Conics: Circles and Ellipses

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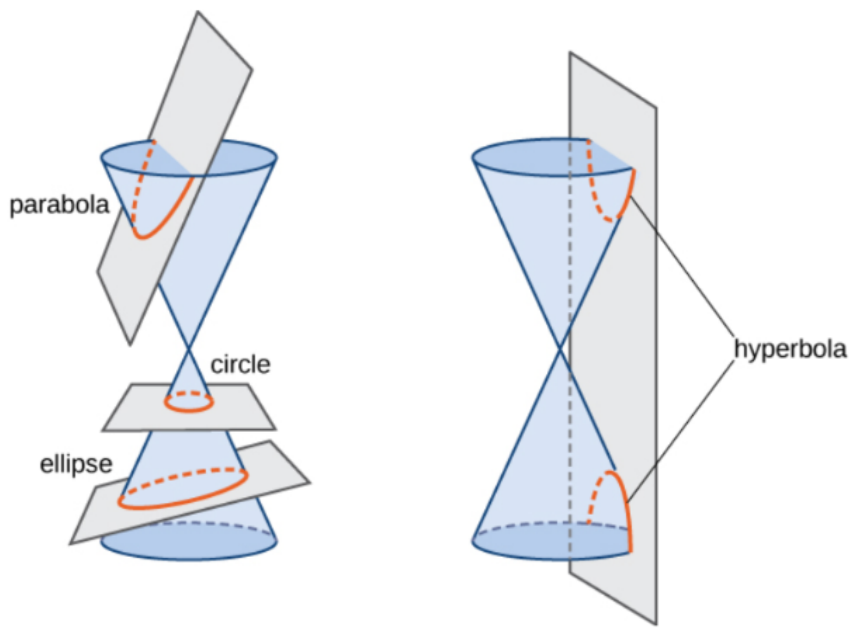

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1 Circles and Ellipses

1.1 The Ellipse Labelled

DEFINITION An *ellipse* is the set of all points for which the sum of their distances, d_1 and d_2, from two fixed points (the foci, $F\prime$ and F) is constant.

For ellipses with a center $(0, 0)$:

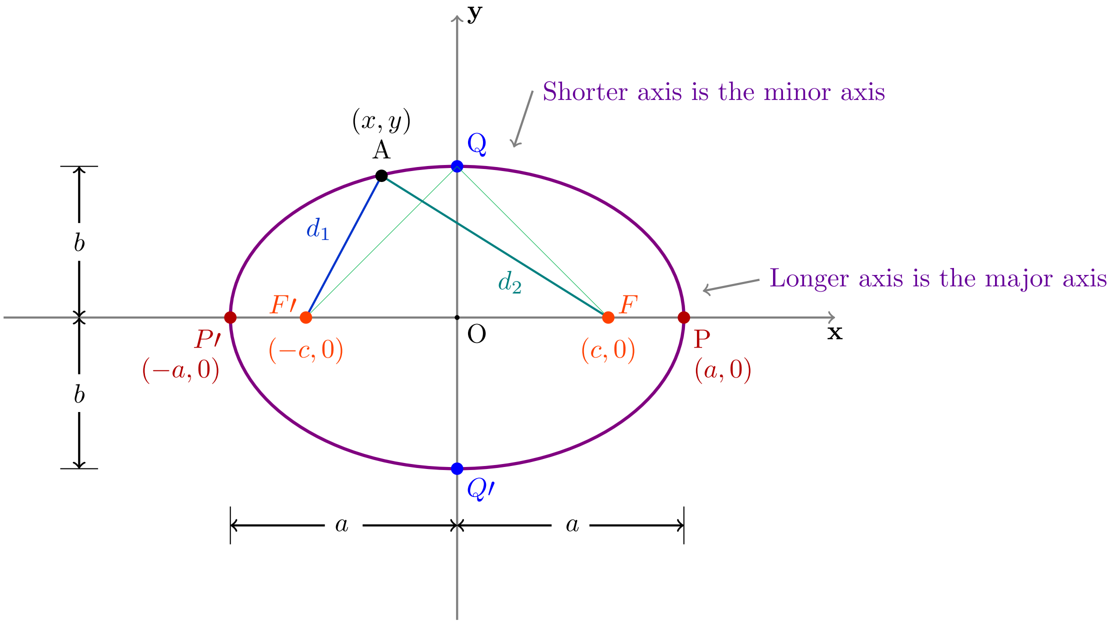

Figure 1: The ellipse labelled with its foci, vertices, and components.

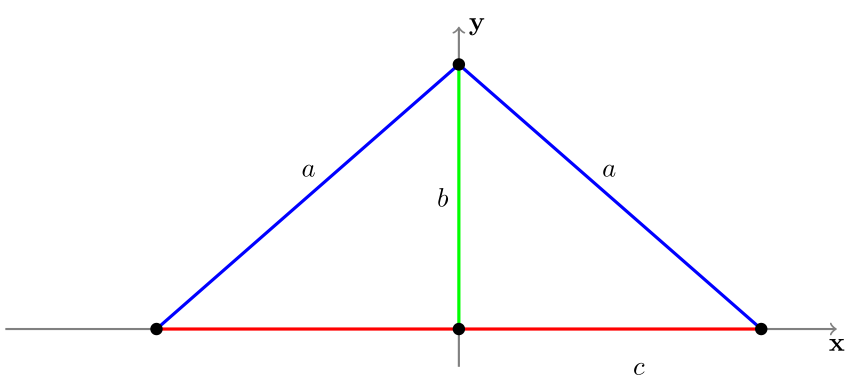

Figure 2: The a, b, and c components of an ellipse.

As seen in Figure 2, the ellipse has an a, b, and c component. Let the string length be $2a$. For ellipses, using Pythagorean Theorem, the equation for any side, given the other two, becomes

$$a^2 = b^2 + c^2$$

1.2 Deriving the Ellipse's Equation

Using the two distances, d_1 and d_2,

$$d_1 + d_2 = 2a$$

$$\begin{aligned}
\sqrt{(x-c)^2+(y-0)}^2 + \sqrt{(x-c)^2+(y-0)^2} &= 2a \\
\sqrt{(x+c)^2+y^2})^2 &= (2a - \sqrt{(x-c)^2+y^2})^2 \\
(x+c)^2+y^2 &= 4a^2 - 4a\sqrt{(x-c)^2+y^2} + ((x-c)^2+y^2) \\
x^2+2cx+c^2+y^2 &= 4a^2 - 4a\sqrt{(x-c)^2+y^2} + x^2 - 2cx + c^2 + y^2
\end{aligned}$$

Cancelling like-terms will give you this:

$$(4a\sqrt{(x-c)^2+y^2})^2 = (4a^2 - 4cx)^2$$

Again, cancel like-terms and factor out a four to get

$$a^2((x-c)^2+y^2) = a^4 - 2a^2cx + c^2x^2$$

Further simplification will become

$$\begin{aligned} a^2(x^2-2cx+c^2+y^2) &= a^4 - 2a^2cx + c^2x^2 \\ a^2x^2 - 2a^2cx + a^2c^2 + a^2y^2 &= a^4 - 2a^2cx + c^2x^2 \end{aligned}$$

Cancel like-terms again and you will get

$$\begin{aligned} (a^2-c^2)x^2 + a^2y^2 &= a^4 - a^2c^2 \\ (a^2-c^2)x^2 + a^2y^2 &= a^2(a^2-c^2) \\ \frac{(a^2c^2)x^2}{a^2(a^2-c^2)} + \frac{a^2y^2}{a^2(a^2-c^2)} &= 1 \\ \frac{x^2}{a^2} + \frac{y^2}{a^2-c^2} &= 1 \end{aligned}$$

Thus, the *final equation for an ellipse* is:

$$\boxed{\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1}$$

1.3 Ellipse Examples

1.3.1 Example 1

Graph, locate *foci* and state *eccentricity*

$$\frac{x^2}{25} + \frac{y^2}{16} = 1$$

$$a = 5 \text{ and } b = 4$$

$$\begin{aligned} a^2 &= b^2 + c^2 \\ 25 &= 16 + c^2 \\ 9 &= c^2 \\ 3 &= c \end{aligned}$$

Now, let's find the eccentricity

$$e = \frac{c}{a} = \frac{3}{5}$$

Finally, using the c value, you can calculate where the foci and vertices are. Graphing this ellipse will give you:

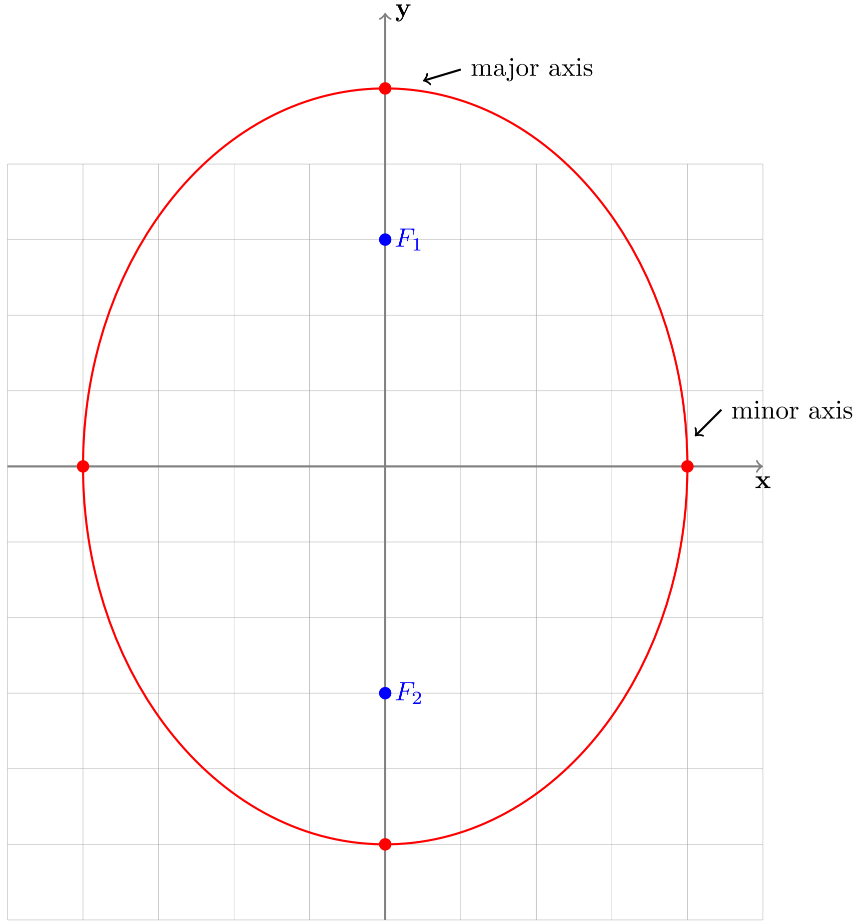

Figure 3: The correct graph of this example.

1.4 Ellipses with Center (h, k)

Standard Forms For an ellipse centered at (h, k), the standard form is

$$\boxed{\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1}$$

i.e., in Figure 4, the center of the ellipse and the ellipse's a and b values result in this ellipse:

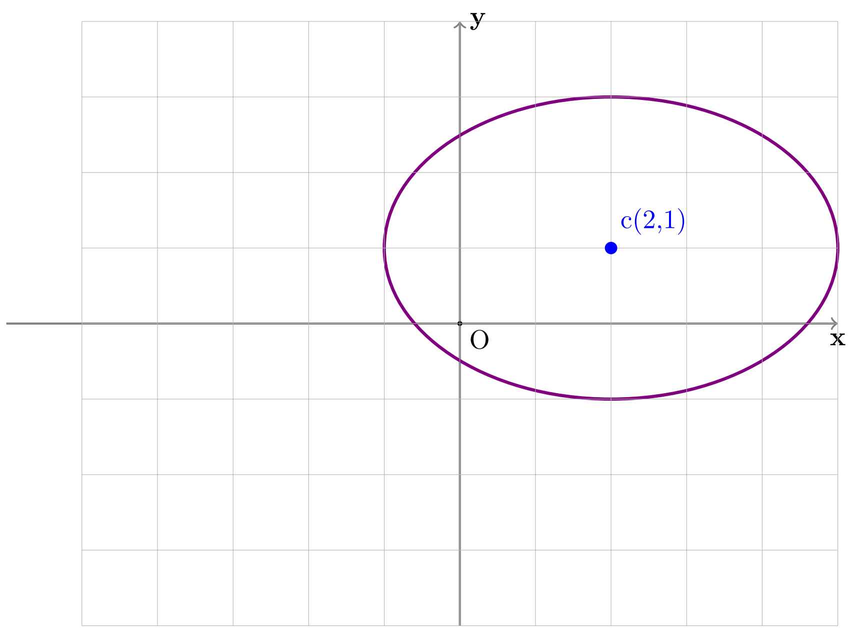

Figure 4: An example of an ellipse centered at (h, k).

In Figure 4, the h is 2 and the k is 1.

1.4.1 Example 2 (*An ellipse centered at* (h, k))

Graph, locate *foci* and state *eccentricity*.

$$\frac{(x-1)^2}{16} + \frac{(y+2)^2}{9} = 1$$

$$c(h,k) = (1,-2)$$

$$v_1(1+4,-2) = (5,-2)$$

$$v_2(1-4,-2) = (-3,-2)$$

$$a^2 = 16$$

$$b^2 = 9$$

$$cv_1(1,-2+3) = (1,1)$$

$$c^2 = a^2 - b^2$$

$$cv_2(1,-2-3) = (1,-5)$$

$$c^2 = 7$$

$$F_1(1+\sqrt{7},-2)$$

$$c = \sqrt{7}$$

$$F_2(1-\sqrt{7},-2)$$

$$e = \frac{c}{a} = \frac{\sqrt{7}}{4}$$

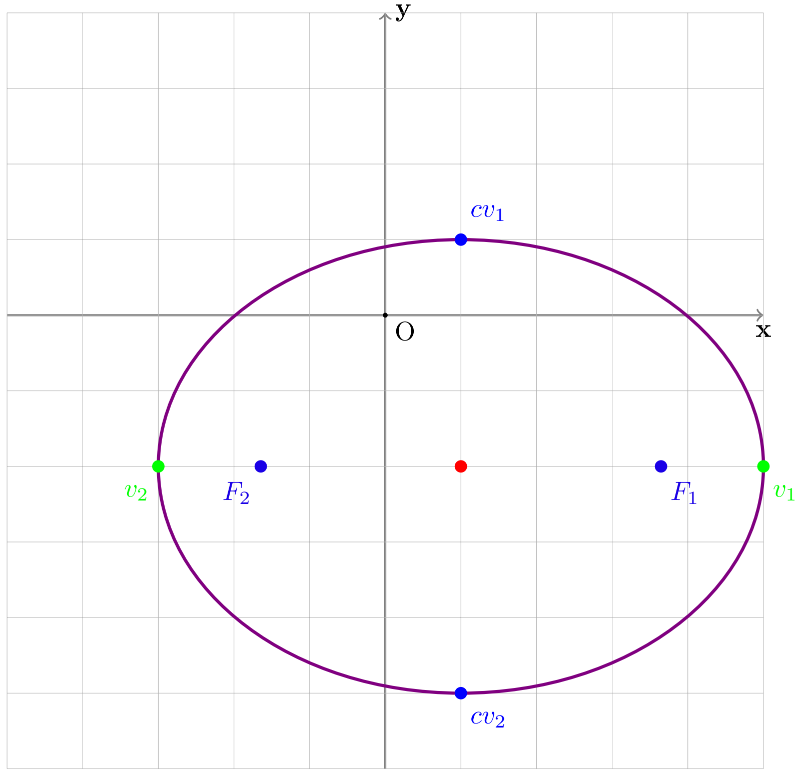

Figure 5: The major points and the finished graph of this example.

1.5 The Equation of a Circle

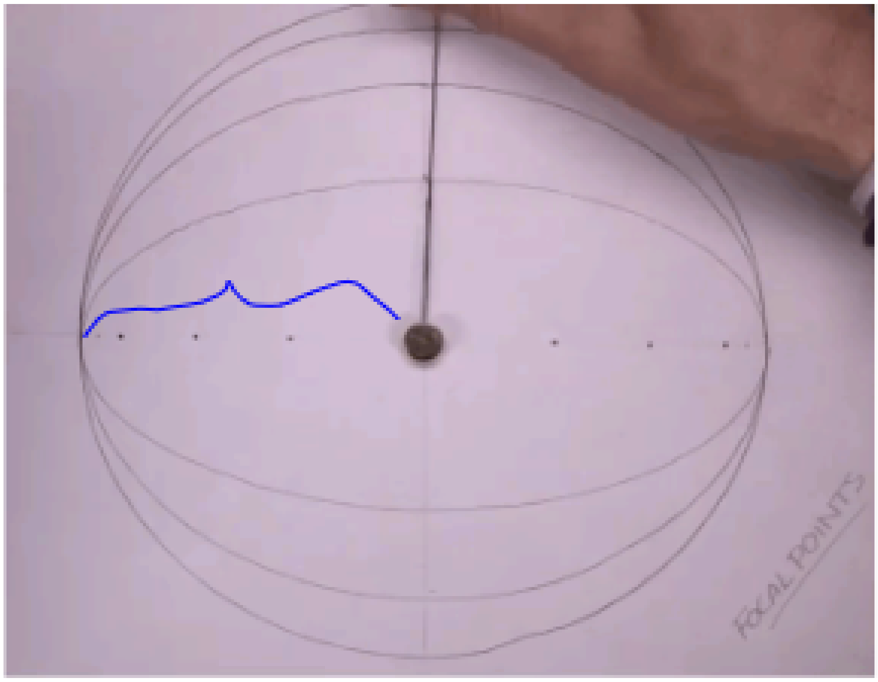

Figure 6: A circle (the radius is blue)

$$a = b = r$$

$$\frac{(x-h)^2}{r^2} + \frac{(y-k)^2}{r^2} = 1$$

$$(x-h)^2 + (y-k)^2 = r^2$$

2 Completing the Square

In some cases, completing the square is necessary to rewrite the given equation into standard form.

For example, a standard polynomial would look as so

$$x^2 + ax + (\tfrac{1}{2}a)^2 - (\tfrac{1}{2}a)^2$$

In terms of areas, the green area is what completing the square is finding (literally completing the overall square).

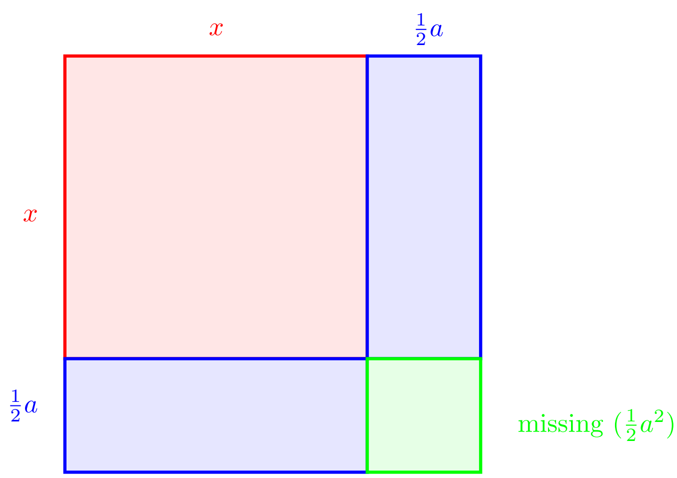

Figure 7: The polynomial visualized.

2.1 Example 3

Convert each equation to standard form by <u>completing the square</u> in x and y. Graph, locate *foci* and state *eccentricity*.

$$9x^2 + 25y^2 - 36x + 50y - 164 = 0$$

$$(9x^2 - 36x) + (25y^2 + 50y) = 164$$

$$9(x^2 - 4x + 4) + 25(y^2 + 2y + 1) = 164 + 36 + 25$$

$$\frac{9(x-2)^2}{225} + \frac{25(y+1)^2}{225} = \frac{225}{225}$$

$$\frac{(x-2)^2}{36} + \frac{(y+1)^2}{9} = 1$$

3 Eccentricity

Eccentricity is the “weirdness” of a circle; for example, a circle’s eccentricity is always 0.

For *hyperbolas*, $e > 1$

ellipses, $0 < e < 1$

circles, $e = 0$

parabola, $e = 1$

The equation for eccentricity is

$$\boxed{e = \frac{c}{a}}$$

An infinite eccentricity results in a line.

4 Hyperbolas

DEFINITION A *hyperbola* is the set of all points where the difference between their distances from two fixed points (the foci) is constant.

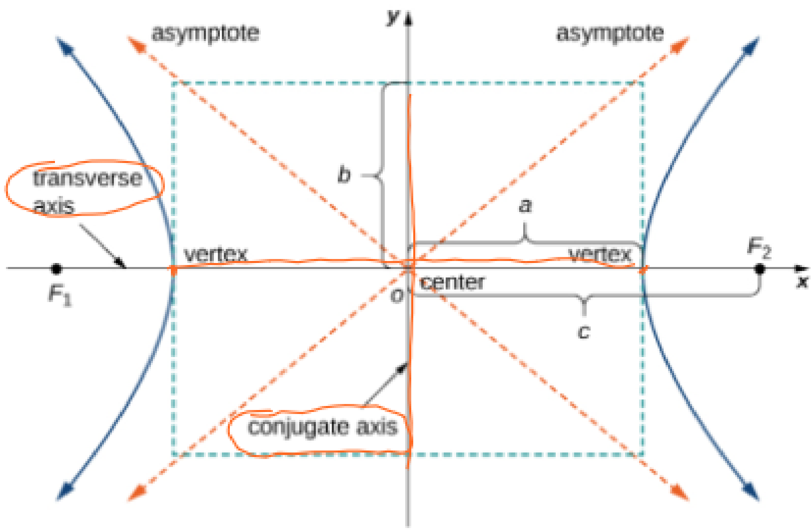

Figure 8: The different aspects of the hyperbola, labelled.

INTRO (x, y) is an *ellipse* if $d_1 + d_2$ is a constant. However, (x, y) is a *hyperbola* if $|d_1 - d_2|$ is a constant.

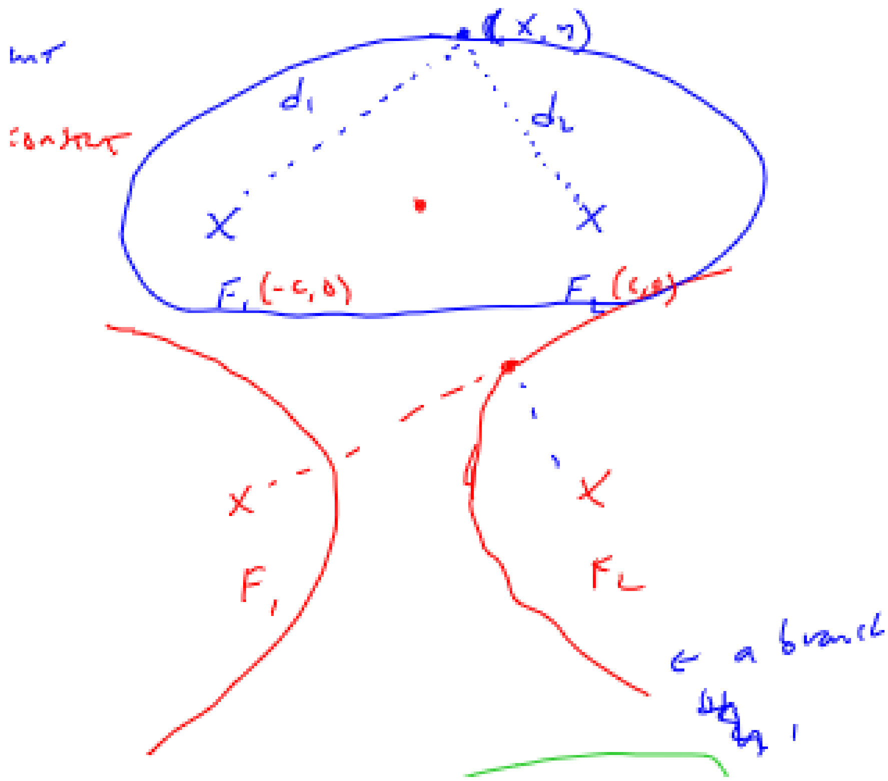

Figure 9: The top figure is the ellipse while the bottom figure displays how the hyperbola is labelled in comparison. As seen in the figure, one arc of the hyperbola is called a branch.

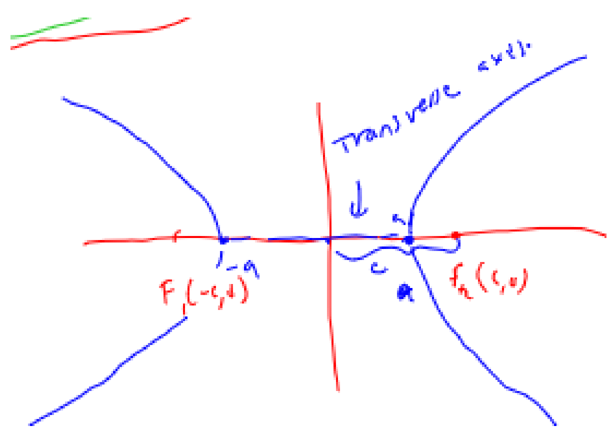

Figure 10: As seen here, the transverse axis is labelled.

The transverse axis is the equivalent to an ellipse's major axis; the foci and vertices are located on the axis. Its length is $2a$, so it doesn't extend past the foci.

4.1 The Hyperbola's Equations

4.1.1 Deriving the Hyperbola's Standard Form

$$
\begin{aligned}
|d_1 - d_2| &= 2a \\
\sqrt{(x+c)^2 + (y-0)^2} - \sqrt{(x-c)^2 + (y)^2} &= 2a \\
&. \\
&. \\
&.
\end{aligned}
$$

$$\boxed{\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1}$$

In the hyperbola's case,

$$\boxed{b^2 = c^2 - a^2}$$

4.2 The Elements of a Hyperbola

What is the significance of b *in* $(b^2 = c^2 - a^2)$?

Rearranging the standard form of a hyperbola,

$$
\begin{aligned}
\frac{x^2}{a^2} - \frac{y^2}{b^2} &= 1 \\
\frac{y^2}{b^2} &= \frac{x^2}{a^2} - 1 \\
\frac{y^2}{b^2} &= \frac{x^2}{a^2}(1 - \frac{a^2}{x^2})
\end{aligned}
$$

You will get this. As $x \to \infty$, the behavior of $(1 - \frac{a^2}{x^2})$ will approach 0.

Then, the equation will start to look like

$$
\begin{aligned}
\frac{y^2}{b^2} &= \frac{x^2}{a^2} \\
y^2 &= \frac{b^2}{a^2}x^2
\end{aligned}
$$

$$\boxed{y \pm \frac{b}{a}x}$$

Which is the equation for both asymptote lines in a hyperbola.

4.3 Example 1

Use the vertices and asymptotes to graph the hyperbola. Locate the *foci*, find the equation of the asymptotes, and state *eccentricity*.

$$\frac{x^2}{9} - \frac{y^2}{25} = 1$$

$$a^2 = 9$$
$$b^2 = 25$$
$$c^2 = a^2 + b^2$$
$$c^2 = 34$$
$$c = \sqrt{34}$$

$$c(0,\ 0)$$
$$v_1(0 + a,\ 0) = (3,\ 0)$$
$$v_2(0 - a,\ 0) = (-3,\ 0)$$
$$cv_1(0,\ 0 + 5) = (0,\ 5)$$
$$cv_2(0,\ 0 - 5) = (0,\ -5)$$

$$F_1(0 + \sqrt{34},\ 0) = (\sqrt{34},\ 0)$$
$$F_2(0 - \sqrt{34},\ 0) = (-\sqrt{34},\ 0)$$
$$e = \frac{c}{a} = \frac{\sqrt{34}}{3}$$

$$y = \frac{5}{3}x$$
$$y = \frac{-5}{3}x$$

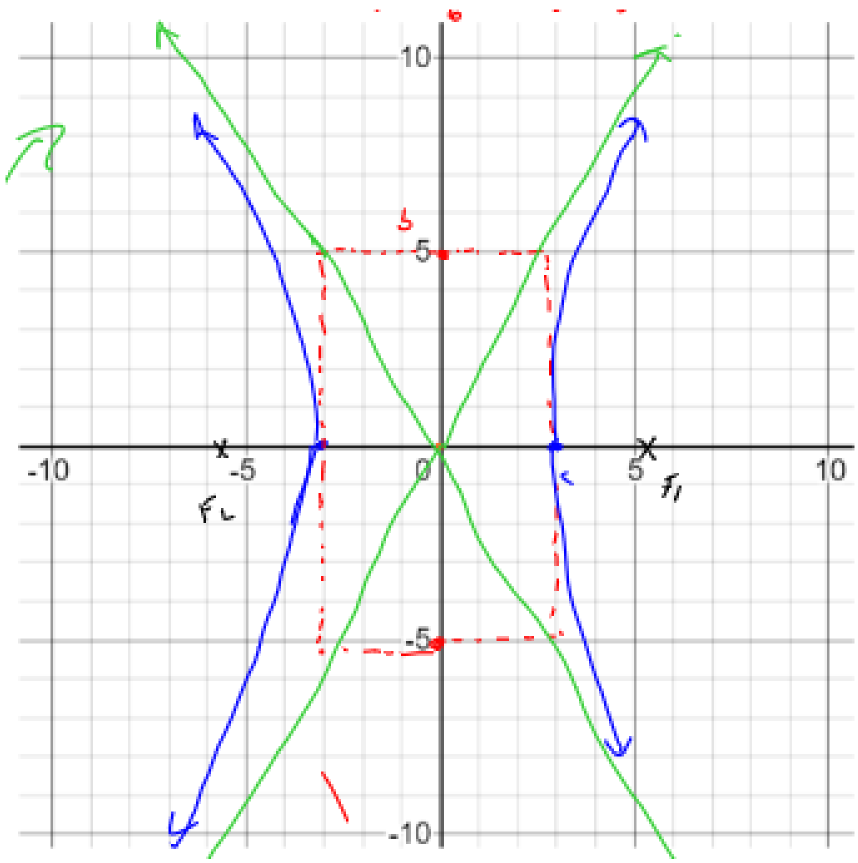

Figure 11: The correct graph of example 1.

5 Parabolas

DEFINITION A *parabola* is the set of all points whose distance from a fixed point, called the focus, is equal to the distance from a fixed line, called the *directrix*. The point halfway between the focus and directrix is called the vertex of the parabola.

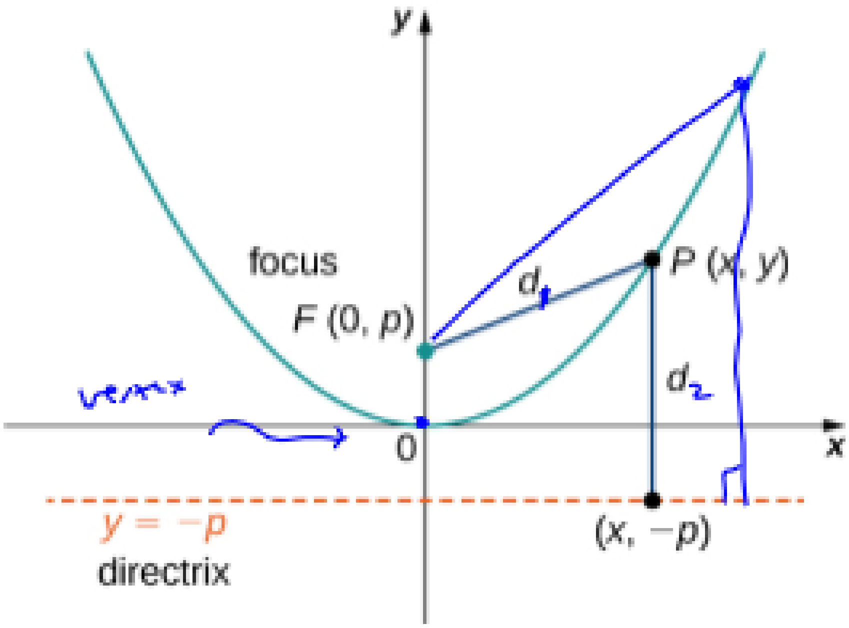

Figure 12: The different components of the parabola labelled.

5.1 Deriving the Parabola's Equation

Given $F(0,\ P)$ and $y\ =\ -P$,

$$\begin{aligned} d_1 &= \sqrt{(x - 0)^2 + (y - P)^2} \\ d_2 &= (y - (-P)) = y + P \end{aligned}$$

$$\begin{aligned} d_1 &= d_2 \\ \sqrt{(x - 0)^2 + (y - P)^2} &= y + P \\ x^2 + (y - P)^2 &= (y + P)^2 \\ x^2 + y^2 - 2Py + P^2 &= y^2 + 2Py + P^2 \end{aligned}$$

Cancelling the terms y^2 and P^2, the resulting equation for a parabola centered at $(0\ ,\ 0)$ is

$$\boxed{x^2 = 4Py}$$

Here, $|P|$ is the distance from vertex to focus or directrix.

However, if the vertex were at $(h\ ,\ k)$, then the equation for the standard form would be

$$(x\ -\ h)^2\ =\ 4P(y\ -\ k)$$

or

$$(y\ -\ k)^2\ =\ 4P(x\ -\ h)$$

5.2 Example 1

Find the foci and equation of the *directrix* for this parabola:

$$y^2 \qquad = \qquad 12x$$

$$\begin{aligned} (y\ -\ 0)^2\ &=\ 4P(x\ -\ 0) \\ v(0,\ 0)& \\ 4P\ &=\ 12 \\ P\ &=\ 3 \\ F(0\ +\ P,\ 0)\ &=\ (3,\ 0) \\ directrix:\ x\ &=\ P\ =\ -3 \end{aligned}$$

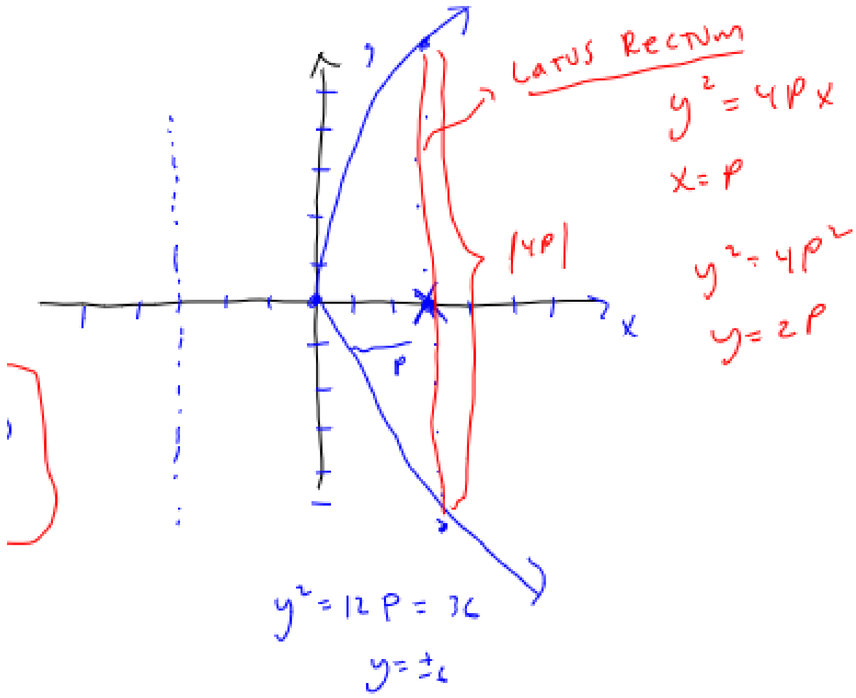

Figure 13: The correct graph of this problem.

5.3 Latus Rectum

DEFINITION The Latus Rectum of a parabola is the line segment that passes through the foci, and it is parallel to the directrix. It is labelled in Figure 13.

5.4 Rotation of Axes

Here is the commonly seen polynomial:

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

This is a second degree polynomial. Each constant is denoted by A, B, C, etc. The **cross-term** is the term Bxy.

5.4.1 Discriminant Test

The quadratic curve $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ is

I. a *parabola* if $B^2 - 4AC = 0$

II. an *ellipse* if $B^2 - 4AC < 0$

III. a *hyperbola* if $B^2 - 5AC > 0$