# Quadric Surfaces

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# Contents

# 1 Intro

- A **quadric surface** is the graph of a second degree equation in three variables. The general form:

$$Ax^2 + By^2 + Cz^2 + \underbrace{Dx'y' + Ex'z' + Fy'z'}_{\text{cross terms}} + Gx + Hy + Iz + J = 0$$

where $A, B ... J$ are constants.

## 1.1 Examples

- $x + y = 2$ quadric graph? $\rightarrow$ No. No 2nd degree term.
- $x + y + z = 2$ quadric graph? $\rightarrow$ No. No 2nd degree term.
- $x^2 + y + z = 5$ quadric graph? $\rightarrow$ Yes.
- $x' + y' = 1$ quadric graph? $\rightarrow$ Yes.

# 2 Ellipsoids

$$\frac{x^2}{1} + \frac{y^2}{4} + \frac{z^2}{9} = 1$$

suppose $x = 0$ ($zy$ plane):

$$\frac{0^2}{1} + \frac{y^2}{4} + \frac{z^2}{9} = 1$$
$$\frac{y^2}{4} + \frac{z^2}{9} = 1$$

let $y = 0$ ($xz$ plane):

$$\frac{x^2}{1} + \frac{z^2}{9} = 1$$

let $z = 0$ ($xy$ plane):

$$\frac{x^2}{1} + \frac{y^2}{4} = 1$$

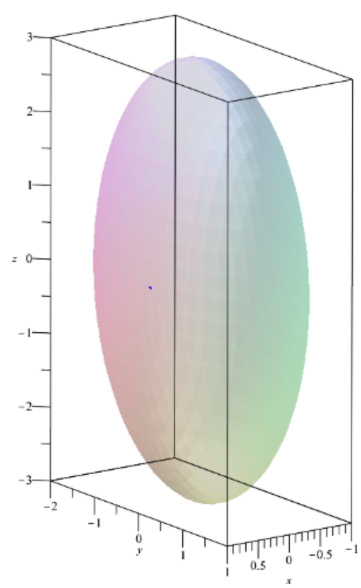


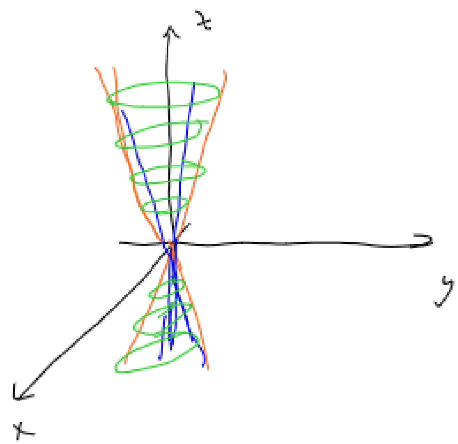
Figure 1: egg
Notes: the traces (like slices of the figure) are ellipses.

let $y = 1$

$$\frac{x^2}{1} + \frac{1}{4} + \frac{z^2}{9} = 1$$
$$\frac{x^2}{1} + \frac{z^2}{9} = \frac{3}{4}$$
$$\frac{x^2}{\frac{3}{4}} + \frac{z^2}{\frac{27}{4}} = 1$$

## 2.1 Cones

$$\frac{x^2}{16} + \frac{y^2}{81} = \frac{z^2}{256}$$
$$\Rightarrow \quad \frac{x^2}{16} + \frac{y^2}{81} - \frac{z^2}{256} = 0 \quad \checkmark$$




let $x = 0$:

$$\frac{y^2}{81} = \frac{z^2}{256}$$
$$\frac{y}{9} = \pm\frac{z}{16}$$
$$z = \pm\frac{16}{9}y$$
$$z = \frac{-16}{9}y$$

$z = c$

$$\frac{x^2}{16} + \frac{y^2}{81} = \frac{c^2}{256}$$

$\underline{y=0}$

$$\frac{x^2}{16} = \frac{z^2}{256} \quad \Rightarrow \quad z = \pm\frac{16}{4}x$$

### 2.2 Equations

- **Elliptic Paraboloid:**
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z}{c}$$
- **Circular Paraboloid:**
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z}{c}$$
- **Elliptic Cone:**
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z^2}{c^2}$$

## 3 Hyperboloids

### 3.1 1 Sheet Elliptical Hyperboloid

$$\frac{x^2}{16} + \frac{y^2}{81} - \frac{z^2}{256} = 1$$

traces:

$z = 0$
$$\frac{x^2}{16} + \frac{y^2}{81} = 1$$
(ellipse)

$y = 0$
$$\frac{x^2}{16} - \frac{z^2}{256} = 1$$
(hyperbola)

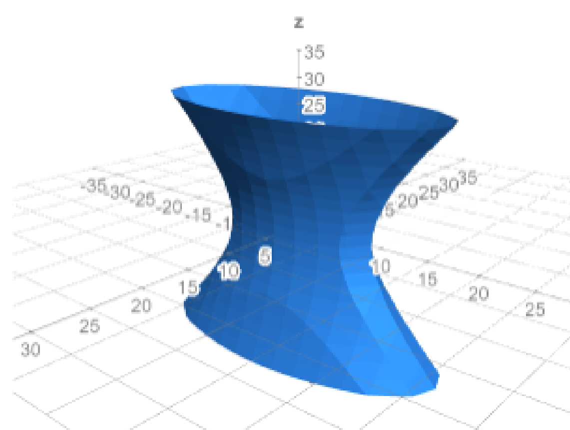



Figure 2: “elliptical hyperboloid” & 1 sheet

$x = 0$ $$\frac{y^2}{81} - \frac{z^2}{256} = 1$$

(hyperbola)

$z = 16$ $$\begin{aligned} \frac{x^2}{16} + \frac{y^2}{81} - 1 &= 1 \\ \frac{x^2}{16} + \frac{y^2}{81} &= 2 \\ \frac{x^2}{32} + \frac{y^2}{162} &= 1 \end{aligned}$$

(hyperbola)

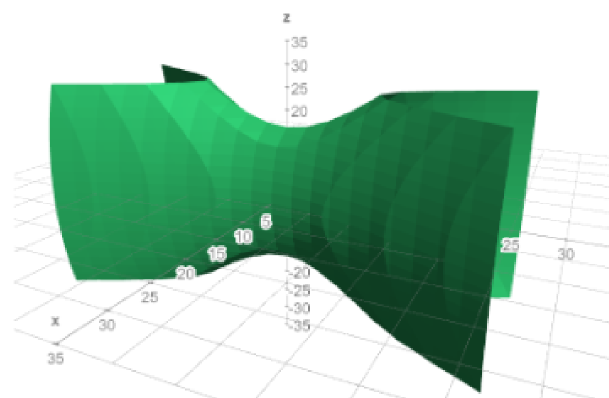
What if we put the minus sign on the $y$ term?

$$\frac{x^2}{16} - \frac{y^2}{81} + \frac{z^2}{256} = 1$$

$z = 0$ $$\frac{x^2}{16} - \frac{y^2}{81} = 1$$

$y = 0$ $$\frac{x^2}{16} - \frac{z^2}{256} = 1$$

$x = 0$ $$\frac{z^2}{256} - \frac{y^2}{81} = 1$$



### 3.1.1 2 Sheet Elliptical Hyperboloid

$$\frac{x^2}{16} - \frac{y^2}{81} - \frac{z^2}{256} = 1$$

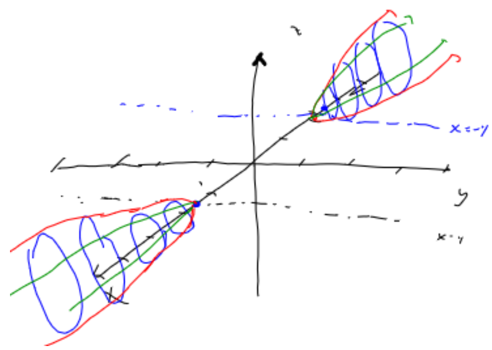


Figure 3: 2 sheets because there's 2 "sheets" between them.

let $x = 0$:

$$\frac{y^2}{81} - \frac{z^2}{256} = 1$$

No solutions.

$$\frac{y^2}{81} - \frac{z^2}{256} = 1 - \frac{x^2}{16}$$

No solutions when $x > 4$.

let $y = 0$:

$$\frac{x^2}{16} - \frac{z^2}{256} = 1$$

let $z = 0$:

$$\frac{x^2}{16} - \frac{y^2}{81} = 1$$

## 3.2 Equations

- **1 Sheet Hyperboloid:**

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

- **2 Sheet Hyperboloid:**

$$\frac{z^2}{c^2} - \frac{x^2}{a^2} - \frac{y^2}{a^2} = 1$$

- **Hyperbolic Paraboloid:**

$$\frac{y^2}{b^2} - \frac{x^2}{a^2} = \frac{z}{c}, \qquad c > 0$$

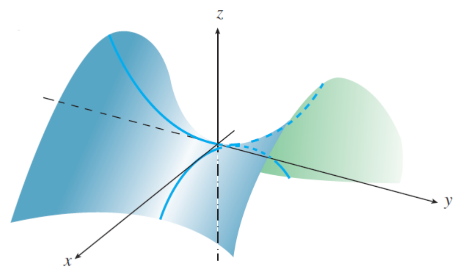



Figure 4: A hyperbolic paraboloid.