TTK4551 Technical Cybernetics - Specialization Project

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Contents

Acronyms

AUR Lab	Applied Underwater Robotics Lab
AUV	Autonomous Underwater Vehicle
COLAMD	Column Approximate Minimum Degree
EKF	Extended Kalman Filter
GNSS	Global Navigation Satellite System
iSAM	Iterative Smoothing And Mapping
LAUV	Light Autonomous Underwater Vehicle
MAP	Maximum A Posteriori
SLAM	Simultaneous Localization And Mapping
SSS SLAM	Side Scan Sonar Simultaneous Localization And Mapping

1 Introduction

1.1 Goal

This specialization project studies how marine robots navigate on their own, with a focus on SLAM. The aim is to build a solid, practical understanding of modern navigation and SLAM algorithms, how current SLAM systems are structured, and which methods are used in practice. The work centers on sonar-based SLAM, in particular Side Scan Sonar (SSS) SLAM for AUVs, because side scan sonar is widely available on mature platforms and is useful for mapping large seabed areas. I will work with real AUV datasets (for eks from NTNUs AUR Lab platforms such as LAUV Harald) to evaluate how well SSS SLAM supports navigation tasks, and to see what it takes to run the pipeline close to real time. The project builds on prior work, including the 2023 master thesis by Vegard Haraldstad on a sidescan sonar pipeline [1] and the 2024 Intelligent Robots and Systems (IROS) paper on side scan sonar based landmark detection for underwater vehicles [2]. The plan is to study these methods, reimplement the key parts, adapt them where needed and make SSS SLAM run in real time on real hardware.

1.2 Motivation

Marine robots must make decisions without constant human input, often far from easy access. They need to know where they are and what surrounds them to move safely and do useful work. A prior map helps, but the ocean changes over time. Currents, waves, moving vessels, new structures, and shifting seabeds make static maps go out of date. GNSS data is weak or unavailable underwater, and dead reckoning drifts over time. Even in coastal areas, fjords and valleys can block signals, and ships operating in shallow water want to avoid the bottom with good margins. Because of this, robots need to build and update their own map while they localize in it, this is a SLAM problem. SLAM fuses sensors, handles loop closures to correct drift, and provides a consistent pose and an up to date map during the mission. Sonar is a key sensor for robust navigation underwater, and side scan sonar in particular provides wide swath, high resolution imagery that can reveal seafloor structure and landmarks suitable for data association. Focusing on SSS SLAM lets us study a concrete, relevant problem, how to extract stable features from side scan sonar, associate them across passes, close loops, and feed this information into a SLAM back-end that remains fast and stable over long runs. This matters both for precise mapping and for safe, efficient navigation when external positioning is unreliable.

The motivation is to run state of the art SLAM on real hardware in real time and see how it actually performs. We want to measure accuracy, robustness, and runtime on real AUV data, not just on papers or simulations, and learn what changes are needed to make it reliable at sea. In short, take modern SLAM, make it work on the robot, and judge it by field results.

1.3 Side Scan Sonar SLAM Architecture

SOme images here of basic overvies

Talk abit on SLAM

Then a picture of complex overview

Talk a bit more in depth on slam

2 Sensors

3 Data Processing

4 Preintegration

5 Data Association

6 Optimizers

6.1 Introduction

Optimizers are the engine behind the SLAM update step. Sensors add constraints, but the optimizer decides how the state moves to satisfy them. In practice we solve a Maximum A Posteriori (MAP) problem, we want the most likely trajectory (and landmarks) given all measurements and priors. With some assumptions we will discuss later like Gaussian noise and a first order linearization, MAP turns into nonlinear least squares problem that we solve iteratively. At each iteration we linearize the residuals around the current estimate and compute an increment that improves the state. This "correction" is what lets SLAM reduce drift, enforce loop closures, and keep the map consistent. [3]

There are two dominant families for doing state estimation in SLAM, filtering and smoothing. Filtering methods, like EKF-SLAM (Extended Kalman Filter) and particle-filter SLAM, maintain a rolling belief over the current state only (or a short window). They update online/live as measurements arrive by propagating the state and compressing all past information into the filter's covariance or a set of weighted particles. This is simple and has low memory, but it throws away structure in old constraints and it can become inconsistent after many linearizations, especially when revisiting places (loop closures) or when correlations span long time intervals. Particle filters can represent multi modal beliefs but scale poorly with dimension and often need heavy resampling and clever proposal distributions to avoid degeneracy, something that is difficult to achieve in practice. [4][5]

Smoothing methods keep a dense record of the variables we care about (eks: the whole robot trajectory and, if needed, landmarks) and all the measurement factors that tie them together. Instead of only "where am I now?", smoothing asks "what is the entire trajectory and map that best fits everything we have ever seen?". This global view tends to produce better accuracy and consistency, especially when closing loops or fusing many asynchronous sensors. Computationally, smoothing exposes sparse structure, meaning each measurement only touches a few variables, so the global normal equations are large but very sparse. Modern linear algebra plus careful data structures exploit that sparsity and outperform classical filters on realistic SLAM workloads. That is why smoothing has become the predominant approach in modern SLAM systems.

We keep an estimate of the unknowns, call it θ (robot poses, and landmarks). Each measurement gives an error, or residual, that says how far our current estimate is from what the sensor expects. Close to the current estimate, we can approximate how those residuals change if we nudge θ a little. This is a first order (linear) approximation. Stacking all residuals together, that approximation looks like this [6]:

$$r(\theta + \Delta\theta) \approx A\Delta\theta - b$$

Here A is the Jacobian, it tells us how each residual changes with each variable. The vector b is the residual at the current estimate (with a sign convention so the equation above points toward reducing error). The small vector $\Delta\theta$ is the "correction" we want to compute. [6]

The update step chooses $\Delta\theta$ that reduces all residuals as much as possible. The best practice here is to use MAP approach. Here if we assume measurement noise is Gaussian (it is not always true and we will later discuss how to solve for non Gaussian noise, for now this assumption will suffice). After linearizing the residuals around the current estimate, the MAP problem becomes a least-squares fit [6]:

$$\Delta\theta^\star = \arg\min_{\Delta\theta} \; \|A\Delta\theta - b\|^2 \tag{1}$$

Optimize this estimate so that change in $\Delta\theta^\star$ is equla to 0, ie linearize. When linearized, equation (1) can be simplified to a so called normal equation [6]:

$$A^T A\Delta\theta = A^T b$$

This equation system can be then be solved by Cholesky decomposition of $A^T A$ or by optimization algorithms we will be discussing down bellow. Solve this linear system for $\Delta\theta$, then update/correct the estimate [6]:

$$\theta = \theta + \Delta\theta$$

For stability on harder problems we can add Levenberg-Marquardt damping, but the core idea stays the same across these optimizer algorithms. [7]

Classical batch smoothing forms the full information matrix, eliminates variables in a chosen order, and solves for all states together. That is accurate but not ideal for online use. Every new measurement would, in principle, require rebuilding and refactoring a large system, with cost growing with mission length. Real robots need real-time behavior, so we prefer iterative, incremental smoothing that reuses previous computation. The idea is to keep the factorization of the linearized problem in a data structure that can be updated locally when new factors arrive, only touching the parts of the graph that actually change.

This is where Iterative Smoothing and Mapping (iSAM and iSAM2) methods come in. They exploit that SLAM data are very sparse and mostly locally connected, a new odometry or measurement links a pose to a neighbor pose or a nearby landmark, not to everything. iSAM maintains a square-root factor (via QR) and updates it incrementally using Givens rotations, with occasional reordering to control fill in. It keeps uncertainty queries fast and avoids full resolves, except when needed. iSAM2 goes further by expointing factor graphs and organizing the factorization into a Bayes tree data type (a directed tree of cliques). On new measurements, only the impacted cliques are relinearized and refactored, and variables are reordered incrementally. As a result, work scales with the local update rather than the entire graph, this makes update step “fluid”.

In modern SLAM the hard part isn’t “doing SLAM”, it’s solving the SLAM optimization fast as data grows. Most methods use the same MAP correction loop. Linearize, solve for $\Delta\theta^\star$, update θ. The real difference is how we represent and update the problem. Smart data structures and good variable ordering keep data structures sparse and decoupled, and solves quick. Meanwhile bad data structure representation of data causes slowdowns.

This is exactly why, for SLAM on marine vessels, especially AUVs with tight space, power, and compute budgets but strict real-time needs, iterative smoothing methods like iSAM2 are a strong fit. They reuse prior factorizations, add new measurements as local factors, relinearize and refactor only the affected cliques, and reorder variables incrementally. In practice that means low latency, bounded memory and CPU load, and accuracy close to batch solutions, even on long missions.

6.2 iSAM

6.2.1 Getting to SLAM update step

Before we can compute a good estimate, we need a simple model of how the robot moves and how sensors observe the world. We use a motion model for state evolution and a measurement model for sensor readings. States are x_i (robot poses), controls are u_i, and measurements z_k (landmarks). We stack all unknowns into θ (poses and landmarks).

Motion (process) model:

$$\begin{aligned} x_i &= f_i(x_{i-1}, u_i) + w_i \\ w_i &\sim N(0, Q_i) \end{aligned}$$

Given the previous state x_{i-1} and control u_i, the next state x_i comes from a model f plus uncertainty noise in the model itself w_i. This uncertainty captures things like currents, slip, and actuator errors. f_i can be a discrete time dynamics update or use plain odometry. Assuming Gaussian w_i is a handy start so MAP becomes least squares. Later we can switch uncertainty model to robust or heavy tailed noise if needed.

Measurement model:

$$\begin{aligned} z_k &= h_k(x_{i_k}, l_{j_k}) + v_k \\ v_k &\sim N(0, R_k) \end{aligned}$$

Each measurement z_k depends on state x_{i_k} and landmarks l_{j_k} transformed using measurement transform function $h_k(\cdot)$, this allows state estimate to become estimated measurement position. In addition this measurement has noise v_k witch we model ad Gaussian noise for simplifications later on when calculating.

Prior:

$$x_0 \sim N(\mu_0, \Sigma_0)$$

A prior anchors the graph (otherwise the problem is underdetermined up to a global transform). It can encode GPS at the start, a known dock pose, or simply a weak "zero" prior to fix gauge.

We want our predictions to match measurements. In a perfect world, every residual (prediction minus measurement) would be zero. In reality we have model errors and sensor noise, so residuals are nonzero. Estimation is about choosing the state update that makes all residuals as small and as statistically consistent as possible.

This is where MAP algorithm comes in. MAP (Maximum A Posteriori) is the principled way to fuse everything we know. A prior on the state, the motion model, and all measurements. It combines them through probability, weighting each residual by its uncertainty. With Gaussian noise, the negative log posterior becomes a sum of squared (weighted) residuals. That gives us a single objective to minimize, where more reliable terms (small covariance) count more. This is better than ad hoc weighting and naturally handles many sensors.

Our motion and measurement functions are nonlinear (angles, rotations, ranges). Minimizing the nonlinear MAP cost directly is hard. Linearization lets us solve it iteratively. At the current estimate we approximate the nonlinear functions by their first order Taylor expansion, solve a linear least squares problem for a small increment, update the estimate, and repeat. This is all shown in the iSAM paper [6] where linearized forms of our system becomes:

$$\begin{aligned} f_i(x_{i-1}, u_i) - x_i &\approx (F_i^{i-1}\Delta x_{i-1} - \Delta x_i) - a_i \\ F_i^{i-1} &:= \left.\frac{\partial f_i(x_{i-1}, u_i)}{\partial x_{i-1}}\right|_{x_{i-1}^0} \\ a_i &= x_{i-1}^0 - f_i(x_{i-1}^0, u_i) \end{aligned} \tag{2}$$

$$\begin{aligned} h_k(x_{i-1}, u_i) - z_k &\approx (H_k^{i_k} \Delta x_{i_k} - J_k^{j_k} \Delta l_{j_k}) - c_k \\ H_k^{i_k} &:= \left. \frac{\partial h_k(x_{i_k}, l_{j_k})}{\partial x_{i_k}} \right|_{(x_{i_k}^0, l_{j_k}^0)} \\ J_k^{j_k} &:= \left. \frac{\partial h_k(x_{i_k}, l_{j_k})}{\partial l_{j_k}} \right|_{(x_{i_k}^0, l_{j_k}^0)} \\ c_k &= z_k - h_k(x_{i_k}^0, l_{j_k}^0) \end{aligned} \tag{3}$$

Now we can plug the linearized forms of the odometry (2) and measurement models (3) into a single least squares estimator. After linearization, the problem becomes “find the small state change $\Delta\theta$ that best reduces all residuals at once”. To keep the notation uniform, we treat the current pose increment like any other variable (that’s why $G_i^i = -I$ appears). Next, we whiten each term, ie pre multiply by the square root of the inverse covariance so every residual has unit variance. For scalars, this is just dividing by the measurement standard deviation. Whitening turns all the weighted (Mahalanobis) errors into plain Euclidean errors, so we can drop the covariance symbols and stack everything into one big, sparse least squares system.

$$\Delta\theta^* = \arg\min_{\Delta\theta} \left\{ \sum_{i=1}^{M} \| F_i^{i-1} \Delta x_{i-1} + G_i^i \Delta x_i - a_i \|_{\Lambda_i}^2 + \sum_{k=1}^{K} \| H_k^{i_k} \Delta x_{i_k} + J_k^{j_k} \Delta l_{j_k}) - c_k \|_{\Gamma_k}^2 \right\} \tag{4}$$

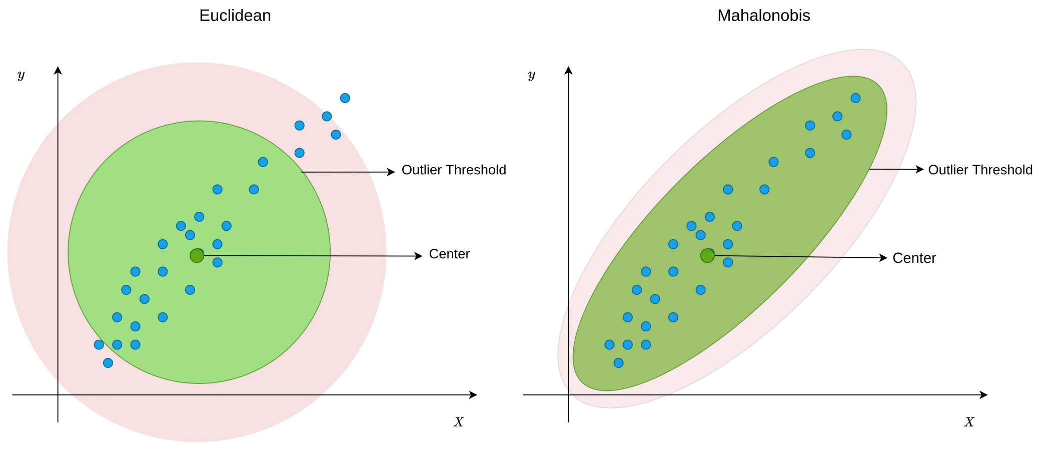


Figure 1: Euclidean vs. Mahalanobis residual contours. *Left:* isotropic (equal) weighting yields circular inlier regions. *Right:* a covariance Σ skews and scales the contours into an ellipse whose axes/tilt follow the noise correlations, whitening this with $\Sigma^{-1/2}$ maps this ellipse back to a circle.[8]

Mahalanobis distance is just “error measured in the units of its noise” (See Figure 1). If a residual has high variance, we shouldn’t punish it as much, if two components are correlated, we shouldn’t treat them as independent. That’s what the covariance does. In the left plot (Euclidean), all directions are weighted equally so the inlier region is a circle. In the right plot (Mahalanobis), directions with low uncertainty are tighter and correlated axes tilt the ellipse. In our SLAM cost, each residual (process or measurement) is evaluated with its own covariance. Small reliable noises count more, whilst large noisy ones count less. When two parts of a measurement drift together, their error isn’t along x or y alone, it’s along some tilted direction. Mahalanobis tilts the “penalty shape” to match that direction. We are punished less along noisy directions and more where the sensor is precise.

Equation (4) is a sum of Mahalanobis residuals (process terms use Λ_i, measurement terms use Γ_k). To turn that into one clean least squares system, we “whiten” each residual so its noise is unit, for scalars divide by the standard deviation, for vectors apply the covariance's square root inverse to the residual and its Jacobians Σ^{-1}. After whitening, all errors are ordinary Euclidean ones, so we drop the covariance symbols, stack the Jacobians into one big sparse matrix A, stack the prediction errors into b, and solve the standard least squares problem (5).

$$\Delta\theta^\star = \arg\min_{\Delta\theta} \ \|A\Delta\theta - b\|^2 \tag{5}$$

Here, θ stacks all unknowns (robot poses x and landmarks l), A is the single large, sparse (whitened) measurement Jacobian formed by stacking the block Jacobians F, G, H, and J from the linearized motion and measurement models, and b is the stacked prediction error vector that collects the current odometry errors a and measurement errors c with a consistent sign convention. Intuitively, A describes how residuals change for small state perturbations, b encodes the present mismatch between predictions and measurements, and solving equation (5) yields the best local correction $\Delta\theta^\star$ used to update the estimate.

In the linearized setting, the optimal increment $\Delta\theta^\star$ is found by setting the gradient of the least squares objective to zero. This yields the normal equations according to iSAM paper [6]:

$$A^T A\Delta\theta = A^T b$$

Solving this system is typically performed using a numerically stable square root method (QR/Cholesky) rather than forming an explicit inverse. This gives the optimal correction $\Delta\theta^\star$. The state estimate is then updated as follows:

$$\theta \leftarrow \theta + \Delta\theta^\star$$

6.2.2 Incremental QR for fast updates (iSAM)

We solve the linearized SLAM subproblem by least squares. Solving the normal equations $(A^\top A)\Delta\theta = A^\top b$ with Cholesky can be fast but very unstable and ill conditioned as the problem grows (it squares the condition number and increases fill in). iSAM avoids this by working directly with the whitened Jacobian A using QR factorization, and by updating that factorization incrementally when new factors arrive.

Batch square root form (QR on the Jacobian) can be shown in iSAM paper [6] to be of form:

$$A \ = \ Q\begin{bmatrix} R \\ 0 \end{bmatrix}, \qquad Q^\top Q = I, \qquad R\text{: upper triangular}$$

$$\begin{bmatrix} d \\ e \end{bmatrix} \ = \ Q^\top b$$

$$\|A\Delta\theta - b\|^2 \ = \ \|R\Delta\theta - d\|^2 + \|e\|^2$$

The iSAM paper [6] shows that after QR we have:

$$A\Delta\theta - b \ = \ \begin{bmatrix} R \\ 0 \end{bmatrix}\Delta\theta - \begin{bmatrix} d \\ e \end{bmatrix}, \quad \Rightarrow \quad \|A\Delta\theta - b\|^2 = \|R\Delta\theta - d\|^2 + \|e\|^2.$$

Put simply, once we do QR, the error splits into two parts. To make the total error as small as possible, we make the first part zero by solving:

$$R\Delta\theta^\star = d \tag{6}$$

leaving $\|e\|^2$ as the (minimal) residual norm. If R has full rank, this linearized system has one singular unique solution $\Delta\theta^\star$.

In iSAM the matrix R is upper triangular, so we solve equation (6) by back substitution (no matrix inverse). This gives a fast, numerically stable way to compute the correction and update the state $\theta \leftarrow \theta + \Delta\theta^\star$ without heavy compute.

6.2.3 What is R? The square root information matrix

At the end of QR, the triangular factor R satisfies the following form:

$$R^\top R \;=\; A^\top A.$$

This means $A^\top A$ (the information matrix obtained by linearization) is represented by the "square root" R. Working with R keeps all the curvature of the problem but in a form that is easier to use and numerically safer because R is upper triangular, so computations reduce to cheap substitution methods instead of expensive matrix inverses. Uncertainty can also be extracted directly from R. The state covariance is given by:

$$\Sigma \;=\; (A^\top A)^{-1} \;=\; (R^\top R)^{-1},$$

We never build a dense inverse. When we need entries of the uncertainty Σ, we solve small triangular systems with $R^\top$ and R and read only the pose and pose to landmark blocks we care about. Since R is sparse and triangular, this is fast and stable, and we avoid forming $A^\top A$. (See 6.2.8 Data Association from R)

6.2.4 Matrix Factorization for building QR (Givens rotations)

We use *Givens rotations* to build an upper triangular factor R from the (whitened) Jacobian A by zeroing entries below the diagonal, one at a time. This yields a QR factorization without forming $A^\top A$ and without explicitly storing Q.

A Givens rotation is a 2×2 orthogonal transform applied to two rows (or two columns) to annihilate one chosen entry. Givens rotation matrix is defined as:

$$G(\varphi) = \begin{bmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{bmatrix} \tag{7}$$

Start at the leftmost non zero column of the A matrix and sweep to the right, one column at a time. In each column, pick two rows, *"k"* (the current pivot row) and *"i"* (a row below it), and apply the small "rotate and combine" equation (7) so the entry under the diagonal in that column becomes zero. Only those two rows are mixed, the new row *"k"* becomes a bit of the old row *"k"* plus a bit of row *"i"*, and the new row *"i"* becomes a bit of the old row *"i"* minus a bit of row *"k"*. Repeat down the column until all subdiagonal entries are gone, then move to the next column on the right. (see Figure 2 down bellow for a visual of one Givens step)

As we sweep the columns of A, the matrix is transformed into an upper triangular form, this is R and we never need to build the full Q. Apply the same row rotations to b as you eliminate entries so the right hand side stays consistent. After the initial factorization of A matrix, new measurements don't require rebuilding A. We will illustrate later that we can just append the new (whitened) rows under the current R and apply a short sequence of the same row rotations to re triangularize R matrix again. In other words, updates operate directly on R (and b), we bypass A entirely for incremental steps.

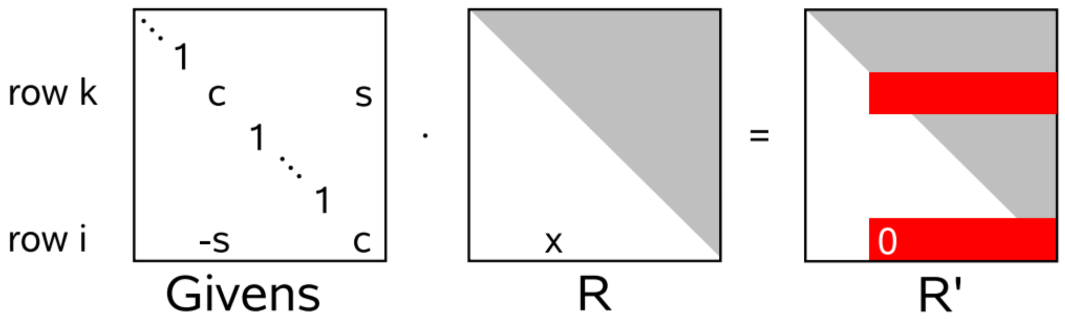


Figure 2: One Givens step in QR. The entry marked "x" is eliminated by rotating two rows, only the entries shown in red are modified, and the exact pattern depends on sparsity. Repeating this column wise (left to right) turns the matrix into an upper triangular R. Apply the same rotation to the b vector to keep the least squares system consistent.[6]

In order to make R upper triangular, we need to get perfect φ value to zero a single sub diagonal entry in preliminary matrix, either be it A matrix on batch step or R matrix on iterative steps. we choose a rotation angle φ from the two numbers we want to combine in the current column, the pivot $x = a_{kk}$ and the subdiagonal $y = a_{ik}$.

$$r = \sqrt{x^2 + y^2} = \sqrt{a_{kk}^2 + a_{ik}^2}$$
$$c = \cos\varphi = \frac{x}{r} = \frac{a_{kk}}{r}$$
$$s = \sin\varphi = \frac{y}{r} = \frac{a_{ik}}{r}$$

Solving for φ gives us the following answer, where $\alpha = x = a_{kk}$ and $\beta = y = a_{ik}$:

$$(\cos\varphi,\ \sin\varphi) = \begin{cases} (1,\,0), & \text{if } \beta = 0, \\ \left(-\frac{\alpha}{\beta}\frac{1}{\sqrt{1+(\alpha/\beta)^2}},\ \frac{1}{\sqrt{1+(\alpha/\beta)^2}}\right), & \text{if } |\beta| > |\alpha|, \\ \left(\frac{1}{\sqrt{1+(\beta/\alpha)^2}},\ -\frac{\beta}{\alpha}\frac{1}{\sqrt{1+(\beta/\alpha)^2}}\right), & \text{otherwise.} \end{cases} \quad \text{with } \alpha := a_{kk},\ \beta := a_{ik}. \tag{8}$$

These coefficients in equation (8) give the same rotation as (7). They guarantee the (i, k) entry in the working matrix becomes zero, and they do it without changing lengths first for the two number pair $[x,\, y]^\top$ we rotate, and, when embedded, for the affected parts of the two rows (and the matching entries of b). In practice, embed $G_{(i,k)}(\varphi)$ so it acts only on rows k and i, and apply the same rotation to b to keep the least squares system consistent.

6.2.5 Incremental Updating

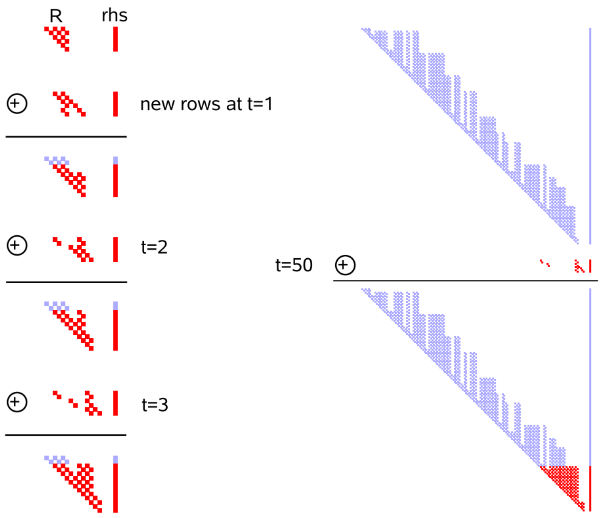


Figure 3: Incremental update of the factored system. A new whitened row $w^\top$ and RHS (Right Hand Side) entry γ are appended beneath the current R and d. A short sequence of Givens rotations restores the upper triangular form, yielding updated R' and d'. Unchanged entries are shown in light color, only a small stencil is touched each step, so update cost stays bounded.[6]

After the initial QR factorization, we maintain the solution in "square root" form, an upper triangular matrix R and a transformed right hand side d. Here, R is the triangular factor that satisfies $R^\top R = A^\top A$ (the Gauss Newton information), and d is the top part of $Q^\top b$. When a new measurement arrives, we first whiten it (divide by its standard deviation or apply the square root information of its covariance) so it has unit variance. The whitened measurement contributes a new row $w^\top$ to the Jacobian and a new scalar γ to the RHS (Right Hand Side). Notice that we do NOT rebuild A. Instead, we append $w^\top$ under the current R, and γ under the current d, which produces a system that is "almost" triangular but has one non triangular row at the bottom.

$$R' = \begin{bmatrix} R \\ w^\top \end{bmatrix}, \qquad d' = \begin{bmatrix} d \\ \gamma \end{bmatrix}$$

Next, we re-triangularize locally with Givens rotations (7). We only touch the columns where the new whitened Jacobian row $w^\top$ has nonzeros (i.e, the variables that this new factor actually connects to, such as a pose x_i or a landmark l_j). Starting from the leftmost such column, each rotation mixes the current pivot row with the new bottom row to kill one sub diagonal entry. We repeat until the entire bottom row is zero and the matrix is upper triangular again. The equation would look something like this:

$$\begin{bmatrix} R \\ w^\top \end{bmatrix} \xrightarrow{\text{Givens rotation on affected columns}} \begin{bmatrix} R' \\ 0 \end{bmatrix}$$

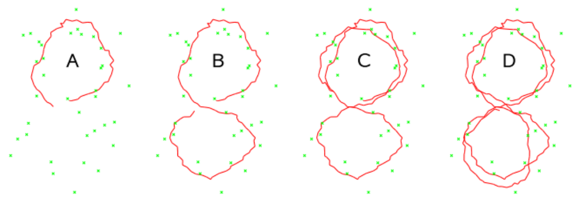
While we rotate the matrix, we apply the same rotations to the right hand side so that the least squares system stays consistent. Here d is the transformed RHS (Right Hand Side) before the update and γ is the new whitened RHS entry that pairs with $w^\top$. After the rotations, the top block becomes the updated RHS d' used for solving, and the final bottom entry becomes a small leftover error e_{new} that adds to the total residual.

$$\begin{bmatrix} d \\ \gamma \end{bmatrix} \xrightarrow{\text{same rotations}} \begin{bmatrix} d' \\ e_{\text{new}} \end{bmatrix}$$

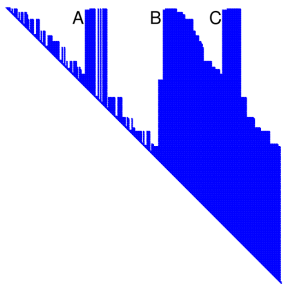
Intuitively, the new row $w^\top$ is "folded up" into the triangular structure by a short chain of 2x2 rotations that only touch the connected variables, everything else is left alone. We then get the correction by a fast back substitution on the updated matrix R' and vector d':

$$R'\Delta\theta^\star = d'$$

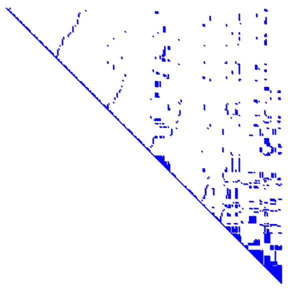
6.2.6 Loop Closure



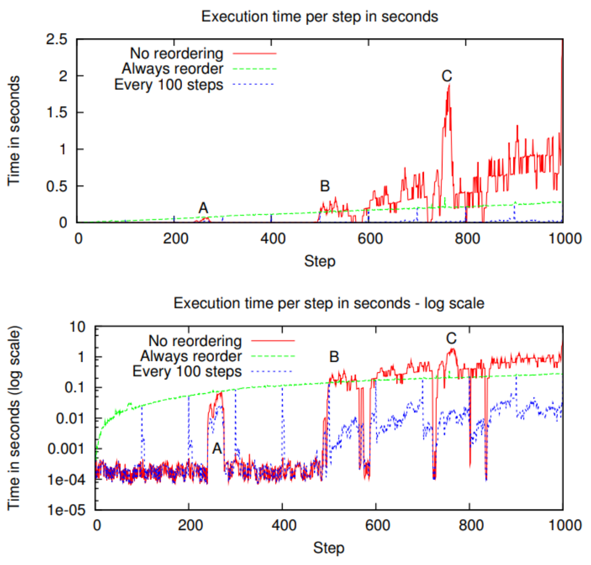
(a) Simulated double 8 loop at key loop closure moments.[6]



(b) Upper triangular factor R after several closures shows fill-in.[6]



(c) The same R after variable reordering (COLAMD) becomes sparser again.[6]



(d) Per step execution time for three strategies. A: no reordering, B: reorder every step, C: reorder every 100 steps, shown in linear (top) and log (bottom) scale. Periodic reordering (C) limits spikes and keeps runtime predictable between loop closures.[6]

Figure 4: Effect of loop closures and variable reordering (iSAM).[6]

Loop closures tie together far-apart parts of the trajectory (and landmarks), which makes previously separate columns interact. In QR terms this creates fill-in, R gets extra non zeros, so updates, back substitution, and selected covariance queries get slower and memory grows. We fix this by variable reordering. What we do is pick a new elimination order that preserves sparsity. In practice we run a heuristic like COLAMD (Column Approximate Minimum Degree) (often the block version for pose/landmark blocks) on the Jacobian's sparsity matrix A, then do batch factorization on the whole A matrix using rotations (7) with that order [6]. Reordering costs time because we rebuild the factor with a new permutation, but it pays back by making the next many updates cheap again.

Because reordering is expensive, we don't do it every step. Instead we do periodic reorder every N steps (eks, 50 - 200) so cost stays predictable. In marine AUV runs this keeps compute bounded. Between reorders incremental updates are fast and local. After a loop closure, we accept one spike (reorder + refactor), then return to low latency. Practical tips when doing this step is to keep poses as blocks (block COLAMD) to reduce fill-in, align reordering with planned relinearization passes, and monitor simple stats (nonzeros in R, update time) to decide when N is too small (wasting time reordering) or too large (letting fill in snowball).

6.2.7 Re-Linearization

Re-linearization keeps the local model honest. All the QR/update tricks we have gone through now assume the system is locally linear around the current estimate, but with angles, 3D motion, and nonlinear sensors that linearization drifts as the robot moves. If we never refresh it, increments stop being small, the optimizer biases the map, and loop closures can break the solution. The fix is to re-linearize, recompute Jacobians at the current state for the factors that matter. Doing this for every factor at every step is too expensive, so in practice we always linearize new factors (fresh odometry and measurements) and refresh older ones only when needed.

In iSAM the practical schedule is to perform incremental updates between maintenance cycles, then do a full re-linearization at a fixed interval N. Between cycles we do not re-linearize old factors, we only whiten and insert the new ones and update R incrementally. At the cycle boundary when we hit N steps, we re-linearize the entire problem at the current estimate (conceptually rebuild the full Jacobian A), run a variable reordering to restore sparsity using COLAMD, and refactor using equation (7) to get a fresh triangular R. Thats why we usually bunch variable reordering with batch linearization in the same N step.

We must choose N carefully, by balancing freshness vs compute. If N is too large, the linearization point drifts far from reality, Jacobians no longer match the true geometry, corrections become biased, loop closures pull hard, and the map can warp (a classic "stale linearization" issue). If N is too small, we keep stopping to relinearize and refactor, burning CPU and power and hurting real-time throughput.

6.2.8 Data Association from R

For data association we often use a Mahalanobis based approach instead of plain nearest neighbour Approach. Nearest neighbour measures raw Euclidean distance and ignores sensor noise and correlations. Mahalanobis measures the innovation in the units of its uncertainty, so noisy directions count less, precise directions count more, and correlated components are handled correctly (See Figure 1). For a candidate match between current pose x_i and landmark l_j, form the innovation ν_k and score:

$$d_k^2 = \nu_k^\top \, \Xi_k^{-1} \, \nu_k, \qquad \Xi_k = J_k \, \Sigma \, J_k^\top + \Gamma_k$$

where $J_k = [\, H^{x_i} \quad H^{l_j} \,]$ is the linearized measurement Jacobian, Γ_k is the sensor noise, and Σ is the state covariance.

This type of data association often uses gating with a chi-square test. The test keeps matches that are within $d_k^2 \le \chi^2_{m,\alpha}$ (right dimension m, chosen confidence α). From the survivors, we pick the "minimum cost" one (or solve a global assignment using d_{ij}^2 as the cost matrix if several features compete). One thing to note about this approach is that we don't need the full dense $\Sigma = (R^\top R)^{-1}$ to do our data association, we only require local covariances touched by the measurement. These types of covariances can be pull directly from the square root information matrix R. Here we can get pose, and pose to landmark terms are available live. Landmark terms on the other hand are either a approximate fast conservative estimate or computed exactly on demand. This keeps the association real-time for the most part. Here are the partitioned forms. Full state covariance (poses vs. landmarks):

$$\Sigma \;=\; \begin{bmatrix} \Sigma_{xx} & \Sigma_{xL} \\ \Sigma_{Lx} & \Sigma_{LL} \end{bmatrix}$$

where Σ_{xx} is pose to pose, Σ_{LL} is landmark to landmark, and $\Sigma_{xL} = \Sigma_{Lx}^\top$ is pose to landmark.

The 2×2 submatrix needed for a single candidate (x_i, l_j):

$$\Sigma_{\{x_i,l_j\}} \;=\; \begin{bmatrix} \Sigma_{x_ix_i} & \Sigma_{x_il_j} \\ \Sigma_{l_jx_i} & \Sigma_{l_jl_j} \end{bmatrix}$$

Fast marginals from the square root factor (online/live)
In iSAM paper [6], they propose keeping the current pose last in the ordering. Then the covariances needed for association, the pose variance $\Sigma_{x_ix_i}$ and the pose to landmark cross terms $\Sigma_{x_il_j}$ come straight from the square root factor R with two small triangular solves:

$$R^\top Y = B, \qquad RX = Y$$

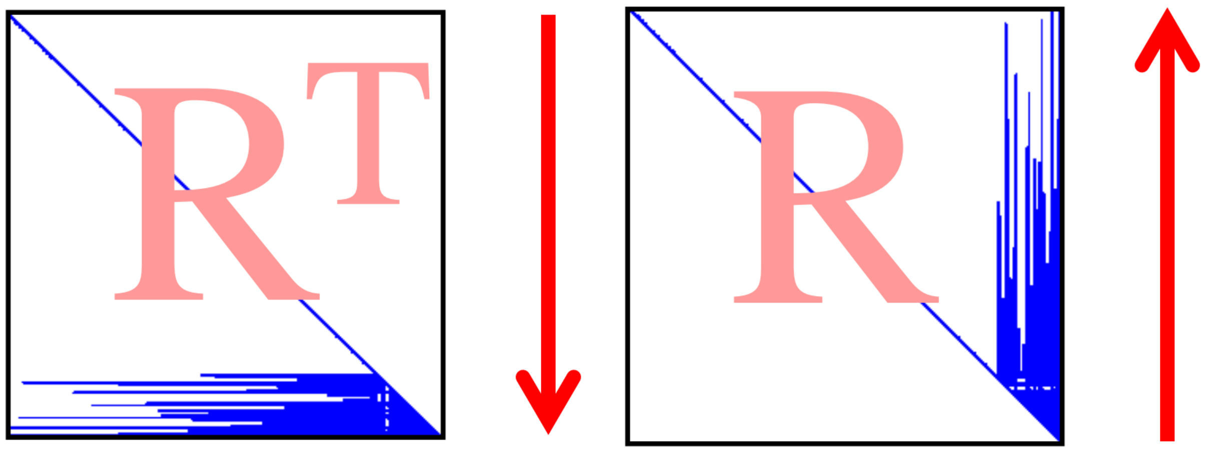
where $B = \begin{bmatrix} 0 \\ I_{d_x} \end{bmatrix}$ simply selects the last pose block of size d_x. Because R is upper triangular and B is zero above the last block, the forward solve gives:

$$Y = \begin{bmatrix} 0, \ldots, 0, & R_{ii}^{-1} \end{bmatrix}^\top$$

This Y preliminary matrix is the clue, where only d_x back substitutions are needed to get full X vector. Reading the result X yields, in one pass:

$$\Sigma_{x_ix_i} \;\; \text{(bottom–right block of } \Sigma) \qquad \Sigma_{l_jx_i} = \Sigma_{x_il_j}^\top \;\; \text{for connected } l_j$$

Intuitively we "solve up" then "solve down" on R for the last pose, and we get exactly the columns of $\Sigma = (R^\top R)^{-1}$ that matter for gating in data association. This can be done every step without forming any dense inverses, only using R square root information matrix iteratively. (See Figure 1)



(a) $R^\top Y = B$ (forward substitution). Sweep "down" the matrix to form Y from a selector B that picks the last pose block.

(b) $RX = Y$ (back substitution). Sweep "up" the matrix to obtain the desired columns $X = \Sigma_{:,x_i}$ (pose and pose to landmark).

Figure 5: We grab the needed covariances in two quick steps: first solve "down" with $R^\top$, then solve "up" with R. We only touch entries near the last pose block, so each update stays fast and cheap.[6]

Conservative landmark covariances (online/live)
Exact landmark landmark blocks Σ_{jj} (or old pose to landmark $\Sigma_{(i-n)j}$) are expensive to extract at every step. Therefore in iSAM paper [6] they propose using a safe, conservative bound built from the current pose covariance and the measurement noise via the linearized back projection:

$$\tilde{\Sigma}_{jj} \;=\; \bar{J} \begin{bmatrix} \Sigma_{ii} & 0 \\ 0 & \Gamma \end{bmatrix} \bar{J}^\top$$

where $\bar{J}$ is the Jacobian of the local inverse measurement model, Σ_{ii} is the current pose covariance, and Γ is the measurement noise. This upper bounds landmark uncertainty (never over confident), is fast, and works well for online Mahalanobis gating. As more measurements of a landmark arrive, this bound typically tightens.

An important caveat to mention is that we usually don't know the true Γ (measurement noise) exactly. The iSAM recipe is to choose Γ conservatively so we don't get over confident. That keeps things safe but can make the gate too tight, causing the data association to reject more matches than it should. Later, we will discuss a more confident way to approximate Γ from measurement characteristics without being overly confident (eks: range/resolution for sonar and landmark confidence) so the gate is realistic without being risky.

Exact landmark covariances (on demand)
When we need accuracy (eks: on risky loop closure, conflicting hypotheses), Data Association can recover exact Σ_{jj} and $\Sigma_{(i-n)j}$ without forming the full dense inverse information matrix $\Sigma = (A^T A)^{-1} = (R^T R)^{-1}$. Because the covariance is the inverse of the information matrix:

$$\Sigma = (R^\top R)^{-1}, \qquad R^\top R\,\Sigma = I$$

We can get the needed entries without forming the full inverse by solving for two triangular matrixes:

$$R^\top Y = I, \qquad R\Sigma = Y$$

We only follow the "nonzeros" of R when solving. That means we don't touch the whole matrix, just the parts that matter. iSAM walks backwards along those non zero links and gives us exactly the covariance numbers σ_{ij} Data Association ask for. If R is mostly banded, this is near linear time. We should only use this exact method when we really need it (eks: a few Σ_{jj} blocks to check on a loop closure). It's slower than the conservative shortcut, but still much faster than inverting the whole matrix.

Alternative way to finding Γ
There's another angle. Instead of being very conservative with our Γ measurement noise. Uncertainty used in Data Association should reflect the sensor, for example a sonar for scanning the sea floor. With sonar we can make the measurement noise grow with range and use the transducer's resolution to set per axis variances $N(0, \Sigma_{sonar})$. The landmark detection can also contribute its own uncertainty $N(0, \Sigma_{landmark})$. In practice we could combine "sensor noise" and "landmark estimate noise" to get the prediction uncertainty for the residual we score. In that case we can say:

$$\Gamma = Var(h(x,l)) = \Sigma_{sonar} + \Sigma_{landmark}$$

This can be more informative than a one size fits all setting. However we must be very careful with this approach, if we make these noises too optimistic, the gate in Data Association gets too tight, associations become brittle, and the system can go unstable. Never the less this is a lot better approach for building more accurate maps and estimating robots track.

6.2.9 Algorithm

At each time step we absorb new information, linearize around the current estimate, solve a small least squares, and update. Periodically we refresh linearization and variable order. Concretely:

1. **Add factors (whiten first):** take the new odometry/measurements and add them to the graph. "Whiten"" them so every residual has unit noise (as described before (5)).
2. **Linearize at the current guess θ:** turn the nonlinear motion/measurement models into local linear ones using the Jacobians in (2) and (3). This gives the summed Mahalanobis cost (4), which after whitening becomes one least squares problem (5).
3. **Keep a triangular system up to date (QR):** append the new (whitened) rows and apply a few Givens rotations (7) so the matrix stays upper triangular R. Update the right hand side b vector the same way (see "Incremental Updating").
4. **Solve for the small change:** because R is triangular, solve (6) by back substitution (fast) to get the correction .
5. **Update the estimate:** replace the old state with the improved one, $\theta \leftarrow \theta + \Delta\theta^\star$.

6. **Every N steps (maintenance):** refresh accuracy by re-linearizing all factors at the new θ, reorder variables (eks: COLAMD algorithm) to keep things sparse, and refactor with Givens (7) to get a clean R.
7. **Data Association update:** read the needed covariances from R (pose and pose to landmark live, landmark blocks conservative or exact on demand) and run Mahalanobis gating in Data Association for next matches.

6.2.10 Limitations

iSAM is fast between updates but has practical downsides. They come from the "data structure", not the underlying SLAM algorithm. iSAM keeps a single, global square root information matrix R (from $A^\top A$) and does periodic maintenance (reordering + re-linearization). This makes updates simple, but couples cost to global structure and variable ordering instead of just local changes.

- **Latency spikes at maintenance:** Periodic global variable reordering and re-linearization trigger stalls (especially after loop closures), since R must be refactored end to end.
- **Fill in growth between reorders:** Incremental QR on a fixed order accumulates fill in in R, touching more entries per update and increasing time/memory step by step.
- **All or nothing re-linearization:** iSAM typically refreshes many factors at maintenance even if most variables barely moved, wasting Jacobian recomputations.
- **Global refactor on ordering changes:** Any change to elimination order implies a large refactor of the global R, regardless of how small the new information is.
- **Broad marginal queries are costly:** Last pose and nearby cross terms can be pulled quickly from R, but wide Σ blocks (eks: many landmarks or older poses) require multiple triangular solves and can be costly.
- **Schedule sensitivity:** Choosing "every N steps" for reorder/re-linearize is heuristic, too small wastes time, too large lets fill in and linearization error grow, causing jitter and warp in the map.
- **Numerical robustness vs simplicity:** Working with A and R avoids explicit $A^\top A$, but long incremental runs plus fill in can still hurt conditioning and stability if ordering lags.

These limitations motivated **iSAM2**, which replaces the single monolithic R that is very static, with a Bayes tree representation that is dynamic and updates only the affected parts. We address iSAM2 and how it mitigates the issues above in the next chapter.

6.3 iSAM2

iSAM2 stuff here

6.4 GTSAM

GTSAM stuff here Shortest chapter Big library that includes iSAM2 Its more for educational purposes so the algorithms aren't fully optimized However is more that god enough

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