Closed Form Price of American Digital Call Option

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1. Introduction

An American digital option is a kind of exotic options that allows the option holders to exercise a digital option at any time before the maturity. Use the call option as an example, with the payoff of the option A, when the underlying aseet price reaches the strike price m, the option holder(longer) can receive fixed payoff: A dollar. In an American context, the longer can decide to use the exercising right at any time during the option contract horizon. In fact, the rational option longer should exercise the option once the underlying price reaches the strike price immediately because there does not exist a greater payoff if the holder choose to continue to wait in the remaining horizon. **So we can say the American property in digital option is not a real American property. This property is more close to a barrier property that the longer receive the payoff once the underlying price touches the barrier and this contract hence comes to an end. This structure is essentially the automatic knocking-out part in autocallables.** Next we prove the analytical price of American digital call option based on the underlying dynamics of geometric brownian motion.

2. Proof

Under the geometric brownian motion hypothesis, the underlying price follows the following stochastic differential equation:

$$dS_t = rS_t dt + \sigma S_t dW_t^{\mathbb{Q}} \tag{1}$$

Where r is the risk free rate, σ is the volatility, $W_t^{\mathbb{Q}}$ is a standard brownian motion under $\mathbb{Q}$ measure.

With Ito's lemma, the price of the underlying asset at time t is:

$$S_t = S_0 e^{(r-\frac{1}{2}\sigma^2)t+\sigma W_t^{\mathbb{Q}}} \tag{2}$$

According to risk neutral pricing theory, the price of the finite horizon American digital call option at initial time should be:

$$c_0 = \mathbb{E}^{\mathbb{Q}}\left[Ae^{-r\tau_m}\mathbb{I}_{\{\tau_m<T\}}\right] \tag{3}$$

Where $\tau_m = min\{t|S_t = m\}$ is the stopping time that the underlying price first hits the barrier(strike) price $m > S_0$.

Then the pricing formula can be rewritten as:

$$c_0 = A\int_0^T e^{-r\tau}p(\tau)d\tau \tag{4}$$

So now, what we need to do is to find the distribution of the first passage time of the underlying asset price S_t (i.e. the distribution of τ_m:$p(\tau)$) .

As the exponential function is a strictly ascending function, the underlying price reaches m is equivalent to the borwnian motion with drift: $\mu t + \sigma W_t^{\mathbb{Q}}\left(\mu = r - \frac{\sigma^2}{2}\right)$ reaches $m' = ln\frac{m}{S_0} > 0$.

Assume a brownian motion with drift:

$$X_t = \mu t + \sigma W_t \tag{5}$$

We can give out the analytical distribution of the first passage time of this brownian motion with drift without a proof. The proof details can be found in [Bhattacharya and Waymire].

$$p(\tau) = \frac{m'}{\sqrt{2\pi\sigma^2\tau^3}}e^{-\frac{(m'-\mu\tau)^2}{2\sigma^2\tau}}, \quad \tau > 0 \tag{6}$$

Where $m' > 0$ is the first passage barrier.

This is an inverse-gaussian distribution with parameters $\left(\frac{m'}{\mu}, \left(\frac{m'}{\sigma}\right)^2\right)$.Thus, the price of the option is:

$$c_0 = A\int_0^T e^{-r\tau}\frac{m'}{\sqrt{2\pi\sigma^2\tau^3}}e^{-\frac{(m'-\mu\tau)^2}{2\sigma^2\tau}}d\tau \tag{7}$$

Merge and rewrite the exponential term, we obtain:

$$c_0 = Ae^{\frac{m'(\mu-\sqrt{\mu^2+2\sigma^2 r})}{\sigma^2}}\int_0^T \frac{m'}{\sqrt{2\pi\sigma^2\tau^3}}e^{-\frac{(m'-\sqrt{\mu^2+2\sigma^2 r}\tau)^2}{2\sigma^2\tau}}d\tau \tag{8}$$

So the formula in integral part is also an inverse-gaussian density with parameters $\left(\frac{m'}{\sqrt{\mu^2+2\sigma^2 r}}, \left(\frac{m'}{\sigma}\right)^2\right)$. With the property of inverse-gaussian distribution, we have:

$$\int_0^T \frac{m'}{\sqrt{2\pi\sigma^2\tau^3}}e^{-\frac{(m'-\sqrt{\mu^2+2\sigma^2 r}\tau)^2}{2\sigma^2\tau}}d\tau = N\left(\frac{m'}{\sigma\sqrt{T}}\left(\frac{T\sqrt{\mu^2+2\sigma^2 r}}{m'}-1\right)\right)+e^{\frac{2m'\sqrt{\mu^2+2\sigma^2 r}}{\sigma^2}}N\left(-\frac{m'}{\sigma\sqrt{T}}\left(\frac{T\sqrt{\mu^2+2\sigma^2 r}}{m'}+1\right)\right) \tag{9}$$

Where $N(\cdot)$ is the CDF of a standard normal distribution.

As a result, the closed form price of a non-perpetual American digital call option with payoff A, barrier price m and maturity T is:

$$c_0 = A\left(e^{\frac{m'(\mu-\sqrt{\mu^2+2\sigma^2 r})}{\sigma^2}}N\left(\frac{m'}{\sigma\sqrt{T}}\left(\frac{T\sqrt{\mu^2+2\sigma^2 r}}{m'}-1\right)\right)+e^{\frac{m'(\mu+\sqrt{\mu^2+2\sigma^2 r})}{\sigma^2}}N\left(-\frac{m'}{\sigma\sqrt{T}}\left(\frac{T\sqrt{\mu^2+2\sigma^2 r}}{m'}+1\right)\right)\right) \tag{10}$$

Plug $\mu = r-\frac{\sigma^2}{2}, m' = ln\frac{m}{S_0}$ into the formula and note that $\sqrt{\mu^2+2\sigma^2 r} = r+\frac{\sigma^2}{2}$, we eventually have:

$$c_0 = A\left(\frac{S_0}{m}N\left(\frac{ln\frac{S_0}{m}+\left(r+\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right)+\left(\frac{S_0}{m}\right)^{-\frac{2r}{\sigma^2}}N\left(\frac{ln\frac{S_0}{m}-\left(r+\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right)\right) \tag{11}$$

Furthermore, we can also derive the price of a perpetual American digital call option.

This is equivalent to calculate:

$$c_0 = \mathbb{E}^{\mathbb{Q}}\left[Ae^{-r\tau_m}\right] = A\mathbb{E}^{\mathbb{Q}}\left[e^{-r\tau_m}\right] = A\int_0^{+\infty} e^{-r\tau}p(\tau)d\tau \tag{12}$$

This can be obtained by letting $T\to+\infty$ in (10).

Then finally we have the price of perpetual American digital call option:

$$c_0 = Ae^{\frac{m'(\mu-\sqrt{\mu^2+2\sigma^2 r})}{\sigma^2}} \tag{13}$$

Plug $\mu = r-\frac{\sigma^2}{2}, m' = ln\frac{m}{S_0}$ into the formula, we have:

$$c_0 = Ae^{-m'} = \frac{AS_0}{m} \tag{14}$$

3. Reference:

1. Bhattacharya and Waymire - Random Walk, Brownian Motion, and Martingales.