_2 \\ \epsilon_3 & \eta & -\epsilon_1 \\ -\epsilon_2 & \epsilon_1 & \eta \end{bmatrix}.$$

Together, H_x and $X_{\delta x}$ form the complete measurement Jacobian $H = H_x X_{\delta x}$, which links measurement residuals to the linearized error state and enables consistent correction of both translational and rotational components during the update phase of the ESKF.

5.5.6 Filter Algorithm

The ESKF estimation process consists of several distinct stages, each responsible for different aspects of the nominal and error state propagation. The filter propagates the nonlinear nominal state using the INS equations, while maintaining a linearized covariance model of the small error dynamics. Unlike the EKF, the ESKF does not directly propagate the error-state mean, as it is always reset to zero after each correction. Instead, only the nominal state and the error covariance are propagated during the prediction stage.

Prediction step:

At the beginning of each iteration, the error state mean is initialized to zero:

$$\delta\hat{\mathbf{x}}_{k|k-1} = \mathbf{0}$$

The nominal state is propagated through the nonlinear discrete time INS process model:

$$\hat{\mathbf{x}}_{k|k-1} = f_d(\hat{\mathbf{x}}_{k-1|k-1}, \mathbf{u}_{k-1})$$

Simultaneously, the covariance of the error state is propagated using the linearized system matrices F_k and Q_k , derived from the Jacobians of the continuous time error dynamics:

$$P_{k|k-1} = F_k P_{k-1|k-1} F_k^\top + Q_k$$

This step predicts the evolution of uncertainty around the nominal trajectory, incorporating the effects of process and sensor noise. The mean of the error state remains zero since the nominal state already represents the best estimate of the system.

Update step:

When a new measurement \mathbf{z}_k becomes available, the error state is updated using the linearized measurement model:

$$\begin{aligned} K &= P_{k|k-1} H^\top (H P_{k|k-1} H^\top + R)^{-1} \\ \delta\hat{\mathbf{x}}_{k|k} &= K[\mathbf{z}_k - h(\hat{\mathbf{x}}_{k|k-1})] \\ P_{k|k} &= (I - KH)P_{k|k-1} \end{aligned}$$

Here, H denotes the Jacobian of the measurement function with respect to the error state. The correction term $\delta\hat{\mathbf{x}}_{k|k}$ represents a small estimated deviation between the predicted nominal state and the measurement-derived state.

Injection step:

After computing the correction, it is injected back into the nominal state using the nonlinear composition operator \oplus , which updates the nominal quantities according to the estimated error:

$$\hat{\mathbf{x}}_{k|k} \leftarrow \hat{\mathbf{x}}_{k|k-1} \oplus \delta\hat{\mathbf{x}}_{k|k}$$

For translational states, this operation corresponds to vector addition, whereas for attitude states represented by quaternions, it involves quaternion multiplication between the nominal quaternion and the small-angle correction quaternion derived from $\delta\theta$.

Reset step:

Following the injection, the error state mean is reset to zero:

$$\delta\hat{\mathbf{x}}_{k|k} \leftarrow 0$$

To maintain a consistent uncertainty representation after the nominal state update, the covariance must be transformed using the reset Jacobian G :

$$P_{k|k} \leftarrow GP_{k|k}G^\top, \quad G = \begin{bmatrix} I_6 & 0 & 0 \\ 0 & I - [\frac{\delta\theta}{2}]_\times & 0 \\ 0 & 0 & I_6 \end{bmatrix}$$

Here, $S = [\frac{\delta\theta}{2}]_\times$ denotes the small angle correction matrix related to the attitude quaternion update. This transformation ensures that the new error state is correctly defined relative to the updated nominal trajectory and that the covariance remains properly aligned, preventing the accumulation of linearization errors over time.

Quaternion normalization:

Finally, quaternion normalization is applied to maintain a valid unit quaternion representation:

$$\mathbf{q}_{k|k} \leftarrow \frac{\mathbf{q}_{k|k}}{\|\mathbf{q}_{k|k}\|}$$

This step guarantees that the attitude estimate remains on the unit hypersphere, preventing numerical drift and preserving a consistent rotation representation.

The combined structure of the prediction, update, injection, and reset stages allows the ESKF to maintain high numerical stability and consistency. By separating the nonlinear nominal state from the small linear error state, the filter achieves superior accuracy and robustness for inertial navigation systems involving rotational dynamics compared to the standard EKF formulation.

5.5.7 EKF vs ESKF

Overall, the ESKF provides significantly improved numerical stability, robustness, and estimation consistency compared to the EKF. By estimating only small, locally linear error quantities while maintaining a nonlinear nominal state propagation, it achieves an effective balance between accuracy and computational efficiency. This separation ensures that orientation updates remain well conditioned and free from singularities, even during highly dynamic motion. As a result, the ESKF has become the preferred framework for modern inertial navigation and sensor fusion systems, particularly in applications involving rotational dynamics and quaternion based attitude estimation.

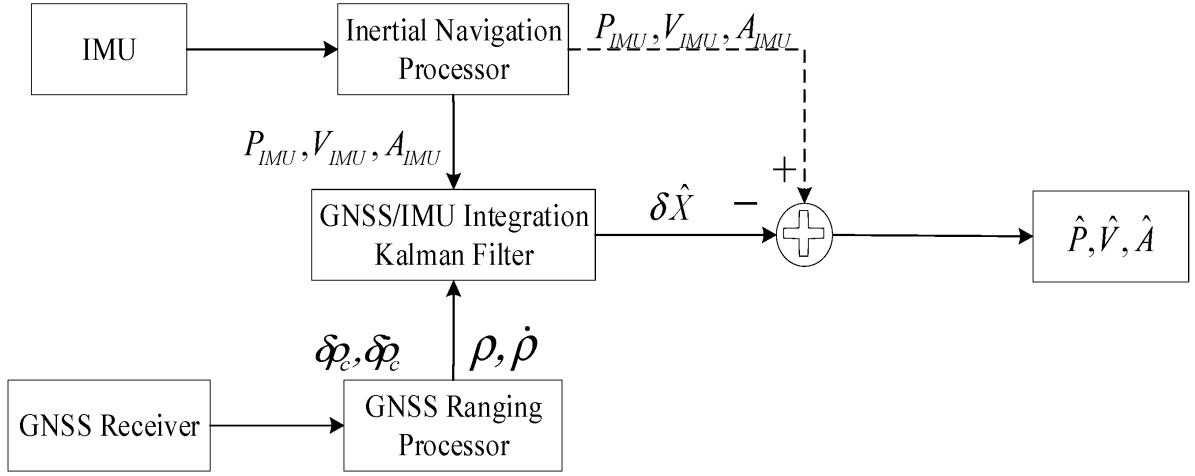


Figure 26: Functional overview of the ESKF architecture. The filter propagates and estimates the small error state based on IMU data, updates it using GNSS measurements, and then injects the correction into the nominal state. After injection, the error state is reset to zero. This loop ensures stable and consistent estimation by combining fast IMU propagation with slower GNSS updates. Image taken from a paper on the Error State Kalman Filter.^[24]

5.6 Unscented Kalman Filter on Manifolds

5.6.1 Concept and Motivation

Up to this point, all Kalman Filter variants, including the EKF and ESKF, approximate nonlinear systems locally through 1st order linearization. Although effective for mildly nonlinear dynamics, these methods rely on Jacobians, which introduce errors when the system exhibits strong nonlinearity or discontinuous dynamics. An alternative approach to local linearization is sampling based nonlinear approximation. A well known technique of this class is the “*Unscented Transform (UT)*”, which represents a Gaussian distribution using a small, deterministically chosen set of sample points (called sigma points) and then propagates them through the nonlinear function to capture the transformed mean and covariance up to the third order for Gaussian inputs [19].

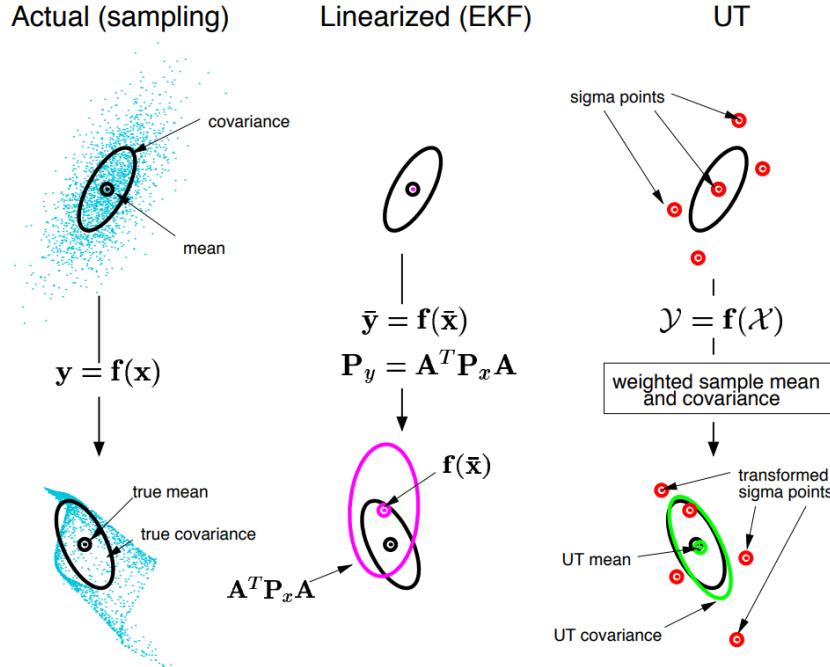


Figure 27: Comparison of different methods for nonlinear mean and covariance propagation. The figure contrasts (a) direct sampling of the true nonlinear distribution, (b) linearization using the Extended Kalman Filter (EKF), and (c) the Unscented Transform (UT), which captures the mean and covariance more accurately without linearization. Image taken from Unscented Kalman Filter paper.^[19]

5.6.2 Unscented Kalman Filter

The Unscented Transform is defined by generating a set of $2L + 1$ sigma points from the prior mean \mathbf{x} and covariance P , where L denotes the dimensionality of the state vector. Each sigma point represents a deterministic sample capturing the local mean and covariance structure of the state distribution, ensuring accurate nonlinear propagation up to the second order for any nonlinearity.

$$\begin{aligned} \chi_0 &= \mathbf{x} \\ \chi_i &= \mathbf{x} + (\sqrt{(L + \lambda)P_i}) \quad i = 1, \dots, L \\ \chi_i &= \mathbf{x} - (\sqrt{(L + \lambda)P_{i-L}}) \quad i = L + 1, \dots, 2L \\ \lambda &= \alpha^2(L + \kappa) - L \end{aligned}$$

where λ is a scaling parameter controlling the spread of the sigma points. Each sigma point is assigned associated weights W_i^m and W_i^c for mean and covariance reconstruction. These weights ensure that both the central and surrounding sigma points contribute correctly to the nonlinear mean and covariance

propagation according to their statistical significance. The weights are defined as

$$\begin{aligned} W_0^m &= \frac{\lambda}{L + \lambda} \\ W_0^c &= \frac{\lambda}{L + \lambda} + (1 - \alpha^2 + \beta) \\ W_i^m = W_i^c &= \frac{1}{2(L + \lambda)} \quad i = 1, \dots, 2L \end{aligned}$$

The parameters α , β , and κ govern the spread, scaling, and higher order accuracy of the sigma point distribution. According to the original Unscented Kalman Filter formulation by Julier and Uhlmann [19], these parameters should be tuned to balance numerical stability and approximation accuracy. The parameter α determines the overall spread of the sigma points around the mean, it is typically chosen as a small positive value ($10^{-3} \leq \alpha \leq 1$), with smaller values resulting in sigma points closer to the mean and larger values increasing the nonlinear coverage at the cost of potential numerical instability. The parameter κ acts as a secondary scaling term that adjusts the effective spread of the sigma points; it is often set to 0 for simplicity or $3 - L$ to guarantee positive semi definiteness of the covariance. The parameter β encodes prior knowledge of the underlying distribution, for Gaussian distributions, $\beta = 2$ is recommended, as it ensures optimal 2nd order accuracy in the covariance reconstruction.

Together, these parameters define how the sigma points are positioned and weighted to best approximate the true nonlinear mean and covariance transformation while maintaining numerical stability across a wide range of system non-linearities.

Using these sigma points and weights, the propagated mean and covariance are computed as

$$\begin{aligned} \hat{\mathbf{x}}^- &= \sum_{i=0}^{2L} W_i^m f_d(\chi_i, \mathbf{u}) \\ P^- &= \sum_{i=0}^{2L} W_i^c [f_d(\chi_i, \mathbf{u}) - \hat{\mathbf{x}}^-] [f_d(\chi_i, \mathbf{u}) - \hat{\mathbf{x}}^-]^\top \end{aligned}$$

This process effectively replaces linearization and analytical Jacobian computation with a deterministic sampling of the nonlinear function. A similar procedure is applied during the measurement update step, where the sigma points are propagated through the nonlinear measurement model $h(\mathbf{x})$ to compute the predicted observation mean and covariance.

During the measurement update step, each predicted sigma point χ_i^- is passed through the nonlinear measurement model $h(\mathbf{x})$ (for example, the GNSS measurement model in Equation 6) to get predicted measurement samples:

$$\mathcal{Z}_i = h(\chi_i^-)$$

The predicted measurement sample mean is computed as

$$\hat{\mathbf{z}} = \sum_{i=0}^{2L} W_i^m \mathcal{Z}_i$$

The corresponding innovation covariance and cross covariance are then obtained as

$$\begin{aligned} S &= \sum_{i=0}^{2L} W_i^c (\mathcal{Z}_i - \hat{\mathbf{z}})(\mathcal{Z}_i - \hat{\mathbf{z}})^\top + R \\ P_{xz} &= \sum_{i=0}^{2L} W_i^c (\chi_i^- - \hat{\mathbf{x}}^-)(\mathcal{Z}_i - \hat{\mathbf{z}})^\top, \end{aligned}$$

where R is the measurement noise covariance matrix.

The Kalman gain is then computed as

$$K = P_{xz} S^{-1}.$$

The state and covariance are updated according to

$$\hat{\mathbf{x}} = \hat{\mathbf{x}}^- + K(\mathbf{z} - \hat{\mathbf{z}}),$$

$$P = P^- - KSK^\top.$$

Finally, the quaternion component of the state is normalized to maintain unit length and ensure a valid rotation representation:

$$\mathbf{q} \leftarrow \frac{\mathbf{q}}{\|\mathbf{q}\|}.$$

This completes the classical UKF update stage, where non-linearities are handled through sigma point sampling rather than analytic Jacobian linearization.

5.6.3 Manifold Operators

The classical UKF assumes that all system states evolve in Euclidean space \mathbb{R}^n . However, many real world systems and motion models, such as INS motion model 4, include quantities that lie on nonlinear manifolds. A common example is the attitude represented by unit quaternions $\mathbf{q} \in \mathbb{S}^3$, which form a curved space where addition and averaging are not globally valid operations. Applying the standard UKF directly to such states can lead to inconsistencies, since linear updates may move the estimate off the manifolds surface.

The “*Unscented Kalman Filter on Manifolds (UKF-M)*” [20] extends the standard UKF by performing all statistical operations within the tangent space of the manifold. The Unscented Transform is carried out locally in this Euclidean tangent space, and the resulting sigma points are mapped back to the manifold using the exponential and logarithmic maps introduced in Equations 1 and 2. This approach ensures that all propagated and updated states remain geometrically consistent.

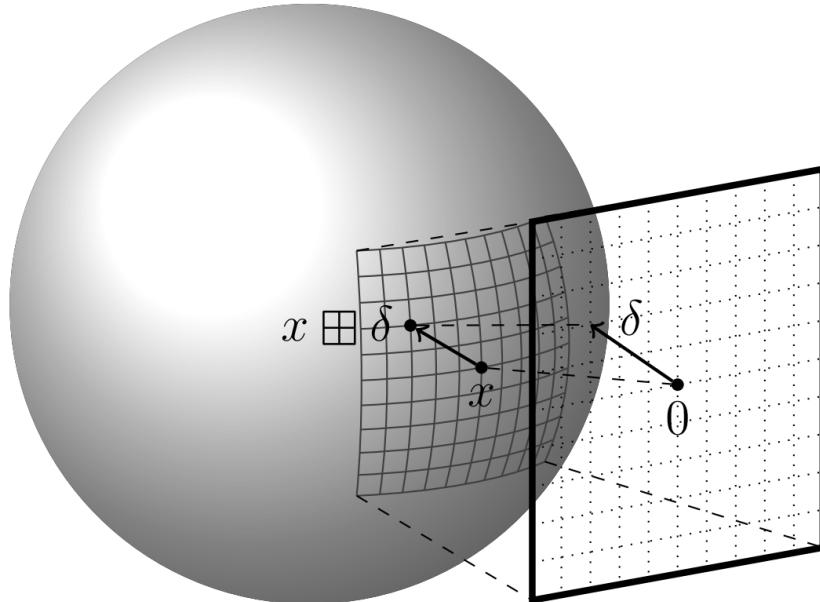


Figure 28: Mapping a local neighborhood in the state space (here: on the unit sphere \mathbb{S}^2) into \mathbb{R}^n (here: the plane) allows for the use of standard sensor fusion algorithms without explicitly encoding the global topological structure. Image and figure text taken from UKF-M paper.^[20]

To enable consistent state updates on a nonlinear manifold, the UKF-M introduces two generalized operators, \boxplus and \boxminus , which replace standard addition and subtraction. They relate the manifold to its tangent space:

$$\mathbf{x}' = \mathbf{x} \boxplus \boldsymbol{\delta}$$

$$\boldsymbol{\delta} = \mathbf{y} \boxminus \mathbf{x}$$

Note that δ is *not* the error state as used in the ESKF formulation. Instead, δ represents a small perturbation defined in the local tangent space at \mathbf{x} , and can take any arbitrary value within that space.

Here, \mathbf{x} and \mathbf{y} are elements on the manifold \mathcal{M} , while $\delta \in T_{\mathbf{x}}\mathcal{M}$ is the corresponding vector in the tangent space. The \boxplus operator applies a perturbation from the tangent space to move the state along the manifold, producing an updated manifold element \mathbf{x}' . Conversely, the \boxminus operator computes the minimal difference between two manifold states by mapping that displacement back into the tangent space.

Together, these two operators define a consistent way to move back and forth between the manifold \mathcal{M} and its tangent space $T_{\mathbf{x}}\mathcal{M}$, enabling the UKF-M to perform all statistical operations (eks, mean and covariance propagation) in a locally Euclidean space while keeping the final state representation on the true manifold.

In Euclidean space, these operators reduce to standard vector addition and subtraction, i.e. $\mathbf{x}' = \mathbf{x} \boxplus \delta = \mathbf{x} + \delta$ and $\delta = \mathbf{y} \boxminus \mathbf{x} = \mathbf{y} - \mathbf{x}$. However, on curved manifolds like \mathbb{S}^3 , directly adding vectors can move the state off the manifold, violating its geometric constraints. To address this, the \boxplus and \boxminus operators rely on the exponential and logarithmic maps that move between the manifold and its tangent space:

$$\begin{aligned}\mathbf{x} \boxplus \delta &= \exp(\delta) \circ \mathbf{x} && (\text{Tangent} \rightarrow \text{Manifold}) \\ \mathbf{y} \boxminus \mathbf{x} &= \log(\mathbf{y} \circ \mathbf{x}^{-1}) && (\text{Tangent} \leftarrow \text{Manifold})\end{aligned}$$

where \circ denotes the group composition operator. The exponential map projects a tangent space perturbation onto the manifold, while the logarithmic map computes the smallest displacement between two manifold elements within the tangent space.

These mappings are directly analogous to the Lie group relations (See Equations 1-2 and Figure 18)

$$\begin{aligned}R &= \exp(\boldsymbol{\omega}^\times) && (\text{Tangent} \rightarrow \text{Manifold}) \\ \boldsymbol{\omega} &= \log(R) && (\text{Tangent} \leftarrow \text{Manifold})\end{aligned}$$

which connect a rotation matrix $R \in SO(3)$ and its corresponding rotation vector $\boldsymbol{\omega} \in \mathbb{R}^3$. In the UKF-M framework, the same concept generalizes to full state vectors composed of multiple manifold and Euclidean components.

5.6.4 Manifold Operators for the Process Model

In the UKF-M framework, the process model defined in Equation 4 evolves the system state over time using inertial measurements. Since the state vector

$$\mathbf{x} = \begin{bmatrix} \mathbf{p}_{b/O}^n & \mathbf{v}_{b/O}^n & \mathbf{q} & \mathbf{a}_b & \omega_b \end{bmatrix}^\top$$

contains both Euclidean components ($\mathbf{p}, \mathbf{v}, \mathbf{a}_b, \omega_b$) and a manifold component ($\mathbf{q} \in \mathbb{S}^3$), the standard addition and subtraction operations used in classical UKF cannot be directly applied. The attitude quaternion lies on the unit sphere \mathbb{S}^3 , which is a nonlinear manifold, meaning that the linear operations used for Euclidean states are not globally valid.

To handle this properly, the UKF-M defines two special manifold consistent operators, \boxplus and \boxminus , that map between the nonlinear manifold and its local tangent space. These operators are used throughout the prediction and update stages whenever states or covariances are combined or compared.

\boxplus : Mapping a tangent space perturbation to the manifold

The \boxplus operator defines how a state on the manifold is updated using a perturbation vector ξ that lives in the tangent space. In this formulation, \mathbf{x} represents the nominal state on the manifold, while ξ is a small local offset expressed in the linear tangent space around \mathbf{x} . The tangent space acts as a flat, Euclidean approximation of the manifold near the current mean, allowing linear algebra operations such as addition and covariance computation to be performed safely.

Taking an example of sigma point generation from the UKF-M algorithm discussed later down below takes form

$$\mathbf{x} = \mathbf{x} \boxplus \boldsymbol{\xi}$$

where \mathbf{x} is the current estimated state and $\boldsymbol{\xi}$ represents a small perturbation drawn from the covariance in tangent space. The vector $\boldsymbol{\xi}$ has the same structure as the state but replaces the quaternion with a 3D rotation vector. Explicitly,

$$\mathbf{x} = \begin{bmatrix} \mathbf{p}_{b/O}^n & \mathbf{v}_{b/O}^n & \mathbf{q} & \mathbf{a}_b & \omega_b \end{bmatrix}^\top \quad \boldsymbol{\xi} = [\xi_p \ \ \xi_v \ \ \xi_\theta \ \ \xi_{a_b} \ \ \xi_{\omega_b}]^\top$$

Here, the quaternion \mathbf{q} is an element of the unit hypersphere \mathbb{S}^3 , while the vector $\boldsymbol{\xi}_\theta \in \mathbb{R}^3$ is a tangent space representation of a small rotation around the mean orientation.

In tangent space, quaternions do not exist, only 3D rotation vectors do. The vector $\boldsymbol{\xi}_\theta$ simply points in the direction and magnitude of the local rotational perturbation, describing how the attitude should change infinitesimally around the current estimate.

When converting this local perturbation back to the manifold, the rotation vector must be mapped to a unit quaternion on \mathbb{S}^3 . This is done through the exponential map, which transforms a small 3D vector into a valid quaternion rotation. The mapping is

$$\mathbf{q}_{\boldsymbol{\xi}_\theta} \triangleq \text{Exp}\left(\frac{1}{2}\boldsymbol{\xi}_\theta\right) = \begin{bmatrix} \cos\left(\frac{|\boldsymbol{\xi}_\theta|}{2}\right) \\ \frac{\boldsymbol{\xi}_\theta}{|\boldsymbol{\xi}_\theta|} \sin\left(\frac{|\boldsymbol{\xi}_\theta|}{2}\right) \end{bmatrix}$$

The division by two appears because quaternion rotations encode angles that are twice the corresponding rotation vector magnitude. Using this, the complete state perturbation becomes

$$\mathbf{x} \boxplus \boldsymbol{\xi} = \begin{bmatrix} \mathbf{p}_{b/O}^n + \boldsymbol{\xi}_p \\ \mathbf{v}_{b/O}^n + \boldsymbol{\xi}_v \\ \mathbf{q} \otimes \text{Exp}\left(\frac{1}{2}\boldsymbol{\xi}_\theta\right) \\ \mathbf{a}_b + \boldsymbol{\xi}_{a_b} \\ \omega_b + \boldsymbol{\xi}_{\omega_b} \end{bmatrix}$$

The linear components like position, velocity, and biases are updated using simple vector addition, since they exist in Euclidean space. The quaternion, however, must be updated multiplicatively using the exponential map to remain on the manifold \mathbb{S}^3 . This ensures that every sigma point maintains unit norm and represents a valid orientation.

In essence, $\boldsymbol{\xi}$ is never a physical state but a local linear displacement defined in the tangent space. It expresses how much the manifold state should move in each direction, including rotation, to represent local uncertainty. When mapped back through \boxplus , the translational components shift linearly, while the rotational component “bends” along the hypersphere surface via the exponential map. This process allows the UKF-M to handle quaternions and other manifold states consistently, performing uncertainty propagation and updates in linear tangent space while always keeping the resulting states valid on their nonlinear manifold.

\boxminus : Mapping a manifold difference to the tangent space

The \boxminus operator defines how the difference between two states on the manifold, \mathbf{x}_1 and \mathbf{x}_2 , is expressed as a perturbation vector in the tangent space. While \boxplus projects a local perturbation from the tangent space onto the manifold, \boxminus performs the inverse operation, it measures how far one manifold element is from another and maps that difference back into the linear tangent domain. This operation is essential for computing residuals, innovations, and covariance updates within the UKF-M, since all statistical quantities must reside in Euclidean tangent space.

For two states

$$\mathbf{x}_1 = [\mathbf{p}_1 \ \ \mathbf{v}_1 \ \ \mathbf{q}_1 \ \ \mathbf{b}_{a_1} \ \ \mathbf{b}_{\omega_1}]^\top, \quad \mathbf{x}_2 = [\mathbf{p}_2 \ \ \mathbf{v}_2 \ \ \mathbf{q}_2 \ \ \mathbf{b}_{a_2} \ \ \mathbf{b}_{\omega_2}]^\top$$

the manifold consistent subtraction is defined as

$$\mathbf{x}_1 \boxminus \mathbf{x}_2 = \begin{bmatrix} \mathbf{p}_1 - \mathbf{p}_2 \\ \mathbf{v}_1 - \mathbf{v}_2 \\ 2 \operatorname{Log}(\mathbf{q}_2^{-1} \otimes \mathbf{q}_1) \\ \mathbf{b}_{a_1} - \mathbf{b}_{a_2} \\ \mathbf{b}_{\omega_1} - \mathbf{b}_{\omega_2} \end{bmatrix}$$

Here, the linear quantities (position, velocity, and sensor biases) can be subtracted directly, as they live in Euclidean space. However, the quaternion part requires special handling because it resides on the unit hypersphere \mathbb{S}^3 . To obtain a meaningful rotational difference between \mathbf{q}_1 and \mathbf{q}_2 , the relative rotation

$$\mathbf{q}_{\text{relative}} = \mathbf{q}_2^{-1} \otimes \mathbf{q}_1$$

is computed, representing the rotation that brings \mathbf{q}_2 to \mathbf{q}_1 .

This relative quaternion is then converted into its corresponding tangent space vector using the logarithmic map, which projects the quaternion from \mathbb{S}^3 to \mathbb{R}^3 . For a quaternion $\mathbf{q}_{\text{relative}} = [q_w, \mathbf{q}_v]^\top$ where $\mathbf{q}_v = [q_x, q_y, q_z]^\top$ composed of scalar q_w and vector \mathbf{q}_v parts, the mapping is defined as

$$\boldsymbol{\theta}_{\text{relative}} \triangleq \operatorname{Log}(\mathbf{q}_{\text{relative}}) = \begin{cases} \frac{2 \arctan 2(\|\mathbf{q}_v\|, q_w)}{\|\mathbf{q}_v\|} \mathbf{q}_v, & \text{if } \|\mathbf{q}_v\| \neq 0 \\ \mathbf{0}, & \text{otherwise} \end{cases}$$

The result of this operation is a 3D rotation vector $\boldsymbol{\theta}_{\text{relative}} \in \mathbb{R}^3$ that lies in the tangent space of the unit quaternion manifold. This vector points along the rotation axis and has a magnitude equal to the rotation angle between the two orientations. It therefore provides a minimal, linear representation of the orientation error. The factor of two ensures consistency with the quaternion exponential map used in the \boxplus operator.

Intuitively, \boxminus answers the question: “What small rotation and translation in tangent space would transform \mathbf{x}_2 into \mathbf{x}_1 ?” For Euclidean components, the answer is simply the difference between vectors. For rotational components, it is the smallest rotation vector on the manifold surface that aligns \mathbf{q}_2 with \mathbf{q}_1 . By performing this mapping, the UKF-M can compute orientation residuals, innovations, and covariance deviations in a consistent, linearized space without ever violating the unit quaternion constraint.

Interpretation

The \boxplus and \boxminus operators together define the mathematical bridge between the nonlinear manifold and its locally linear tangent space, ensuring that attitude, position, and bias states are updated consistently. The \boxplus operator applies a perturbation from tangent space to the manifold, producing a new valid state on \mathbb{S}^3 , while the \boxminus operator performs the reverse, projecting the difference between two manifold states back into tangent space where linear operations such as mean and covariance can be computed.

Within the UKF-M process framework, \boxplus is used when generating sigma points or updating the state estimate after correction, effectively moving the state along the manifold surface in response to perturbations. Conversely, \boxminus is used when comparing predicted and updated states or computing innovations, allowing differences in orientation and position to be expressed in a linearized space suitable for statistical computation.

Together, these mappings preserve the unit quaternion constraint and maintain full geometric consistency of the nonlinear attitude dynamics described by the process model 4, preventing normalization errors or distortion of uncertainty during repeated prediction and update cycles.

5.6.5 Manifold Operators for the Measurement Model

The measurement model in Equation 6 maps the nonlinear navigation state \mathbf{x} into a measurable quantity \mathbf{z} composed of GNSS position and yaw. The measurement manifold therefore combines both Euclidean and circular components:

$$\mathbf{z} = \begin{bmatrix} \mathbf{p}_{b/O}^n \\ \psi \end{bmatrix}, \quad \mathbf{z} \in \mathbb{R}^3 \times \mathbb{S}^1$$

where ψ is the yaw angle, living on the circle manifold \mathbb{S}^1 .

The process $f(\cdot)$ and measurement $h(\cdot)$ manifolds thus differ in structure, while the state includes quaternion orientation $\mathbf{q} \in \mathbb{S}^3$, the measurement only includes a single angular degree of freedom (yaw) and the position.

\boxminus : Mapping a tangent space perturbation to the manifold

The measurement residual must be computed in the tangent space of the measurement manifold. This ensures that angular differences are represented as the shortest signed distance on the circle, avoiding discontinuities around $\pm\pi$.

Given an actual measurement \mathbf{z} and predicted measurement $\hat{\mathbf{z}} = h(\hat{\mathbf{x}})$, the innovation is

$$\boldsymbol{\nu} = \mathbf{z} \boxminus \hat{\mathbf{z}} = \begin{bmatrix} \mathbf{z}_p - \hat{\mathbf{z}}_p \\ \text{wrap_to_pi}(\mathbf{z}_\psi - \hat{\mathbf{z}}_\psi) \end{bmatrix}$$

The function `wrap_to_pi`(\cdot) ensures that angular differences are represented within the principal interval $[-\pi, \pi]$, preventing discontinuities when the heading crosses the $\pm\pi$ boundary. In practice, this operation computes the smallest signed angular distance between the measured and predicted yaw angles. Without this normalization, an orientation difference between 179° and -179° would be interpreted as a large 358° error, even though the true discrepancy is only 2° . By wrapping the result, the innovation $\boldsymbol{\nu}$ remains continuous and physically meaningful across the entire rotation range, ensuring numerical stability and consistency in the filter update. The resulting innovation vector thus lies in the tangent space $\mathbb{R}^3 \times \mathbb{R}$, allowing it to be used directly in the covariance update and gain computation.

\boxplus : Mapping a manifold difference to the tangent space

In certain filtering frameworks (such as smoothing or iterative measurement correction), a \boxplus operation may be required to reapply a correction in measurement space. For example, if the predicted measurement $\hat{\mathbf{z}}$ must be adjusted by a correction $\delta\mathbf{z}$, the operation is defined as

$$\hat{\mathbf{z}} \boxplus \delta\mathbf{z} = \begin{bmatrix} \hat{\mathbf{z}}_p + \delta\mathbf{z}_p \\ \text{wrap_to_pi}(\hat{\mathbf{z}}_\psi + \delta\mathbf{z}_\psi) \end{bmatrix}$$

ensuring that angular corrections respect the periodic geometry of \mathbb{S}^1 .

Interpretation

In the measurement model, the roles of \boxminus and \boxplus differ from those in the process model. The \boxminus operator is the one predominantly used, as it computes the innovation between the actual and predicted measurements in a geometrically consistent way. This operation ensures that the angular residuals, such as yaw differences, are always represented as the smallest signed angle, maintaining continuity across the $\pm\pi$ boundary.

The \boxplus operator, although formally defined for completeness, is rarely used in practical state estimation filters such as the UKF-M. It becomes relevant only in specific cases, such as iterative measurement refinement or optimization based frameworks, where perturbations in measurement space must be mapped back onto the manifold.

Together, these operators maintain the consistency between the nonlinear measurement manifold $\mathbb{R}^3 \times \mathbb{S}^1$ and its tangent space. The \boxminus operator enables proper innovation computation for covariance updates, while \boxplus ensures that potential corrections respect the manifold geometry. This separation guarantees that both position and angular measurements are handled accurately within their respective geometric domains, preserving filter stability and correctness.

5.6.6 MeanOfSigmaPoints() on Manifolds

When sigma points lie on a manifold, their average cannot be computed using standard Euclidean addition. Unlike vector spaces, manifolds do not support global linear operations such as direct addition or subtraction. Therefore, the mean must be defined geometrically as the point that minimizes the

total squared distance to all sigma points when those distances are measured in the local tangent space. Formally, the mean μ is given by

$$\mu = \operatorname{argmin}_{\mathbf{x}} \sum_{i=0}^{2L} \|\mathcal{X}_i \boxminus \mathbf{x}\|^2$$

This expression defines the manifold mean as the point that minimizes the weighted sum of squared tangent space distances between each sigma point \mathcal{X}_i and the candidate mean \mathbf{x} . The \boxminus operator computes the local difference between two manifold elements and maps that difference back into the tangent space, where distances are represented as standard Euclidean vectors.

Intuitively, this means that rather than directly averaging points on the manifold, which would produce a result that leaves the manifold surface, the mean is found through a search process that balances all sigma points in the tangent space. The final μ lies on the manifold and represents the most symmetric and geometrically correct center of the sigma point cloud.

To compute this mean in practice, we use an iterative fixed point algorithm:

$$\begin{aligned} \mu_0 &= \mathcal{X}_0 \\ \mu_{k+1} &= \mu_k \boxplus \left(\frac{1}{2L+1} \sum_{i=0}^{2L} (\mathcal{X}_i \boxminus \mu_k) \right) \\ \mu &= \lim_{k \rightarrow \infty} \mu_k \end{aligned} \tag{11}$$

This iterative method refines an initial estimate μ_0 (often chosen as the first sigma point or previous state estimate) until the mean converges to a point around which all sigma points are symmetrically distributed in the tangent space. Each iteration alternates between mapping manifold elements to the tangent space with \boxminus and updating the mean on the manifold with \boxplus . The result is a process that moves back and forth between the curved manifold and its locally linear tangent space until equilibrium is reached.

The first operation, $(\mathcal{X}_i \boxminus \mu_k)$, measures how far each sigma point \mathcal{X}_i lies from the current mean estimate μ_k along the manifold. Since subtraction is undefined on a curved surface, \boxminus projects this geometric difference into the tangent space at μ_k , producing small linear displacement vectors that can be averaged safely. The average of these displacements,

$$\frac{1}{2L+1} \sum_{i=0}^{2L} (\mathcal{X}_i \boxminus \mu_k)$$

represents the average direction and magnitude by which the current mean estimate μ_k is offset from the true geometric center of the sigma points.

It is important to note that in this formulation, the explicit sigma point weights W_i^m and W_i^c that appear in the classical UKF are no longer required. The UKF-M relies instead on the geometric symmetry of the sigma point distribution, where each positive perturbation has an equal negative counterpart. Because the sigma points are generated symmetrically around the mean using $\pm\sqrt{P}$, their contribution to the average naturally balances out. This symmetry ensures that the mean and covariance reconstruction are unbiased without needing explicit weighting factors. The normalization factor $\frac{1}{2L+1}$ simply ensures equal contribution from all points, maintaining the geometric consistency of the averaging process. In other words, this method depends purely on the manifolds local geometry rather than statistical weights designed for Euclidean Gaussian moments.

If the averaged deviation in the tangent space is zero, it means that all sigma points are symmetrically arranged around μ_k , and the iteration has converged. If not, the mean must be corrected. The update step is carried out using the \boxplus operator,

$$\mu_{k+1} = \mu_k \boxplus \Delta\mu, \quad \text{where} \quad \Delta\mu = \frac{1}{2L+1} \sum_{i=0}^{2L} (\mathcal{X}_i \boxminus \mu_k)$$

Here, $\Delta\mu$ is a small vector in the tangent space describing how to shift the current mean estimate toward the geometric center of the sigma points. The \boxplus operator then maps this small correction vector from the

tangent space back onto the manifold, updating μ_k into a new manifold element μ_{k+1} . Each iteration therefore involves two transformations:

1. \boxminus : Tangent \leftarrow Manifold, then measure displacements in a linear domain.
2. \boxplus : Tangent \rightarrow Manifold, then update the mean in the curved domain.

Repeating this process ensures that the mean estimate remains on the manifold at every step while progressively converging toward the true center in tangent space terms. The algorithm stops once the magnitude of the correction vector becomes sufficiently small:

$$\|\Delta\mu\| = \left\| \frac{1}{2L+1} \sum_{i=0}^{2L} (\mathcal{X}_i \boxminus \mu_k) \right\|$$

When $\|\Delta\mu\|$ drops below a user defined tolerance ϵ (typically 10^{-6} or 10^{-8}), the iteration is terminated:

stop if $\|\Delta\mu\| < \epsilon$

This convergence criterion ensures that further updates would no longer produce meaningful changes, meaning the mean has stabilized.

Conceptually, this iterative mean calculation behaves like computing an expected value, but in a manifold consistent manner. Instead of adding the sigma points directly, which would violate manifold geometry, the algorithm repeatedly measures deviations in tangent space, computes an average there, and re-projects the correction back to the manifold using \boxplus . Through successive iterations, the mean converges to the point that minimizes the sum of squared tangent space residuals to all sigma points. The final μ lies exactly on the manifold, is geometrically correct, and provides the proper reference for subsequent covariance computation.

This entire operation is conveniently encapsulated by the function call

$$\mu = \text{MeanOfSigmaPoints}(\mathcal{X}_i)$$

which internally performs the iterative procedure described above, providing a consistent and geometrically valid manifold mean.

5.6.7 Filter Algorithm: Prediction Step

In the Unscented Kalman Filter on Manifolds (UKF-M), the propagation follows the same general structure as the classical Unscented Kalman Filter but introduces one essential geometric correction. In the UKF-M, all statistical operations such as averaging, difference calculation, and covariance computation are performed in the tangent space of the manifold rather than directly on the manifold itself. The reason is that linear algebra operations such as addition or subtraction are only defined in Euclidean vector spaces, while a manifold is curved and does not support these operations globally.

The tangent space is a locally flat linear approximation of the manifold that touches it at the current state estimate $\hat{\mathbf{x}}$. Inside this tangent space, we can safely perform vector operations and treat perturbations as if the system were linear, while on the manifold itself, only composition and mapping operations are valid. This separation is the key idea of the UKF-M, where the state lives on the manifold, but the uncertainty around that state, represented by the covariance matrix P , lives in the tangent space.

An important difference from the standard UKF is that in the UKF-M, no explicit sigma point weights W_i^m or W_i^c are used during the mean and covariance computations. Instead, the algorithm relies purely on the geometric symmetry of the sigma point distribution. In the classical UKF, the weights are necessary because the sigma points are designed to reproduce the exact first and second moments of a Gaussian distribution in Euclidean space, which requires careful weighting. However, in the UKF-M, the sigma points are constructed symmetrically around the mean in the tangent space, and this symmetry ensures that the unweighted average of their spread already provides a correct and unbiased representation of the local uncertainty. Each positive perturbation has a corresponding negative one, canceling any bias in the mean and producing a covariance that correctly represents the local shape of the uncertainty region. This geometric symmetry removes the need for explicit weighting and simplifies both

the mean and covariance reconstruction.

The propagation begins by generating these symmetric sigma points in the tangent space around the current mean $\hat{\mathbf{x}}$,

$$\begin{aligned}\boldsymbol{\xi}_0 &= \mathbf{0} \\ \boldsymbol{\xi}_i &= \sqrt{P_i} \quad i = 1, \dots, L \\ \boldsymbol{\xi}_i &= -\sqrt{P_{i-L}} \quad i = L+1, \dots, 2L\end{aligned}\tag{12}$$

Each $\boldsymbol{\xi}_i$ is a perturbation vector that expresses a small local offset in the tangent space, describing how the true state might deviate from the mean. The scaling factor is absorbed into the square root, and all sigma points are equally important due to their symmetric layout.

Because the process model $f_d(\cdot)$ is defined directly on the manifold, these perturbations cannot be used as-is. They must first be mapped to actual states on the manifold. This is done using the \boxplus operator,

$$\mathcal{X}_i = \hat{\mathbf{x}} \boxplus \boldsymbol{\xi}_i$$

The \boxplus operator takes the current state $\hat{\mathbf{x}}$ on the manifold and a tangent vector $\boldsymbol{\xi}_i$ in its local tangent space and produces a new manifold element. It effectively “moves” the state along the manifold surface in the direction described by $\boldsymbol{\xi}_i$. This step ensures that all sigma points \mathcal{X}_i are valid manifold elements, such as normalized quaternions or valid pose transformations.

Each of these manifold sigma points is then propagated through the nonlinear discrete dynamics model,

$$\mathcal{X}_i^- = f_d(\mathcal{X}_i, \mathbf{u})$$

After propagation, all \mathcal{X}_i^- remain on the manifold because $f_d(\cdot)$ defines evolution rules that preserve the manifold structure. For example, quaternion propagation keeps unit norm, and rigid-body motion remains on SE(3).

The next step is to determine the predicted mean state $\hat{\mathbf{x}}^-$. Since the propagated sigma points \mathcal{X}_i^- now lie on the manifold, their mean must also lie on the same manifold. Because manifolds do not support standard addition or averaging, the mean cannot be computed as a simple arithmetic sum. Instead, we define a dedicated function that estimates the manifold mean of a set of sigma points,

$$\hat{\mathbf{x}}^- = \text{MeanOfSigmaPoints}(\mathcal{X}_i^-)$$

This function represents a general procedure that finds the best mean state on the manifold based on the geometry of the sigma points. Internally, it performs the necessary manifold-consistent averaging using the \boxminus operation to measure distances in the tangent space, returning a valid and accurate mean estimate. The resulting $\hat{\mathbf{x}}^-$ lies on the manifold and becomes the new predicted state mean.

Once the mean is found, we must express the uncertainty of the state distribution around this mean. This cannot be done directly on the manifold because addition and subtraction between manifold elements are not globally valid. Therefore, all propagated sigma points \mathcal{X}_i^- are mapped back into the tangent space centered at $\hat{\mathbf{x}}^-$ using the \boxminus operator,

$$\boldsymbol{\xi}_i^- = \mathcal{X}_i^- \boxminus \hat{\mathbf{x}}^-$$

This step is crucial because the covariance P represents uncertainty in a linear space. It encodes how much the state may vary locally around the mean. If we attempted to compute P directly on the manifold, we would violate its geometry, as covariance addition and subtraction would move the result off the manifold. By using the \boxminus operator, each sigma point is projected back into the tangent space, effectively flattening the curved neighborhood around $\hat{\mathbf{x}}^-$ into a locally linear space where covariance computation is valid.

The predicted covariance is then reconstructed from these tangent space deviations using an unweighted symmetric formulation,

$$P^- = \frac{1}{2} \sum_{i=1}^{2L} \boldsymbol{\xi}_i^-(\boldsymbol{\xi}_i^-)^\top + Q$$

Here, P^- represents the predicted covariance of the state in the tangent space, while the mean $\hat{\mathbf{x}}^-$ remains strictly on the manifold. The covariance P^- captures the spread and orientation of uncertainty in local linear coordinates, effectively describing how much and in which directions the state can vary around the mean within the manifolds neighborhood.

In this formulation, explicit weighting coefficients such as W_i^m or W_i^c , which are typically used in the classical UKF are no longer present. The reason is that in the UKF-M, sigma points are generated in a geometrically symmetric way around the mean, using $\pm\sqrt{P}$. This symmetric structure inherently guarantees that the contributions from all sigma points are balanced and unbiased. Each positive perturbation has an equal and opposite negative counterpart, ensuring that the overall distribution remains centered around zero in the tangent space. As a result, the average and covariance of these symmetric deviations can be computed without statistical weighting because the geometry itself enforces equilibrium.

In contrast, the traditional UKF relies on asymmetric scaling and weighting to preserve the exact first and second statistical moments of a Gaussian distribution in Euclidean space. Those weights compensate for the uneven influence of the central sigma point and its surrounding samples. However, on a manifold, this probabilistic interpretation is less meaningful since addition and subtraction are not globally defined. The UKF-M instead treats the sigma points as a symmetric geometric sampling of local uncertainty rather than as discrete weighted samples of a probability distribution. Consequently, the covariance is reconstructed purely from geometry, using uniform contributions from each tangent space deviation. The factor $\frac{1}{2}$ in the equation serves as a normalization constant that reflects the fact that each dimension contributes both a positive and a negative sigma point of equal magnitude.

This geometric treatment greatly simplifies the covariance reconstruction and ensures that the resulting P^- is always consistent with the local curvature of the manifold. The tangent-space deviations ξ_i^- provide valid Euclidean displacement vectors that can be combined using standard linear algebra, while the mean $\hat{\mathbf{x}}^-$ remains on the nonlinear manifold surface. In this way, the UKF-M cleanly separates the roles of geometry and statistics: $\hat{\mathbf{x}}^-$ captures the nonlinear evolution of the state itself, and P^- describes the uncertainty in a locally flat, linearized representation.

This separation between the manifold domain for the state and the tangent-space domain for the covariance is fundamental to the design of manifold-based filters. The \boxplus operator maps tangent perturbations onto the manifold to form valid states, while the \boxminus operator maps manifold differences back into the tangent space for linear operations. These two operators together maintain full geometric consistency throughout the propagation process. Without this dual-space treatment, adding or subtracting states directly would produce inconsistencies—potentially leading to drift, loss of normalization, or distorted covariance when operating in curved spaces such as rotations or poses.

Therefore, in the UKF-M, the covariance reconstruction step is not just a numerical simplification but a direct consequence of respecting the manifold geometry. By leveraging the symmetry of the sigma point distribution, the filter avoids the need for weighting coefficients, preserves local geometric balance, and keeps the statistical representation consistent with the curvature of the underlying manifold. The state mean $\hat{\mathbf{x}}$ always resides on the manifold, while the covariance P always resides in its tangent space, ensuring that both the geometry and uncertainty of the system are correctly and consistently represented throughout propagation.

5.6.8 Filter Algorithm: Correction Step

After the propagation step, the Unscented Kalman Filter on Manifolds (UKF-M) incorporates new sensor information to refine the predicted state estimate. The goal of the update stage is to compare the predicted measurement with the actual sensor measurement and then correct the state estimate in a way that is consistent with the manifold geometry.

The overall process closely mirrors the classical UKF, but all operations that involve addition or subtraction of states or measurements must be performed in the corresponding tangent spaces. The reason is the same as before: the state $\hat{\mathbf{x}}$ lives on a nonlinear manifold, while both measurement innovations and covariance updates are linear statistical quantities that can only exist in Euclidean vector spaces. Therefore, throughout the update step, the UKF-M repeatedly moves between the manifold and

its tangent space using the \boxplus and \boxminus operators.

The update begins with the predicted state mean $\hat{\mathbf{x}}^-$ and covariance P^- . Using these, a new set of sigma points is generated in the tangent space centered at $\hat{\mathbf{x}}^-$,

$$\begin{aligned}\boldsymbol{\xi}_0 &= \mathbf{0} \\ \boldsymbol{\xi}_i &= \sqrt{P_i^-} \quad i = 1, \dots, L \\ \boldsymbol{\xi}_i &= -\sqrt{P_{i-L}^-} \quad i = L + 1, \dots, 2L\end{aligned}\tag{13}$$

Each $\boldsymbol{\xi}_i$ again represents a small perturbation in the tangent space, describing local uncertainty around the current mean. These perturbations are mapped back onto the manifold to form actual state sigma points,

$$\mathcal{X}_i^- = \hat{\mathbf{x}}^- \boxplus \boldsymbol{\xi}_i$$

Each \mathcal{X}_i^- now represents a valid manifold state, for instance, a quaternion with unit norm or a valid pose transformation in SE(3).

Each of these manifold sigma points is then passed through the nonlinear measurement model $h(\cdot)$,

$$\mathcal{Z}_i = h(\mathcal{X}_i^-)$$

which predicts what the measurement would look like for each sigma point. The resulting set of \mathcal{Z}_i points represents a distribution of predicted measurements in the measurement space. Depending on the sensor, this space can itself be Euclidean (for example GNSS position) or also contain manifold elements (for example, angles on \mathbb{S}^1 or quaternions on \mathbb{S}^3).

The predicted measurement mean $\hat{\mathbf{z}}$ is then computed. Since the measurement space may also include manifold valued quantities, such as orientations or angular variables, a direct arithmetic average is not valid. Instead, use the same consistent geometric averaging operation as for the state mean. The mean of the predicted measurement sigma points is obtained using the function

$$\hat{\mathbf{z}} = \text{MeanOfSigmaPoints}(\mathcal{Z}_i)$$

This function performs manifold consistent averaging of all predicted measurement sigma points \mathcal{Z}_i around an initial estimate $\hat{\mathbf{z}}$. It iteratively maps the sigma points into the tangent space using \boxminus , computes their weighted average deviation, and updates the mean on the manifold with \boxplus until convergence. The resulting $\hat{\mathbf{z}}$ lies exactly on the measurement manifold, representing the central point where the sigma points are symmetrically distributed in the tangent space sense, ensuring a geometrically valid and consistent measurement mean.

Once the mean $\hat{\mathbf{z}}$ is obtained, all predicted measurement sigma points are mapped back into the tangent space at $\hat{\mathbf{z}}$ using the \boxminus operator,

$$\boldsymbol{\zeta}_i = \mathcal{Z}_i \boxminus \hat{\mathbf{z}}$$

These tangent-space deviations $\boldsymbol{\zeta}_i$ describe how the predicted measurements vary locally around their mean in a linear space, which is necessary for computing covariances. Using these deviations, the measurement innovation covariance is calculated as,

$$S = \frac{1}{2} \sum_{i=0}^{2L} \boldsymbol{\zeta}_i \boldsymbol{\zeta}_i^\top + R$$

where R is the measurement noise covariance matrix. The matrix S thus represents the total uncertainty in the predicted measurement, combining both the propagated state uncertainty and the sensor noise.

Next, the cross-covariance between the state and the measurement is computed. To do this consistently, we express both quantities in their respective tangent spaces. The state deviations are given by $\boldsymbol{\xi}_i^- = \mathcal{X}_i^- \boxminus \hat{\mathbf{x}}^-$, and the measurement deviations are the previously computed $\boldsymbol{\zeta}_i = \mathcal{Z}_i \boxminus \hat{\mathbf{z}}$. The cross-covariance is therefore,

$$P_{xz} = \frac{1}{2} \sum_{i=0}^{2L} \boldsymbol{\xi}_i^- \boldsymbol{\zeta}_i^\top$$

This matrix expresses how variations in the state tangent space correspond to variations in the predicted measurement tangent space, effectively linking the two spaces through their local linear approximations.

The Kalman gain is now obtained as,

$$K = P_{xz} S^{-1}$$

which defines how much the state estimate should be corrected based on the measurement innovation.

The innovation itself is computed as the difference between the actual measurement \mathbf{z} and the predicted mean $\hat{\mathbf{z}}$,

$$\boldsymbol{\nu} = \mathbf{z} \boxminus \hat{\mathbf{z}}$$

Note that this operation also uses the manifold-aware subtraction, because the measurement may include angular or orientation components. The innovation vector $\boldsymbol{\nu}$ thus lives in the tangent space of the measurement manifold.

After computing the Kalman gain, the correction step refines the predicted mean and covariance based on the new measurement.

The first step is to compute the correction vector in the tangent space of the state manifold,

$$\boldsymbol{\delta} = K\boldsymbol{\nu}$$

This vector represents the optimal adjustment of the predicted state in the local tangent space. The updated uncertainty of this correction is obtained as

$$P' = P^- - KSK^\top$$

which represents the new covariance of the correction vector $\boldsymbol{\delta}$ after the measurement information has been incorporated.

Using $\boldsymbol{\delta}$ and P' , a new set of sigma points is created to represent the corrected uncertainty around the predicted mean. In the tangent space, the sigma points are defined as

$$\begin{aligned} \boldsymbol{\xi}'_0 &= \boldsymbol{\delta} \\ \boldsymbol{\xi}'_i &= \boldsymbol{\delta} + \sqrt{P_i^-} \quad i = 1, \dots, L \\ \boldsymbol{\xi}'_{i-L} &= \boldsymbol{\delta} - \sqrt{P_{i-L}^-} \quad i = L + 1, \dots, 2L \end{aligned} \tag{14}$$

These sigma points capture the spread of the post update uncertainty in the tangent space.

Each tangent space sigma point is then mapped back onto the manifold around the predicted mean using the \boxplus operator,

$$\mathcal{X}'_i = \hat{\mathbf{x}}^- \boxplus \boldsymbol{\xi}'_i$$

This produces a new set of manifold sigma points \mathcal{X}'_i that describe the updated state distribution after the measurement correction.

The corrected state mean is then obtained by computing the manifold consistent average of these sigma points,

$$\hat{\mathbf{x}}^+ = \text{MeanOfSigmaPoints}(\mathcal{X}'_i)$$

This operation finds the manifold point that minimizes the weighted squared tangent-space distances to all sigma points, ensuring that $\hat{\mathbf{x}}^+$ lies exactly on the manifold and represents the most likely corrected state.

Finally, the updated covariance is computed from the spread of the new sigma points around the corrected mean in the tangent space,

$$P^+ = \frac{1}{2} \sum_{i=0}^{2L} [\mathcal{X}'_i \boxminus \hat{\mathbf{x}}^+] [\mathcal{X}'_i \boxminus \hat{\mathbf{x}}^+]^\top$$

This final covariance P^+ captures the remaining uncertainty in the tangent space centered at $\hat{\mathbf{x}}^+$ and ensures full geometric consistency between the state and its uncertainty.

For systems that include quaternions or other unit-norm quantities, it is common to re-normalize the quaternion after the update step to correct for small numerical drift,

$$\mathbf{q}^+ \leftarrow \frac{\mathbf{q}^+}{\|\mathbf{q}^+\|}$$

This normalization step ensures that orientation remains physically valid and consistent with the manifold structure.

In summary, the update step of the UKF-M carefully keeps the separation between the manifold and its tangent spaces. The state mean $\hat{\mathbf{x}}$ always remains on the manifold, while the covariance P and all perturbations live in the tangent space. The \boxplus operator moves perturbations from tangent to manifold, while \boxminus maps differences from manifold back to tangent. This bidirectional mapping ensures that the state update remains geometrically consistent while allowing covariance and innovation computations to be performed in a locally linear space. The mean recomputation for both $\hat{\mathbf{z}}$ and $\hat{\mathbf{x}}^+$ follows the same principle as before: they are defined as the manifold elements minimizing the weighted residuals of their sigma points in their respective tangent spaces. Through these mappings, the UKF-M performs a statistically correct and geometrically sound measurement correction step, preserving both the nonlinear geometry and the probabilistic consistency of the filter.

5.6.9 Advantages and Practical Aspects

UKF-M provides several key advantages over traditional linearization based filters such as EKF and ESKF. Unlike these approaches, UKF-M does not rely on Jacobians, thereby eliminating the need for analytical derivations, symbolic differentiation, and the discretization errors that often occur in complex nonlinear systems. This not only simplifies implementation but also removes a common source of modeling mistakes. Furthermore, the UKF-M preserves full 2nd order accuracy during nonlinear mean and covariance propagation, resulting in more consistent and stable estimates in systems with strong non-linearities.

Because the filter operates directly on manifolds, it naturally supports state representations that live in non Euclidean spaces such as rotation groups $\text{SO}(3)$, unit quaternions \mathbb{S}^3 , and rigid body poses $\text{SE}(3)$. This allows seamless integration of mixed state variables like positions, velocities, orientations, and sensor biases within a single unified framework. Orientation handling is intrinsically consistent, avoiding the small angle approximations often required in EKF and ESKF formulations. The manifold formulation also removes the need for tuning heuristic scaling parameters such as α , β , and κ , which in the classical UKF can significantly affect numerical stability and convergence. By relying purely on symmetric sigma point geometry, the UKF-M achieves a robust representation of uncertainty on curved spaces.

Compared to the standard UKF, the manifold formulation is more general and geometrically consistent. It can be applied to arbitrary state spaces, including Lie groups and hybrid Euclidean manifold models, without modification. This generality makes the UKF-M particularly powerful in robotics and navigation, where states often include rotations, orientations, or transformations that cannot be treated with standard linear algebra.

The main trade off of the UKF-M lies in computational cost. While it avoids the analytical burden of Jacobians, it replaces it with additional numerical computation. Each propagation and update step requires evaluating the nonlinear dynamics and measurement models for multiple sigma points, as well as performing iterative manifold averaging to compute the new mean. This sample based approach demands more processing than linearized filters, but modern hardware can easily accommodate it. Thus, while mathematically heavier, UKF-M is often simpler and more maintainable from a programming perspective, since all model functions can be implemented directly without differentiation.

In applications such as SLAM and inertial navigation, this balance of simplicity, robustness, and geometric correctness makes the UKF-M a preferred choice. It provides accurate estimation of both position and orientation without relying on Jacobians or linearization, offering similar or superior accuracy

to ESKF while being easier to maintain and extend. Although the increased computational demand can be a limitation for resource constrained systems, it is negligible for modern embedded platforms equipped with SBC. In the context of this thesis, the microAmpere ASV provides sufficient processing power, meaning that computation and memory are not bottlenecks for state estimation. Instead, the primary computational challenge lies in managing and optimizing the ever expanding SLAM map, not in running the filter itself.

For these reasons, the UKF-M is adopted in this work as the primary state estimation framework. It offers a strong combination of theoretical rigor, implementation simplicity, and robust performance in nonlinear and manifold constrained systems, making it ideal for the requirements of marine autonomy, SLAM, and inertial navigation on the microAmpere ASV.

5.7 Tuning and Filter Verification

- Describe the role of process (Q) and measurement (R) noise covariance tuning.
- Explain methods such as innovation monitoring and adaptive noise estimation.
- Introduce the Normalized Innovation Squared (NIS) test for filter consistency validation.
- Mention residual monitoring and gating for outlier rejection.
- Summarize empirical tuning steps performed using real-world test data.

Final State Definition maybe just shortly mention at the very end here nothing much is needed and mention that UKF-M is the best way forward

- Present the final ESKF state vector:

$$\mathbf{x} = [\mathbf{p} \quad \mathbf{v} \quad \mathbf{q} \quad \mathbf{b}_a \quad \mathbf{b}_g]^\top$$

- Explain that position, velocity, and attitude are expressed in the navigation frame, while accelerometer and gyroscope biases evolve in the body frame.
- Conclude that this representation provides an optimal balance between model realism, computational efficiency, and estimator robustness.

6 Preintegration

Uses motion model $f()$ here I think

For SLAM update step when a Sonar picture is made it usually takes 10 seconds or more, in this time span if we fed State estimate data directly into the optimizer we would generate hundreds of odometry factors that are useless as they are just connected to each other for the most part until new sonar image has been generated. This would mean optimizer part in the backend would have to sift and optimize for unnecessary hundreds of factors that don't contribute much by themselves.

Instead what smart people have figured out is that one can do method called preintegration where you use smart ways to sum up all the inertial measurements to create 1 single precise odometry factor that gets generated at the same time as the sonar picture is taken, so that we get a single odometry factor that can be connected with multiple landmark factors. This is significantly faster as now only the essential data is compressed without wasting optimizers time as well as still relaying all the important information. Preintegration is therefore crucial step to making SLAM efficient.

Also talk about how odometry factors are generated here

7 Local Map Generation

All use State estimates and measuremnt model $h()$ here

Also when generating map locally we build it from sonar 1D images to a 2D recontruction, this takes time and this image is what is then processed and fed into SLAM In adition next step we take with the last 1/3rd or the old image and generate a new image that is 2/3 new, this then becomes new 3/3 image that gets fed into SLAM and so forth This is done so that data asocaiton we dont cut off crucial info between frames and Data Sdociation and landmark detection can extract features again here.

Swath Processing

Cartesian Mapping

Feature Extraction (Landmark Detection)

8 Data Association

Gating (Mahalanobis)
matching/tracking
Also talk about loop closures
produce measurement factors for optimizer

9 Optimizers

9.1 Introduction

Optimizers are the engine behind the SLAM update step. Sensors add constraints, but the optimizer decides how the state moves to satisfy them. In practice, the problem is posed as a Maximum A Posteriori (MAP) problem, seeking the most likely trajectory and landmarks given all measurements and priors. With standard assumptions such as Gaussian noise and first order linearization, the MAP problem becomes a nonlinear least squares problem solved iteratively. At each iteration, the residuals are linearized around the current estimate and an increment that improves the state is computed. This “correction” is what lets SLAM reduce drift, enforce loop closures, and keep the map consistent. [25]

There are two dominant families for doing state estimation in SLAM, filtering and smoothing. Filtering methods, like EKF-SLAM (Extended Kalman Filter) and particle-filter SLAM, maintain a rolling belief over the current state only (or a short window). They update online/live as measurements arrive by propagating the state and compressing all past information into the filter’s covariance or a set of weighted particles. This is simple and has low memory, but it throws away structure in old constraints and it can become inconsistent after many linearizations, especially when revisiting places (loop closures) or when correlations span long time intervals. Particle filters can represent multi modal beliefs but scale poorly with dimension and often need heavy resampling and clever proposal distributions to avoid degeneracy, something that is difficult to achieve in practice. [26][27]

Smoothing methods keep a dense record of the variables we care about (eks: the whole robot trajectory and, if needed, landmarks) and all the measurement factors that tie them together. Instead of only “where am I now?”, smoothing asks “what is the entire trajectory and map that best fits everything we have ever seen?”. This global view tends to produce better accuracy and consistency, especially when closing loops or fusing many asynchronous sensors. Computationally, smoothing exposes sparse structure, meaning each measurement only touches a few variables, so the global normal equations are large but very sparse. Modern linear algebra plus careful data structures exploit that sparsity and outperform classical filters on realistic SLAM workloads. That is why smoothing has become the predominant approach in modern SLAM systems.

An estimate of the unknowns is maintained, denoted θ (robot poses, and landmarks). Each measurement yields a residual that indicates how far the current estimate is from the expected sensor reading. Close to the current estimate, a small nudge in θ gives an approximate change in the residuals. This is a first order (linear) approximation. Stacking all residuals together, that approximation looks like this [28]:

$$r(\theta + \Delta\theta) \approx A\Delta\theta - b$$

Here A is the Jacobian, it describes how each residual changes with each variable. The vector b is the residual at the current estimate (with a sign convention so the equation above points toward reducing error). The small vector $\Delta\theta$ is the “correction” to be computed. [28]

The update step chooses $\Delta\theta$ that reduces all residuals as much as possible. The best practice here is to use MAP approach. In this section, measurement noise is assumed Gaussian (this is an approximation, not always true, and non Gaussian cases are discussed later). After linearizing the residuals around the current estimate, the MAP problem becomes a least-squares fit [28]:

$$\Delta\theta^* = \arg \min_{\Delta\theta} \|A\Delta\theta - b\|^2 \quad (15)$$

Optimize this estimate so that change in $\Delta\theta^*$ is equal to 0, ie linearize. When linearized, equation (15) can be simplified to a so called normal equation [28]:

$$A^T A \Delta\theta = A^T b$$

This equation system can be then be solved by Cholesky decomposition of $A^T A$ or by optimization algorithms that will be discussing down below. Solve this linear system for $\Delta\theta$, then update/correct the estimate [28]:

$$\theta = \theta + \Delta\theta$$

For stability on harder problems Levenberg-Marquardt damping can be added, however the core idea stays the same across these optimizer algorithms. [29]

Classical batch smoothing forms the full information matrix, eliminates variables in a chosen order, and solves for all states together. That is accurate but not ideal for online use. Every new measurement would, in principle, require rebuilding and refactoring a large system, with cost growing with mission length. Real robots need real-time behavior, so iterative methods are prefer, incremental smoothing that reuses previous computation. The idea is to keep the factorization of the linearized problem in a data structure that can be updated locally when new factors arrive, only touching the parts of the graph that actually change.

This is where Iterative Smoothing and Mapping (iSAM and iSAM2) methods come in. They exploit that SLAM data are very sparse and mostly locally connected, a new odometry or measurement links a pose to a neighbor pose or a nearby landmark, not to everything. iSAM maintains a square-root factor (via QR) and updates it incrementally using Givens rotations, with occasional reordering to control fill in. It keeps uncertainty queries fast and avoids full resolves, except when needed. iSAM2 goes further by expointing factor graphs and organizing the factorization into a Bayes tree data type (a directed tree of cliques). On new measurements, only the impacted cliques are relinearized and refactored, and variables are reordered incrementally. As a result, work scales with the local update rather than the entire graph, this makes update step “fluid”.

In modern SLAM the hard part isn’t “doing SLAM”, it’s solving the SLAM optimization fast as data grows. Most methods use the same MAP correction loop. Linearize, solve for $\Delta\theta^*$, update θ . The real difference lies in how the problem is represented and updated. Smart data structures and good variable ordering keep data structures sparse and decoupled, and solves quick. Meanwhile bad data structure representation of data causes slowdowns.

This is exactly why, for SLAM on marine vessels, especially AUVs with tight space, power, and compute budgets but strict real-time needs, iterative smoothing methods like iSAM2 are a strong fit. They reuse prior factorizations, add new measurements as local factors, relinearize and refactor only the affected cliques, and reorder variables incrementally. In practice that means low latency, bounded memory and CPU load, and accuracy close to batch solutions, even on long missions.

9.2 iSAM

9.2.1 Getting to SLAM update step

Before computing a good estimate, defining simple models for robot motion and sensor observations is crucial. The motion model describes state evolution, and the measurement model describes sensor readings. States are x_i for robot poses, controls are u_i , and measurements are z_k for landmarks. Stack all unknowns into θ , poses and landmarks.

Motion (process) model:

$$\begin{aligned} x_i &= f_i(x_{i-1}, u_i) + w_i \\ w_i &\sim N(0, Q_i) \end{aligned}$$

Given the previous state x_{i-1} and control u_i , the next state x_i comes from a model f plus uncertainty noise in the model itself w_i . This uncertainty captures things like currents, slip, and actuator errors. f_i can be a discrete time dynamics update or use plain odometry. Assuming Gaussian w_i is a handy start so MAP becomes least squares. Later this uncertainty model can be switched to robust or heavy tailed noise model if needed.

Measurement model:

$$\begin{aligned} z_k &= h_k(x_{i_k}, l_{j_k}) + v_k \\ v_k &\sim N(0, R_k) \end{aligned}$$

Each measurement z_k depends on state x_{i_k} and landmarks l_{j_k} transformed using measurement transform function $h_k(\cdot)$, this allows state estimate to become estimated measurement position. In addition this measurement has noise v_k which is modeled as Gaussian noise for simplifications later on when calculating.

Prior:

$$x_0 \sim N(\mu_0, \Sigma_0)$$

A prior anchors the graph (otherwise the problem is underdetermined up to a global transform). It can encode GPS at the start, a known dock pose, or simply a weak “zero” prior to fix gauge.

Predictions should match measurements. In a perfect world, every residual (prediction minus measurement) would be zero. In practice, model errors and sensor noise make the residuals nonzero. Estimation is about choosing the state update that makes all residuals as small and as statistically consistent as possible.

This is where MAP algorithm comes in. MAP (Maximum A Posteriori) is the principled way to fuse everything we know. A prior on the state, the motion model, and all measurements. It combines them through probability, weighting each residual by its uncertainty. With Gaussian noise, the negative log posterior becomes a sum of squared (weighted) residuals. That gives us a single objective to minimize, where more reliable terms (small covariance) count more. This is better than ad hoc weighting and naturally handles many sensors.

Motion and measurement functions are nonlinear (angles, rotations, ranges). Minimizing the nonlinear MAP cost directly is hard. Linearization lets us solve it iteratively. At the current estimate approximate the nonlinear functions by their first order Taylor expansion, solve a linear least squares problem for a small increment, update the estimate, and repeat. This is all shown in the iSAM paper [28] where linearized forms of the system becomes:

$$\begin{aligned} f_i(x_{i-1}, u_i) - x_i &\approx (F_i^{i-1} \Delta x_{i-1} - \Delta x_i) - a_i \\ F_i^{i-1} &:= \left. \frac{\partial f_i(x_{i-1}, u_i)}{\partial x_{i-1}} \right|_{x_{i-1}^0} \\ a_i &= x_{i-1}^0 - f_i(x_{i-1}^0, u_i) \end{aligned} \tag{16}$$

$$\begin{aligned}
h_k(x_{i-1}, u_i) - z_k &\approx (H_k^{i_k} \Delta x_{i_k} - J_k^{j_k} \Delta l_{j_k}) - c_k \\
H_k^{i_k} &:= \frac{\partial h_k(x_{i_k}, l_{j_k})}{\partial x_{i_k}} \Big|_{(x_{i_k}^0, l_{j_k}^0)} \\
J_k^{j_k} &:= \frac{\partial h_k(x_{i_k}, l_{j_k})}{\partial l_{j_k}} \Big|_{(x_{i_k}^0, l_{j_k}^0)} \\
c_k &= z_k - h_k(x_{i_k}^0, l_{j_k}^0)
\end{aligned} \tag{17}$$

Plug the linearized odometry (16) and measurement (17) models into a single objective over the stacked increment vector $\Delta\theta$ (all pose and landmark updates). The goal is to pick the small change $\Delta\theta^*$ that jointly reduces all linearized residuals. Each factor becomes a linear row in the relevant increments.

For an odometry factor i , the linearized residual is

$$r_i^{\text{odo}} = F_i^{i-1} \Delta x_{i-1} + G_i^i \Delta x_i - a_i,$$

where F and G are the odometry Jacobians. Because odometry constrains the relative change between x_{i-1} and x_i , the block on Δx_i is $-I$ ($G_i^i = -I$). Here a_i is the current odometry prediction error.

For a measurement factor k connecting pose x_{i_k} to landmark l_{j_k} , the residual is

$$r_k^{\text{meas}} = H_k^{i_k} \Delta x_{i_k} + J_k^{j_k} \Delta l_{j_k} - c_k,$$

with H and J the measurement Jacobians with respect to the involved pose and landmark, and c_k the corresponding prediction error.

Each residual is measured with a Mahalanobis norm $\|r\|_{\Sigma}^2 := r^\top \Sigma^{-1} r$, using its own covariance, Λ_i for odometry and Γ_k for measurements. This matters because Mahalanobis distance “bakes in” uncertainty. Directions the sensor is confident about are penalized more. Noisy or correlated directions are penalized less, and the metric tilts along correlated axes. As a result, the errors are not judged in plain Euclidean meters/radians but in “standard-deviation units” tailored to each factor. Intuitively, this turns “Euclidean space + covariance” into Mahalanobis space, where the residual ellipses already encode the right weighting. That is why the covariance symbols appear inside the cost, uncertainty is not ignored, it’s embedded in how distance is measured. With this, the whole objective of equation (18) is just “add up all these linearized residuals, each judged fairly in its own noise units, and pick the $\Delta\theta^*$ that makes the total smallest”. Intuitively, factor can be visualized as spring pulling on the variables. Mahalanobis scaling makes the springs stiff along low noise directions and soft along high noise ones, so the solution balances all pulls by their reliability.

Collecting all linearized factors with their covariances, the MAP update $\Delta\theta^*$ is obtained by minimizing the following Mahalanobis-weighted least-squares objective:

$$\Delta\theta^* = \arg \min_{\Delta\theta} \left\{ \sum_{i=1}^M \|F_i^{i-1} \Delta x_{i-1} + G_i^i \Delta x_i - a_i\|_{\Lambda_i}^2 + \sum_{k=1}^K \|H_k^{i_k} \Delta x_{i_k} + J_k^{j_k} \Delta l_{j_k} - c_k\|_{\Gamma_k}^2 \right\} \tag{18}$$

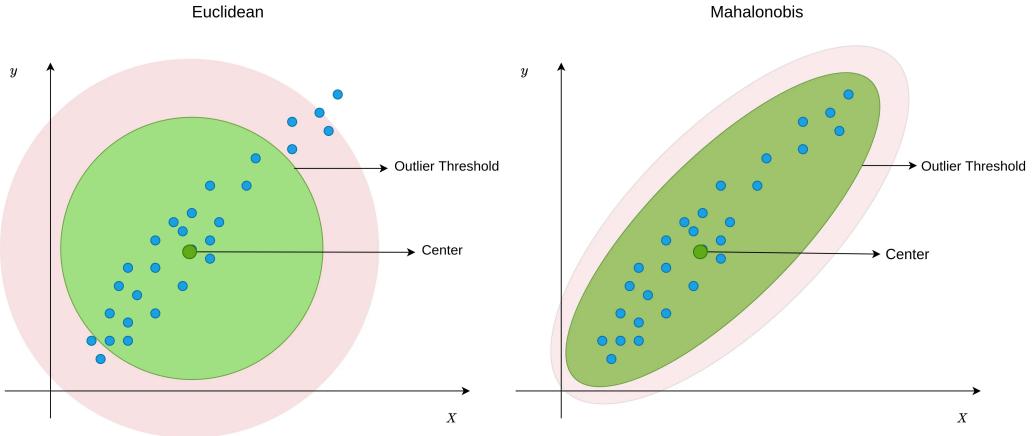


Figure 29: Euclidean vs. Mahalanobis residual contours. *Left:* isotropic (equal) weighting yields circular inlier regions. *Right:* a covariance Σ skews and scales the contours into an ellipse whose axes/tilt follow the noise correlations, whitening this with $\Sigma^{-1/2}$ maps this ellipse back to a circle.^[30]

Mahalanobis distance is just “error measured in the units of its noise” (See Figure 29). If a residual has high variance, it should be penalized less. If two components are correlated, they should not be treated as independent. That’s what the covariance does. In the left plot (Euclidean), all directions are weighted equally so the inlier region is a circle. In the right plot (Mahalanobis), directions with low uncertainty are tighter and correlated axes tilt the ellipse. In SLAM cost function, each residual (process or measurement) is evaluated with its own covariance. Small reliable noises count more, whilst large noisy ones count less. When two parts of a measurement drift together, their error isn’t along x or y alone, it’s along some tilted direction. Mahalanobis tilts the “penalty shape” to match that direction. Penalties are smaller along noisy directions and larger where the sensor data is precise.

Equation (18) is a sum of Mahalanobis residuals (process terms use Λ_i , measurement terms use Γ_k). To turn that into one clean least squares system, first step is to “whiten” each residual so its noise is unit, for scalars divide by the standard deviation, for vectors apply the covariance’s square root inverse to the residual and its Jacobians Σ^{-1} . After whitening, all errors are ordinary Euclidean ones, so the covariance symbols can be dropped, stack the Jacobians into one big sparse matrix A , stack the prediction errors into b , and solve the standard least squares problem (19).

$$\Delta\theta^* = \arg \min_{\Delta\theta} \|A\Delta\theta - b\|^2 \quad (19)$$

Here, θ stacks all unknowns (robot poses x and landmarks l), A is the single large, sparse (whitened) measurement Jacobian formed by stacking the block Jacobians F, G, H , and J from the linearized motion and measurement models, and b is the stacked prediction error vector that collects the current odometry errors a and measurement errors c with a consistent sign convention. Intuitively, A describes how residuals change for small state perturbations, b encodes the present mismatch between predictions and measurements, and solving equation (19) yields the best local correction $\Delta\theta^*$ used to update the estimate.

In the linearized setting, the optimal increment $\Delta\theta^*$ is found by setting the gradient of the least squares objective to zero. This yields the normal equations according to iSAM paper [28]:

$$A^T A \Delta\theta = A^T b$$

Solving this system is typically performed using a numerically stable square root method (QR/Cholesky) rather than forming an explicit inverse. This gives the optimal correction $\Delta\theta^*$. The state estimate is then updated as follows:

$$\theta \leftarrow \theta + \Delta\theta^*$$

9.2.2 Incremental QR for fast updates (iSAM)

The linearized SLAM subproblem is solved by least squares. Solving the normal equations $(A^\top A)\Delta\theta = A^\top b$ with Cholesky can be fast but very unstable and ill conditioned as the problem grows (it squares the condition number and increases fill in). iSAM avoids this by working directly with the whitened Jacobian A using QR factorization, and by updating that factorization incrementally when new factors arrive.

Batch square root form (QR on the Jacobian) can be shown in iSAM paper [28] to be of form:

$$A = Q \begin{bmatrix} R \\ 0 \end{bmatrix}, \quad Q^\top Q = I, \quad R: \text{upper triangular}$$

$$\begin{bmatrix} d \\ e \end{bmatrix} = Q^\top b$$

$$\|A\Delta\theta - b\|^2 = \|R\Delta\theta - d\|^2 + \|e\|^2$$

The iSAM paper [28] shows that after QR the equation is:

$$A\Delta\theta - b = \begin{bmatrix} R \\ 0 \end{bmatrix} \Delta\theta - \begin{bmatrix} d \\ e \end{bmatrix}, \quad \Rightarrow \quad \|A\Delta\theta - b\|^2 = \|R\Delta\theta - d\|^2 + \|e\|^2.$$

Put simply, once QR factorization is performed, the error splits into two parts. To make the total error as small as possible, set the first term to zero and solve:

$$R\Delta\theta^* = d \tag{20}$$

leaving $\|e\|^2$ as the (minimal) residual norm. If R has full rank, this linearized system has one singular unique solution $\Delta\theta^*$.

In iSAM the matrix R is upper triangular, so equation (20) is solved by back substitution (no matrix inverse). This gives a fast, numerically stable way to compute the correction and update the state $\theta \leftarrow \theta + \Delta\theta^*$ without heavy compute.

9.2.3 What is R? The square root information matrix

At the end of QR, the triangular factor R satisfies the following form:

$$R^\top R = A^\top A.$$

This means $A^\top A$ (the information matrix obtained by linearization) is represented by the “square root” R . Working with R keeps all the curvature of the problem but in a form that is easier to use and numerically safer because R is upper triangular, so computations reduce to cheap substitution methods instead of expensive matrix inverses. Uncertainty can also be extracted directly from R . The state covariance is given by:

$$\Sigma = (A^\top A)^{-1} = (R^\top R)^{-1},$$

A dense inverse is never built. In reality, when entries of the uncertainty Σ are needed, solve small triangular systems with R^\top and R , and read only the pose blocks and the pose to landmark blocks of interest. Since R is sparse and triangular, this is fast and stable, and it avoids forming $A^\top A$. (See 9.2.8 Data Association from R)

9.2.4 Matrix Factorization for building QR (Givens rotations)

Here *Givens rotations* is used to build an upper triangular factor R from the (whitened) Jacobian A by zeroing entries below the diagonal, one at a time. This yields a QR factorization without forming $A^\top A$ and without explicitly storing Q .

A Givens rotation is a 2×2 orthogonal transform applied to two rows (or two columns) to annihilate one chosen entry. Givens rotation matrix is defined as:

$$G(\varphi) = \begin{bmatrix} \cos \varphi & \sin \varphi \\ -\sin \varphi & \cos \varphi \end{bmatrix} \tag{21}$$

Start at the leftmost non zero column of the A matrix and sweep to the right, one column at a time. In each column, pick two rows, “ k ” (the current pivot row) and “ i ” (a row below it), and apply the small “rotate and combine” equation (21) so the entry under the diagonal in that column becomes zero. Only those two rows are mixed, the new row “ k ” becomes a bit of the old row “ k ” plus a bit of row “ i ”, and the new row “ i ” becomes a bit of the old row “ i ” minus a bit of row “ k ”. Repeat down the column until all subdiagonal entries are gone, then move to the next column on the right. (see Figure 30 down bellow for a visual of one Givens step)

As the algorithm sweeps the columns of A , the matrix is transformed into the upper triangular form, this is R , and the full Q doesn’t need to be formed to get to the result. Apply the same row rotations to b as you eliminate entries so the right hand side stays consistent. After the initial factorization of A matrix, new measurements don’t require rebuilding A . Later it is shown that new whitened rows can be appended beneath the current R , then a short sequence of row rotations re-triangularizes R . In other words, updates operate directly on R and b , and A is bypassed for incremental steps.

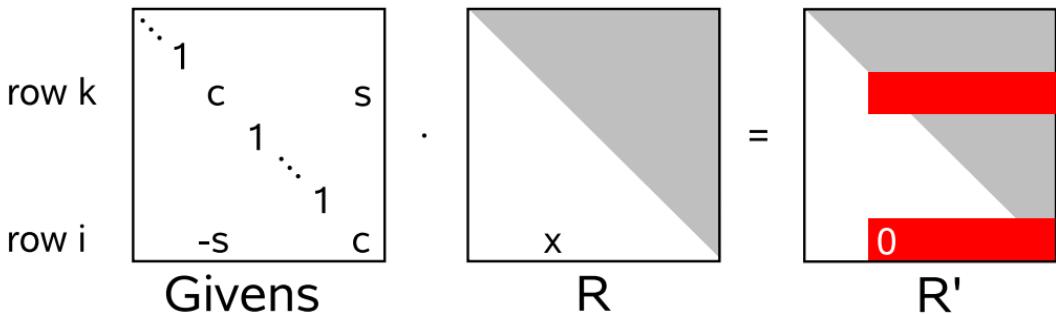


Figure 30: One Givens step in QR. The entry marked “ x ” is eliminated by rotating two rows, only the entries shown in red are modified, and the exact pattern depends on sparsity. Repeating this column wise (left to right) turns the matrix into an upper triangular R . Apply the same rotation to the b vector to keep the least squares system consistent.^[28]

In order to make R upper triangular, φ value must be chosen precisely to zero out a single sub diagonal entry in preliminary matrix, either be it A matrix on batch step or R matrix on iterative steps. The rotation angle φ is computed from the two entries in the current column, the pivot $x = a_{kk}$ and the subdiagonal $y = a_{ik}$.

$$\begin{aligned} r &= \sqrt{x^2 + y^2} = \sqrt{a_{kk}^2 + a_{ik}^2} \\ c &= \cos \varphi = \frac{x}{r} = \frac{a_{kk}}{r} \\ s &= \sin \varphi = \frac{y}{r} = \frac{a_{ik}}{r} \end{aligned}$$

Solving for φ gives the following answer, where $\alpha = x = a_{kk}$ and $\beta = y = a_{ik}$:

$$(\cos \varphi, \sin \varphi) = \begin{cases} (1, 0), & \text{if } \beta = 0, \\ \left(-\frac{\alpha}{\beta} \frac{1}{\sqrt{1 + (\alpha/\beta)^2}}, \frac{1}{\sqrt{1 + (\alpha/\beta)^2}} \right), & \text{if } |\beta| > |\alpha|, \\ \left(\frac{1}{\sqrt{1 + (\beta/\alpha)^2}}, -\frac{\beta}{\alpha} \frac{1}{\sqrt{1 + (\beta/\alpha)^2}} \right), & \text{otherwise.} \end{cases} \quad \text{with } \alpha := a_{kk}, \beta := a_{ik}. \quad (22)$$

These coefficients in equation (22) give the same rotation as (21).

Givens rotations guarantee that the (i, k) entry of the working matrix becomes zero, and they preserve lengths. First the two numbers $[x, y]^\top$ are rotated. When embedded in the full matrix, the same rotation is applied to the affected parts of the two rows and to the matching entries of b . In practice, embed $G_{(i,k)}(\varphi)$ so it acts only on rows k and i , and apply the same rotation to b to keep the least squares system consistent.

9.2.5 Incremental Updating

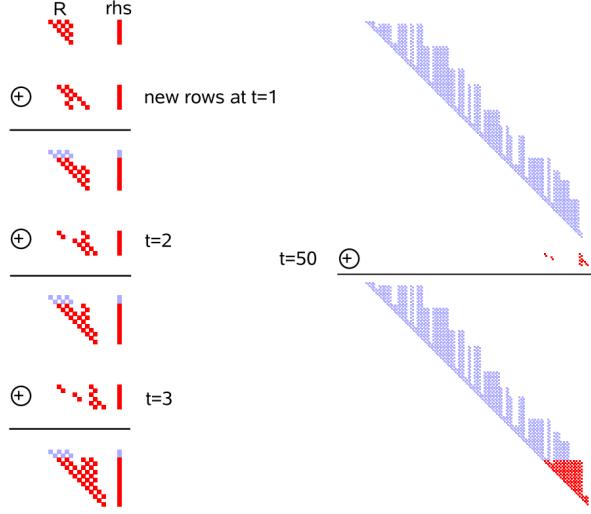


Figure 31: Incremental update of the factored system. A new whitened row w^\top and RHS (Right Hand Side) entry γ are appended beneath the current R and d . A short sequence of Givens rotations restores the upper triangular form, yielding updated R' and d' . Unchanged entries are shown in light color, only a small stencil is touched each step, so update cost stays bounded.^[28]

After the initial QR factorization, maintain the solution in “square root” form, an upper triangular matrix R and a transformed right hand side d . Here, R is the triangular factor that satisfies $R^\top R = A^\top A$ (the Gauss Newton information), and d is the top part of $Q^\top b$. When a new measurement arrives, first whiten it (divide by its standard deviation or apply the square root information of its covariance) so it has unit variance. The whitened measurement contributes a new row w^\top to the Jacobian and a new scalar γ to the RHS (Right Hand Side). Notice that A is NOT rebuild. Instead, append w^\top under the current R , and γ under the current d , which produces a system that is “almost” triangular but has one non triangular row at the bottom.

$$R' = \begin{bmatrix} R \\ w^\top \end{bmatrix}, \quad d' = \begin{bmatrix} d \\ \gamma \end{bmatrix}$$

Next, re-triangularize locally with Givens rotations (21). Only touching the columns where the new whitened Jacobian row w^\top has nonzeros (i.e, the variables that this new factor actually connects to, such as a pose x_i or a landmark l_j). Starting from the leftmost such column, each rotation mixes the current pivot row with the new bottom row to kill one sub diagonal entry. Repeat the process until the entire bottom row is zero and the matrix is upper triangular again. The equation would look something like this:

$$\begin{bmatrix} R \\ w^\top \end{bmatrix} \xrightarrow{\text{Givens rotation on affected columns}} \begin{bmatrix} R' \\ 0 \end{bmatrix}$$

While the matrix is rotates, apply the same rotations to the right hand side so that the least squares system stays consistent. Here d is the transformed RHS (Right Hand Side) before the update and γ is the new whitened RHS entry that pairs with w^\top . After the rotations, the top block becomes the updated RHS d' used for solving, and the final bottom entry becomes a small leftover error e_{new} that adds to the total residual.

$$\begin{bmatrix} d \\ \gamma \end{bmatrix} \xrightarrow{\text{same rotations}} \begin{bmatrix} d' \\ e_{\text{new}} \end{bmatrix}$$

Intuitively, the new row w^\top is “folded up” into the triangular structure by a short chain of 2×2 rotations that only touch the connected variables, everything else is left alone. Then get the correction by a fast back substitution on the updated matrix R' and vector d' :

$$R' \Delta \theta^* = d'$$

9.2.6 Loop Closure

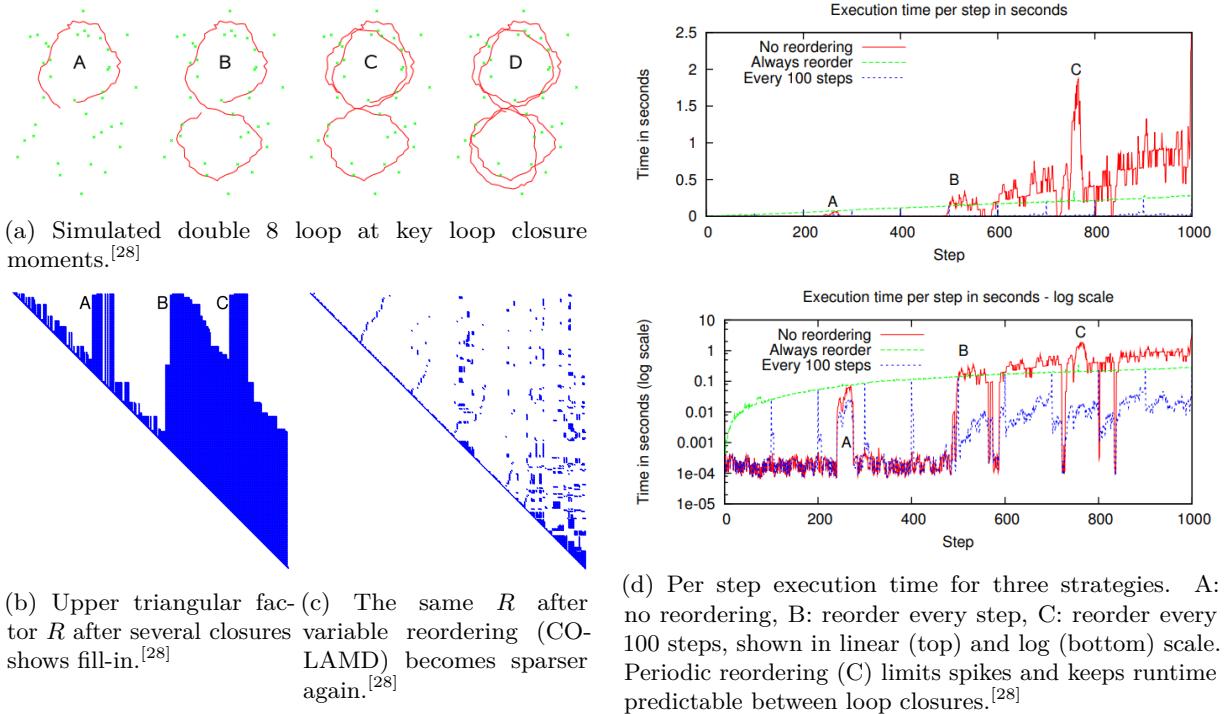


Figure 32: Effect of loop closures and variable reordering (iSAM).^[28]

Loop closures tie together far-apart parts of the trajectory (and landmarks), which makes previously separate columns interact. In QR terms this creates fill-in, R gets extra non zeros, so updates, back substitution, and selected covariance queries get slower and memory grows. This can be fixed by variable reordering. The goal is to pick a new elimination order that preserves sparsity. In practice run a heuristic like COLAMD (Column Approximate Minimum Degree) (often the block version for pose/landmark blocks) on the Jacobian's sparsity matrix A , then do batch factorization on the whole A matrix using rotations (21) with that order [28]. Reordering costs time because the factor must be rebuilt with a new permutation, but it pays back by making subsequent updates cheap again.

Because reordering is expensive, it is not done at every step. Instead, reordering is performed periodically every N steps (e.g., 50 - 200) so the cost stays predictable. In marine AUV and ASV runs this keeps compute bounded. Between reorders incremental updates are fast and local. After a loop closure, one spike occurs (reorder + refactor), then the system returns to low latency. Practical tips when doing this step is to keep poses as blocks (block COLAMD) to reduce fill-in, align reordering with planned relinearization passes, and monitor simple stats (nonzeros in R , update time) to decide when N is too small (wasting time reordering) or too large (letting fill in snowball).

9.2.7 Re-Lineralization

Re-linearization keeps the local model valid. The QR and update methods assume the system is locally linear around the current estimate, but with angles, three dimensional motion, and nonlinear sensor data, that approximation drifts as the robot moves. If it is not refreshed, increments grow, the optimizer biases the map, and loop closures can fail. The remedy is to re-linearize and recompute Jacobians at the current state for the factors that matter. Doing this for every factor at every step is too expensive, so in practice new factors are always linearized, and older ones are refreshed only when needed.

In iSAM the practical schedule is to run incremental updates between maintenance cycles, then perform a full re-linearization every N steps. Between cycles, old factors are not re-linearized, new factors are whitened and inserted, and R is updated incrementally. At the cycle boundary after N steps, the entire problem is re-linearized at the current estimate, the full Jacobian A is rebuilt conceptually, variables are reordered with COLAMD to restore sparsity, and the system is refactored using equation

(21) to obtain a fresh triangular R . For this reason variable reordering is usually grouped with batch re-linearization at the same N step.

N must be chosen carefully, by balancing freshness vs compute. If N is too large, the linearization point drifts far from reality, Jacobians no longer match the true geometry, corrections become biased, loop closures pull hard, and the map can warp (a classic “stale linearization” issue). If N is too small, the system stops often to re-linearize and refactor, wasting CPU and power, and reducing real-time throughput.

9.2.8 Data Association from R

For data association, Mahalanobis based approach is often used instead of plain nearest neighbour Approach. Nearest neighbour measures raw Euclidean distance and ignores sensor noise and correlations. Mahalanobis measures the innovation in the units of its uncertainty, so noisy directions count less, precise directions count more, and correlated components are handled correctly (See Figure 29). For a candidate match between current pose x_i and landmark l_j , form the innovation ν_k and score:

$$d_k^2 = \nu_k^\top \Xi_k^{-1} \nu_k, \quad \Xi_k = J_k \Sigma J_k^\top + \Gamma_k$$

where $J_k = [H^{x_i} \ H^{l_j}]$ is the linearized measurement Jacobian, Γ_k is the sensor noise, and Σ is the state covariance.

This type of data association often uses gating with a chi-square test. The test keeps matches that are within $d_k^2 \leq \chi_{m,\alpha}^2$ (right dimension m , chosen confidence α). From the survivors, pick the “minimum cost” one (or solve a global assignment using d_{ij}^2 as the cost matrix if several features compete). One thing to note about this approach is that the full dense $\Sigma = (R^\top R)^{-1}$ is not required for data association, only the local covariances influenced by the measurement. These covariances can be extracted directly from the square root information matrix R . Pose blocks and pose to landmark blocks are available online as well. Landmark terms on the other hand are either a approximate fast conservative estimate or computed exactly on demand. This keeps the association real-time for the most part. Here are the partitioned forms. Full state covariance (poses vs. landmarks):

$$\Sigma = \begin{bmatrix} \Sigma_{xx} & \Sigma_{xL} \\ \Sigma_{Lx} & \Sigma_{LL} \end{bmatrix}$$

where Σ_{xx} is pose to pose, Σ_{LL} is landmark to landmark, and $\Sigma_{xL} = \Sigma_{Lx}^\top$ is pose to landmark.

The 2×2 submatrix needed for a single candidate (x_i, l_j) :

$$\Sigma_{\{x_i, l_j\}} = \begin{bmatrix} \Sigma_{x_i x_i} & \Sigma_{x_i l_j} \\ \Sigma_{l_j x_i} & \Sigma_{l_j l_j} \end{bmatrix}$$

Fast marginals from the square root factor (online/live)

In iSAM paper [28], they propose keeping the current pose last in the ordering. Then the covariances needed for association, the pose variance $\Sigma_{x_i x_i}$ and the pose to landmark cross terms $\Sigma_{x_i l_j}$ come straight from the square root factor R with two small triangular solves:

$$R^\top Y = B, \quad RX = Y$$

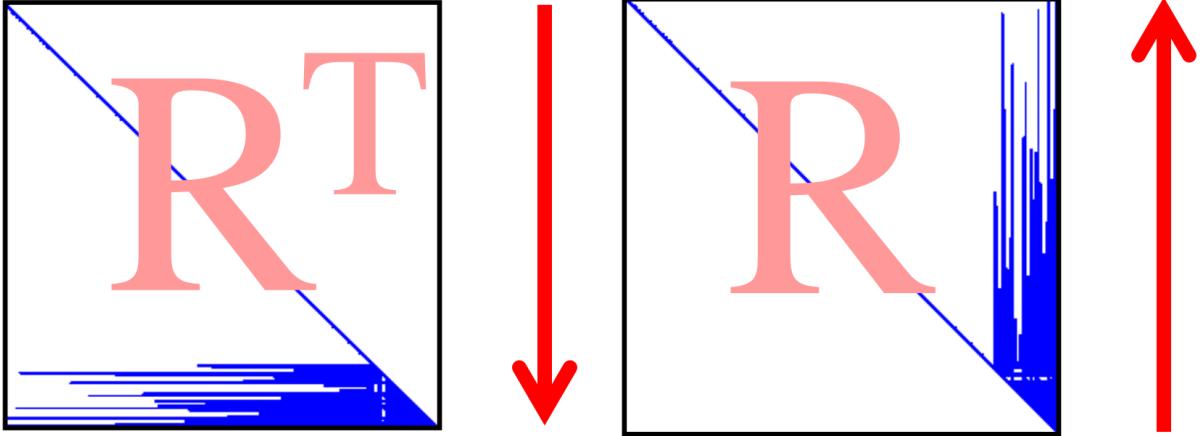
where $B = \begin{bmatrix} 0 \\ I_{d_x} \end{bmatrix}$ simply selects the last pose block of size d_x . Because R is upper triangular and B is zero above the last block, the forward solve gives:

$$Y = [0, \dots, 0, R_{ii}^{-1}]^\top$$

This Y preliminary matrix is the clue, where only d_x back substitutions are needed to get full X vector. Reading the result X yields, in one pass:

$$\Sigma_{x_i x_i} \text{ (bottom-right block of } \Sigma) \quad \Sigma_{l_j x_i} = \Sigma_{x_i l_j}^\top \text{ for connected } l_j$$

Intuitively “solve up” then “solve down” on R for the last pose, and get exactly the columns of $\Sigma = (R^\top R)^{-1}$ that matter for gating in data association. This can be done every step without forming any dense inverses, only using R square root information matrix iteratively. (See Figure 33)



(a) $R^\top Y = B$ (forward substitution). Sweep “down” the matrix to form Y from a selector B that picks the last pose block.

(b) $RX = Y$ (back substitution). Sweep “up” the matrix to obtain the desired columns $X = \Sigma_{:,x_i}$ (pose and pose to landmark).

Figure 33: Grab the needed covariances in two quick steps: first solve “down” with R^\top , then solve “up” with R . Only touch entries near the last pose block, so each update stays fast and cheap.^[28]

Conservative landmark covariances (online/live)

Exact landmark landmark blocks Σ_{jj} (or old pose to landmark $\Sigma_{(i-n)j}$) are expensive to extract at every step. Therefore in iSAM paper [28] they propose using a safe, conservative bound built from the current pose covariance and the measurement noise via the linearized back projection:

$$\tilde{\Sigma}_{jj} = \bar{J} \begin{bmatrix} \Sigma_{ii} & 0 \\ 0 & \Gamma \end{bmatrix} \bar{J}^\top$$

where \bar{J} is the Jacobian of the local inverse measurement model, Σ_{ii} is the current pose covariance, and Γ is the measurement noise. This upper bounds landmark uncertainty (never over confident), is fast, and works well for online Mahalanobis gating. As more measurements of a landmark arrive, this bound typically tightens.

An important caveat to mention is that the true Γ (measurement noise) is usually unknown. The iSAM recipe is to choose Γ conservatively so the algorithms doesn’t get overly confident. That keeps things safe but can make the gate too tight, causing the data association to reject more matches than it should. Later, a more reliable way to approximate Γ from measurement characteristics is described. It avoids overconfidence and uses cues such as sonar range and resolution and landmark confidence, so the gate is realistic and not risky.

Exact landmark covariances (on demand)

When accuracy is needed (eks: on risky loop closure, conflicting hypotheses), Data Association can recover exact Σ_{jj} and $\Sigma_{(i-n)j}$ without forming the full dense inverse information matrix $\Sigma = (A^\top A)^{-1} = (R^\top R)^{-1}$. Because the covariance is the inverse of the information matrix:

$$\Sigma = (R^\top R)^{-1}, \quad R^\top R \Sigma = I$$

Needed entries can be extracted without forming the full inverse by solving for two triangular matrixes:

$$R^\top Y = I, \quad R\Sigma = Y$$

The solve uses only the “nonzero” entries of R , so computation touches only the parts that matter, not the whole matrix. iSAM walks backwards along those non zero links and gives us exactly the covariance numbers σ_{ij} Data Association ask for. If R is mostly banded, this is near linear time. This exact method

should only be used when its really needed (eks: a few Σ_{jj} blocks to check on a loop closure). It's slower than the conservative shortcut, but still much faster than inverting the whole matrix.

Alternative way to finding Γ

There's another angle. Instead of being very conservative with Γ measurement noise. Uncertainty used in Data Association should reflect the sensor, for example a sonar for scanning the sea floor. With sonar the measurement noise can grow with range, and the transducer resolution can set per axis variances $N(0, \Sigma_{sonar})$. The landmark detection can also contribute its own uncertainty $N(0, \Sigma_{landmark})$. In practice these are combined to form the prediction uncertainty for the residual that is scored. In that case the effective covariance is:

$$\Gamma = \text{Var}(h(x, l)) = \Sigma_{sonar} + \Sigma_{landmark}$$

This can be more informative than a one size fits all setting. However, this approach requires care. If the noise terms are too confident, the data association gate becomes too tight, matches become brittle, and the system can become unstable. Nevertheless, it often yields more accurate maps and a better estimate of the robot's track.

9.2.9 Algorithm

At each time step absorb new information, linearize around the current estimate, solve a small least squares, and update. Periodically refresh linearization and variable order. Concretely:

1. **Add factors (whiten first):** take the new odometry/measurements and add them to the graph. “Whiten” them so every residual has unit noise (as described before (19)).
2. **Linearize at the current guess θ :** turn the nonlinear motion/measurement models into local linear ones using the Jacobians in (16) and (17). This gives the summed Mahalanobis cost (18), which after whitening becomes one least squares problem (19).
3. **Keep a triangular system up to date (QR):** append the new (whitened) rows and apply a few Givens rotations (21) so the matrix stays upper triangular R . Update the right hand side b vector the same way (see “Incremental Updating”).
4. **Solve for the small change:** because R is triangular, solve (20) by back substitution (fast) to get the correction .
5. **Update the estimate:** replace the old state with the improved one, $\theta \leftarrow \theta + \Delta\theta^*$.
6. **Every N steps (maintenance):** refresh accuracy by re-linearizing all factors at the new θ , reorder variables (eks: COLAMD algorithm) to keep things sparse, and refactor with Givens (21) to get a clean R .
7. **Data Association update:** read the needed covariances from R (pose and pose to landmark live, landmark blocks conservative or exact on demand) and run Mahalanobis gating in Data Association for next matches.

9.2.10 Limitations

iSAM is fast between updates but has practical downsides. They come from the “data structure”, not the underlying SLAM algorithm. iSAM keeps a single, global square root information matrix R (from $A^\top A$) and does periodic maintenance (reordering + re-linearization). This makes updates simple, but couples cost to global structure and variable ordering instead of just local changes.

- **Latency spikes at maintenance:** Periodic global variable reordering and re-linearization trigger stalls (especially after loop closures), since R must be refactored end to end.
- **Fill in growth between reorders:** Incremental QR on a fixed order accumulates fill in in R , touching more entries per update and increasing time/memory step by step.
- **All or nothing re-linearization:** iSAM typically refreshes many factors at maintenance even if most variables barely moved, wasting Jacobian recomputations.
- **Global refactor on ordering changes:** Any change to elimination order implies a large refactor of the global R , regardless of how small the new information is.

- **Broad marginal queries are costly:** Last pose and nearby cross terms can be pulled quickly from R , but wide Σ blocks (e.g.: many landmarks or older poses) require multiple triangular solves and can be costly.
- **Schedule sensitivity:** Choosing “every N steps” for reorder/re-linearize is heuristic, too small wastes time, too large lets fill in and linearization error grow, causing jitter and warp in the map.
- **Numerical robustness vs simplicity:** Working with A and R avoids explicit $A^\top A$, but long incremental runs plus fill in can still hurt conditioning and stability if ordering lags.

These limitations motivated **iSAM2**, which replaces the single monolithic R that is very static, with a Bayes tree representation that is dynamic and updates only the affected parts. We address iSAM2 and how it mitigates the issues above in the next chapter.

9.3 iSAM2

9.3.1 Introduction and Motivation

iSAM2 was developed to overcome the practical limitations of iSAM. The core optimization method in iSAM (nonlinear least squares solved through incremental QR factorization) is sound and provides accurate solutions. The bottlenecks arise not from the underlying algorithms, but from the static data structure used to maintain the problem. In iSAM the system is stored as a single global square root information matrix R . While this representation is compact and efficient for batch updates, it leads to several issues during incremental operation.

First, the R matrix is “global”. Adding new factors or loop closures often changes many rows and columns, which requires expensive refactorization. This causes latency spikes, especially when loop closures occur and large parts of the trajectory suddenly become coupled. Second, relinearization in iSAM is also global. To maintain accuracy, the system periodically refreshes Jacobians for all factors, which forces full reconstruction of the R matrix. Third, variable reordering to reduce fill in and keep R sparse is again an all or nothing operation, with cost proportional to the entire problem size. Together, these properties mean that iSAM, while efficient between updates, still suffers from periodic heavy computation that disrupts real time performance.

iSAM2 addresses these limitations by introducing a new data structure, the “Bayes tree”. Instead of representing the system as a static R matrix, iSAM2 leverages the factor graph formulation of SLAM, applies variable elimination, and interprets the resulting structure as a chordal Bayes net. This Bayes net can then be compactly represented as a Bayes tree, which retains all probabilistic information while enabling local updates. With the Bayes tree, adding new measurements or relinearizing states only modifies the affected cliques in the tree, leaving the rest untouched. This local property eliminates the global refactorization bottlenecks of iSAM, smooths out computation over time, and makes the algorithm scalable to large and long term mapping problems. [29, 31]

9.3.2 Factor Graphs

A factor graph is a bipartite graph that connects variable nodes (poses and landmarks) to factor nodes (priors, motion, and measurements). It encodes the same estimation problem as in iSAM, but makes sparsity and locality explicit because each factor touches only a few variables.

Let the variables be robot poses x_1, \dots, x_M and landmarks l_1, \dots, l_N , and let $\Theta = \{x_1, \dots, x_M, l_1, \dots, l_N\}$. According to the iSAM2 papers [29, 31] the posterior factorizes as

$$f(\Theta) = \prod_i f_i(\Theta_i),$$

Here, each factor f_i depends only on its adjacent variables Θ_i . How each factor is modeled depends on the situation. Because odometry and measurements are uncertain in the real world, factors should be represented with a probabilistic model. One of the easiest probabilistic models is Gaussian, which is easy to model and will work well later when working in Mahalanobis form to optimize the problem and solve it in simple manner (See Figure 29 for Mahalanobis form). Factor functions can therefore be modeled as follows:

$$\text{Prior: } f_p(x_0) \propto \exp\left(-\frac{1}{2}\|\mu_0 - x_0\|_{P_0}^2\right),$$

$$\text{Odometry: } f_i(x_{i-1}, x_i) \propto \exp\left(-\frac{1}{2}\|f_i(x_{i-1}, u_i) - x_i\|_{Q_i}^2\right),$$

$$\text{Measurement: } f_k(x_{i_k}, l_{j_k}) \propto \exp\left(-\frac{1}{2}\|h_k(x_{i_k}, l_{j_k}) - z_k\|_{R_k}^2\right).$$

This is in a way very similar to how iSAM models the system, however here in iSAM2, the system is represented as factor graphs. This graph view exposes conditional independence directly in the topology. Each factor only connects nearby variables in time or space, which later yields a sparse linear system.

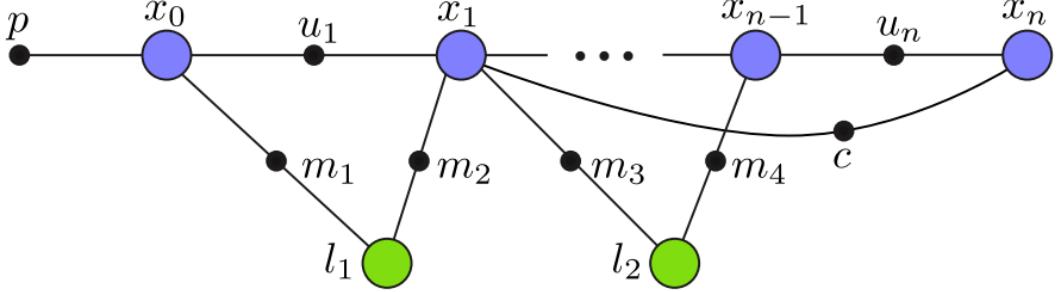


Figure 34: Picture from iSAM2 paper [29] describes factor graph formulation of the SLAM problem. Variable nodes (large circles) are poses x_0, \dots, x_n and landmarks l_1, l_2 . Factor nodes (small solid circles) represent a prior p , odometry u_i , landmark measurements m_i , and a loop closure constraint c . This approach can represent any cost function, including factors that connect more than two variables.

This example shows how local connectivity induces sparsity. This will be useful when solving the optimization problem later down below. Each factor touches only its adjacent variables, so after linearization (to compute a local optimum) the Jacobian rows have nonzeros only in those columns. The resulting matrix is sparse, which helps the optimizer.

9.3.3 From Factor Graphs to the SLAM Optimization Problem

Here SLAM is represented with a factor graph $G = (\mathcal{F}, \Theta, \mathcal{E})$. There are two node types, factor nodes $f_i \in \mathcal{F}$ that encode pieces of information (prior, odometry, measurements, loop closures), and variable nodes $\theta_j \in \Theta$ that hold unknowns (poses, landmarks, calibration). An edge $e_{ij} \in \mathcal{E}$ is drawn only when factor f_i depends on variable θ_j . This wiring is important, missing edges mean independence, and the pattern of edges controls which variables interact in the estimation problem.

The graph specifies how a global objective splits into simple parts:

$$f(\Theta) = \prod_i f_i(\Theta_i),$$

Here Θ_i collects only the variables that touch factor f_i . In the SLAM setting, a prior p anchors the first pose, odometry factors u relate consecutive poses, landmark factors m couple a pose with a landmark, and loop closure factors c link poses that see the same place again (see Picture 34). The same framework can also handle factors that involve three or more variables, for example a factor that ties a pose to a landmark and a camera intrinsics block (calibration), or “separator” variables shared in cooperative mapping. The key point is that the graph can host any cost term as long as it states which variables it touches.

Under Gaussian measurement models as defined in the previous subsection, each factor has the form:

$$f_i(\Theta_i) \propto \exp\left(-\frac{1}{2} \|h_i(\Theta_i) - z_i\|_{\Sigma_i}^2\right),$$

This Gaussian form will simplify calculations as Gaussian is nice to work with. Here $h_i(\cdot)$ predicts what the sensor should see from the current variables Θ_i , z_i is the actual measurement, and Σ_i is the measurement covariance. The notation $\|e\|_{\Sigma}^2 \triangleq e^\top \Sigma^{-1} e$ is the squared Mahalanobis distance, which measures error in “units of its noise” (directions with low variance are penalized more). (See Picture 29)

To combine all information, simply multiply the factor likelihoods. Products are awkward to optimize, instead take a negative logarithm to turn the product into a sum. Terms that do not depend on Θ drop out, and each Gaussian factor becomes a squared Mahalanobis residual weighted by its covariance. The result is one scalar objective that collects the prior, all odometry factors, all landmark measurements, and any loop closures. Small covariances make a factor count more, large covariances count less. In short, multiply the factors and take the negative log to get a single sum of squared errors. This can be written in compact form as:

$$\Theta^* = \arg \min_{\Theta} \frac{1}{2} \sum_i \|h_i(\Theta_i) - z_i\|_{\Sigma_i}^2$$

Here Θ collects all unknowns, such as poses, landmarks, and other parameters. Each factor h_i depends only on a small subset Θ_i , so each residual couples only those variables. After linearization this gives a sparse system, because most variables do not appear together in any single residual.

This is our MAP function in nonlinear form.

This nonlinear cost is not solved in one shot because the measurement transform $h_i(\cdot)$ is nonlinear (because of angles, ranges, bearing-only sensors, etc...). As in the iSAM system, linearize around the current estimate and solve iteratively. Choose a current estimate Θ^0 and look for a small update $\Delta\theta$ that improves the fit. For each factor take a 1st-order Taylor expansion around Θ^0 , which turns that factor into a simple linear residual in the increment $\Delta\theta$ that touches only its adjacent variables. After whitening by $\Sigma_i^{-1/2}$, stack all linearized factors into one sparse least-squares problem:

$$\Delta\Theta^* = \arg \min_{\Delta\Theta} (-\log(f(\Delta\Theta))) = \Delta\theta^* = \arg \min_{\Delta\theta} \|A\Delta\theta - b\|^2 \quad (23)$$

Here $A \in \mathbb{R}^{m \times n}$ is the measurement Jacobian (one row block per factor), b stacks the whitened prediction errors, and $\Delta\theta$ is the n -dimensional increment. The sparsity pattern of A is dictated by the factor graph, a row has nonzeros only in the columns of the variables that appear in that factor.

Here the linear least squares problem is solved in a numerically stable manner. One route is the normal equations $A^\top A \Delta\theta = A^\top b$ and a Cholesky factorization $A^\top A = R^\top R$ followed by forward and back substitution to recover $\Delta\theta$. Another route is QR factorization on A , which yields an upper triangular system $R\Delta\theta = d$ that can be solved by back substitution. Both routes are standard, in practice QR factorization methods are preferred for stability.

After solving for $\Delta\theta$, update the state $\theta \leftarrow \theta + \Delta\theta$ and repeat, linearize, solve, update. This is Gauss-Newton. If the problem is difficult (poor linearization, strong nonlinearity), add Levenberg-Marquardt damping and instead solve $(A^\top A + \lambda I)\Delta\theta = A^\top b$, which blends Gauss-Newton with a trust-region step to keep updates safe. Stop when $\Delta\theta$ is small or the cost no longer decreases.

This is the same least squares core as in iSAM. The difference is the data structure. iSAM works with the Jacobian A and its triangular factor R . iSAM2 keeps the factor graph as the main object, which carries the same information as A from iSAM but in a graph form. Eliminate variables to get a Bayes net and store it as a Bayes tree. This lets iSAM2 update only the cliques touched by new factors instead of refactoring everything. The next subsection shows how this is done.

9.3.4 From Factor Graphs to Bayes Networks

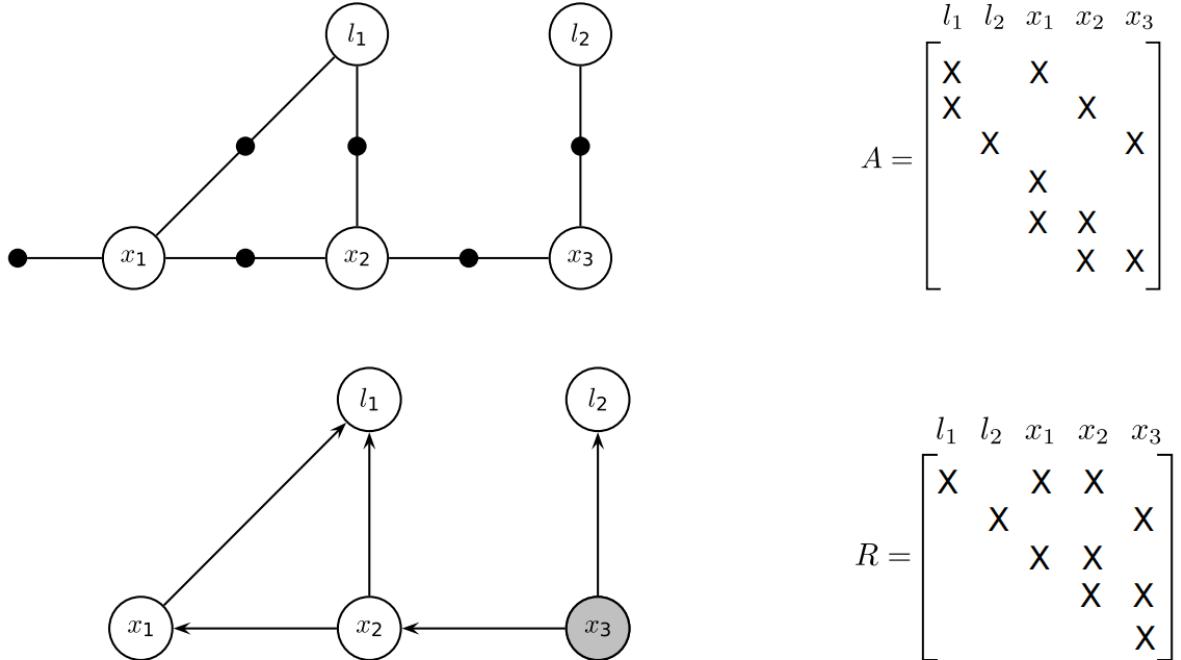


Figure 35: Picture from iSAM2 paper [29]. (Top) factor graph and associated Jacobian A for a small SLAM example with poses x_1, x_2, x_3 , landmarks l_1, l_2 , and a prior on x_1 (Bottom) the chordal Bayes net and the square root factor R obtained by eliminating in the order l_1, l_2, x_1, x_2, x_3 . The last eliminated variable (the root) is shaded. [29]

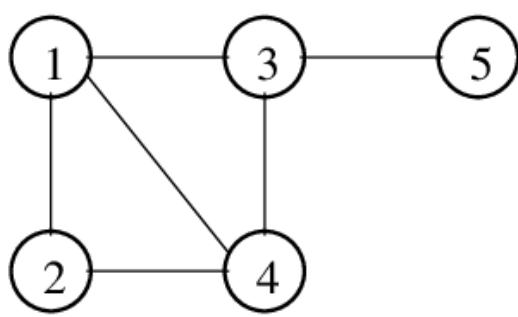
Starting from the least squares form in (23), the whitened measurement matrix A is factored as in iSAM. An elimination order is chosen, (for the example in Picture 35 it is l_1, l_2, x_1, x_2, x_3), and sparse QR with Givens rotations (21) is applied. This produces an orthogonal Q and an upper triangular R . Because R is triangular, solving by back substitution proceeds variable by variable in the chosen order. That solve can be read as a chain of simple Gaussian conditionals, one per variable, where the off diagonal entries to the right of each pivot indicate which previously eliminated variables that conditional depends on. Drawing arrows from the parent variables to the current variable turns the same structure into a directed graphical model. In short, for a chosen ordering, sparse QR on the factor graph yields an R that encodes a Bayes network, this is what the bottom panel of Picture 35 shows.

Instead of running sparse QR on the whitened Jacobian, the same result can be obtained by eliminating variables directly on the factor graph using bipartite elimination game methods [32]. Starting from (23), choose an order, then for each variable collect its adjacent factors, combine them, marginalize that variable out, and attach the resulting factor to the remaining neighbors. Repeat until all variables are removed. For any fixed ordering, the purely graphical elimination produces the same dependency pattern and the same square root information factor as numeric QR factorization. This procedure forms a Bayes network with chordal properties, and in the matrix view yields an upper triangular R with $R^\top R = A^\top A$ (see Picture 35). The advantage is that A does not need to be assembled, and Givens rotations and explicit QR factorization are avoided. The computation stays on the graph, the ordering controls fill, and the desired chordal structure is obtained. In practice this is the same QR algebra, just done in a graph aware way that avoids unnecessary intermediate fill and work according to Good Column Orderings for Sparse QR Factorization paper [32].

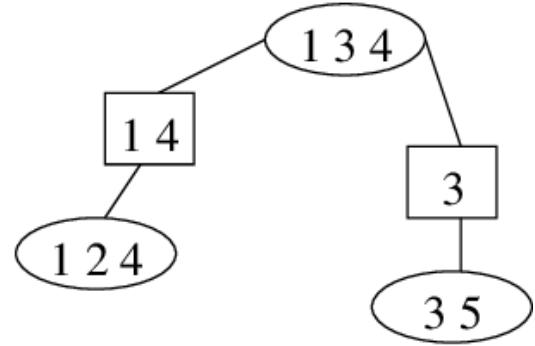
This observation is the bridge to iSAM2. A chordal Bayes network groups naturally into cliques, this can be represented as a Bayes tree. The next subsections uses this tree, rather than a single global R , to support local updates and avoid global re-factorizations.

9.3.5 R as a Bayes tree data structure

A key outcome of variable elimination on the SLAM factor graph is a chordal Bayes network. Chordal means that in the moralized (undirected) view every long cycle has a shortcut edge, which keeps parents of a variable grouped in small cliques and prevents excessive fill in during elimination. This property is what makes the square root factor R sparse and tractable, and it also sets up a clean bridge to a tree representation of the same information. [29, 31]



chordal graph



junction tree

Figure 36: Picture from Robert Castelo paper [33] shows an example of a simple Chordal graph (Left side) and corresponding small cliques/junction tree (Right side). Eliminating in a good order produces a chordal Bayes net whose moralized graph can be grouped into cliques/junction trees. This is the structure that keeps R matrix sparse.

When linearizing and eliminating in some order, the resulting triangular system R still satisfies $R^\top R = A^\top A$, but each row block of R matrix now carries a specific meaning. Each row block of R belongs to the variable that was just eliminated. That row says, in Gaussian form, how this variable depends on a few variables that were eliminated earlier. Think of those earlier variables as its parents. When solving $R\Delta\theta = d$ by back substitution, start at the last row and move upward. That is the same as computing each variable from its parents in turn. So R is not just numbers in a triangle. It is the same set of conditional relationships as the Bayes network, written in matrix form.

Because the Bayes net is chordal, its conditionals naturally group into cliques. Consecutive row blocks of R that share the same separator form a clique. Cliques that share a separator are connected to form a Bayes tree. Each clique stores frontal conditionals, child given the parent separator, and each edge carries only the shared separator variables. The tree encodes the same numeric information as R , but it is organized by locality rather than by a single global ordering.

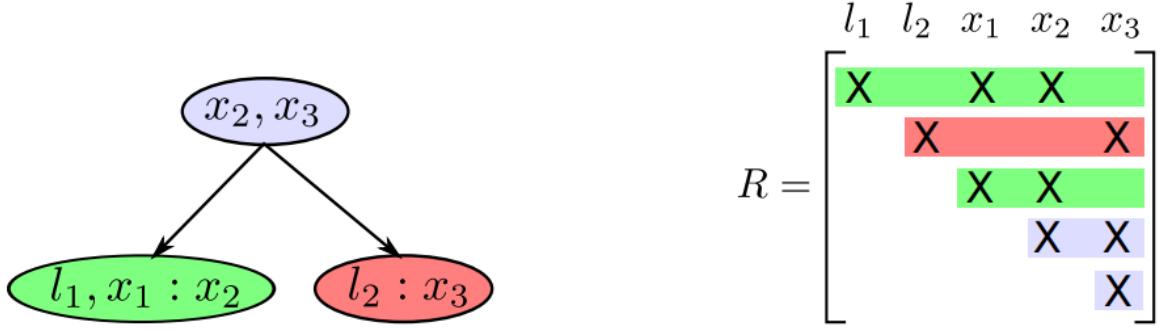


Figure 37: Picture from Bayes tree paper [31] shows Bayes tree (left) and its matching rows in the square root factor R (right). Colors indicate which contiguous row blocks of R belong to each clique.

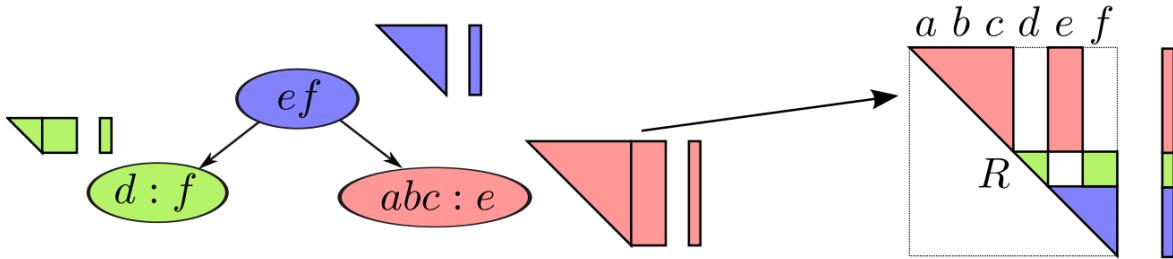


Figure 38: Picture from iSAM2 paper [29] shows each clique contains the conditional of its frontal variables given the separator. The same entries appear in the corresponding rows/columns of R . Back substitution on R mirrors evaluating the tree from leaves to root, while marginal queries follow the few cliques that touch the queried variables.

The takeaway is simple. Eliminating a factor graph gives a chordal Bayes net. Grouping its conditionals yields a Bayes tree. The square root factor R contains the very same conditionals in matrix form. Thus R can be viewed as a Bayes tree data structure in matrix form. iSAM2 stores and updates this Bayes tree directly instead of one global R , which preserves the numerical benefits of the square root form while enabling strictly local updates. This means when new measurements arrive or when some variables must be re-linearized, only the cliques on a small subtree are touched and the rest of the structure stays unchanged, making computational complexity manageable and the data set grows.

9.3.6 Incremental Updates directly on Bayes tree

iSAM2 never rebuilds the whole system when new data arrives. A new odometry or landmark factor only touches a few variables in the factor graph, so only the matching subtree in the Bayes tree is modified. All other branches stay exactly the same. This is the key difference from iSAM, where adding a factor could force a global re-factorization of the single R matrix.

Two simple rules explain why updates stay local. First is that information flows upward in the tree during elimination, so changes propagate only from the touched variables toward the root. Second, a factor becomes active when the first variable in its local elimination order is eliminated, so only the paths from those variables up to the root can be affected, while unrelated subtrees remain untouched.

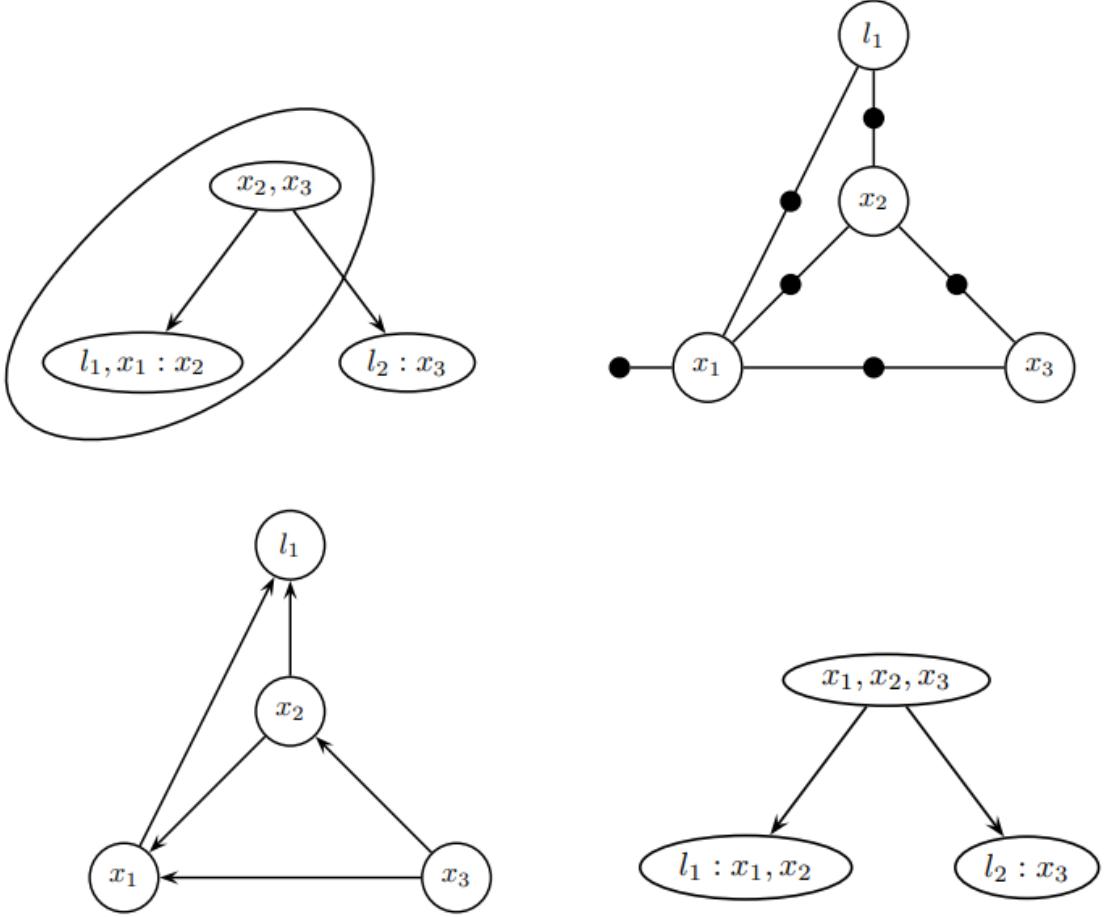


Figure 39: Picture taken from Bayes tree paper [31] shows how to update a Bayes tree with a new factor.
 (Top left) Affected cliques when adding a factor between x_1 and x_3 , the right branch is unaffected.
 (Top right) Local factor graph rebuilt from those cliques plus the new factor.
 (Bottom left) Local chordal Bayes net after elimination.
 (Bottom right) New Bayes subtree with the untouched “orphan” subtree reattached.

One update works as follows. first, locate the cliques that contain the variables touched by the new factor and follow their ancestors up toward the root, this marks the only region that needs work, while the rest of the tree becomes “orphans” that stay valid and untouched. Next, convert the conditionals stored in those marked cliques back into a small local factor graph and insert the new factor. Then re-eliminate just this local graph (using the same graph aware elimination as discussed previously) to produce a new chordal Bayes net and its updated Bayes subtree. Finally, reattach the orphan subtrees at the proper separators. Only this small subtree changes, everything else is reused. (See Picture 39)

The result is incremental and predictable computation. iSAM2 edits only the cliques touched by the new information, runs a small local elimination, and solves by back substitution along that subtree. There are no global re-factorization spikes, and accuracy is maintained by re-linearizing only the variables in the affected region when needed.

9.3.7 Loop Closure and Incremental Reordering

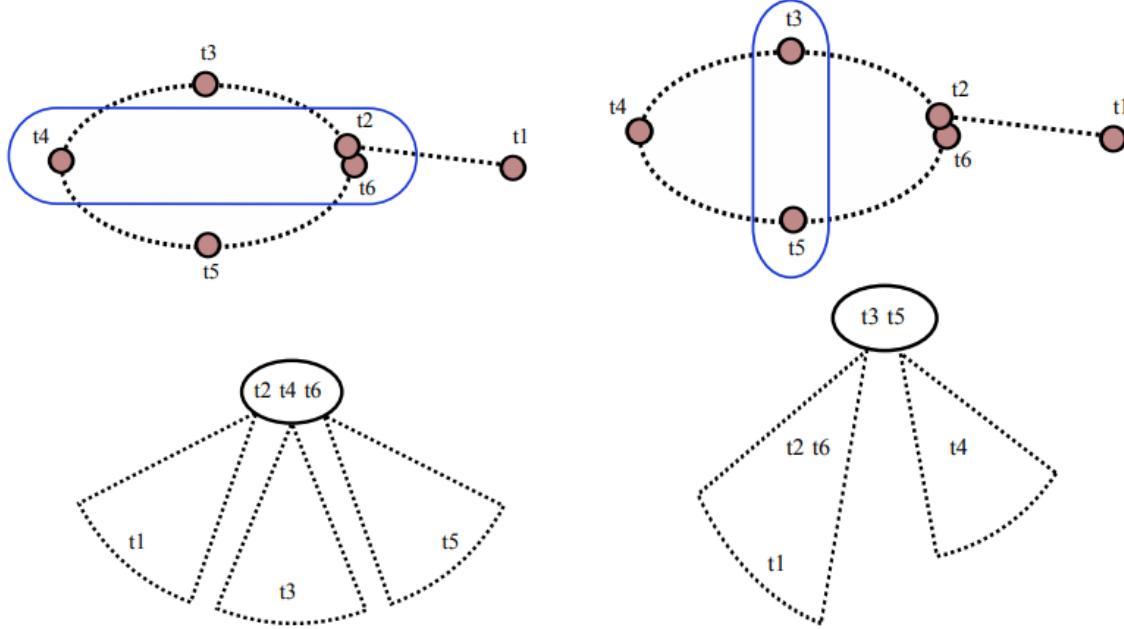


Figure 40: Picture taken from Bayes tree paper [31] shows loop closure with incremental reordering. Two batch optimal orderings (top) yield different Bayes trees (bottom). For online operation, the ordering should keep the newest variables near the root, so future updates affect only a small subtree.

Loop closures add a factor between two far apart poses. In a Bayes tree this never forces a global re-factorization. Only the cliques on the (unique) paths from those two pose cliques up to the root are affected, all other subtrees are untouched “orphans” and are reused as is. The affected top region is converted back to a local factor graph, the loop closure factor is added, and the region is re-eliminated to produce a new local chordal Bayes net and Bayes subtree. The unchanged subtrees are then reattached at the matching separators. This keeps updates local and predictable, unlike iSAMs global R re-factorization after loop closure. [29, 31]

A good variable ordering is still crucial because it controls fill in (clique sizes) during elimination. iSAM2 performs incremental reordering only over the affected variables, rather than periodic global reorderings. A simple and effective rule is to force the most recent variables (the ones new factors usually touch) to be eliminated last (i.e. near the root). Practically, this is implemented with a constrained COLAMD heuristic. Here the newest pose blocks are kept at the end of the order while letting COLAMD algorithm find a sparse order for the rest. This produces small, stable updates at each step, even when loops close. [31]

Batch orderings found by nested dissection (or similar heuristics) can look equally good for a one time solve because they produce comparable sparsity. For online SLAM the next update matters. The preferred ordering therefore leaves the newest poses at or near the Bayes tree root, so the next odometry or loop closure factor changes only a small subtree. If the newest pose lies deep in the tree, the same update must rewrite many cliques. For this reason iSAM2 uses constrained reorderings, keeping recent variables last in the order near the root and letting the heuristic arrange the rest. This keeps updates local and cheap. [31]

This constrained COLAMD heuristic will not yield a globally optimal ordering, however it reliably reduces fill in and the amount of work near the update, keeps recent variables near the root, and avoids large latency spikes. In practice it delivers close to batch sparsity while saving compute time, and when needed it is reapplied only to the affected subtree, so the cost stays proportional to that small region rather than the whole tree.

9.3.8 Fluid Re-Linearization

iSAM2 stops doing periodic global “re-linearize everything”. Instead iSAM2 only refresh (re-linearize) the parts of the problem that truly need it, right when they need it. The Bayes tree makes this easy, new measurements change only a small subtree, so iSAM2 recomputes just that piece and leave the rest of the tree alone. This keeps the math accurate without the big stalls that happened in iSAM after loop closures or long runs. [29, 31]

How it works in practice is simple. Always keep a running correction vector Δ from the latest linear solve. If a variable’s change is tiny, treat its current linearization point as “good enough” and do not touch it. If a variable’s change is larger than a small threshold (call it β), mark that variable for re-linearization. Then mark the cliques that contain those variables and their ancestors in the Bayes tree. Only those cliques are rebuilt, go back to the original nonlinear factors for those cliques, recompute their Jacobians at the new linearization point, add cached “marginal” factors from untouched children, and eliminate again to update just the top of the tree. Everything else remains as it was. Note that this re-linearization threshold can be set per state. Positions (x, y) can use a looser (higher) threshold so they are refreshed less often, while the heading angle, being more nonlinear should use a tighter (lower) threshold. [31]

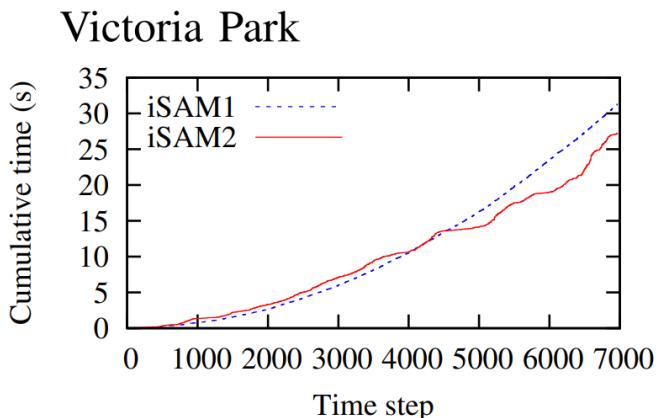
Solving for $\Delta\theta$ is also done only where needed. First solve on the modified top of the tree. Then walk into children only if any parent update exceeded a small propagation threshold (call it α). This “only follow when it matters” rule avoid needless work in distant parts of the map while still keeping accuracy where the robot is and where measurements arrived. Together, the two thresholds (β to decide who to re-linearize, and α to decide where to propagate the solve) give iSAM2 its fluid, real-time/online behavior. Accuracy when and where it matters, speed everywhere else.

This local, threshold strategy removes the need for heavy batch steps and keeps runtime smooth over long missions just like incremental reordering does the same. In essence what iSAM2 does is it exploits Factor graphs representations of the map and Bayes tree data structure to do dynamic variable reordering and re-linearization, no need for periodic batch calculations like in iSAM. On standard datasets (Victoria Park) the cumulative computation of iSAM2 grows much more slowly than iSAM, because re-linearization and re-elimination are confined to small subtrees rather than the full graph. And over time the discrepancy will only grow where iSAM will slow down whilst iSAM2 will hold its compute much more efficient. (See Picture 41)

Victoria Park



Victoria Park SLAM map used in the original iSAM benchmarks.



Cumulative time vs. step for iSAM (periodic batch) and iSAM2 (fluid). iSAM2 stays faster as the dataset grows.

Figure 41: Picture from the Bayes tree paper [31]. In iSAM2, relinearization is fluid, only the affected cliques are recomputed and the rest of the tree is reused.

9.3.9 Sparse Factor Graphs

Over long missions the factor graph can accumulate many near duplicate constraints (revisits of the same place, repeated landmark sightings from similar viewpoints). This “Eiffel Tower effect” slowly densifies the graph and enlarges cliques in the Bayes tree, which increases update cost. The fix is sparsification. Here keep the informative constraints and summarize or drop the redundant data points while preserving the important information flow. In practice this is done locally where data arrive. Retain the freshest odometry and a few diverse loop closures, and replace discarded constraints by a light summary on the small separator set where they would have entered the tree. Intuitively, keep the “shape” of the information at the boundary and forget interior details that are now redundant. Simple policies such as keyframing (keeping only selected poses as variables), pruning highly correlated measurements, and limiting per landmark observation count work well. The result is a graph that stays sparse, cliques that stay small, and a Bayes tree that remains cheap to update even in very long runs.

9.3.10 Beyond Gaussian Assumptions (Robust Estimators)

Pure Gaussian residuals are fragile in the face of outliers (bad data association, spurious loop closures, moving objects, changing environment). Robust estimators fix this by replacing the quadratic loss with a robust loss that grows slower than a square. Common choices include Huber (quadratic near zero, linear in the tails), Cauchy, or Tukey. In iSAM2 this is a drop in change at the factor level. Each robust loss yields a weight for its residual, updated as the estimate improves (iteratively re-weighted least squares). Factors that fit well keep high weight, inconsistent ones are down weighted, so they no longer dominate the solution. This makes incremental updates and fluid re-linearization safer because a single wrong constraint will not trigger large edits high up in the tree. For hard loop closures, one can also use switchable or graduated penalties that let the optimizer “turn off” a suspect factor until there is enough supporting evidence. The Bayes tree concept stays the same, only local factor weights and linearization adapt, so robustness comes with little extra complexity.

9.3.11 Data Association from the Bayes Tree

The same as in iSAM, iSAM2 scores candidate matches with a Mahalanobis distance, which measures the innovation in units of its uncertainty. However in iSAM2 the needed covariances are read directly from the Bayes tree without forming a dense matrix. Pose and nearby pose to landmark covariances are obtained efficiently by following only the few cliques that contain those variables and their separators. This keeps online/real-time gating fast for data association. Queries involving far away landmarks or many old poses may touch a larger portion of the tree and therefore cost more, but they remain practical on demand. In practice this combines a cheap, conservative bound (for routine gating) with exact small block queries when decisions are critical (e.g. verifying a loop closure). The result is reliable association with predictable compute cost during operation.

9.3.12 Algorithm

At each step iSAM2 absorbs new measurements, touch only the relevant cliques in the Bayes tree, solve for a small increment, and update the state. iSAM2 combines the linear update with fluid re-linearization, so work stays local and predictable. Concretely, one iteration looks like this (matching the structure summarized in the iSAM2 and Bayes tree papers [29, 31])

1. **Add new data:** Insert the new factors (odometry, measurements, loop closures) into the factor graph. If new states appear, add them to the estimate Θ .
2. **Mark what to refresh (fluid re-linearization):** Keep the last increment $\Delta\theta$. If a state moved more than a small threshold β , mark it for re-linearization. Only pass this mark to neighbors if the parent changed more than a smaller threshold α . This finds the small set that really needs work now.
3. **Build a small local problem:** Take just the cliques in the Bayes tree that touch the marked states (and their ancestors up to the root) and turn them back into a tiny factor graph, everything else becomes reusable “orphans”.
4. **Order and eliminate locally:** Find a sparse order for this small graph using a constrained COLAMD (keep the newest states last, near the root), then eliminate to make a new local Bayes subtree.

5. **Reattach orphans:** Connect the untouched subtrees back at the correct separators. Only the edited subtree changed, the rest is reused.
6. **Solve where needed:** Back solve on the updated top of the tree to get a new increment $\Delta\theta$. Propagate the solve into children only when the parent's change is big enough (same α rule).
7. **Update the estimate:** Apply the increment on the whole factor graph, $\Theta \leftarrow \Theta \oplus \Delta\Theta$ (where $\theta = \Theta$ and $\Delta\theta = \Delta\Theta$). Keep $\Delta\Theta$ for the next steps re-linearization test.
8. **Keep variable ordering healthy (incremental):** When a loop closure grows cliques near the root, run constrained COLAMD again, but only on that small region to reduce fill and keep the newest states near the root.
9. **Data association update:** Fetch the needed local covariances from the Bayes tree, compute far away covariances only on demand, and compare predicted and observed features using chosen Data Association method.

9.3.13 Limitations

While iSAM2 removes the big computation spikes seen in iSAM, some practical and theoretical limits remain.

- **Ordering is heuristic, not optimal:** Choosing a variable order that minimizes fill in is NP-hard, so iSAM2 relies on constrained COLAMD and related heuristics. These give good, stable performance online but cannot guarantee the globally best sparsity.
- **Clique growth in dense areas.** Heavy revisiting of the same places (“Eiffel Tower effect”) or many near duplicate constraints can enlarge cliques near the root. Updates remain local, but the cost of each local elimination grows with clique size. In long runs, sparsification/keyframing is often needed to keep the graph light.
- **Threshold tuning for fluid re-linearization:** The accuracy/speed trade off depends on two small thresholds (who to re-linearize and how far to propagate the solve). These must be tuned for the sensor and motion model. Too loose can delay accuracy, too tight does extra work.
- **Faraway covariances can be expensive:** Exact marginal/covariance queries are done by recursive message passing on the tree (dynamic programming style). Queries that span long paths or large separators touch more cliques and therefore cost more.
- **Gaussian least-squares core:** Outliers and non Gaussian effects are not handled by iSAM2 alone. Robust losses or switchable constraints must be added at the factor level to down weight bad data, otherwise accuracy can degrade during long missions.
- **Nonlinearity still matters:** Poor initial guesses or highly nonlinear measurements can require multiple Gauss-Newton/Levenberg-Marquardt steps. iSAM2 just makes each step local.
- **Engineering complexity and memory:** Compared to a single global R in iSAM, the Bayes tree in iSAM2 adds much more moving parts (cliques, separators, cached/orphan subtrees, incremental reordering). Correct, efficient implementations are more involved, and memory still grows with map size unless one prunes or summarizes.

In practice, most of these limits can be mitigated with careful design. Using keyframing/sparsification to cap clique size, robust losses or switchable constraints to handle outliers, and constrained incremental reordering to keep updates local. The main hurdle is engineering complexity. This is where the open source **GTSAM** library (from the Georgia Tech team behind iSAM/iSAM2) is invaluable. It ships a production quality Bayes tree/iSAM2 implementation, clean factor graph APIs, robust noise models, and utilities for ordering and re-linearization, making it a practical starting point for both research and deployment.

9.4 GTSAM

Georgia Tech Smoothing and Mapping, GTSAM, is a BSD licensed C++ library for modeling estimation problems as factor graphs and solving them efficiently with batch optimizers and with incremental iSAM2 and Bayes tree methods. The core idea is to represent knowledge as a product of small factors, each involving only a few variables, and then exploit sparsity and variable elimination to compute fast and stable MAP estimates. In practice this gives a single, uniform toolkit for SLAM in 2D and 3D, visual odometry/SLAM, structure from motion, calibration, and related inference tasks, all built on the same mathematical foundation described in the iSAM2 and Bayes tree papers [34, 29, 31].

Design philosophy: graph first, values separate: GTSAM cleanly separates the “*model*” from the “*state*”. A `NonlinearFactorGraph` stores factors (priors, odometry, landmark/vision measurements, loop closures), while a `Values` container holds one current assignment to the unknowns (poses, landmarks, intrinsics, etc...). The estimate can be changed without touching the graph, and factors can be added or removed without invalidating unrelated variables. This mirrors the math formulation $f(\Theta) = \prod_i f_i(\Theta_i)$. The graph captures structure and sparsity. A particular Θ provides an assignment that can be evaluated or optimized. [34]

Core building blocks: Variables use compact “*keys*” (e.g: `Symbol('x', i)` for pose x_i , `Symbol('l', j)` for landmark l_j). You build the problem by adding small, typed “*factors*”, each encoding one piece of sensor information plus its noise model. For example `PriorFactor<Pose2>`, `BetweenFactor<Pose2>`, `BearingRangeFactor2D`, camera factors like `GenericProjectionFactor<Cal3_S2>`, and many others. Noise models are explicit and first class (`noiseModel::Isotropic`, `noiseModel::Diagonal`, robust M-estimators), so units and weighting are clear and consistent. There are different ways to represent rotations in 3D space, however GTSAM has defined poses in Lie groups, this is a mathematical way to describe 3D space including rotation in an easy and intuitive to handle way. Because poses live on curved rotation/pose spaces (SE(2)/SE(3)), GTSAM updates them using “*local coordinates*” and a “*retraction*” operator. GTSAM computes a small 3D/6D increment in a flat tangent space and then maps it back to a valid pose, avoiding angle wrap around and keeping rotations proper. In practice, Gauss-Newton/Levenberg-Marquardt linearization methods “just works” with orientations, no ad hoc hacks needed. The end result is that writing SLAM code feels like drawing the factor graph, add one factor per measurement between the variables it touches, then optimize. GTSAM handles sparsity, ordering, and iSAM2s incremental updates under the hood. [34]

Batch optimization and linear algebra under the hood: In GTSAM, SLAM is posed as a `NonlinearFactorGraph` with initial `Values`. At each optimizer step, GTSAM linearize the factors at the current estimate to obtain a linear system. Rather than forming one huge matrix to represent this linearized system, GTSAM solves this linear step by *variable elimination*. Choose an order, combine the factors that touch the next variable, eliminate it, and keep going. The result can be viewed as a Bayes net, grouping by shared separator variables yields a Bayes tree, through which the solution is recovered by simple back substitution. For online use, iSAM2 keeps that Bayes tree and, when new measurements arrive or some states need re-linearization, it rebuilds only the small subtree that is affected and reuses the rest unchanged. Speed and memory depend on the elimination order, so GTSAM provides practical heuristics (e.g: COLAMD and constrained COLAMD) that reduce fill in and keep the newest poses near the root so updates stay local. In short, add small typed factors, GTSAM handles linearization and sparse elimination, and with iSAM2 it updates only where needed. [34]

iSAM2 and the Bayes tree in GTSAM: For real-time use, GTSAM iSAM2 keeps a Bayes tree data structure, instead of one giant monolithic R matrix. When a new measurement arrives, it usually touches only a few variables, so iSAM2 edits just that small part of the tree, it pulls out the affected piece, re-solves that tiny subproblem, and snaps it back in place while leaving the rest untouched. Re-linearization is local and on-demand, only refreshing variables if it moves past a small (per-state) threshold, and only push the solve down the tree if a parent changed a lot. The variable order is maintained incrementally (via constrained COLAMD heuristics) so the newest poses stay near the root, which keeps future updates local and fast. In code simply add factors and initial guesses, call `isam.update(...)`, and read out the current estimate (and local covariances) without the big compute spikes of global re-factorization. [34]

What using GTSAM looks like: Create a `NonlinearFactorGraph` and a `Values` with first initial guesses. As new sensor data arrives, add the right factors (odometry, landmark/vision, loop closures) and insert any new variables. For a batch solve, run `GaussNewtonOptimizer` or `LevenbergMarquardtOptimizer` and read the improved `Values`. For real-time use, keep an `ISAM2` object, call `update(newFactors, newValues)` each step, then get the current best estimate with `isam.calculateEstimate()`. When uncertainty is required for data association or validation, compute marginal covariances only for the variables that matter, no giant matrix inverse needed. `Pose2/Pose3`, `Point2/Point3`, camera calibration, and robust noise models can be mixed in the same graph, and all follow the same pattern. [34]

Educational and practical: GTSAM is designed to match the mathematics in the iSAM and iSAM2 papers [28, 29]. Each measurement becomes a small, typed “*factor*”. Current guesses live in a `Values` container that understands poses and rotations, and the solvers make sparsity and variable ordering visible. GTSAM provides clear examples and MATLAB and Python bindings, so ideas can be tested quickly and results plotted with minimal setup. The focus is clarity and research, not necessarily efficiency. There are a lot of abstractions, and being pedagogical and educational comes before optimizations to the code and specific hardware for CPU and GPU maximum performance. The core methods from the iSAM and iSAM2 papers [28, 29] are exactly the same here behind the algorithms, square-root solving, variable elimination, Bayes trees, and iSAM2. GTSAM has powered real robots and vision systems, so for most projects GTSAM is “good enough” as a reliable back end that can be read, extended, and trusted. [34]

What GTSAM does NOT do (and how to fill the gap): GTSAM is a back-end optimizer, it does not detect features, track them, perform loop detection, or decide data association for you. Those front-end tasks live outside and feed GTSAM through factors. Robustness to outliers (bad matches, spurious loops, moving objects) is handled by choosing robust loss functions or switchable/graduated penalties at the factor level. Over very long runs, mitigate graph growth (the “Eiffel Tower” effect) with keyframing and sparsification so cliques stay small and updates remain local. For distant covariance queries, expect higher cost because more of the tree is touched. Use approximate or local covariances for routine gating, and reserve exact queries for critical decisions. With clever design these gaps can be resolved. [34]

Takeaway: GTSAM provides a principled, factor graph centric way to model estimation problems and couples it with high quality batch and incremental solvers built on iSAM2 and the Bayes tree. It preserves the numerical strengths of square root factorization while delivering the locality needed for real-time operation. GTSAM provides a practical implementation of iSAM2 ideas for direct use in SLAM systems. [34]

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