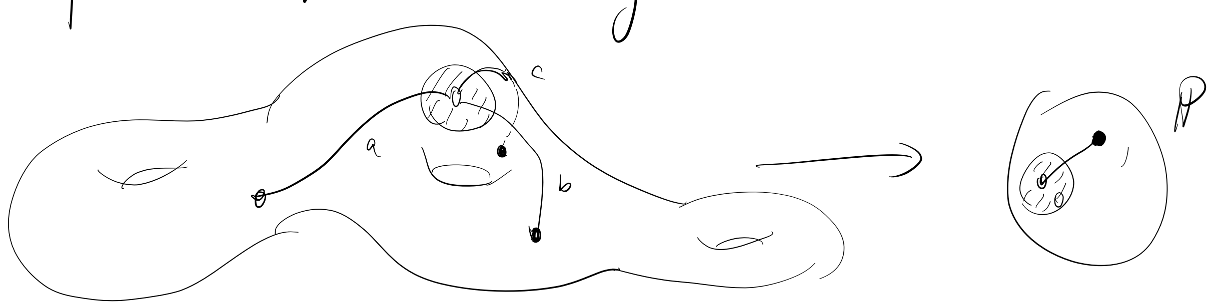
Dessin d'Enfant — Notes

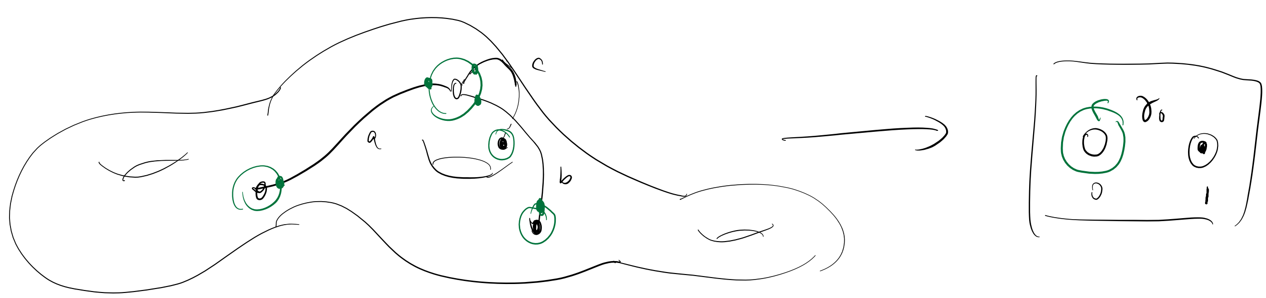
Consider a branching map f from an RS : X to $\mathbb{P}^1$ with finite degree d. Suppose there're only ramification values in the set $\{0, 1, \infty\}$, and consider the premage of $[0,1]$. We can produce a graph : the vertex set $\Delta = f^{-1}\{0,1\}$, where we dye $f^{-1}(0)$ with white color and dye $f^{-1}(1)$ with black, and the edge set $= f^{-1}(0,1)$. It has property that each white vertex is connected with only some black vertexes and vice versa.

The map restricted on $X - \Delta$ is an covering map to its image $\mathbb{P} - \{0, 1, \infty\}$, whose fundamental group is F_2, i.e. a free group of order 2. Then $\pi_1(X - \Delta)$ is induced as a subgroup of F_2 with index d. Noticing that there's an 1-1 correspondence between subgroups with index d and transitive representation $F_2 \to S_d$ up to conjugacy equivalence, f can be represented by a conjugacy class of $\mathrm{Hom}(F_2, S_d)$, or equivalently, assignments to each of $\{0,1\}$ an element of S_d, up to a common conjugation in S_d. Geometrically it arises from the way the local sheets suboardinated to a ramification value winding around.



It's obvious to see that the graph of the map has exactly g edges and we label them by $\{1, \dots, d\}$.

Let us revisit the previous process – called monodromy – explicitly. $\pi_1(X - \Delta)$ is now viewed as a subgroup of F_2 with index d. The way F_2 acting on $\{1, \dots, d\}$ can be expressed as follow: we only need to describe how γ_0 acts. Let radius of γ_0 small enough that it is in the local chart around 0. Suppose b is a white vertex and $a_1 \dots a_k$ are black vertexes closed to b, and the edges connecting them is denoted by $e_1 \dots e_k$. There're unique points on each e_i crossing with $f^{-1}\gamma_0$ and is denoted by p_i. By lifting γ_0 from initial point p_i we reach another p_j, and we map e_i to e_j, and repeating the process we obtain a cycling of $\{e_1 \dots e_k\}$. Do the same to all white points we obtain a permutation of $\{1, \dots, d\}$.



So we attach the permutation structure to the graph and obtain the definition of "dessin d'enfant":

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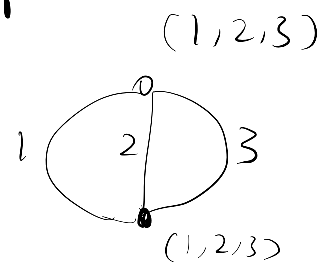
Def: A "dessin d'enfant" is a connected graph with:

1. Each vertex is assigned one of 2 (black & white) colors and two ends of every edge are colored differently.
2. Each vertex is attached by a cyclic permutation of the edges meeting it.

We've seen that such a map f can lead to a dessin d'enfest. Conversely, from a dessin d'enfant we can revert to the monodromy on F_2 and then by the Riemann's existing theorem, we can create a branch covering.

And also, we could get the cellular decomposition of an RS from a dessin: vertexes being the 0-skelecton, the graph being 1-skelecton, and components of $f^{-1}([0,1]^c)$, attached on the 1-skelecton, forming the 2-cell structure.

Examples:



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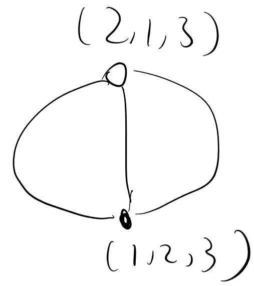
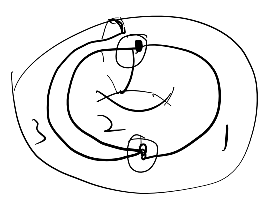
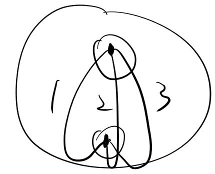


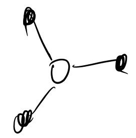
$g_x = 1$



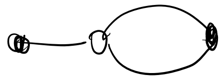
$g_x = 0$



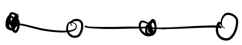
Since every compact RS is an algebraic manifold, i.e. biholomorphic with a projective variety, for certain dessins we could figure out the algebraic expression for f. Here are some examples:



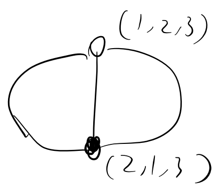
$X = \mathbb{P}^1, \quad f(x) = x^3$



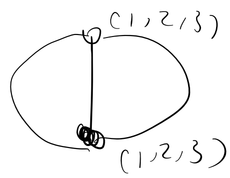
$X = \mathbb{P}^1, \quad f(x) = \dfrac{4(x-1)^3}{27x}$



$X = \mathbb{P}^1, \quad f(x) = \dfrac{x^3+3x^2}{4}$



$X = \mathbb{P}^1, \quad f(x) = \dfrac{x^3}{x^3-1}$



$X: y^2 - x^3 - 1 = 0, \quad f(x,y) = \frac{1}{2}(1+y)$.

We now enter the realm of arithmetic algebraic geometry with the pleasant observation: Any dessin arises from a finite covering of $\mathbb{P}^1$ that can be defined over the algebraic number field $\overline{\mathbb{Q}}$. It's a consequence of Weil's descent theory (?).

We've just seen that dessins corresponds to certain algebraic varieties and coverings of $\mathbb{P}^1$ defined over $\overline{\mathbb{Q}}$, but we don't know what alg curves arises in this may. Belyi's theorem show's that every algebraic curve over $\bar{\mathbb{Q}}$ can be represented as a covering of $\mathbb{P}$ ramified over at most 3 points, i.e. it can be arised from a dassin.